ANNALES DE L'I. H. P., SECTION A

RICHARD KERNER

Approximate mass formula for mesons : a model with quarks interacting by means of the Yang-Mills field

Annales de l'I. H. P., section A, tome 23, nº 4 (1975), p. 379-388
<http://www.numdam.org/item?id=AIHPA_1975__23_4_379_0>

Ann. Inst. Henri Poincaré,
Vol. XXIII, n° 4, 1975, p. 379-388.

Section A :
Physique théorique.

Approximate mass formula for mesons: a model with quarks interacting by means of the Yang-Mills field

by

Richard KERNER
Département de Mécanique, Laboratoire de Mécanique relativiste,
Université de Paris VI, Tour 56-66, 4, place Jussieu, Paris-5^e, France

1. INTRODUCTION. NOTATIONS AND PRELIMINARIES

Since the introduction of quarks into elementary particle physics by Gell-Mann, Ne'eman, Zweig, Okubo and others, a lot of models had been tried with more of less success. Most of these models do not deal explicitly with any precise form of interaction between the quarks constituting an elementary particle: the essence of the theory lies in the algebraic symmetry properties of these interactions, which lead in elegant and simple way to the mass formulæ, as well as to the algebraic relations between the cross-sections of the different scattering processes.

In the model proposed below we try to describe a system composed of two quarks interacting by means of the Yang-Mills field. In order to be honest we have to say already that we don't yet know how to handle in a similar way a system composed of three quarks, so that the resulting mass formulæ will apply to the mesons only. The plausibility of using the Yang-Mills fields for describing an intermediate boson has been discussed in many papers, and it does not seem worthwhile to justify it here once more.

The objection one could make that the Yang-Mills fields are in principle massless and therefore do describe a long-range force is serious, but it had been also shown that one can introduce a mass, term, and then reestablish thus broken symmetry (Higgs-Kibble, Veltman, t'Hooft), at least in some order of approximation.

Our example, though on the quantum mechanical level only, follows a similar line of thought. In the Hamiltonian describing our system we will first add an arbitrary mass-splitting term, therefore breaking the gauge symmetry. Then we perform a generalized Foldy-Wouthuysen transformation in order to diagonalize the Hamiltonian in the first few orders of approximation. The mass splitting is still present in the final result. Its general form is the same as in the Gell-Mann-Okubo mass formula for SU(3), but the coefficients can be in principle deduced by feeding in some particular form of the Yang-Mills field potential. For the sake of simplicity, our Yang-Mills field will be generated by the groups SU(2) or SU(3).

Our notations will be the following:

The Minkowskian space-time: M_4, $x \in M_4$: $x = x_\mu$, μ, $\nu = 0, 1, 2, 3$, or $x = (x^0, \vec{x}) = (x^0, x^i)$, with $i, j, \ldots = 1, 2, 3$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \quad , \quad p_k = -i\frac{\partial}{\partial x^k} = -i\partial_k; \qquad \hbar = c = 1 \tag{1}$$

The gauge group, supposed to be compact and semi-simple: G, dim G = N; $a, b, \ldots = 1, 2, \ldots$ N.

$C^a_{bc} = -C^a_{cb}$ are the structure constants of the group G.

The basis of the Lie algebra of G: Q_a, verifying:

$$[Q_a, Q_b] = C^c_{ab} Q_c \tag{2}$$

The Yang-Mills field potential will be denoted by:

$$A^a_\mu = (A^a_0, A^a_i) = (A^a_0, \vec{A}^a) \tag{3}$$

The Yang-Mills field tensor has the form

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a_{bc} A^b_\mu A^c_\nu \tag{4}$$

and the field equations are:

$$\partial^\mu F^a_{\mu\nu} + C^a_{bc} A^{b\mu} F^c_{\mu\nu} = J^a_\nu \tag{5}$$

and

$$\partial^\mu \check{F}^a_{\mu\nu} + C^a_{bc} A^{b\mu} \check{F}^c_{\mu\nu} = 0 \tag{6}$$

where

$$\check{F}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\varkappa\lambda} F^a_{\varkappa\lambda} \tag{7}$$

and J^a_ν is the conserved external current, null in the vacuum.

Dirac's equation reads as follows:

$$H\psi = E\psi \tag{8}$$

with:

$$\mathrm{H} = \beta m + \vec{\alpha}(\vec{p} - e\vec{\mathrm{A}}) + e\Phi \qquad (\Phi = \mathrm{A}_0) \tag{9}$$

and the matrices β, α_i verify

$$\beta^2 = \mathrm{I}d, \qquad \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}\mathrm{I}d, \qquad \alpha_i\alpha_j - \alpha_j\alpha_i = 2i\epsilon_{ijk}\Sigma_k, \tag{10}$$

Σ_k meaning the matrix $\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}$, σ_k being the usual 2×2 Pauli matrices; in what follows, we shall use the notation $\vec{\sigma}$ instead of $\vec{\Sigma}$ where there is no risk of confusion; $\mathrm{I}d$ stays for the 4×4 identity matrix.

The equation (9) is naturally generalized for the case of the Yang-Mills field interaction. It becomes:

$$\mathrm{H}\psi = \mathrm{E}\psi, \qquad \mathrm{H} = \beta m + \vec{\alpha}(\vec{p} - \mathrm{Q}_a\vec{\mathrm{A}}^a) + \mathrm{Q}_a\Phi^a \tag{11}$$

The whole difference is that now the generalized charges Q_a do not commute (2); we suppose, however, that they commute with β and α_i. In other words, our wave function ψ belongs now to a product space,

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_\mathrm{N}\end{pmatrix} \tag{12}$$

where every component is a 4-spinor. The matrices β, α_i, Q_a are therefore in a product representation and are composed by 4×4 blocks; the dimension is 4N:

$$\beta \to \underbrace{\begin{pmatrix}\beta & & \\ & \beta & \\ & & \ddots \ \beta\end{pmatrix}}_{\leftarrow \mathrm{N}\ \text{blocks} \rightarrow} \qquad \vec{\alpha} \to \underbrace{\begin{pmatrix}\vec{\alpha} & & \\ & \vec{\alpha} & \\ & & \ddots \ \vec{\alpha}\end{pmatrix}}_{\leftarrow \mathrm{N}\ \text{blocks} \rightarrow} \tag{13}$$

Q_a are the matrices τ_i of the SU(2) group or the λ_a of the SU(3), with each unity replaced by a 4×4 identity matrix.

Our basic equations (8), (11) being now defined, we can proceed to the next problem, which is to diagonalize the Hamiltonian.

2. THE FOLDY-WOUTHUYSEN TRANSFORMATION

The diagonalization of the Hamiltonian (11) can not be performed exactly at once, but we can diagonalize it in any order of $1/m$ by means of the

Foldy-Wouthuysen procedure. The unitary transformation needed is of the form:

$$\mathrm{H} \to \mathrm{H}' = e^{i\mathrm{S}}\left(\mathrm{H} - i\frac{\partial}{\partial t}\right)e^{-i\mathrm{S}} \tag{14}$$

where S is the some unknown hermitian matrix.

First we try to diagonalize H in the spin space, *i. e.* to get rid of the expression containing the non-diagonal matrices $\vec{\alpha}$. Let us call

$$\begin{aligned} \vec{\alpha}(\vec{p} - \mathrm{Q}_a\vec{\mathrm{A}}^a) &= \vartheta \qquad \text{(the odd part)} \\ \mathrm{Q}_a\Phi^a &= \varepsilon \qquad \text{(the even part)} \end{aligned} \tag{15}$$

so that

$$\mathrm{H} = \beta m + \vartheta + \varepsilon$$

Supposing that the matrix S if of order $1/m$, the expression (14) can be developed as follows:

$$\begin{aligned} \mathrm{H}' = \mathrm{H} &+ i[\mathrm{S}, \mathrm{H}] - \frac{1}{2}[\mathrm{S}, [\mathrm{S}, \mathrm{H}]] - \frac{i}{6}[\mathrm{S}, [\mathrm{S}, [\mathrm{S}, \mathrm{H}]]] \\ &+ \frac{1}{24}[\mathrm{S}, [\mathrm{S}, [\mathrm{S}, [\mathrm{S}, \beta m]]]] - \dot{\mathrm{S}} - \frac{i}{2}[\mathrm{S}, \dot{\mathrm{S}}] + \frac{1}{6}[\mathrm{S}, [\mathrm{S}, \dot{\mathrm{S}}]] + \dots \end{aligned} \tag{16}$$

The only 0-order term in $1/m$ will be:

$$i[\mathrm{S}, \beta m] \tag{17}$$

and we want it to be equal to the 0-order odd part:

$$i[\mathrm{S}, \beta m] = -\vartheta \tag{18}$$

This is obtained by putting

$$\mathrm{S} = -\frac{i\beta\vartheta}{2m} \tag{19}$$

Now our Hamiltonian will take on the form:

$$\mathrm{H}' = \beta m + \vartheta' + \varepsilon' \tag{20}$$

where the new odd term is of order $1/m$. Performing the next unitary transformation of the same type, with

$$\mathrm{S}' = -\frac{i\beta\vartheta'}{2m} \tag{21}$$

we shall obtain:

$$\mathrm{H}'' = \beta m + \vartheta'' + \varepsilon'' \tag{22}$$

with the odd part ϑ'' of order $1/m^2$, and so on.

The result calculated up to the order $1/m^3$ is as follows:

$$\begin{aligned} \mathrm{H} \approx \beta\left(m + \frac{(\vec{p} - \mathrm{Q}_a\vec{\mathrm{A}}^a)^2}{2m} - \frac{\vec{p}^4}{8m^3}\right) + \mathrm{Q}_a\Phi^a - \frac{\mathrm{Q}_a}{2m}\beta\vec{\sigma}.\vec{\mathrm{B}}^a \\ - \frac{i\mathrm{Q}_a}{8m^2}\vec{\sigma}.(\vec{\nabla} \times \vec{\mathrm{E}}^a) - \frac{\mathrm{Q}_a}{4m^2}\vec{\sigma}.(\vec{\mathrm{E}}^a \times \vec{p}) - \frac{\mathrm{Q}_a}{8m^2}(\vec{\nabla}.\vec{\mathrm{E}}^a) \end{aligned} \tag{23}$$

where by definition:

$$\text{(24)}\qquad \begin{gathered} B_i^a = \frac{1}{2}\epsilon_{ijk}F_{jk}^a, \qquad E_i^a = F_{oi}^a, \\ (\vec{\nabla} \times \vec{E}^a)_i = \epsilon_{ijk}(\partial_j E_k^a + C_{bc}^a A_j^b E_k^c) \\ (\vec{\nabla} . \vec{E}^a) = \partial^i \vec{E}_i^a + C_{bc}^a A^{bi} E_i^c \end{gathered}$$

However, our Hamiltonian still contains the block-non-diagonal matrices Q_a, and we have to perform a similar transformation in order to diagonalize it completely. The case when $G = SU(2)$ and Q_a are the three τ_i is resolved very easily. We add to our Hamiltonian the term $M\tau_3$. Then, because of the relations

$$\text{(25)}\qquad \{\tau_i, \tau_j\}_+ = 2\delta_{ij}I$$

we can perform the same kind of the transformation, the matrix S being now equal to

$$\text{(26)}\qquad S = -\frac{i\tau_3\vartheta}{2M}$$

where ϑ means now the sum of all terms of H containing the non-diagonal matrices τ_1 and τ_2. In order to make our approximation valid, we have to assume M of the same order of magnitude that m. The result, up to the order $1/M^3$ (or $1/Mm^2$, $1/mM^2$, etc.) is as follows:

$$\text{(27)}\qquad \begin{aligned} H \approx {} & \tau_3\left[M + \frac{\vartheta^2}{2M}\right] + \beta\left(m + \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3} - \tau_3\frac{\vec{A}^3.\vec{p}}{m}\right) + \tau^2\left[\frac{\vec{A}^a\vec{A}_a}{2m} + \frac{\Phi^a\Phi_a}{2M}\right] \\ & - \tau_3\frac{\beta}{2m}\vec{\sigma}.\vec{B}^3 - \tau_3^2\frac{\Phi_3^2}{2M} - \tau_3\frac{\vec{\nabla}.\vec{E}^3}{8m^2} - \tau_3\frac{\vec{\sigma}}{4m^2}\vec{E}^3 \times \vec{p} - \frac{i\tau_3}{8m^2}\vec{\sigma}.(\vec{\nabla}.\vec{E}^3) \end{aligned}$$

and after developping ϑ^2 explicitly becomes

$$\text{(28)}\qquad H \approx (27) + \frac{\tau^2}{4m^2M}\{\epsilon_{ijk}B_i^1B_j^2\sigma_k + i\sigma_j[A_i^1(p_iB_j^2) - A_i^2(p_iB_j^1)]\}$$

The last formula becomes even more interesting when we feed in some simple static solution of the Yang-Mills field equations. The Coulomb-like potential gives:

$$\text{(29)}\qquad H = \tau_3 M + \beta\left(m + \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^2}\right) + \tau^2\frac{e_ae^a}{2Mr^2} - \tau_3^2\frac{e_ae^a}{2Mr^2} - \frac{e^3\tau_3}{4m^2}\frac{\vec{\sigma}.\vec{L}}{r^3}$$

with $\Phi^a = e^a/r$, $\vec{B}^a = 0$.

The solution corresponding to the generalized magnetic monopole

$$\text{(30)}\qquad \vec{B}^a = g^a\frac{\vec{x}}{r^3}, \quad \Phi^a = 0$$

gives:

$$(31)\quad \mathrm{H} \approx \tau_3 \mathrm{M} + \beta\left(m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^2}\right) + \frac{\tau^2}{2m}\vec{\mathrm{A}}^a\vec{\mathrm{A}}_a + \frac{\tau_3\beta}{m}\left[\frac{g^3\vec{\sigma}.\vec{r}}{2r^3} + \vec{\mathrm{A}}^3.\vec{p}\right]$$

and the magnetic spin-monopole (see *e. g.* [*4*], [*5*]), with

$$(32)\qquad \mathrm{A}_i^a = \epsilon_{aik}\frac{x^k}{r^2}$$

yields the formula:

$$(31\,a)\quad \mathrm{H} \approx \mathrm{M}\tau_3 + \beta\left(m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3}\right) + \frac{\tau^2}{2\mathrm{M}}\left[\vec{\mathrm{A}}^a\vec{\mathrm{A}}_a + \frac{1}{2m^2}\left(\frac{\sigma_3}{r^4} - \frac{3n_3}{r^4}\right)\right] + \tau_3\left[\frac{\beta}{2m}\vec{\sigma}.\vec{n}\frac{n_3}{r^4} + \frac{\beta}{m}\frac{(\vec{r}\times\vec{p})_3}{r^2}\right]$$

Here the mass splitting depends on the direction; this proves that the spin-monopole potential is unphysical.

Besides $\mathrm{M}\tau_3$ added at the beginning of the diagonalization procedure, the mass splitting persists with respect to the eigenvalues of the operator τ^2, *i. e.* $\mathrm{I}(\mathrm{I}+1)$ and the eigenvalue of the operator τ_3, *i. e.* I_z. However, we are still far from the Gell-Mann-Okubo formula, which, as we know, is quite well verified. In order to obtain a formula of this type, we have to generalize our procedure for the SU(3) group.

3. DIAGONALIZATION OF THE SU(3) HAMILTONIAN

Let us start with the Hamiltonian

$$(33)\quad \mathrm{H} \approx \beta\left(m + \frac{(\vec{p} - \mathrm{Q}_a\vec{\mathrm{A}}^a)^2}{2m} - \frac{\vec{p}^4}{8m^3}\right) - \frac{\mathrm{Q}_a}{2m}\beta\vec{\sigma}.\vec{\mathrm{B}}^a - \frac{i\mathrm{Q}_a}{8m^2}\vec{\sigma}.(\vec{\nabla}\times\vec{\mathrm{E}}^a) - \frac{\mathrm{Q}_a}{4m^2}\vec{\sigma}.(\vec{\mathrm{E}}^a\times\vec{p}) - \frac{\mathrm{Q}_a}{4m^2}(\vec{\nabla}.\vec{\mathrm{E}}^a) + \mathrm{Q}_a\Phi^a$$

in which the Q_a's are the generators of the SU(3) Lie algebra. In the simplest 3×3 representation we have $(a, b, \ldots\ 1, 2, \ldots\ 8)$

$$(34)\qquad \begin{aligned} [\mathrm{Q}_a, \mathrm{Q}_b] &= 2if_{abc}\mathrm{Q}_c \\ \{\mathrm{Q}_a, \mathrm{Q}_b\}_+ &= 2d_{abc}\mathrm{Q}_c + \frac{4}{3}\delta_{ab}\mathrm{I}d \end{aligned}$$

The idea of the approximate diagonalizing procedure is essentially the same. We shall first illustrate it taking the roughest approximation, in which

$$(35)\qquad \mathrm{H} \approx \beta\left(m + \frac{(\vec{p} - \mathrm{Q}_a\vec{\mathrm{A}}^a)^2}{2m}\right) + \mathrm{Q}_a\Phi^a - \frac{\mathrm{Q}_a}{2m}\beta\vec{\sigma}.\vec{\mathrm{B}}^a$$

i. e. conserving the terms of order $1/m$ only. After expanding the term $(\vec{p} - Q_a\vec{A}^a)^2$ we obtain:

$$(36) \quad \begin{aligned}(\vec{p} - Q_a\vec{A}^a)^2 &= \vec{p}^2 - 2Q_a\vec{A}^a.\vec{p} + Q_aQ_b(\vec{A}^a\vec{A}^b) \\ &= \vec{p}^2 - 2Q_a\vec{A}^a.\vec{p} + d_{abc}Q_c\vec{A}^a\vec{A}^b + \frac{2}{3}\vec{A}^a.\vec{A}_a\end{aligned}$$

and the Hamiltonian becomes:

$$(37) \quad \begin{aligned}H = \beta\left(m + \frac{\vec{p}^2}{2m}\right) + Q_a\Phi^a + \frac{\beta}{3m}\vec{A}^a.\vec{A}_a - \frac{Q_a\beta}{m}\vec{A}^a.\vec{p} \\ + d_{abc}\frac{\beta Q_c}{2m}\vec{A}^a.\vec{A}^b - \frac{\beta}{3m}\vec{A}^a.\vec{A}_a + \vartheta + \varepsilon\end{aligned}$$

Now, here we have two diagonal operators Q_3 and Q_8, and six non-diagonal ones, $Q_1, Q_2, \ldots, Q_7$. Let us introduce the indices A, B, ..., taking on the values 1, 2, 4, 5, 6, 7. Then we can write our Hamiltonian symbolically as

$$(38) \quad H = \beta\left(m + \frac{\vec{p}^2}{2m}\right) + \frac{\beta}{3m}\vec{A}^a.\vec{A}_a + \vartheta + \varepsilon$$

Here $\vartheta = \vartheta^A Q_A$ is the sum of all the terms containing the non-diagona matrices Q_A, and $\varepsilon = \varepsilon^3 Q_3 + \varepsilon^8 Q_8$ is the diagonal term. Now we add to our Hamiltonian the symmetry-breaking terms:

$$(39) \quad H \rightarrow H + M_3Q_3 + M_8Q_8$$

As above, M_3 and M_8 should be great enough to ensure good convergence of the approximation series.

Next we have to find a matrix S which shall verify:

$$(40) \quad i[S, H] = -\vartheta + O\left(\frac{1}{M}\right)$$

in order to get rid of the non-diagonal part up to the order $1/m$. It is obvious that in (40) the lowest-order term will be

$$(41) \quad i[S, M_3Q_3 + M_8Q_8]$$

other terms being at least of order $1/m$. The matrix S is therefore defined up to an arbitrary term $S_3Q_3 + S_8Q_8$ commuting with $M_3Q_3 + M_8Q_8$. We put this term equal to zero, of course, and therefore can write $S = S^A Q_A$. So we have

$$(42) \quad i[S^AQ_A, M_3Q_3 + M_8Q_8] = -\vartheta^B Q_B$$

Developing this commutator we get explicitly

$$(43) \quad S^A[2M_3f_{A3C} + 2M_8f_{A8C}] = \vartheta^c Q_c$$

Defining

$$(44) \quad B_{CA} = 2(M_3f_{A3C} + M_8f_{A8C})$$

we obtain

$$B_{CA}S_A = \vartheta_C \tag{45}$$

and therefore

$$S_A = B_{AC}^{-1}\vartheta_C \tag{45 a}$$

if $\det B \neq 0$.

The 6×6 matrix B_{AC} has the following form:

$$B_{AC} = \left(\begin{array}{cc|cc|cc} 0 & 2M_3 & & & & \\ -2M_3 & 0 & & & & \\ \hline & & 0 & M_3+\sqrt{3}M_8 & & \\ & & -(M_3+\sqrt{3}M_8) & 0 & & \\ \hline & & & & 0 & \sqrt{3}M_8-M_3 \\ & & & & M_3-\sqrt{3}M_8 & 0 \end{array}\right) \tag{46}$$

and its determinant is equal to

$$\det B = 4M_3^2(M_3^2 - 3M_8^2)^2 \tag{47}$$

This determinant is different from zero if and only if $M_3 \neq 0$ *and* $M_3^2 \neq 3M_8^2$. The last condition could be postponed if we assume that the field A_i^a, B_i^a has no components 6 and 7, but this is equivalent to the reducing our symmetry group to $SU(2) \times U(1)$ which is not interesting. Moreover, one should keep in mind that the difference $M_3 - \sqrt{3}M_8$ cannot be too small, because if it was, the development in orders of $(M_3 - \sqrt{3}M_8)^{-1}$ which occurs afterwards is not valid.

Assuming that these conditions are verified, we can proceed farther. We have to calculate the expression

$$i\left[S, \frac{\vec{p}^2}{2m}\right] + i\left[S, \frac{1}{m}\vec{A}^a\vec{A}_a\right] + i[S, \vartheta] + i[S, \varepsilon] \tag{48}$$

(supposing for simplicity that $\frac{\partial S}{\partial t} = 0$). The expression (48) will be of order $1/m^2$; its non-diagonal part can then be removed by the next similar step, after which the non-diagonal parts will be of order $\left(\frac{1}{m}\right)^3$ which we shall drop out. Therefore, we can be interested only in the diagonal components of the (48). It is easy to see that neither the first, nor the second term do not yield such components, so we are left only with two terms $i[S, \vartheta] + i[S, \varepsilon]$, which give explicitly:

$$i[S, \vartheta] = -2B_{AC}^{-1}\vartheta_C\vartheta_B(f_{AB3}Q_3 + f_{AB8}Q_8) + \text{non-diagonal terms} \tag{49}$$

and

$$i[S, \varepsilon] = iS_A\varepsilon_3[Q_A, Q_3] + iS_A\varepsilon_8[Q_A, Q_8] = \text{all non-diagonal} \tag{49 a}$$

The final calculus depends now on the kind of interaction we shall assume. In the case when only scalar part of the potential is different from zero, it is easy to see that the final formula will be of the form

$$\text{(50)} \qquad H = H_0 + aQ_3 + bQ_8$$

i. e. there will be only linear mass-splitting present.

In the case when there is only vector potential, and the only non-vanishing part of scalar potential is $Q_3M_3 + Q_8M_8$, the approximated diagonalized Hamiltonian will take on the form:

$$\text{(51)} \qquad \begin{aligned} H = \beta\left(m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3}\right) + \frac{3\beta}{m}\vec{A}^a\vec{A}_a\left(Q_3^2 - \frac{Q_8^2}{4}\right) \\ + \frac{\beta}{2m}\vec{A}^a.\vec{A}^b(d_{ab3}Q_3 + d_{ab8}Q_8) - \frac{\beta}{m}[Q_3\vec{A}^3 + Q_8\vec{A}^8].\vec{p} \\ - \frac{\beta\vec{\sigma}}{2m}[Q_3\vec{B}^3 + Q_8\vec{B}^8] - \frac{2}{m^2}B_{AC}^{-1}\vartheta_C\vartheta_B(f_{AB3}Q_3 + f_{AB8}Q_8) \end{aligned}$$

which can be written symbolically as

$$\text{(52)} \qquad H = H_0 + aQ_3 + bQ_8 + c\left(Q_3^2 - \frac{Q_8^2}{4}\right)$$

This is similar to the Gell-Mann-Okubo formula, with a, b, c being now some complicated functions of A^a, and $\vec{\sigma}$.

DISCUSSION

In this very simple-minded approach we have chosen the external field approximation for the interaction between quarks. Of course, this choice is quite arbitrary: all the dynamics is contained in the quarks themselves, whereas the field of interaction has no dynamics of its own, being purely kinematic in nature. But this kind of arbitrary separation of dynamical and kinematical features is all the way present in any theory of interactions, and the actual difference between the dynamical and kinematical quantities depends on the physical aspects of the system described and mostly on the intuition.

Moreover, the diagonalization technique used here is based essentially on the non-relativistic approximation, which we are not at all sure is valid for the description of the interaction between the quarks. But then, the fact that the meson masses obey the sum rules deduced from the linear representations of SU(3) group are no less mysterious, because should the elementary particles be composed of quarks, the underlying dynamics would have to be highly non-linear. It seems interesting to obtain here not only the right qualitative mass formula (52), of the same kind that the Gell-Mann-Okubo one, but also the spin dependence for the coefficients.

This kind of formula would be impossible to obtain for the totally SU(3)-symmetric Hamiltonian: it is obvious that without the symmetry-breaking term $M_3Q_3 + M_8Q_8$ any unitary transformation performed on H_0 would yield no mass-splitting at all. Therefore, what we have demonstrated, is the following: if for any reason the Hamiltonian describing two quarks interacting by means of the Yang-Mills field contains a linear symmetry-breaking term (this initial assymetry being due *e. g.* to the mass-difference between non-interacting quarks), then in the approximation given by the Foldy-Wuthoysen diagonalization procedure we obtain the formula (52). The approximation is the better, the heavier are the assumed « naked » masses of the quarks and their differences. Another possible interpretation is to assume that the term $M_3Q_3 + M_8Q_8$ describes in the linear approximation the mass of the intermediate boson.

We think that the same kind of calculus, however much more complicated, can be performed for the baryons constitued of three quarks, in which case we should derive the gauge-invariant Hamiltonian from the Faddeev equations, or the Bethe-Salpeter equation.

ACKNOWLEDGMENT

The author expresses many thanks to Prof. M. Flato for enlightening discussions and remarks.

REFERENCES

[1] C. N. Yang and R. L. Mills, *Phys. Rev.*, t. **96**, 1954, p. 191.
[2] Bjorken and Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, New York, 1964.
[3] R. Kerner, *Ann. Inst. H. Poincaré*, vol. IX, n° 2, 1968.
[4] R. Kerner, *Comptes Rendus Acad. Sci. Paris*, t. **270**, 1970, p. 1467.
[5] A. Joseph, *Phys. Rev.*, t. **5**, 1972, p. 313.
[6] P. A. M. Dirac, *Phys. Rev.*, t. **74**, 1948, p. 817.
[7] B. Feld, *Models of Elementary Particles*, Blaisdell Publishing Company, 1969.
[8] M. Gell-Mann, *Phys. Letters*, t. **8**, 1964, p. 214.
[9] G. Morpurgo, *Physics*, t. **2**, 1965, p. 95.
[10] O. W. Greenberg, *Phys. Rev. Letters*, t. **13**, 1964, p. 598.
[11] Y. Ne'eman, *Algebraic Theory of Strong Interactions*, Benjamin, New York, 1968.