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Conformally covariant field equations

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ABSTRACT. — Conformally covariant field equations are defined in an analogous manner to relativistically invariant field equations with non-zero mass. To obtain a meaningfull definition we take the fields as Banach valued functions on the conformal closure of the Minkowski space. A standard conformal operator is a conformally covariant operator which is also a non-decomposable relativistically invariant operator in the sense of Naimark. The first and second order standard conformal operators and the corresponding differential equations are studied in detail.

1. INTRODUCTION

The present work is devoted to conformally covariant field equations, i. e. to an approach in field theory in which the conformal group plays a role of a symmetry group of space-time. (For more details as well as for other approaches of the conformal symmetry in particle physics see *[1]* *[3]* *[6]* *[10]* *[14]* *[16]*.)

There are at least four sources of motivations for such a study. Firstly, the conformal group is the largest group which (beyond singularities) preserves the light-cone and thus may be treated as a possible generaliza-

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tion of the Poincaré group—the symmetry group of the special relativity. Secondly, there have been attempts to use a certain homogeneous space (conformal closure of the Minkowski space) of the conformal group as a model of the Universe in cosmology; in particular, this model has been used for calculating the redshift and microwave background [*15*]. The third motivation is originated from an interesting indication that « any theory invariant simultaneously with respect to the special linear and conformal groups will be invariant also with respect to the general covariance group of the general relativity » [*2*]. Finally, from the experimentally observed phenomena of a « scaling law » in a deep inelastic electron-proton scattering the idea of an approximate scale invariant field theory arose and, since scale invariance implies conformal invariance for a class of Lagrangian field theories, also the idea of constructing conformally covariant field theories (the effect of symmetry breaking is considered afterwards).

We shall try to deal with conformally covariant field equations in an analogous manner to relativistically invariant field equations. Let us therefore first study the latter case and then try to find a fruitful generalization to the former.

Let M be the Minkowski space and let E be a complex Banach space. For the sake of simplicity we shall assume that E is finite-dimensional. Let $g \mapsto R(g)$ be a representation of $SL(2, \mathbb{C})$, the universal covering group of the Lorentz group $SO(3, 1)$, in the vector space E. Let $C^\infty(M, E)$ be the linear space of all C^∞-mappings $\psi : M \to E$. We construct the (non-unitarily) induced representation T of $\bar{P}$ in $C^\infty(M, E)$, the universal covering group of the Poincaré group P, by the rule

$$(T(g)\psi)(x) := R(g_x^{-1} g\, g_{g^{-1}x})\psi(g^{-1}x),$$

where $g \in \bar{P}$, $x \in M = P/SO(3, 1)$, $\psi \in C^\infty(M, E)$ and g_x is a translation, $g_x 0 = x$. This representation is continuous if the space $C^\infty(M, E)$ is equipped with the Schwartz topology. The group $\bar{P}$ acts through the natural homomorphism $\bar{P} \to P$ on M. Let us consider a first order field equation

$$L^\nu \partial_\nu \psi + \varkappa \psi = 0, \qquad \text{(sum over } \nu),$$

where $L^\nu : E \to E$ $(\nu = 0, 1, 2, 3)$ are linear operators, $\psi \in C^\infty(M, E)$, ∂_ν is a derivative with respect to the co-ordinate x^ν, and $\varkappa \neq 0$ is a constant which is connected with the mass [*7*]. As usual, we define the field equation to be Poincaré invariant if $T(g)L^\nu\partial_\nu = L^\nu\partial_\nu T(g)$ for each $g \in \bar{P}$. It is well-known that this condition is equivalent with the condition

$$[m_{\nu\mu}, L_\lambda] = g_{\mu\lambda}L_\nu - g_{\nu\lambda}L_\mu,$$

where the operators $m_{\nu\mu}$ represent the Lie algebra of $SL(2, \mathbb{C})$ in E, corresponding to the representation R of $SL(2, \mathbb{C})$.

Next we would like to define a conformally invariant field equation in a similar way. The first obstacle is the simple fact that the conformal group SO(4, 2) does not act in the Minkowski space. If g_c is a special conformal transformation in the direction $c = (c_0, c_1, c_2, c_3)$, then

$$g_c x = \frac{x - x^2 c}{1 - 2c \cdot x + c^2 x^2}, \qquad x \in \mathrm{M},$$

but this is not defined when $1 - 2c \cdot x + c^2 x^2 = 0$. In order to overcome this difficulty, one must use the method developed in [*5*] or to work with the conformal algebra as in [*11*] and [*12*] or to construct a field theory in the conformal closure $\bar{\mathrm{M}}$ of the Minkowski space [*8*]. In this paper we shall study the last possibility. Roughly speaking, $\bar{\mathrm{M}}$ is the Minkowski space + a light-cone at the infinity (for a precise definition, see section 2). The conformal group acts in a C^∞-way on $\bar{\mathrm{M}}$. This space was introduced by Veblen already in 1933 [*17*]. When going from M to $\bar{\mathrm{M}}$, the time axis is replaced by a circle and the space by a sphere S^3. However, if the radii are large it is realistic to expect that elementary particle physics in $\bar{\mathrm{M}}$ is almost the same as in M because the distances and time intervals are usually very small. Instead of $\bar{\mathrm{M}}$, one can also consider a certain covering $\tilde{\mathrm{M}}$ of $\bar{\mathrm{M}}$ (as is done in [*15*]) on which one can introduce the notion of causality; this change has no effect on the results of the present work.

If $g \mapsto \mathrm{R}(g)$ is now a representation of the universal covering group $\bar{\mathrm{W}}$ of the Weyl group $\mathrm{W} \subset \mathrm{SO}(4, 2)$ in a vector space E, then we can construct the induced representation $g \mapsto \mathrm{T}(g)$ of $\overline{\mathrm{SO}(4, 2)}$ in $\mathrm{C}(\bar{\mathrm{M}}, \mathrm{E})$ in an analogous manner to the case of the Poincaré group. We come now to the second obstacle. Namely, there are no physically interesting differential operators which commute with $\mathrm{T}(g)$ for every $g \in \overline{\mathrm{SO}(4, 2)}$. Therefore we modify the definition of an invariant operator. Let $\mathscr{D}$ be a first-order differential operator in $\mathrm{C}(\bar{\mathrm{M}}, \mathrm{E})$. We say that $\mathscr{D}$ is conformally covariant if $\mathrm{T}(g)\mathscr{D}\mathrm{T}(g^{-1}) = \Omega_g^{-1}\mathscr{D}$ for each $g \in \overline{\mathrm{SO}(4, 2)}$ where Ω_g is a real valued function on $\bar{\mathrm{M}}$ determined by a certain one-dimensional representation $g \mapsto \Omega(g)$ of $\bar{\mathrm{W}}$ (section 3). The physical meaning of this definition is the following: if we take a conformal transformation g, then in the « new co-ordinate system » we have the field $\psi' = \mathrm{T}(g)\psi$ and the mass $\varkappa'(x) = \Omega_g(x)\varkappa(g^{-1}x)$ where ψ is the field and $\varkappa$ is the mass in the « old co-ordinate system ». Note that the mass $\varkappa$ depends on the position x. If g is a dilatation by a factor $\rho > 0$ then $\varkappa' = \rho^{-1}\varkappa$. If ψ is a solution, $\mathscr{D}\psi + \varkappa\psi = 0$, then ψ' is a solution when $\varkappa$ is replaced by $\varkappa'$, $\mathscr{D}\psi' + \varkappa'\psi' = 0$.

In section 3 we consider first-order conformally covariant field equations. The results of section 3 have been derived earlier by Kotecký and Niederle in a formal way [*11*]. In section 4 we study second-order field equations. Under some technical assumptions we find the most general finite-component conformally covariant equation of the type $(\mathrm{A}^{\nu\mu}\bar{\partial}_\nu\bar{\partial}_\mu + \mathrm{B}^\nu\bar{\partial}_\nu + \varkappa^2)\psi = 0$, where $\bar{\partial}_\nu$ is the extension of the partial derivative ∂_ν to $\bar{\mathrm{M}}$. The Klein-Gordon operator $\bar{\partial}^\nu \bar{\partial}_\nu + \varkappa^2$ as been studied by Go and Mayer in the conformal space $\bar{M}$ [9].

In the Appendix the reader can find some simple facts about tensor operators of SL(2, $\mathbb{C}$) which are used in the text.

2. PRELIMINARIES

We denote by G the conformal group SO(4, 2) and by $\mathcal{G}$ the Lie algebra of G. A basis of $\mathcal{G}$ consists of the generators $M_{\nu\mu}(\nu, \mu = 0, 1, 2, 3; M_{\nu\mu} \equiv -M_{\mu\nu})$ of homogeneous Lorentz transformations, of the translation generators P_ν, of the generators K_ν of the so-called special conformal transformations and of the dilatation generator D. The commutation relations are:

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho} \\
[P_\rho, M_{\mu\nu}] &= g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu \\
[P_\nu, P_\mu] &= 0 \\
[D, M_{\mu\nu}] &= 0 \\
[D, P_\nu] &= P_\nu \\
[D, K_\nu] &= -K_\nu \\
[K_\nu, K_\mu] &= 0 \\
[K_\mu, P_\nu] &= 2(g_{\mu\nu}D - M_{\mu\nu}) \\
[K_\rho, M_{\mu\nu}] &= g_{\mu\rho}K_\nu - g_{\nu\rho}K_\mu .
\end{aligned}
$$

Here $g_{\mu\nu} = 0$ when $\nu \neq \mu$ and $-g_{00} = g_{11} = g_{22} = g_{33} = -1$. The elements $M_{\nu\mu}$, D and $K_\nu (\nu, \mu = 0, 1, 2, 3)$ generate an eleven-parameter subgroup W of G, called the Weyl group.

The quotient space $\bar{M} := G/(W \times \mathbb{Z}_2)$ is sometimes called the conformal closure of the Minkowski space M. The group $\mathbb{Z}_2$ consists of the matrices $\mathbb{1}$ and $-\mathbb{1}$. We can give a geometrical construction of $\bar{M}$ as follows [4] [15].

We take first the cone

$$C := \{ (x_0, x_1, \ldots, x_5) \neq 0 \mid x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 + x_5^2 = 0 \} \subset \mathbb{R}^6 .$$

We claim that $\bar{M} = C/\sim$ where « $\sim$ » is the equivalence relation $\underline{x} \sim \underline{y}$ iff $\underline{x} = \lambda \underline{y}$ for some real number λ; $\underline{x}, \underline{y} \in C$. The quotient space $C/\sim$ is dressed with the natural C^∞-structure inherited from $\mathbb{R}^6$. It is clear that $C/\sim$ is isomorphic with $(S^3 \times S^1)/Z_2$ where S^1 is a circle and S^3 is a sphere in $\mathbb{R}^4$; $\mathbb{Z}_2$ identifies the antipodal points.

The group SO(4, 2) acts in a natural way on $\mathbb{R}^6$. This action determines a C^∞-action of SO(4, 2) in $C/\sim$ by the rule

$$g[\underline{x}] := [g\underline{x}] ,$$

where $[\underline{x}]$ is the equivalence class $\{ \lambda\underline{x} \mid \lambda \in \mathbb{R} - \{0\} \}$; $\underline{x} = (x_0, x_1, \ldots, x_5) \in C$.

It is easily seen that G acts transitively on $C/\sim$; in fact the subgroup $SO(4) \times SO(2) \subset G$ already acts transitively. Let $\hat{x} = [(0, 0, 0, 0, 1, 1)]$. Then the isotropy subgroup of $\hat{x}$ is generated by:

a) The Lorentz transformations SO(3, 1) leaving x_4 and x_5 fixed.

b) The dilatations, which are hyperbolic rotations in the (4,5)-plane:

$$x'_4 = x_4 \cosh \lambda + x_5 \sinh \lambda ,$$
$$x'_5 = x_4 \sinh \lambda + x_5 \cosh \lambda ,$$

and $x'_\nu = x_\nu$ for $\nu = 0, 1, 2, 3$.

c) The special conformal transformations

$$x'_\nu = x_\nu + c_\nu(x_4 - x_5) ,$$

$$x'_4 = x_4 + c \cdot x + \frac{1}{2} c^2(x_4 - x_5) ,$$

$$x'_5 = x_5 + c \cdot x + \frac{1}{2} c^2(x_4 - x_5) ,$$

where $c = (c_0, c_1, c_2, c_3) \in \mathbb{R}^4$. For any vector $a = (a_0, a_1, a_2, a_3)$ we define $a^0 = a_0$, $a^k = - a_k$, $k = 1, 2, 3$. We put $a \cdot b := a^\nu b_\nu$ (we take a sum over repeated indices) and $a^2 := a \cdot a$.

d) The reflection $\underset{\sim}{x} \mapsto - \underset{\sim}{x}$.

The isotropy group of $\hat{x}$ is thus isomorphic with $W \times \mathbb{Z}_2$ and therefore $C/\sim \, = G/(W \times \mathbb{Z}_2) = \bar{M}$.

Next we note that the Minkowski space M can be embedded in $\bar{M}$ as a dense open submanifold. We define first the map $\varphi : M \to C$ by

$$\varphi(x_0, x_1, x_2, x_3) := \left(x_0, x_1, x_2, x_3, \frac{1}{2}(1 + x^2), \frac{1}{2}(1 - x^2) \right).$$

The image of M in C is equal to the intersection of C with the plane $x_4 + x_5 = 1$. We then define the mapping $\bar{\varphi} : M \to \bar{M}$ by

$$\bar{\varphi}(x) := [\varphi(x)] .$$

It is clear that $\bar{M} - \bar{\varphi}(M)$ consists of the classes $[(x_0, \ldots, x_5)]$ for which $x_4 + x_5 = 0$. The map $\bar{\varphi} : M \to \bar{\varphi}(M) \subset \bar{M}$ is a C^∞-diffeomorphism. In SO(4, 2) there is a commutative four-parameter subgroup T_4 defined by

$$x'_\nu = x_\nu + t_\nu(x_4 + x_5) ,$$

$$x'_4 = x_4 + t \cdot x + \frac{1}{2} t^2(x_4 + x_5) ,$$

$$x'_5 = x_5 - t \cdot x - \frac{1}{2} t^2(x_4 + x_5) ,$$

where $t = (t_0, t_1, t_2, t_3) \in \mathbb{R}^4$. This action of T_4 on $\mathbb{R}^6$ induces the following action on $\varphi(M)$:

$$x'_\nu = x_\nu + t_\nu ,$$
$$x'_4 = x_4 + t\cdot x + \frac{1}{2}t^2 ,$$
$$x'_5 = x_5 - t\cdot x - \frac{1}{2}t^2 .$$

If we identify M with $\bar{\varphi}(M)$, then the action of T_4 on M is given by $x'_\nu = x_\nu + t_\nu$ so that T_4 is the translation subgroup of G. Because of

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 + x_5^2 = x^2 - (x_4 + x_5)(x_4 - x_5),$$

we can consider $\bar{M} - M$ (where $x_4 + x_5 = 0$) as « a light-cone $x^2 = 0$ at the infinity ».

Let E be a complex Banach space and let ξ be a C^∞-vector field on a C^∞-manifold X. We denote by $C^\infty(X, E)$ the space of C^∞-mappings from X into E. The vector field ξ acts in $C^\infty(X, E)$ in a natural way,

$$(\xi\psi)(x) := (T_x\psi)(\xi(x)), \qquad \psi \in C^\infty(X, E), \qquad x \in X,$$

where $T_x\psi : T_x(X) \to E$ is the tangent mapping; $T_x(X)$ is the tangent space to X at x. If $X = M$ then we can write

$$(\xi\psi)(x) = a^\nu(x)\partial_\nu\psi(x),$$

where $a^\nu \in C^\infty(M, \mathbb{C})$ ($\nu = 0, 1, 2, 3$) and ∂_ν is the partial derivative with respect to x^ν.

Any one-parameter group $t \mapsto g(t)$ acting on X in a C^∞-manner induces a vector field ξ_g on X, i. e. the derivative of the mapping $t \mapsto g(t)x$, $x \in X$, at $t = 0$. In particular, if $X = \bar{M}$ and $t \mapsto g(t) \in G$ is the group of translations in the x_ν-direction, we denote this vector field by $\bar{\partial}_\nu$. It is clear that the restriction of $\bar{\partial}_\nu$ to M is equal to ∂_ν. Because M is dense in $\bar{M}$, any C^∞-vector field on $\bar{M}$ is of the form $a^\nu\bar{\partial}_\nu$ where $a^\nu \in C^\infty(\bar{M}, \mathbb{C})$.

By an n^{th} order differential operator in $C^\infty(\bar{M}, E)$ we mean a linear combination of the operators of the type $A\xi_1\xi_2 \dots \xi_k$ where $k = 1, 2, \dots, n$ and $A(x)$ is a linear operator which is defined on a dense invariant subspace E_0 of E for all $x \in \bar{M}$. The product $\xi_1 \dots \xi_k$ of C^∞-vector fields on $\bar{M}$ is to be interpreted as the composite mapping $\xi_1 \circ \xi_2 \circ \dots \circ \xi_k$ acting on $C^\infty(\bar{M}, E)$. From the previous remarks it is clear that the most general n^{th} order differential operator in $C^\infty(\bar{M}, E)$ is of the form

$$\sum_{k=1}^{n} A_k^{\nu_1\dots\nu_k}\bar{\partial}_{\nu_1}\bar{\partial}_{\nu_2} \dots \bar{\partial}_{\nu_k},$$

$\nu_i = 0, 1, 2, 3$; $i = 1, 2, \dots,$

3. FIRST-ORDER COVARIANT FIELD EQUATIONS

Let a continuous representation $g \mapsto R(g)$ of $\bar{W}$, the universal covering group of W, be given in the Banach space E. For any $x \in \bar{M}$ we fix an element g_x of the universal covering group $\bar{G}$ of G such that

$$g_x\hat{x} = x\,; \qquad \hat{x} := [(0, 0, 0, 0, 1, 1)]\,.$$

The group $\bar{G}$ acts through the natural homomorphism $\bar{G} \to G$ on $\bar{M}$. We construct an induced representation $g \mapsto T(g)$ of $\bar{G}$ in $C(\bar{M}, E)$, the space of all continuous mappings $\psi : \bar{M} \to E$, by putting

$$(T(g)\psi)(x) := R(g_x^{-1} g g_{g^{-1}x})\psi(g^{-1}x)\,.$$

T is a continuous representation of $\bar{G}$ if we define in $C(\bar{M}, E)$ the topology induced by the norm

$$\| \psi \| := \sup_{x \in \bar{M}} \| \psi(x) \|_E\,.$$

Let $g \mapsto \Omega(g)$ be a one-dimensional representation of $\bar{W}$ such that $\Omega(g) = 1$ for Lorentz transformations and special conformal transformations and $\Omega(g(t)) = \exp(-t)$ for the dilatations $g(t) = \exp tD$, $g(t)x = e^t x$ $(x \in M)$. We denote by Ω_g the function $\Omega_g(x) := \Omega(g_x^{-1} g g_{g^{-1}x})$ where $g \in \bar{G}$ and $x \in \bar{M}$.

DEFINITION 3.1. — The first-order differential operator $\mathscr{D} = L^\nu \partial_\nu$ is conformally covariant if

(*i*) there exists a dense subspace $E_0 \subset E$ which is invariant under R and the operators $L^\nu(x)$,

(*ii*) the differential dR of R is defined on E_0 and E_0 is invariant under dR,

(*iii*) $T(g)\mathscr{D}T(g^{-1})\psi = \Omega_g^{-1}\mathscr{D}\psi$ for all $\psi \in C^\infty(\bar{M}, E_0)$ and $g \in \bar{G}$.

REMARK 3.2. — In the case g is a translation $x_\nu \mapsto x_\nu + t_\nu$ equation (*iii*) reads

$$L^\nu(x + t)\partial_\nu\psi(x) = L^\nu(x)\partial_\nu\psi(x) \qquad \text{for} \qquad x \in M\,.$$

Thus L^ν must be constant on M and therefore also constant on $\bar{M}$ because M is dense in $\bar{M}$.

REMARK 3.3. — For Lorentz transformations we have $T(g)\mathscr{D}T(g^{-1}) = \mathscr{D}$ (on $C^\infty(\bar{M}, E_0)$), i. e. $\mathscr{D}$ is Lorentz invariant in the usual sense; see Naimark [*13*].

Let $\varkappa$ be a positive function on $\bar{M}$, which we call the mass. We consider the first-order differential equation

$$\mathscr{D}\psi + \varkappa\psi = 0\,, \tag{1}$$

where $\mathscr{D}$ is a covariant operator. Let ψ be a solution of Eq. (1) and let $\psi' = T(g)\psi$ for some $g \in \bar{G}$. Then

$$0 = \Omega_g T(g)\mathscr{D}T(g^{-1})T(g)\psi + \Omega_g T(g)\varkappa\psi = \mathscr{D}\psi' + \varkappa'\psi'\,,$$

where $\varkappa'(x) = \Omega_g(x)\varkappa(g^{-1}x)$. Thus also ψ' is a solution of the equation (1) but with the mass $\varkappa'$. This is just what is usually meant by saying that Eq. (1) is conformally covariant.

Note that Definition 3.1 is taillored only to deal with equations of type (1) in which $\varkappa$ transforms according to representation $\Omega_g(x)$, i. e. as a mass. In a more general case in which, for example, $\varkappa$ is a matrix (even singular), (*iii*) in Definition 3.1 should be replaced by

$$T(g)\mathscr{D}T(g^{-1})\psi = N_g^{-1}\mathscr{D}\psi,$$

where N_g is a certain matrix function of g and x. It is generally true that higher order differential equations can be written as a system of first order equations. However, because we are not studying the most general first order equation, there are higher order equations which are not contained in the list of the first order equations treated below; thus we study second order equations separately in Section 4.

Next we consider the differential of dT of the representation T of G; $dT(G)$ acts on the dense subspace $C^\infty(M, E_0) \subset C(M, E)$ when dR acts on E_0. As $g \mapsto T(g)$ is a continuous representation of G in the Banach space C(M, E), the differential $dT(G)$ can be exponentiated to give the original representation T of G; this will be used in the proof of Lemma 3.4. Let $v \in \mathscr{G}$ and let $\alpha(t) = \exp(tv)$. Then we have

$$(dT(v)\psi)(x) = \frac{d}{dt}\psi(\alpha(-t)x)\,|_{t=0} + \frac{d}{dt}R\big(g_x^{-1}\alpha(t)g_{\alpha(-t)x}\big)\,|_{t=0}\,\psi(x).$$

In the following we shall often denote by the same symbol an element v of $\mathscr{G}$ and its representative $dT(v)$. We define $dR(M_{\nu\mu}) = m_{\nu\mu}$, $dR(D) = d$ and $dR(K_\nu) = k_\nu$. After a straight-forward calculation we obtain the formulæ

$$\begin{aligned}
M_{\nu\mu} &= x_\nu\partial_\mu - x_\mu\partial_\nu + m_{\nu\mu},\\
P_\nu &= -\partial_\nu,\\
D &= d - x^\nu\partial_\nu,\\
K_\nu &= k_\nu + 2dx_\nu - 2m_{\nu\mu}x^\mu - 2x_\nu x^\mu\partial_\mu + x^2\partial_\nu,
\end{aligned}$$

which are valid on the dense submanifold $M \subset \bar{M}$. Next we define

$$\omega(v) = \frac{d}{dt}\Omega_{\alpha(t)}^{-1}\,|_{t=0}$$

when $\alpha(t) = \exp tv$, $v \in \mathscr{G}$.

LEMMA 3.4. — The differential operator $\mathscr{D} = L^\nu\bar{\partial}_\nu$ is conformally covariant iff in addition to (*i*) and (*ii*) of Definition 3.1 we have

$$[\mathscr{D}, dT(v)] = -\omega(v)\mathscr{D} \qquad (\text{on } C^\infty(\bar{M}, E_0))$$

for all $v \in \mathscr{G}$.

Proof. — $T(\alpha(t)) = \exp t dT(v)$ for any one-parameter subgroup $\alpha(t) = \exp tv$ of $\tilde{G}$. □

THEOREM 3.5. — The differential operator $\mathscr{D} = L^\nu \bar{\partial}_\nu$ is conformally covariant iff in addition to (*i*) and (*ii*) of Definition 3.1 the following is true:

$$
\begin{aligned}
&(a) && [m_{\nu\mu}, L_\lambda] = g_{\mu\lambda}L_\nu - g_{\nu\lambda}L_\mu\,,\\
&(b) && [d, L_\nu] = 0\,,\\
&(c) && [k_\nu, L_\mu] = 0\,,\\
&(d) && L^\mu m_{\nu\mu} - L_\nu d = 0\,,
\end{aligned}
$$

and $L_\nu = L_\nu(x)$ for all $x \in \bar{M}$.

Proof. — On M we get $\omega(M_{\nu\mu}) = \omega(P_\nu) = 0$, $\omega(D) = 1$ and $\omega(K_\nu) = 2x_\nu$. Putting these and the expressions for $dT(\mathscr{G})$ into the formula

$$[\mathscr{D}, dT(v)] = -\,\omega(v)\mathscr{D}$$

and using the fact that M is dense in $\bar{M}$ we get the desired result. □

DEFINITION 3.6. — We say that the conformally covariant operator $\mathscr{D} = L^\nu \bar{\partial}_\nu$ is a (first-order) standard conformal operator if

(*i*) the representation R of $\bar{W}$ is a direct sum of irreducible representations of the subgroup $SL(2, \mathbb{C}) \subset \bar{W}$,

(*ii*) E does not decompose into a direct sum of two non-trivial closed subspaces which are invariant under $m_{\nu\mu}$ and the operators L^ν,

(*iii*) there exists a basis in E_0 in which d is diagonal.

Because of (*i*) and (*ii*) any standard conformal operator is a Lorentz invariant non-decomposable operator in the sense of Ref. [7] see also [*13*].

THEOREM 3.7. — Let $\Gamma_1 = \frac{1}{4} m^{\mu\nu}m_{\nu\mu}$. Then the non-decomposable Lorentz invariant operator $L^\nu\bar{\partial}_\nu$ is a standard conformal operator iff $d = \lambda\,\mathbb{1}$ for some $\lambda \in \mathbb{C}$, $k_\nu = 0$ and $[\Gamma_1, L_0] = \left(\lambda + \frac{3}{2}\right)L_0$.

Proof. — Since $[d, L_\nu] = [d, m_{\nu\mu}] = 0$ the eigenspaces of d corresponding to different eigenvalues are invariant under L_ν and $m_{\nu\mu}$. It follows that d has exactly one eigenvalue (d is diagonalizable), i. e. $d = \lambda\,\mathbb{1}$, $\lambda \in \mathbb{C}$. From $[d, k_\nu] = k_\nu$ it then follows that $k_\nu = 0$. Because of the commutation relations (*a*) of Theorem 3.5 the equation (*d*) of Theorem 3.5 is equivalent to

$$L^\mu m_{0\mu} - L_0 d = 0\,.$$

Repeatedly using (*a*) in 3.5 we get

$$\begin{aligned} 0 &= [m^{\mu}{}_0, \mathrm{L}_0]m_{0\mu} - \mathrm{L}_0 d \\ &= \frac{1}{2}[m^{\mu}{}_0 m_{0\mu}, \mathrm{L}_0] - \frac{1}{2}[m_{0\mu}, [m^{\mu}{}_0, \mathrm{L}_0]] - \mathrm{L}_0 d \\ &= \frac{1}{2}[m^{\mu}{}_0 m_{0\mu}, \mathrm{L}_0] - \frac{1}{2}[m_{0\mu}, \mathrm{L}^{\mu}] - \mathrm{L}_0 d \\ &= \frac{1}{2}[m^{\mu}{}_0 m_{0\mu}, \mathrm{L}_0] - \frac{3}{2}\mathrm{L}_0 - \mathrm{L}_0 d \\ &= \frac{1}{4}[m^{\mu\nu}m_{\nu\mu}, \mathrm{L}_0] - \left(\lambda + \frac{3}{2}\right)\mathrm{L}_0 \\ &= [\Gamma_1, \mathrm{L}_0] - \left(\lambda + \frac{3}{2}\right)\mathrm{L}_0 . \qquad \square \end{aligned}$$

THEOREM 3.8. — If $\mathscr{D} = \mathrm{L}^{\nu}\bar{\partial}_{\nu}$ is a standard conformal operator with $d = \lambda \ \mathbb{1}$ then the equation $\mathscr{D}\psi + \varkappa\psi = 0$ has only the trivial solution $\psi = 0$ if $\lambda \neq -\frac{3}{2}$ and $\varkappa(x) \neq 0$ for all $x \in \bar{\mathrm{M}}$.

Proof. — The operator Γ_1 is diagonal in the canonical basis of the Appendix,

$$\Gamma_1 e^k(l_0 l_1 \,;\, lm) = \frac{1}{2}(l_0^2 + l_1^2 - 1)e^k(l_0 l_1 \,;\, lm) .$$

Let $\Delta := \lambda + \frac{3}{2}$. From Theorem 3.7 it follows that if $\Gamma_1 x = \gamma x$ then $\Gamma_1 \mathrm{L}_0 x = (\gamma + \Delta)\mathrm{L}_0 x$. Because of $[m_{j0}, \mathrm{L}_0] = \Gamma_j$ $(j = 1, 2, 3)$ we have

$$\Gamma_1 \mathrm{L}_\nu x = (\gamma + \Delta)\mathrm{L}_\nu x, \qquad \nu = 0, 1, 2, 3 .$$

If L_ν has a non-zero matrix between the basis vectors $e^k(l_0 l_1 \,;\, lm)$ and $e^{k'}(l'_0 l'_1 \,;\, l'm')$ then, according to Example 1 in the Appendix, either $l'_0 = l_0 \pm 1$ and $l'_1 = l_1$ or $l'_0 = l_0$ and $l'_1 = l_1 \pm 1$. If $\Gamma_1 e^k(l_0 l_1 \,;\, lm) = \gamma e^k(l_0 l_1 \,;\, lm)$ and γ' is the eigenvalue of Γ_1 when acting on the vector $e^{k'}(l'_0 l'_1 \,;\, l'm')$ then either $\gamma' - \gamma = \pm l_0 + \frac{1}{2} = \Delta$ or $\gamma' - \gamma = \pm l_1 + \frac{1}{2} = \Delta$. Now, Δ is fixed (and assumed to be $\Delta \neq 0$) so one sees that when acting repeatedly on a given vector $e^k(l_0 l_1 \,;\, lm)$ by the operators L_ν, the vectors so obtained correspond to a finite number ($\leqq 4$) of eigenvalues of the operator Γ_1. On the other hand, because $\mathscr{D}$ is standard conformal operator ((*ii*) in Definition 3.6), all eigenvalues of Γ_1 in E must be on the same line in the complex plane, the line being parallel with the vector $\Delta \in \mathbb{C}$. For the same reason and because the number of eigenvalues of Γ_1 is limited to 4 in any subspace which is obtained by acting with the operators L_ν and $m_{\nu\mu}$ on a basis vector, we conclude that $e^k(l_0 l_1 \,;\, lm)$ can be chosen such that it is cyclic with respect to the algebra generated by the operators L_ν and $m_{\nu\mu}$. Let the different eigenspaces of Γ_1 in E be $E_l, \ldots, E_p$ with eigenvalues $\gamma_1, \ldots, \gamma_p$ such that $\gamma_{k+1} = \gamma_k + \Delta$. Let P_k be the projection $E \to E_k$. Now we have

$$0 = P_1(\mathscr{D}\psi)(x) + \varkappa(x)P_1\psi(x) = \varkappa(x)P_1\psi(x)$$

and thus $P_1\psi(x) = 0$ for all $x \in \bar{M}$ when ψ is a solution of our equation. Next we get

$$0 = P_2(\mathscr{D}\psi)(x) + \varkappa(x)P_2\psi(x) = \varkappa(x)P_2\psi(x),$$

because $P_2(\mathscr{D}\psi)(x) = P_2\mathscr{D}P_1\psi(x)$. By induction we get $P_k\psi(x) = 0$, $k = 1, 2, \ldots, p$, and thus $\psi = 0$. □

Note that from Theorems 3.7 and 3.8 it follows that if the equation $L^\nu\bar{\partial}_\nu\psi + \varkappa\psi = 0$ has non-trivial solutions (where $L^\nu\bar{\partial}_\nu$ is a standard conformal operator) then the operators L_ν can connect only the basis vectors for which $l_0^2 + l_1^2 = l_0'^2 + l_1'^2$. From this follows either $\pm 2l_0 + 1 = 0$ or $\pm 2l_1 + 1 = 0$; if the representations are finite-dimensional ($|l_1| - |l_0|$ positive integer) it follows that $l_0 = \pm \frac{1}{2}$. The spin l can have the values $|l_0|, |l_0| + 1, \ldots, |l_1| - 1$; thus we see that the rule $(-1)^{2l} = (-1)^{2\lambda}$ derived in [5] which connects the spin l and the conformal degree λ of the field, is valid in the case of first order equations.

4. SECOND-ORDER COVARIANT FIELD EQUATIONS

In this section we consider a second-order differential operator $\mathscr{D} = A^{\nu\mu}\bar{\partial}_\nu\bar{\partial}_\mu + B^\nu\bar{\partial}_\nu$ where $A^{\nu\mu}$ and B^ν are linear operators defined on the dense invariant subspace E_0; the notation is the same as in sections 2 and 3.

DEFINITION 4.1. — The operator $\mathscr{D}$ is conformally covariant if $T(g)\mathscr{D}T(g^{-1})\psi = \Omega_g^{-2}\mathscr{D}\psi$ for all $\psi \in C^\infty(\bar{M}, E_0)$ and E_0 is invariant under dR, R and $A^{\nu\mu}$, B^ν.

Let us consider the equation ($\mathscr{D}$ is covariant)

$$\mathscr{D}\psi + \varkappa^2\psi = 0, \tag{2}$$

where $\varkappa$ is a positive function on $\bar{M}$. If $\varkappa'(x) = \Omega_g(x)\varkappa(g^{-1}x)$ and $\psi' = T(g)\psi$, then we have

$$\mathscr{D}\psi' + \varkappa'^2\psi' = 0,$$

i. e. Eq. (2) is conformally covariant. In the same way as in the first-order case we get:

LEMMA 4.2. — The second-order operator $\mathscr{D}$ is conformally covariant iff

$$[\mathscr{D}, dT(v)] = -2\omega(v)\mathscr{D} \quad \text{for all} \quad v \in \mathscr{G} \quad (\text{on } C^\infty(\bar{M}, E_0))$$

where E_0 is the invariant subspace in Def. 4.1. Without any real restriction we can assume that $A^{\nu\mu} = A^{\mu\nu}$ because $\bar{\partial}_\nu\bar{\partial}_\mu = \bar{\partial}_\mu\bar{\partial}_\nu$. Through a direct calculation with the help of Lemma 4.2 we arrive at:

THEOREM 4.3. — The operator $A^{\nu\mu}\bar{\partial}_\nu\bar{\partial}_\mu + B^\nu\bar{\partial}_\nu$ is conformally covariant iff (on E_0) we have

(a) $\qquad B^\nu = 0,$

(b) $\qquad [d, A^{\nu\mu}] = [k_\rho, A^{\nu\mu}] = 0,$

(c) $\qquad [m_{\nu\mu}, A_{\lambda\rho}] = g_{\mu\lambda}A_{\nu\rho} - g_{\nu\lambda}A_{\mu\rho} + g_{\mu\rho}A_{\lambda\nu} - g_{\nu\rho}A_{\lambda\mu},$

(d) $\qquad (d - 1)A_{\nu\mu} + \frac{1}{2}A^\rho_\rho \cdot g_{\nu\mu} - A_{\mu\rho}m_\nu{}^\rho = 0.$

We can define a standard second-order conformal operator in the same way as in the first-order case, Definition 3.6. It is easily seen that for a standard conformal operator $\mathcal{D} = A^{\nu\mu}\bar{\partial}_\nu\bar{\partial}_\mu$ we must have $d = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{C}$ and $k_\nu = 0$.

From (c) we see that $A = \{ A_{\nu\mu} \}$ must be a symmetric tensor operator of rank 2 (Appendix). If we like, we can write

$$A_{\nu\mu} = A^{f}_{\nu\mu} + A^{tr}_{\nu\mu},$$

where A^f is a traceless operator, $A^f_{\nu\mu} = A_{\nu\mu} - \frac{1}{4}g_{\nu\mu}A^\rho_\rho$, and A^{tr} is the invariant operator, $A^{tr}_{\nu\mu} = \frac{1}{4}g_{\nu\mu}A^\rho_\rho$. In the notation of the Appendix, A^f is of type $(l_0, l_1) = (0,3)$ and A^{tr} is of type (0,1). If $x \in E^\tau$, where E^τ is the sum of all subspaces of E which carry an irreducible representation $\tau = (l_0, l_1)$ of SL(2, $\mathbb{C}$), the $A_{\nu\mu}x$ can have components only in subspaces $E^{\tau'}$ such that $\tau' = (l'_0, l'_1)$ is one of the following nine pairs: (l_0, l_1), $(l_0 \pm 2, l_1)$, $(l_0, l_1 \pm 2)$, $(l_0 + 1, l_1 \pm 1)$, $(l_0 - 1, l_1 \pm 1)$.

The left-hand side of Eq. (d) is a tensor operator of rank 2. This operator is zero iff its symmetric traceless part, the antisymmetric part and the trace are each equal to zero:

$$(\lambda + 1)A^\rho_\rho = 0; \qquad (3)$$

$$A_{\mu\rho}m_\nu{}^\rho - A_{\nu\rho}m_\mu{}^\rho = 0; \qquad (4)$$

$$2(\lambda - 1)A_{\nu\mu} - \frac{1}{2}(\lambda - 1)g_{\nu\mu}A^\rho_\rho - A_{\mu\rho}m_\nu{}^\rho - A_{\nu\rho}m_\mu{}^\rho = 0. \qquad (5)$$

By a direct calculation of the commutators, the left-hand side of Eq. (5) is equal to

$$-[\Gamma_1, A^f_{\nu\mu}] + 2(\lambda + 1)A^f_{\nu\mu} = 0. \qquad (5)'$$

If now the trace $A^\rho_\rho \neq 0$ then from (3) follows that $\lambda = -1$ and $[\Gamma_1, A^f_{\nu\mu}] = [\Gamma_1, A_{\nu\mu}] = 0$, according to (5)'. On the other hand, if $A^\rho_\rho = 0$ then (3) is automatically satisfied and (5)'. Reduces to

$$[\Gamma_1, A_{\nu\mu}] = 2(\lambda + 1)A_{\nu\mu}. \qquad (5)''$$

Using the same argument as in the proof of Theorem 3.8, one can show that the equation $(A^{\nu\mu}\bar{\partial}_\nu\bar{\partial}_\mu + \varkappa^2)\psi = 0$ can have non-trivial solutions only if $\lambda = -1$. Thus we have the following.

LEMMA 4.4. — If $\mathcal{D} = A^{\nu\mu}\bar{\partial}_\nu\bar{\partial}_\mu$ is a standard conformal operator and the equation $\mathcal{D}\psi + \varkappa^2\psi = 0$ has non-trivial solutions then

(*i*) $d = -\mathbb{1}$, $\quad k_\nu = 0$;
(*ii*) $[\Gamma_1, A_{\nu\mu}] = 0$;
(*iii*) $A_{\mu\rho}m_\nu{}^\rho - A_{\nu\rho}m_\mu{}^\rho = 0$;
(*iv*) $A = \{ A_{\nu\mu} \}$ is a symmetric tensor operator of rank 2.

Any non-decomposable Lorentz invariant operator $\mathcal{D}$ which satisfies (*i*)-(*iv*) is a standard conformal operator.

We assume in the rest of this section that the representations (l_0, l_1) are finite-dimensional. We shall look more closely at the condition (*iii*). Let us denote the left-hand side of Eq. (*iii*) by $T_{\mu\nu}$. Then $T = \{ T_{\mu\nu} \}$ is an antisymmetric tensor operator of rank 2; thus $T_{\mu\nu}$ can have non-zero matrix elements between the subspace $E^{\tau'}$ and E^τ only when $\tau' = \tau$ or $\tau' = (l'_0, l'_1) = (l_0 \pm 1, l_1 \pm 1)$ (four cases). If $T_{\mu\nu}$ connects E^τ and $E^{\tau'}$, then the same must be true for $A_{\nu\mu}$; on the other hand, because of Eq. (*ii*), we must have

$$0 = \frac{1}{2}(l_0^2 + l_1^2 - 1) - \frac{1}{2}(l_0'^2 + l_1'^2 - 1).$$

Using the fact that $| l_0 | < | l_1 |$ and $| l'_0 | < | l'_1 |$ (in the case of finite-dimensional representations) we conclude that there is only the possibility $(l_0, l_1) = (l'_0, l'_1)$. Thus we can write

$$T_{\nu\mu}e^k(l_0l_1 ; lm) = \sum_{k'l'm'} a_{k'l'm'}e^{k'}(l_0l_1 ; l'm'),$$

where the coefficients $a_{k'l'm'}$ depend also on ν, μ, l_0 and l_1. Let $P^{(l_0,l_1)} : E \to E^{(l_0,l_1)}$ be the projection and denote

$$A_{\nu\mu}^{(l_0,l_1)} = P^{l_0,l_1)}A_{\nu\mu}P^{(l_0,l_1)}.$$

According to the Appendix, we can write

$$A_{\nu\mu}^{(l_0,l_1)} = ag_{\nu\mu}\mathbb{1} + b(m_{\nu\rho}m^\rho{}_\mu - m_{\nu\mu})$$

where $a : E^{(l_0,l_1)} \to E^{(l_0,l_1)}$ and $b : E^{(l_0,l_1)} \to E^{(l_0,l_1)}$ are linear operators of the form

$$ae^k(l_0l_1 ; lm) = \sum_{k'} a_{kk'}e^{k'}(l_0l_1 ; lm),$$

$$be^k(l_0l_1 ; lm) = \sum_{k'} b_{kk'}e^{k'}(l_0l_1 ; lm),$$

where the numbers $a_{kk'}$, $b_{kk'}$ depdnd also on l_0 and l_1. Using the definition of $T_{\nu\mu}$ we get

$$T^{(l_0,l_1)}_{\nu\mu} := P^{(l_0,l_1)}T_{\nu\mu}P^{(l_0,l_1)}$$
$$= 2am_{\mu\nu} + b(m_{\nu\lambda}m^{\lambda}{}_{\rho}m^{\rho}{}_{\mu} - m_{\mu\lambda}m^{\lambda}{}_{\rho}m_{\nu}{}^{\rho} - m_{\nu\rho}m^{\rho}{}_{\mu} + m_{\mu\rho}m_{\nu}{}^{\rho}).$$

According to the discussion above, $T_{\nu\mu} = 0$ iff $T^{(l_0,l_1)}_{\nu\mu} = 0$ for each pair (l_0, l_1). By a direct calculation we get:

$$T^{(0,l_1)}_{\nu\mu} = 0 \quad \text{iff} \quad a = -2b\Gamma_1 \quad \text{when} \quad |l_1| > 1,$$
$$T^{(0,1)}_{\nu\mu} \equiv 0 \text{ and}$$
$$T^{(l_0,l_1)}_{\nu\mu} = 0 \quad \text{iff} \quad a = b = 0 \quad \text{when} \quad |l_0| > 0.$$

Thus we have proved the following:

THEOREM 4.5. — Let $\mathcal{D} = A^{\nu\mu}\bar{\partial}_\nu\bar{\partial}_\mu$ be a standard conformal operator in the space $C^\infty(\bar{M}, E)$ (where E is now finite-dimensional) such that the equation $\mathcal{D}\psi + \varkappa^2\psi = 0$ has non-trivial solutions. Then

(*i*) $d = -\mathbb{1}$, $k_\nu = 0$;
(*ii*) $[\Gamma_1, A_{\nu\mu}] = 0$;
(*iii*) $A = \{A_{\nu\mu}\}$ is a symmetric tensor operator of rank 2;
(*iv*) $A^{(l_0,l_1)}_{\nu\mu} = 0$ for all (l_0, l_1) with $|l_0| > 0$,

$$A^{(0,l_1)}_{\nu\mu} = -2b\Gamma_1 g_{\nu\mu} + b(m_{\nu\rho}m^{\rho}{}_{\mu} - m_{\nu\mu}) \quad \text{for} \quad |l_1| > 1$$

where

$$A^{(l_0,l_1)}_{\nu\mu} = P^{(l_0,l_1)}A_{\nu\mu}P^{(l_0,l_1)}, P^{(l_0,l_1)} : E \to E^{(l_0,l_1)}$$

is the projection and $b : E^{(0,l_1)} \to E^{(0,l_1)}$ is a linear operator (which depends on l_1) of the form

$$be^k(0l_1\,;\,lm) = \sum_{k'} b_{kk'}e^{k'}(0l_1\,;\,lm).$$

Conversely, any non-decomposable Lorentz invariant operator $\mathcal{D}$ which satisfies (*i*)-(*iv*) is a standard conformal operator.

EXAMPLE 1. — Let $E = E^{(0,l_1)}$ be irreducible with $|l_1| = 2, 3, \ldots$. In this case we can define

$$A_{\nu\mu} = -2\Gamma_1 \cdot g_{\nu\mu} + m_{\nu\rho}m^{\rho}{}_{\mu} - m_{\nu\mu}.$$

Again $d = -\mathbb{1}$, $k_\nu = 0$ and $A^{\nu\mu}\bar{\partial}_\nu\bar{\partial}_\mu$ is a standard conformal operator.

EXAMPLE 2. — Let $m_{\nu\mu} = 0$, $k_\nu = 0$, $d = -\mathbb{1}$. Then we have the conformally covariant Klein-Gordon operator, $A_{\nu\mu} = g_{\nu\mu} \cdot \mathbb{1}$.

EXAMPLE 3. — Let $E = E^{(1,l_1)} \oplus E^{(1,-l_1)}$ where $E^{(1,l_1)}$ and $E^{(1,-l_1)}$ are representation spaces for the irreducible representations $(1, l_1)$ and $(1, -l_1)$ with $l_1 = 2, 3, \ldots$. Looking at the Clebsch-Gordan series for the direct product $(l_0, l_1) \otimes (0, 3)$ one sees that there exists a symmetric tensor operator $A = \{ A_{\nu\mu} \}$ which maps a space $E^{(l_0, l_1)}$ into a space $E^{(l_0 - 2, l_1)}$ for any finite-dimensional representation (l_0, l_1). In particular,

$$A_{\nu\mu}E^{(1,l_1)} \subset E^{(-1,l_1)} \equiv E^{(1,-l_1)},$$
$$A_{\nu\mu}E^{(1,-l_1)} \subset E^{(-1,-l_1)} \equiv E^{(1,l_1)}.$$

It is easily seen that A satisfies the conditions (*ii*) and (*iii*) of Theorem 4.5 (in the space E). We gain take $d = -\mathbb{1}$ and $k_\nu = 0$. Because neither of the subspaces $E^{(1,l_1)}$ and $E^{(1,-l_1)}$ is invariant under $A_{\nu\mu}$, $A^{\nu\mu}\bar{\partial}_\nu\bar{\partial}_\mu$ is a standard conformal operator.

Note that from Theorem 4.5 and the Clebsch-Gordan series for $(l_0, l_1) \otimes (0, 3)$ given in the Appendix it follows that if the equation $(A^{\nu\mu}\bar{\partial}_\nu\bar{\partial}_\mu + \varkappa^2)\psi = 0$ has non-trivial solutions (where $A^{\mu\nu}\bar{\partial}_\mu\bar{\partial}_\nu$ is a standard conformal operator) and representations τ are finite-dimensional ($|l_1| - |l_0|$ a positive integer) then $l_0 = \pm 1$ or 0 (consequently the spin l can have only integer values: $|l_0|, |l_0| + 1, \ldots, |l_1| - 1$) and the conformal degree λ of the field ψ is -1; thus the rule $(-1)^{2l} = (-1)^{2\lambda}$ derived in [5] is valid in the case of the second order equations.

5. CONCLUSIONS

The definition for a conformally covariant first or second order differential operator was given which is suitable for a description of conformally covariant equations of the type $(\mathscr{D} + \varkappa)\psi = 0$ or $(\mathscr{D} + \varkappa^2)\psi = 0$. Here, $\varkappa$ is non-zero (and is connected with the mass of the field ψ) and transforms under conformal transformations like the mass. (The derived results, however, are also mostly true for $\varkappa = 0$). All conformally covariant first and second order equations (Definitions 3.1 and 4.1) were classified (Theorems 3.7, 3.8 and 4.5). Let us note that the Dirac equation, the Weinberg equation $(\gamma^{\mu\nu}\bar{\partial}_\mu\bar{\partial}_\nu + \varkappa^2)\psi = 0$ with spin 1, the Klein-Gordon equation and the (zero-mass) electromagnetic potential equation $\Box A_\nu - \bar{\partial}_\nu\bar{\partial}^\mu A_\mu = 0$ are conformally covariant in our sense.

It was found that the conformal degree λ is equal to $-\frac{3}{2}$ and $l_0 = \pm\frac{1}{2}$ (so that the spin 1 is half-integer) for all finite-component conformally covariant first-order equations with non-zero mass and that $\lambda = -1$ and $l_0 = \pm 1$ or 0 (so that the spin 1 is integer) for all finite-component conformally covariant second-order equations with non-zero mass. Consequently the rule (derived in [5]) $(-1)^{2\lambda} = (-1)^{2l}$ is valid.

Finally, we stress that the cases where the generator d is not diagonalizable (and k_ν may be different from zero), which we have not studied, are not necessarily unphysical. It it also interesting to consider reducible non-decomposable representations $g \mapsto R(g)$ of SL(2, $\mathbb{C}$) (see Definition 3.6 (*i*)) and especially the zero-mass case which will be done elsewhere.

APPENDIX

TENSOR OPERATORS OF SL(2, $\mathbb{C}$)

Let $g \mapsto R(g)$ be a continuous representation of SL(2, $\mathbb{C}$), the covering group of the Lorentz group, in a Banach space E. Let $E_0 \subset E$ be a dense subspace such that the representation dR (the differential of R) of the Lie algebra of SL(2, $\mathbb{C}$) is defined on E_0 and E_0 is invariant under R and dR. Let V be a finite-dimensional linear subspace of the space of all linear operators on E_0. Let τ be a representation of SL(2, $\mathbb{C}$). We say that V is a tensor operator of type τ on E if

(*i*) $R(g)AR(g^{-1}) \in V$ for all $A \in V$ and $g \in SL(2, \mathbb{C})$;

(*ii*) the rule $(g, A) \mapsto R(g)AR(g^{-1})$ defines a representation of SL(2, $\mathbb{C}$) on V which is equivalent to the representation τ.

If the representation τ is irreducible then V is said to be an irreducible tensor operator.

Let $M_{\nu\mu}$ ($\nu, \mu = 0, 1, 2, 3$) be the generators of the Lie algebra of SL(2, $\mathbb{C}$) (section 2); we denote again $m_{\nu\mu} = dR(M_{\nu\mu})$. Clearly (*i*) and (*ii*) are equivalent to

(*i*)′ $[m_{\nu\mu}, A] \in V$ for all $A \in V$, $\nu, \mu = 0, 1, 2, 3$;

(*ii*)′ $(m_{\nu\mu}, A) \mapsto [m_{\nu\mu}, A]$ is a representation of the Lie algebra of SL(2, $\mathbb{C}$) on V, which is equivalent to $d\tau$, the differential of τ.

We define the linear operators $\Gamma_1 = \frac{1}{4} m^{\nu\mu} m_{\mu\nu}$ and $\Gamma_2 = m_{01}m_{23} + m_{02}m_{31} + m_{03}m_{12}$ which commute with all the $m_{\nu\mu}$'s. The irreducible representations of SL(2, $\mathbb{C}$) which are direct sums of finite-dimensional representations of the subgroup SU(2) are characterized by two numbers l_0 and l_1 (see [7]). The number l_0 is of the form $\pm \frac{n}{2}$, $n = 0, 1, 2, \ldots$, and l_1 is an arbitrary complex number. The representation (l_0, l_1) is equivalent to the representation $(-l_0, -l_1)$. In the representation (l_0, l_1) we have

$$\Gamma_1 = \frac{1}{2}(l_0^2 + l_1^2 - 1)\cdot \mathbb{1}, \qquad \Gamma_2 = i l_0 l_1 \cdot \mathbb{1} .$$

The representation (l_0, l_1) is finite-dimensional iff l_1 is real and $|l_1| - |l_0|$ is a positive integer. The finite-dimensional irreducible representations of SU(2) are characterized by the « spin » l, $l = 0, \frac{1}{2}, 1, \ldots$. If the representation (l_0, l_1) of SL(2, $\mathbb{C}$) is finite-dimensional, then its restriction to SU(2) is the direct sum of unitary irreducible representations characterized by $l = |l_0|, |l_0| + 1, \ldots, |l_1| - 1$. If (l_0, l_1) is infinite-dimensional, then $l = |l_0|, |l_0| + 1, \ldots$.

Let F^τ be a representation space for an irreducible representation of SL(2, $\mathbb{C}$) of the type $\tau = (l_0, l_1)$. Let $\{ e(\tau ; lm) \}_{lm}$ be a normed basis in F^τ such that $\{ e(\tau ; lm) \}_{m=-l,-l+1,\ldots,l}$ is a basis in the subspace of F^τ on which a representation with spin l of SU(2) is defined. Now let F^τ and $F^{\tau'}$ be two finite-dimensional irreducible representation spaces. We consider the tensor product

$$F^\tau \otimes F^{\tau'} = \oplus \sum_{\tau''} F^{\tau''} .$$

We can write

$$e(\tau ; lm) \otimes e(\tau' ; l'm') = \sum_{\tau'' l'' m''} (\tau'' l'' m'' \mid \tau lm ; \tau' l' m') \times e(\tau'' ; l'' m'') ,$$

where the numbers $(\tau''l''m'' \mid \tau lm ; \tau'l'm')$ are Clebsch-Gordan coefficients of SL(2, $\mathbb{C}$). We assume now that the representation R of SL(2, $\mathbb{C}$) on E is finite-dimensional and thus completely reducible,

$$E = \oplus \sum_{\tau,k} F_k^\tau ,$$

where each of the subspaces F_k^τ, $k = 1, \ldots, n_\tau$, carries an irreducible representation of SL(2, $\mathbb{C}$) which is equivalent to the representation τ. We denote $E^\tau = \oplus \sum_{k=1}^{n_\tau} F_k^\tau$; thus $E = \oplus \sum_\tau E^\tau$. In each of the subspaces F_k^τ we construct the canonical basis $\{ e^k(\tau ; lm) \}_{lm}$ which is defined above. Now let V be a tensor operator of type σ in E. Let $V(\sigma ; lm)$ be the canonical basis for V. According to the Wigner-Eckart theorem,

$$V(\sigma ; jn)e^k(\tau ; lm) = \sum_{\tau'l'm'} a^k_{\tau'k'}(\tau'l'm' \mid \sigma jn ; \tau lm)e^{k'}(\tau' ; l'm'),$$

where the numbers $a^k_{\tau'k'}$ depend on the tensor operator V.

EXAMPLE 1. — Tensor operator V of type $\tau = (0,2)$. This is the so-called vector operator; V has four components V_μ ($\mu = 0, 1, 2, 3$; this is *not* the canonical basis),

$$[m_{\nu\mu}, V_\lambda] = g_{\mu\lambda}V_\nu - g_{\nu\lambda}V_\mu .$$

It can be shown (see [7]) that the tensor product of the representation $\tau = (l_0, l_1)$ and of (0,2) is a direct sum $(l_0 - 1, l_1) \oplus (l_0 + 1, l_1) \oplus (l_0, l_1 - 1) \oplus (l_0, l_1 + 1)$ (we identify (l_0, l_1) with $(- l_0, - l_1)$; if $l_1 = l_0 + 1$, then the second and third term are missing). It follows that V_μ can have non-zero matrix elements between the subspaces $E^{(l_0,l_1)}$ and $E^{(l'_0,l'_1)}$ only if $l'_0 = l_0 \pm 1$ and $l'_1 = l_1$ or $l'_0 = l_0$ and $l'_1 = l_1 \pm 1$.

EXAMPLE 2. — The tensor operator V of rank 2. By definition, V transforms according to the representation $(0, 2) \otimes (0, 2)$ of SL(2, $\mathbb{C}$); V has 16 components $V_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$),

$$[m_{\nu\mu}, V_{\alpha\beta}] = g_{\mu\alpha}V_{\nu\beta} - g_{\nu\alpha}V_{\mu\beta} + g_{\mu\beta}V_{\alpha\nu} - g_{\nu\beta}V_{\alpha\mu} .$$

V is not irreducible; we can divide V into four invariant subspaces, namely the antisymmetric tensor operator V^a, $V^a_{\nu\mu} = \frac{1}{2}V_{\nu\mu} - \frac{1}{2}V_{\mu\nu}$, the symmetric traceless operator V^s, $V^s_{\nu\mu} = \frac{1}{2}V_{\nu\mu} + \frac{1}{2}V_{\nu\mu} - \frac{1}{4}g_{\nu\mu}V_{\alpha\beta}g_{\alpha\beta}$, and the trace tensor V^{tr}, $V^{tr}_{\nu\mu} = \frac{1}{4}g_{\nu\mu}V_{\alpha\beta}g_{\alpha\beta}$. The antisymmetric part V^a can be further divided into two invariant subspaces, V^{a_1} and V^{a_2}, which have the basis

$$\{ V^a_{01} + iV^a_{23}, V^a_{02} + iV^a_{31}, V^a_{03} + iV^a_{12} \}$$

and

$$\{ V^a_{01} - iV^a_{23}, V^a_{02} - iV^a_{31}, V^a_{03} - iV^a_{12} \}$$

respectively. As is easily seen, V^s transforms according to the representation $(l_0, l_1) = (0, 3)$, V^{a_1} and V^{a_2} are of type (1,2) and (– 1,2); V^{tr} is of type (0,1), i. e. V^{tr} is invariant, $[m_{\nu\mu}, V^{tr}] = 0$. With the help of [7] one can verify the following Clebsch-Gordan series:

$$(l_0, l_1) \otimes (\ \ 1,2) = (l_0, l_1) \oplus (l_0 + 1, l_1 + 1) \oplus (l_0 - 1, l_1 - 1);$$
$$(l_0, l_1) \otimes (-1,2) = (l_0, l_1) \oplus (l_0 - 1, l_1 + 1) \oplus (l_0 + 1, l_1 - 1);$$
$$(l_0, l_1) \otimes (\ \ 0,3) = (l_0, l_1 - 2) \oplus (l_0 - 1, l_1 - 1) \oplus (l_0 - 2, l_1) \oplus (l_0 + 1, l_1 - 1) \oplus (l_0, l_1) \oplus (l_0 - 1, l_1 + 1) \oplus (l_0 + 2, l_1) \oplus (l_0 + 1, l_1 + 1) \oplus (l_0, l_1 + 2);$$
$$(l_0, l_1) \otimes (\ \ 0,1) = (l_0, l_1).$$

Thus, for example, the vector $V^s_{\nu\mu}e^k(\tau\,;\,lm)$ can have components in at most nine different subspaces $E^{\tau'}$ of E.

The generators $m_{\nu\mu}$ themselves give an example of an antisymmetric tensor operator of rank 2.

We can also construct a symmetric traceless tensor operator

$$S_{\nu\mu} = m_{\nu\rho}m^{\rho}{}_{\mu} - m_{\nu\mu} - \frac{1}{4}g_{\nu\mu}m_{\alpha\beta}m^{\beta\alpha}.$$

Let us consider the action of S_{00} on a vector $e^k(\tau\,;\,l_0l_0)$,

$$\begin{aligned} S_{00}e^k(\tau\,;\,l_0l_0) &= \left(m_{0j}m^{j0} - \frac{1}{4}m_{\alpha\beta}m^{\beta\alpha}\right)e^k(\tau\,;\,l_0l_0) \\ &= \left(\frac{1}{2}m_{\alpha\beta}m^{\beta\alpha} - \frac{1}{2}m_{ij}m^{ji} - \frac{1}{4}m_{\alpha\beta}m^{\beta\alpha}\right)e^k(\tau\,;\,l_0l_0) \\ &= (\Gamma_1 + (m_{12})^2 + (m_{13})^2 + (m_{23})^2)e^k(\tau\,;\,l_0l_0) \\ &= \frac{1}{2}(l_0^2 + l_1^2 - 1 - 2l_0(l_0 + 1))e^k(\tau\,;\,l_0l_0). \end{aligned}$$

Since (l_0, l_1) and $(-l_0, -l_1)$ are equivalent, we can take $l_0 \geqq 0$. Because of $|l_1| \geqq l_0 + 1$, the right-hand side of this equation can be zero only if $|l_1| = l_0 + 1$. If V^s is any symmetric traceless tensor operator of rank 2 such that $V^sE^\tau \subset E^\tau$, then we can write, for some operator a^τ of the form

$$a^\tau e^k(\tau\,;\,lm) = \sum_{k'} a^\tau_{kk'}e^{k'}(\tau\,;\,lm),$$

$$V^s_{\nu\mu}|_{E_\tau} = a^\tau \cdot S_{\nu\mu},$$

if $\tau = (l_0, l_1)$ is such that $|l_1| > l_0 + 1$. With the help of [7] one sees that

$$\begin{aligned} (l_0, \quad l_0 + 1) \otimes (0,3) &= (l_0 - 2, \quad l_0 + 1) \oplus (l_0 - 1, \quad l_0 + 2) \oplus (l_0, \quad l_0 + 3), \\ (l_0, -l_0 - 1) \otimes (0,3) &= (l_0 - 2, -l_0 - 1) \oplus (l_0 - 1, -l_0 - 2) \oplus (l_0, -l_0 - 3). \end{aligned}$$

It follows that we have $a^\tau = 0$ when $\tau = (l_0, l_1)$ with $|l_1| = l_0 + 1$.

Everything that has been said above can be generalized also to the case when the subspaces F^τ_k of E are infinite-dimensional.

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