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M. COMBESCURE-MOULIN

J. GINIBRE

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# Essential Self-adjointness of Many Particle Schrödinger Hamiltonians with Singular Two-Body Potentials

by

**M. COMBESCURE-MOULIN and J. GINIBRE**
Laboratoire de Physique Théorique et Hautes Énergies (*),
Université de Paris-Sud,
91405 Orsay (France)

ABSTRACT. — We study the Schrödinger Hamiltonian H for a system of N particles ($N \geq 3$) in $\mathbb{R}^n$, interacting via translation invariant two-body potentials $V_{ij}$ satisfying the conditions $V_{ij} \in L^2_{loc}(\mathbb{R}^n \backslash \{ 0 \})$ and $V_{ij} \geq c r_{ij}^{-2}$ for a suitable value of $c$. We prove that H is essentially self-adjoint on the space $\mathcal{D}_0$ of $\mathcal{C}^\infty$ functions with compact support contained in the region where no two particles coincide. The value of $c$ for which our proof is valid is the optimal one $c = -n(n-4)/4$ for $n = 1$ and $n = 4$ (all N) and for $N = 3$, $n \leq 6$, and has the correct sign in all cases. Under similar but weaker assumptions on the potentials, we also prove that H defined in a suitable way as a sum of quadratic forms coincides with the Friedrichs extension of its restriction to $\mathcal{D}_0$.

## 1. INTRODUCTION

The problem of essential self-adjointness of the Schrödinger operator $H = -\Delta + V(x)$ in $\mathbb{R}^n$ is an old problem [2] [4] [9]. Until recently, all known results assumed fairly strong local conditions on the potential V, for instance conditions of the Stummel type [9]. Recently however,

(*) Laboratoire associé au Centre National de la Recherche Scientifique.

it was proved by Simon that if the potential V is positive, it is sufficient to assume $V \in L^2_{loc}(\mathbb{R}^n)$ to ensure that H is essentially self-adjoint on the space $\mathscr{D}$ of $\mathscr{C}^\infty$ functions with compact support [12]. Actually, Simon imposed an additional restriction on the growth of the potential at infinity, but the latter was subsequently removed by Kato who generalized Simon's result to a larger class of potentials by an entirely different method [6].

In the applications to two-particle systems, the space variable represents the relative position of the two particles, and the potential represents their interaction. In a large number of cases of physical interest, it is necessary to consider potentials that become highly singular when the two particles come close together, namely at the origin. In such cases, it is a natural question to ask whether H is essentially self-adjoint on the space

$$\mathscr{D}_0 = \mathscr{C}_0^\infty(\mathbb{R}^n \backslash \{ 0 \})$$

of $\mathscr{C}^\infty$ functions with compact support contained in the complement of the origin. This question was considered by Kalf and Walter [3], by Schmincke [11] and subsequently by Simon [13] and Robinson [8] who proved that this is indeed the case if $V \in L^2_{loc}(\mathbb{R}^n \backslash \{ 0 \})$ and if in addition V satisfies the condition

$$V \geq cr^{-2}$$

where $r = | x |$ and $c = -n(n-4)/4$.

The extension of this result to N-particle systems interacting via translation invariant two-body potentials was considered by Robinson *et al.* [1]. In this case, the space variable $x = (x_1, \ldots, x_N)$ represents the set of positions of the N particles, the relevant Hilbert space is $L^2(\mathbb{R}^n)^{\otimes N} = L^2(\mathbb{R}^{nN})$ and the Hamiltonian is defined formally as $H = H_0 + V$, where

$$H_0 = -\frac{1}{2}\sum_{i=1}^{N} \Delta_i \tag{1.1}$$

and

$$V = \sum_{i<j} V_{ij}. \tag{1.2}$$

Here $\Delta_i$ is the Laplacian with respect to the position $x_i$ of particle $i$, and $V_{ij}$ is a two-body potential that is translation invariant, i. e. that depends only on $x_i - x_j$. We have assumed for simplicity that all particles have the same mass $m = 1$. For $N = 2$ and after discarding the center of mass variable, one recovers $-\Delta + V$ where the space variable is the relative position $x_1 - x_2$ of the two particles.

It was then conjectured by Robinson *et al.* that this H is essentially self-adjoint on the space

$$\mathscr{D}_0 = \mathscr{C}_0^\infty(\mathbb{R}^{nN} \backslash S)$$

of $\mathscr{C}^\infty$ functions with compact support contained in the complement of the closed set S where two particles coincide:

$$S = \{ x : x = (x_1, \ldots, x_N) \in \mathbb{R}^{nN} \quad \text{and} \quad x_i = x_j \quad \text{for some } (i,j) \}$$

provided each $V_{ij}$ satisfies the same conditions as in the two-body case, namely $V_{ij} \in L^2_{loc}(\mathbb{R}^n \backslash \{ 0 \})$ and $V_{ij} \geq c r_{ij}^{-2}$ where $r_{ij} = | x_i - x_j |$ and $c = -n(n-4)/4$.

The main purpose of this paper is to try and prove this conjecture. Actually, we prove only a weaker result, in the sense that we do not obtain the correct value of $c$ for all $n$ and N, but only for $n = 1$ and $n = 4$ (all N) and for $N = 3$, $n \leq 6$. For the other values of $n$ and N, the value of $c$ we obtain has nevertheless the correct sign. See theorem 3.1 for a precise statement. The previous result of Simon and Robinson covers the case $N = 2$ (all $n$) and comes out as a special case of our result.

In the two-body case, it is known that H remains semi-bounded down to $c = -(n-2)^2/4$ and one may wonder whether some property weaker than essential self-adjointness remains valid in the interval

$$-(n-2)^2/4 \leq c < -n(n-4)/4 .$$

This is connected with another question raised by Robinson, namely whether H defined in a suitable way as a sum of quadratic forms coincides with the Friedrichs extension of its restriction to $\mathscr{D}_0$. The relevant assumptions on V, and more generally on $V_{ij}$ in the N particle case, are then that $V_{ij} \in L^1_{loc}(\mathbb{R}^n \backslash \{ 0 \})$ and that $V_{ij} \geq c r_{ij}^{-2}$ with $c = -(n-2)^2/4$. Note that $L^1_{loc}$ replaces $L^2_{loc}$ since we now deal with quadratic forms instead of operators. The expected result is not simple to state, because the definition of H as a sum of quadratic forms requires some care. See section 4 for details. Once this is done, the expected result is that the quadratic form associated with H is positive and coincides with the closure of its restriction to $\mathscr{D}_0$ as a quadratic form, under the conditions on the potentials stated above. If in addition $V_{ij} \in L^2_{loc}(\mathbb{R}^n \backslash \{ 0 \})$, this means that H coincides with the Friedrichs extension of the operator $H_0 + V$ defined with domain $\mathscr{D}_0$.

A by-product of our investigation is to prove this second conjecture in a slightly weaker form, in the sense that we obtain the correct value of $c$ only for $n \leq 2$ (all N) and for $N = 2$ (all $n$). For $n \geq 3$ and $N \geq 3$, we obtain only $c = -(n-2)^2/2N$. See theorem 4.1 for a precise statement. A weaker result in the same direction is proved by Robinson *et al.* [*1*], where $L^1_{loc}$ is replaced by $L^\infty_{loc}$ and $c$ by zero.

The method of proof of our results is an extension of that of Kato, in a slightly modified version inspired by the work of Simon. The main new information consists of inequalities and estimates for suitably chosen N particle trial functions.

The paper is organized as follows. In section 2, we introduce the N-particle functions mentioned above and derive the relevant inequalities and estimates. We then prove the results on essential self-adjointness in section 3 and the results on quadratic forms in section 4. Section 5 contains some additional remarks.

## 2. ESTIMATES FOR N-PARTICLE FUNCTIONS

From now on, we consider systems of N particles in the center of mass frame. In particular, the space variable ranges over $\mathbb{R}^{n(N-1)}$ and the Hilbert space is $\mathscr{H} = L^2(\mathbb{R}^{n(N-1)})$. We keep the same notations $H_0$, S and $\mathscr{D}_0$ for the objects associated with this reduced problem, namely

$$H_0 = -\frac{1}{2}\sum_i \Delta_i + \frac{1}{2N}\left(\sum_i \nabla_i\right)^2 \tag{2.1}$$

$$S = \{ x : x \in \mathbb{R}^{n(N-1)} \quad \text{and} \quad x_i - x_j = 0 \quad \text{for some} \quad (i,j) \} \tag{2.2}$$

$$\mathscr{D}_0 = \mathscr{C}_0^\infty(\mathbb{R}^{n(N-1)}\backslash S) \tag{2.3}$$

Let $x_{ij} = x_i - x_j$ and $r_{ij} = |x_{ij}|$ and define $r$ by:

$$r^2 = \sum_{i<j} r_{ij}^2 \tag{2.4}$$

We consider the following class of N-particle functions:

$$\psi = \exp(\omega) \tag{2.5}$$

where:

$$\omega = \alpha \sum_{i<j} \operatorname{Log} r_{ij} - \beta \operatorname{Log} r - r \tag{2.6}$$

with $\alpha$ and $\beta$ arbitrary real numbers.

The first purpose of this section is to derive some inequalities on the function $H_0\psi$. These inequalities will be stated in proposition 2.1. In all this section, such expressions as $H_0\psi$ represent the action on $\psi$ of the ordinary differential operator associated with $H_0$ in the ordinary sense. The function thereby obtained is well defined in the complement of S, since $\psi$ is $\mathscr{C}^\infty$ in this region. It is not claimed that $H_0\psi$ makes sense as the image of a vector in $\mathscr{H}$ under an operator in $\mathscr{H}$.

We need some preliminary definitions and estimates. We define

$$V_0 = \sum_{i<j} r_{ij}^{-2} \tag{2.7}$$

Let now $\sigma$ denote an arbitrary subset of 3 particles (in the case $N \geq 3$).

We define

$$r_\sigma^2 = \sum_{i<j\in\sigma} r_{ij}^2 \tag{2.8}$$

$$W_\sigma = r_\sigma^{-2} \tag{2.9}$$

$$U_\sigma = \frac{1}{2} \sum_{\substack{i,j,k\in\sigma \\ \text{all} \neq}} (x_{ij}.x_{ik}) r_{ij}^{-2} r_{ik}^{-2} \tag{2.10}$$

$$W = \sum_\sigma W_\sigma \tag{2.11}$$

$$U = \sum_\sigma U_\sigma \tag{2.12}$$

where the last two sums run over all 3-particle subsets of $(1, \ldots, N)$.

$U_\sigma$ and $W_\sigma$ are 3-body potentials, and U and W are the corresponding potential energies of the N-particle system. We need the following estimates.

LEMMA 2.1.

(1) Let $N = 3$. Then

$$U = 2(2A)^2 \prod_{i<j} r_{ij}^{-2} \tag{2.13}$$

where A is the area of the triangle with vertices $x_i$. In particular $U \equiv 0$ if $n = 1$.

(2) Let $N = 3$. Then

$$0 \leq 2U \leq 9W \leq V_0 \tag{2.14}$$

(3) Let $N \geq 3$. Then

$$0 \leq 2U \leq 9W \leq (N-2)V_0 \tag{2.15}$$

*Proof*

(1) Let $a_i$ be the angles of the triangle with vertices $x_i$ and A its area. Let $(i, j, k)$ be an arbitrary permutation of $(1, 2, 3)$.

Then

$$U = \sum_i (r_{ij}r_{ik})^{-1} \cos a_i = (4A)^{-1} \sum_i \sin 2a_i$$

$$= A^{-1} \prod_i \sin a_i = 2(2A)^2 \prod_{i<j} r_{ij}^{-2}$$

(2) The first inequality follows from (1) and the third from the inequality $a^{-1} + b^{-1} + c^{-1} \geq 9(a+b+c)^{-1}$ valid for any strictly posi-

tive $a$, $b$, $c$. We now consider the second inequality. With the same notations as in the proof of (1), we obtain:

$$2U = \sum_i (r_{ij}^2 + r_{ik}^2 - r_{jk}^2) r_{ij}^{-2} r_{ik}^{-2}$$

Therefore

$$2Ur^2 = \sum_i [(r_{ij}^2 + r_{ik}^2)^2 - r_{jk}^4] r_{ij}^{-2} r_{ik}^{-2}$$

$$= 9 - \prod_{i<j} r_{ij}^{-2} F$$

where

$$F = \sum_{i<j} r_{ij}^6 - \sum_i r_{ij}^2 r_{ik}^2 (r_{ij}^2 + r_{ik}^2) + 3 \prod_{i<j} r_{ij}^2$$

It remains to be shown that $F \geq 0$ for all $r_{ij}$ compatible with the triangle inequality. This is done easily by noticing that for fixed $r_{ij}^2$ and $(r_{ik}^2 + r_{jk}^2)$, F is a linear function of the variable $(r_{ik}^2 - r_{jk}^2)^2$ and is positive at the two ends of the range of this variable. We omit the details.

(3) is an immediate consequence of (2) and the definitions. This completes the proof of lemma 2.1.

We can now derive the relevant estimates for $H_0\psi$.

PROPOSITION 2.1. — Let $\psi$ be defined by (2.5) and (2.6).

(1) Let $\beta = 0$, $N \geq 2$ and $n = 1$ and let $(N-1)(N\alpha + 1) \geq 1$. Then:

$$\left(H_0 + \alpha(\alpha - 1)V_0 + \frac{1}{2}N\right)\psi \geq 0 \tag{2.16}$$

(2) Let $\beta = 0$, $N \geq 2$ and $n \geq 2$ and let $(N-1)(N\alpha + n) \geq 1$. Then:

$$\left(H_0 + \alpha\left(\frac{1}{2}N\alpha + n - 2\right)V_0 + \frac{1}{2}N\right)\psi \geq 0 \tag{2.17}$$

(3) Let $N = 3$, $n \geq 2$ and $\beta \leq 3\alpha + n - \frac{1}{2}$. Then:

$$\left(H_0 + \alpha(\alpha + n - 2)V_0 + \frac{3}{2}(\beta^2 - 2\beta(3\alpha + n - 1) + 3\alpha^2)r^{-2} + \frac{3}{2}\right)\psi \geq 0 \tag{2.18}$$

*Proof.* — An elementary computation using the identities

$$\sum_i \left(\sum_{j(\neq i)} x_{ij}\right)^2 = Nr^2$$

and

$$2\sum_i \left(\sum_{j(\neq i)} x_{ij} r_{ij}^{-2}\right) \cdot \left(\sum_{k(\neq i)} x_{ik}\right) = N^2(N-1)$$

yields:

$$\left\{ H_0+\alpha(\alpha+n-2)V_0+\alpha^2U-\frac{1}{2}N\beta[(N-1)(N\alpha+n)-2-\beta]r^{-2} \right.$$
$$\left. -\frac{1}{2}N[(N-1)(N\alpha+n)-1-2\beta]r^{-1}+\frac{1}{2}N \right\}\psi=0 \tag{2.19}$$

Proposition 2.1 then follows immediately from lemma 2.1. Indeed (1) follows from the fact that $U \equiv 0$ for $n = 1$, (2) from the last two inequalities in (2.15), and (3) from the second inequality in (2.14). This completes the proof.

It follows in particular from proposition 2.1 that for $N = 3$:

$$\left(H_0 + \alpha(\alpha + n - 2)V_0 + \frac{3}{2}\right)\psi \geq 0$$

in the region defined by

$$\beta \leq 3\alpha + n - \frac{1}{2} \tag{2.20}$$

and

$$\beta^2 - 2\beta(3\alpha + n - 1) + 3\alpha^2 \leq 0 \tag{2.21}$$

The second condition holds in a region limited by an hyperbola with center at $\alpha = \beta = -(n-1)/2$ and asymptotes of slopes $3 \pm \sqrt{6}$ independent of $n$.

The second purpose of the present section is to derive another set of estimates for the class of functions $\psi$ defined by (2.5) and (2.6). We define a function $g_\varepsilon(t)$ of a positive real variable $t$ by

$$\left.\begin{aligned} g_\varepsilon(t) &= 1 \qquad \text{if } \varepsilon \leq t \leq 2\varepsilon \\ &= 0 \qquad \text{otherwise} \end{aligned}\right\} \tag{2.22}$$

Let $\gamma$ be an arbitrary real number. We now look for conditions under which the function

$$\varphi_\varepsilon = g_\varepsilon(r_{12})r_{12}^{-\gamma}\psi \tag{2.23}$$

considered as a vector in $\mathscr{H}$, satisfies the following condition:

(C) $\|\varphi_\varepsilon\|$ is bounded uniformly in $\varepsilon$ near $\varepsilon = 0$.

PROPOSITION 2.2. — Let $\psi$ and $\varphi_\varepsilon$ be defined by (2.5, 6) and (2.23)
(1) Let $N = 3$, $\gamma = 2$. Then condition (C) holds if and only if

$$n + 2\alpha \geq 4, \tag{2.24}$$
$$\beta \leq 3\alpha + n - 2 \tag{2.25}$$

and $(\alpha, \beta) \neq (2 - n/2, 4 - n/2)$.
(At the last point, we have only $\|\varphi_\varepsilon\| \simeq C|\log\varepsilon|$).

(2) Let $N \geq 3$, $\beta = 0$ and $n + \alpha(N - 1) > 0$. Then condition (C) holds if and only if

$$2\gamma \leq 2\alpha + n, \tag{2.26}$$

$$2\gamma \leq (N - 1)(n + N\alpha) \tag{2.27}$$

and $(\alpha, \gamma) \neq (-n/(N + 1), n(N - 1)/2(N + 1))$.
(At the last point we have only $\| \varphi_\varepsilon \| \simeq C | \operatorname{Log} \varepsilon |$).

*Proof.*

(1) We consider

$$\| \varphi_\varepsilon \|^2 = \int g_\varepsilon(r_{12}) r_{12}^{2\alpha-4} dx_{12} \int dx_3 (r_{13} r_{23})^{2\alpha} r^{-2\beta} e^{-2r} \tag{2.28}$$

If $2\beta < 4\alpha + n$, the integral over $x_3$ is bounded uniformly in $r_{12}$ and the conclusion follows iff $2\alpha + n - 4 \geq 0$.
If $2\beta > 4\alpha + n$, the integral over $x_3$ satisfies the estimate

$$\int dx_3 (r_{13} r_{23})^{2\alpha} r^{-2\beta} e^{-2r} \leq \int dx_3 (r_{13} r_{23})^{2\alpha} r^{-2\beta} = C r_{12}^{n+4\alpha-2\beta}$$

by homogeneity. The conclusion follows iff

$$\beta \leq 3\alpha + n - 2 .$$

If $2\beta = 4\alpha + n$, the integral over $x_3$ behaves as $C | \operatorname{Log} r_{12} |$ and the conclusion follows iff $2\alpha + n - 4 > 0$. Part (1) of proposition 2.2 is obtained by collecting these various results.

(2) The case $N = 3$ is trivial and we assume $N \geq 4$.

The proof is elementary but tedious and will only be sketched briefly. With the origin taken at the point $\frac{1}{2}(x_1 + x_2)$ and a suitable normalization of the volume element in $\mathbb{R}^{n(N-1)}$, we get

$$\| \varphi_\varepsilon \|^2 = \int dx_{12} g_\varepsilon(r_{12}) r_{12}^{2\alpha-2\gamma} G \tag{2.29}$$

where

$$G = \int dx_3 \ldots dx_N e^{-2r} \prod_{i<j,(ij)\neq(1,2)} r_{ij}^{2\alpha} \tag{2.30}$$

By power counting, we expect G to be bounded uniformly in $r_{12}$ if $n + (N + 1)\alpha > 0$ and to diverge as the $(n + (N + 1)\alpha)(N - 2)$-th power of $r_{12}$ when $r_{12}$ tends to zero in the opposite case. This suggests the convergence conditions (2.26) in the first case and (2.27) in the second case. Replacing $r_{ij}$ by $r$ for all $(i, j) \neq (1, 2)$ when $\alpha < 0$, we see immediately that these two conditions are necessary. The only non trivial point is their sufficiency in the range $n + \alpha(N - 1) > 0$.

Suppose first that

$$n + (N + 1)\alpha > 0 \tag{2.31}$$

Using the inequalities

$$r^2 \geq \sum_{i>2} r_{1i}^2 + r_{2i}^2 \geq \delta^2\Big(\sum_{i>2} r_i\Big)^2$$

where $r_i = |x_i|$ and $\delta^2 = 2/(N - 2)$, and applying Hölder's inequality, we obtain the estimate

$$G \leq K^{4/(N+1)} L^{(N-3)/(N+1)} \tag{2.32}$$

where

$$K = \int dx_3 \dots dx_N \exp[-2\delta\Sigma r_i] \prod_{i>2} (r_{1i}r_{2i})^{\alpha(N+1)/2} \tag{2.33}$$

$$L = \int dx_3 \dots dx_N \exp(-2\delta\Sigma r_i) \prod_{2<i<j} r_{ij}^{2\alpha(N+1)/(N-3)} \tag{2.34}$$

The integral in K factors as the (N − 2)-th power of the integral

$$\int dx_0 e^{-2\delta r_0}(r_{10}r_{20})^{\alpha(N+1)/2} \tag{2.35}$$

where $x_0$ is a dummy integration variable, $r_0 = |x_0|$ and $r_{i0} = |x_i - x_0|$. This integral converges uniformly in $r_{12}$ near zero under the asumption (2.31). It remains to be shown that L is finite. This is done easily by estimating the integrals over $x_j$ $(3 \leq j \leq N)$ in decreasing order of $j$, for fixed $x_i (i < j)$, by another use of Hölder's inequality. The general form of the inequality to be used is:

$$\begin{aligned}\int dx_N \prod_{i<N} r_{iN}^{2\alpha} &= \int dx_N \prod_{i<j<N} (r_{iN}r_{jN})^{2\alpha/(N-2)} \\ &\leq \prod_{i<j<N} \left\{ \int dx_0 (r_{i0}r_{j0})^{\alpha(N-1)} \right\}^{2/(N-1)(N-2)} \\ &\leq C \prod_{i<j<N} r_{ij}^{2[n+2\alpha(N-1)]/(N-1)(N-2)} \end{aligned} \tag{2.36}$$

which is valid for $-2n < 2\alpha(N - 1) < -n$.

The first inequality follows from Hölder's inequality and the second from homogeneity. C is some constant independent of the $r_{ij}$.

In the present case, we use (2.36) with N replaced by (N − 2) and $\alpha$ by $\alpha(N+1)/(N-3)$, and perform the integrations over $x_j$ for $j = N, N-1, \dots$ until we finally reach a value of $j$ for which the integral over $x_j$ converges uniformly with respect to all $x_i$, $i < j$. This may already occur for $j = N$ if $\alpha$ is sufficiently large (for instance $\alpha \geq 0$) and does certainly occur for some $j \geqslant 4$ if $\alpha > \alpha_1 \equiv - n(N - 3)/(N - 2)(N + 1)$. This proves the finiteness of L and therefore the boundedness of G uniformly in $r_{12}$. Therefore condition (2.26) is sufficient in this case.

We next consider the case

$$n + \alpha(N + 1) \leq 0 \tag{2.37}$$

and estimate the integral G by the same method as above. For $\alpha$ sufficiently small, all the integrals over $x_N, \ldots, x_3$ have a negative power counting and we therefore obtain an estimate

$$G \leq C r_{12}{}^{(n+(N+1)\alpha)(N-2)}$$

for some constant C independent of $r_{12}$. This implies the sufficiency of condition (2.27). This argument turns out to apply in the interval $- n/(N - 1 < \alpha < \alpha_2 \equiv - n(2N - 3)/2N(N - 1)$. In the intermediate interval $\alpha_2 \leqslant \alpha \leqslant \alpha_1$, we estimate G again by integrating over $x_N, \ldots, x_3$ in that order, but we now apply Hölder's inequality in such a way that no factor $r_{12}$ is produced when estimating the integral over $x_j$ for $j \geq 4$. The negative power of $r_{12}$ expected from power counting is then entirely produced by the integration over $x_3$. Again condition (2.27) is found to be sufficient, except in the special case where equality occurs in (2.37). In this case, we obtain only

$$G = O(|\operatorname{Log} r_{12}|)$$

so that strict inequality is required in (2.27) to ensure uniform boundedness in $\varepsilon$. We omit the details.

This completes the proof of proposition 2.2.

In the last part of this section, we combine the estimates contained in propositions 2.1 and 2.2 to solve the following problem: find a real constant $c$ as small as possible such that there exists a $\psi$ in the class defined by (2.5) and (2.6) and a $\lambda \geq 0$ such that $\psi$ satisfies the inequality

$$(H_0 + cV_0 + \lambda)\psi \geq 0 \tag{2.38}$$

and condition (C). We consider only the special cases $\gamma = 2$ and $\gamma = 1$, which will be useful in sections 3 and 4 respectively. For each choice of $n$ and N, we give the values of $\alpha$ and $\beta$ that occur in $\psi$, and the corresponding value of $c$. In all cases one may take any $\lambda \geq N/2$.

PROPOSITION 2.3. — Let $\psi$ and $\varphi_\varepsilon$ be defined by (2.5,6) and (2.23). Let $\gamma = 2$. Then condition (2.38) and condition (C) hold in the following cases:

(1) $n = 1$, $N \geq 2$: take $\alpha = 3/2$, $\beta = 0$, $c = 3/4$.

(2) $n \geq 2$, $N \geq 2$.

If: $(N - 2)(n - 4) \leq 4$

take: $\alpha = 2 - n/2$, $\beta = 0$ and $c = -\frac{1}{4}(n - 4)\left[n + (4 - n)\left(\frac{1}{2}N - 1\right)\right]$

If: $(N - 2)(n - 4) \geq 4$

take: $\alpha = -(n - 2)/N$, $\beta = 0$, and $c = -(n - 2)^2/2N$.

(3) $N = 3$. If $n < 6$, take $\alpha = 2 - n/2$, any $\beta$ such that $3 \leq \beta + n/2 < 4$ and $c = -n(n-4)/4$.
If: $n = 6$, take $\alpha > -1$, $\beta = 1$ and $c = \alpha(\alpha + n - 2) > -3$.

*Proof.*

(1) follows immediately from propositions 2.1.1 and 2.2.2.

(2) follows from propositions 2.1.2 and 2.2.2. We take $\beta = 0$ in this case. Because of (2.17), we can take $c = \alpha\left(\frac{1}{2}N\alpha + n - 2\right)$ where $\alpha$ satisfies the conditions

$$(N-1)(n + N\alpha) \geq 4 \tag{2.39}$$

$$n + 2\alpha \geq 4 \tag{2.40}$$

The optimal value of $\alpha$ is that which minimizes $c$ under these conditions, namely $\alpha_m = -(n-2)/N$ if $\alpha_m$ satisfies (2.39, 40), and the minimal value of $\alpha$ that satisfies (2.39, 40) if $\alpha_m$ does not. Now $\alpha_m \geq 2 - n/2$ if and only if $(N-2)(n-4) \geq 4$, while the condition (2.39) coincides with (2.40) for $N = 2$ and is never relevant for $N \geq 3$. The result follows immediately, except for $N = 3$, $n = 8$ where we hit the exceptional point in proposition 2.2.2 where a logarithmic divergence occurs. This case however is covered by (3) below.

(3) follows from propositions 2.1.3 and 2.2.1. Indeed we may take $c = \alpha(\alpha + n - 2)$ and choose any $(\alpha, \beta)$ such that:

$$\left.\begin{aligned} &\beta \leq 3\alpha + n - 2 \\ &n + 2\alpha \geq 4 \end{aligned}\right\} \tag{2.41}$$

and

$$\beta^2 - 3\beta(3\alpha + n - 1) + 3\alpha^2 \leq 0 \tag{2.42}$$

This allows $\alpha = 2 - n/2$ for $n < 6$. For $n = 6$, the only point with $\alpha = 2 - n/2 = -1$ has also $\beta = 1$ and is therefore the exceptional point in proposition 2.2.1 with a logarithmic divergence. We keep away from it by taking $\alpha > -1$. For $n > 6$, we obtain from (2.41, 42) a value of $c$ that is slightly better (i. e. smaller) than the value $-(n-2)^2/6$ obtained in (2), but is nevertheless different from the expected value $-n(n-4)/4$. This is a negligible improvement and we do not write down the precise value of $c$.

This completes the proof of proposition 2.3.

We finally consider the case $\gamma = 1$.

PROPOSITION 2.4. — Let $\psi$ and $\varphi_\varepsilon$ be defined by (2.5, 6) and (2.23). Let $\gamma = 1$. Then condition (2.38) and condition (C) hold in the following cases:

(1) $n = 1$, $N \geq 2$. Take $\beta = 0$, $\alpha \geq \frac{1}{2}$ and $c = \alpha(\alpha - 1)$

The optimal values are $\alpha = 1/2$, $c = -1/4$.

(2) $n \geq 2$, $N \geq 2$. Take $\beta = 0$, $\alpha \geq -(n-2)/N$ and $c = \alpha\left(\frac{1}{2}N\alpha + n - 2\right)$.

The optimal values are $\alpha = -(n-2)/N$ and $c = -(n-2)^2/2N$.
(3) $n \geq 3$, $N = 3$. Take $\beta = (n-1)(\sqrt{3}-1)/2$,
$\alpha \geq -(n-1)/(3+\sqrt{3})$, $c = \alpha(\alpha + n - 2)$. The optimal values are $\alpha = -(n-1)/(3+\sqrt{3})$, $c = -(n-1)(n-4+\sqrt{3})/6$.

*Proof.* — The proof is similar to that of proposition 2.3 and will be omitted.

## 3. ESSENTIAL SELF ADJOINTNESS

We now state and prove our main result. We consider again a system of N particles ($N \geq 2$) with Hamiltonian $\hat{H} = H_0 + V$ with $H_0$ and V defined by (2.1) and (1.2). We define S and $\mathscr{D}_0$ by (2.2) and (2.3).

THEOREM 3.1. — $\hat{H}$ is essentially self adjoint on $\mathscr{D}_0$ provided each $V_{ij}$ satisfies the following conditions:

(1) $$V_{ij} \in L^2_{loc}(\mathbb{R}^n \backslash \{0\}) \tag{3.1}$$

(2) $$V_{ij} \geq c r_{ij}^{-2} \tag{3.2}$$

where $c$ takes the values given in proposition 2.3, namely

$$c = -n(n-4)/4$$

for $N = 2$, $n \geq 1$ or $N \geq 2$, $n = 1$ or 4, or $N = 3$, $n < 6$.

$$c > -n(n-4)/4 = -3 \qquad \text{for} \qquad N = 3, n = 6.$$

$$c = -(n-2)^2/2N$$

for $N = 3$, $n \geq 7$ or $N \geq 4$, $(N-2)(n-4) \geq 4$.

$$c = -\frac{1}{4}(n-4)\left[n + (4-n)\left(\frac{1}{2}N - 1\right)\right]$$

for $n \geq 2$, $N \geq 4$, $(N-2)(n-4) \leq 4$.

*Proof.* — The proof is very similar to those in [*6*] and [*13*]. Let the potentials $V_{ij}$ satisfy (3.1) and (3.2) where the constant $c$ is left unspecified for the moment. The operator $\hat{H}$ is defined on $\mathscr{D}_0$ and maps $\mathscr{D}_0$ into $\mathscr{H}$. By duality, the adjoint operator $\tilde{H}$ maps $\mathscr{H}$ into $\mathscr{D}'_0$, the space of distributions in $\Omega = \mathbb{R}^{n(N-1)} \backslash S$. $\tilde{H}$ can be described as follows. Any $\varphi \in \mathscr{H}$ belongs to $L^1_{loc}(\Omega)$ and therefore to $\mathscr{D}'_0$. $H_0\varphi$ is defined by applying to it the differential operator of $H_0$ in the sense of distributions. Since $V \in L^2_{loc}(\Omega)$, $V\varphi \in L^1_{loc}(\Omega)$ and therefore $V\varphi \in \mathscr{D}'_0$. One then takes

$$\tilde{H}\varphi = H_0\varphi + V\varphi.$$

The Hilbert space adjoint $\hat{H}^*$ of $\hat{H}$ is the restriction of $\tilde{H}$ to the subspace of those $\varphi \in \mathscr{H}$ such that $\tilde{H}\varphi \in \mathscr{H}$.

In order to prove that $\hat{H}$ is essentially self adjoint on $\mathscr{D}_0$, it suffices to prove that for some $\lambda \geq 0$, $(\lambda + \tilde{H})$ is injective from $\mathscr{H}$ into $\mathscr{D}'_0$, namely that $\varphi \in \mathscr{H}$ and $(\lambda + \tilde{H})\varphi = 0$ imply $\varphi = 0$.

Let therefore $\varphi \in \mathscr{H}$ and $(\lambda + \tilde{H})\varphi = 0$ for some $\lambda \geq \frac{N}{2} + 1$. We want to show that $\varphi = 0$. Now $H_0\varphi = -(\lambda + V)\varphi \in L^1_{loc}(\Omega)$. It then follows from a lemma of Kato ([*6*], lemma A, page 138) that

$$H_0 \mid \varphi \mid \leq \mathrm{Re} \frac{\bar{\varphi}}{\mid \varphi \mid} H_0\varphi \tag{3.3}$$

and therefore, since $V \geq cV_0$ (where $V_0$ is defined by (2.7)),

$$(H_0 + cV_0 + \lambda) \mid \varphi \mid \leq 0 \tag{3.4}$$

in the sense of distributions, i. e. weakly on $\mathscr{D}_0$.

Let us now introduce an auxiliary function $f$ of a real positive variable $t$ such that:

$f \in \mathscr{C}^\infty([0, \infty))$

$f(t) = 0 \quad \text{if} \quad 0 \leq t \leq 1$

$f(t) = 1 \quad \text{if} \quad t \geq 2$

$f(t)$ increases monotonically from 0 to 1 in the interval $[1, 2]$.

For $\varepsilon > 0$, we define a function $f_\varepsilon$ by $f_\varepsilon(t) = f(t/\varepsilon)$ and a function $F_\varepsilon$ by

$$F_\varepsilon(x) = \prod_{i<j} f_\varepsilon(r_{ij})[1 - f_{\varepsilon^{-1}}(r)] \tag{3.5}$$

where $r$ is defined by (2.4). Clearly $F_\varepsilon \in \mathscr{D}_0$. We shall also denote by $F_\varepsilon$ the operator of multiplication by this function in $\mathscr{D}'_0$.

We now assume the existence of a function $\psi$ with the following properties

(A1) $\psi \in \mathscr{C}^\infty(\Omega)$

(A2) $\psi$ is strictly positive almost everywhere

(A3) $(H_0 + cV_0 + \lambda)\psi \geq \psi$ in the ordinary sense in $\Omega$.

(A4) $\psi \in \mathscr{H}$ and $[H_0, F_\varepsilon]\psi$ tends to zero weakly in $\mathscr{H}$ when $\varepsilon$ tends to zero.

($[H_0, F_\varepsilon]\psi$ is defined in the ordinary sense and lies in $\mathscr{D}_0$).

The existence of such a $\psi$ will be established below. Assuming this for the moment, we proceed with the proof of the theorem. Because of (A1), $F_\varepsilon\psi \in \mathscr{D}_0$ and because of (A2), $F_\varepsilon\psi \geq 0$. We then obtain from (3.4):

$$\langle F_\varepsilon\psi, (H_0 + cV_0 + \lambda) \mid \varphi \mid \rangle \leq 0 \tag{3.6}$$

and therefore

$$\langle F_\varepsilon(H_0 + cV_0 + \lambda)\psi, \mid \varphi \mid \rangle + \langle [H_0, F_\varepsilon]\psi, \mid \varphi \mid \rangle \leq 0 \tag{3.7}$$

From (3.7) and (A3) it follows that:

$$\langle F_\varepsilon\psi, \mid \varphi \mid \rangle + \langle [H_0, F_\varepsilon]\psi, \mid \varphi \mid \rangle \leq 0 \tag{3.8}$$

Let now $\varepsilon \downarrow 0$. The first term in the L. H. S. of (3.8) tends to $\langle \psi, \mid \varphi \mid \rangle$ while the second tends to zero by (A4). Therefore $\langle \psi, \mid \varphi \mid \rangle \leq 0$, and therefore $\varphi = 0$ because of (A2).

We complete the proof of theorem 3.1 by establishing the existence of $\psi$ satisfying (A1)-(A4). We take $\psi$ in the form defined by (2.5, 6). (A1) and (A2) are then satisfied. (A3) and (A4) with $\lambda = 1 + N/2$ and the values of $c$ given in theorem 3.1 follow immediately from proposition 2.3 and lemma 3.1 below. The condition $\psi \in \mathcal{H}$ is easily seen to hold for the choices of $\alpha$ and $\beta$ described in proposition 2.3.

LEMMA 3.1. — Let $\psi$ and $\varphi_\varepsilon$ be defined by (2.5, 6) and (2.23) with $\gamma = 2$ and let $\varphi_\varepsilon$ satisfy condition (C). Then $[H_0, F_\varepsilon]\psi$ tends to zero weakly in $\mathcal{H}$ when $\varepsilon$ tends to zero.

*Proof.* — We compute

$$[H_0, F_\varepsilon]\psi = (H_0F_\varepsilon)\psi - \sum_{1 \leq i \leq N} (\nabla_i F_\varepsilon).(\nabla_i \psi) \tag{3.9}$$

Now:

$$H_0F_\varepsilon = \left\{ -\frac{1}{2}\sum_{i \neq j} [f_\varepsilon''(r_{ij}) + (n-1)r_{ij}^{-1} f_\varepsilon'(r_{ij})] f_\varepsilon(r_{ij})^{-1} \right.$$
$$\left. - \sum_{\substack{i,j,k \\ \text{all} \neq}} (x_{ij}.x_{ik})(r_{ij}r_{ik})^{-1} f_\varepsilon'(r_{ij}) f_\varepsilon'(r_{ik}) \Big( f_\varepsilon(r_{ij}) f_\varepsilon(r_{ik}) \Big)^{-1} \right\} F_\varepsilon \tag{3.10}$$

and

$$\sum_i \nabla_i F_\varepsilon . \nabla_i \psi = \sum_{i \neq j} \sum_{k(\neq i)} (x_{ij}.x_{ik}) r_{ij}^{-1} f_\varepsilon'(r_{ij})$$
$$\times f_\varepsilon(r_{ij})^{-1}(\alpha r_{ik}^{-2} - \beta r^{-2} - r^{-1})F_\varepsilon \psi \tag{3.11}$$

Here and below, we neglect the contribution of the last factor in (3.5) which is harmless for any $\alpha$ and $\beta$. From (3.10, 11) we obtain:

$$| H_0F_\varepsilon | \leq \sum_{i<j} \left\{ \left| f_\varepsilon''(r_{ij}) \right| + (n-1)r_{ij}^{-1} f_\varepsilon'(r_{ij}) + 2(N-2) f_\varepsilon'(r_{ij})^2 \right\} \tag{3.12}$$

and

$$\left| \sum_i \nabla_i F_\varepsilon . \nabla_i \psi \right| \leq 2(N-1)(2|\alpha| + |\beta| + 2\varepsilon) \sum_{i<j} r_{ij}^{-1} f_\varepsilon'(r_{ij})\psi \tag{3.13}$$

where we have used the inequality

$$f_\varepsilon'(r_{ij}) r_{ik}^{-1} f_\varepsilon(r_{ik}) \leq 2 r_{ij}^{-1} f_\varepsilon'(r_{ij})$$

which follows from the fact that the L. H. S. has support in the region $\varepsilon \leq r_{ij} \leq 2\varepsilon \leq 2r_{ik}$.

From the definition of $f_\varepsilon$ it follows that the support of $[H_0, F_\varepsilon]\psi$ shrinks to zero when $\varepsilon$ tends to zero. It is therefore sufficient to show that $\| [H_0, F_\varepsilon]\psi \|$

is bounded uniformly in $\varepsilon$ near zero. Now from the definitions of $f_\varepsilon$ and $g_\varepsilon$, it follows that

$$\left.\begin{aligned} f_\varepsilon'(t) &\le Bt^{-1}g_\varepsilon(t) \\ |f_\varepsilon''(t)| &\le Bt^{-2}g_\varepsilon(t) \end{aligned}\right\} \tag{3.14}$$

for some constant B independent of $\varepsilon$. Comparing (3.9), (3.12), (3.13), and (3.14) yields

$$|[H_0, F_\varepsilon]\psi| \le C\sum_{i<j} r_{ij}^{-2}g_\varepsilon(r_{ij})\psi \tag{3.15}$$

for some constant C independent of $\varepsilon$.

Since $\psi$ is symmetric with respect to the N particles, the L. H. S. of (3.15) is uniformly bounded in norm as $\varepsilon \downarrow 0$ if $\varphi_\varepsilon$ is. This completes the proof of lemma 3.1.

## 4. QUADRATIC FORMS AND FRIEDRICHS EXTENSION

In this section, we shall consider possible definitions of the Hamiltonian through the use of quadratic forms. In the whole section, we assume that the potentials $V_{ij}$ satisfy the conditions $V_{ij} \in L^1_{\text{loc}}(\mathbb{R}^n\backslash\{0\})$ and $V_{ij} \ge cr_{ij}^{-2}$ for some suitable $c$. We consider first the case of general values of $n$ and N. Possible improvements in the special case $N = 3$, $n \ge 3$ will be mentioned in remark 4.1 below.

We first derive some identities and inequalities for operators defined on $\mathcal{D}_0$ and the associated quadratic forms defined on $\mathcal{D}_0 \times \mathcal{D}_0$. Let

$$\hat{D}_i = \nabla_i - \nabla_i\omega \tag{4.1}$$

where the subscript $i$ labels the particles $(1 \le i \le N)$, $\alpha$ takes any real value, and

$$\omega = \alpha\sum_{i<j} \operatorname{Log} r_{ij} \tag{4.2}$$

Define

$$\hat{H}(\alpha) = H_0 + \alpha(\alpha + n - 2)V_0 \qquad \text{for} \qquad n = 1 \quad \text{or} \quad N = 2 \tag{4.3}$$

$$\hat{H}(\alpha) = H_0 + \alpha\left(\frac{1}{2}N\alpha + n - 2\right)V_0 \qquad \text{for} \qquad n \ge 2,\ N \ge 2 \tag{4.4}$$

where $V_0$ is defined by (2.7). All these operators are defined with domain $\mathcal{D}_0$. An elementary computation, almost identical with that in the proof of proposition 2.1, yields:

$$\hat{H}(\alpha) = \frac{1}{2}\sum_i \hat{D}_i^*\hat{D}_i \qquad \text{for} \qquad n = 1 \quad \text{or} \quad N = 2 \tag{4.5}$$

$$\hat{H}(\alpha) = \frac{1}{2}\sum_i \hat{D}_i^*\hat{D}_i + \alpha^2\left(\left(\frac{1}{2}N - 1\right)V_0 - U\right) \qquad \text{for} \qquad n \ge 2,\ N \ge 3 \tag{4.6}$$

The last term in (4.6) is positive by lemma 2.1.3. Let now

$$\alpha_m = -(n-2)/2,\ c(\alpha) = \alpha(\alpha + n - 2) \qquad \text{for} \qquad n = 1 \quad \text{or} \quad N = 2$$

$$\alpha_m = -(n-2)/N,\ c(\alpha) = \alpha\left(\frac{1}{2}N\alpha + n - 2\right) \qquad \text{for} \qquad n \geq 2,\ N \geq 2$$

and $c_m = c(\alpha_m)$ in both cases. $c(\alpha)$ is minimum for $\alpha = \alpha_m$ and increasing for $\alpha \geq \alpha_m$. Then for any $\alpha' \geq \alpha > \alpha_m$, the following inequality holds (between quadratic forms on $\mathscr{D}_0 \times \mathscr{D}_0$):

$$0 \leq (c(\alpha) - c_m)V_0 \leq \hat{H}(\alpha) \leq \hat{H}(\alpha') \leq \frac{c(\alpha') - c_m}{c(\alpha) - c_m}\hat{H}(\alpha) \tag{4.7}$$

In particular

$$H_0 \geq [(n-2)^2/4]V_0 \qquad \text{for} \qquad n = 1 \quad \text{or} \quad N = 2 \tag{4.8}$$

$$H_0 \geq [(n-2)^2/2N]V_0 \qquad \text{for} \qquad n \geq 2,\ N \geq 2 \tag{4.9}$$

All these inequalities follow from the fact that $\hat{D}_i^*\hat{D}_i \geq 0$ for any $\alpha$. The last one for general N can also be proved directly from the special case $N = 2$ as follows: let $p_i = -i\nabla_i$. Then from (2.1):

$$H_0 = \frac{1}{2}\sum_i p_i^2 - \frac{1}{2N}\left(\sum_i p_i\right)^2 = \frac{1}{2N}\sum_{i<j}(p_i - p_j)^2 \tag{4.10}$$

Now for $N = 2$, (4.9) is the inequality $p_{ij}^2 \geq (n-2)^2/4r_{ij}^2$ where $p_{ij} = \frac{1}{2}(p_i - p_j)$ is the relative momentum between the two particles. Substituting this inequality into (4.10) yields (4.9) for general N.

We next define operators $D_i$ and $H(\alpha)$ by extending $\hat{D}_i$ and $\hat{H}(\alpha)$ in a natural way. We recall that there is a one-to-one correspondence between positive self-adjoint operators and closed positive quadratic forms, such that the domain of the closed form associated with the operator A is $\mathscr{D}(\sqrt{A})$ ([5], chap. 6). This domain will be called the form domain of A and will be denoted Q(A). If A and B are two positive self adjoint operators, we denote by $A +_q B$ their form sum, i. e. the unique positive self-adjoint operator whose associated quadratic form is the sum of those of A and B. In particular $Q(A +_q B) = Q(A) \cap Q(B)$ ([5], chap. 6).

We now define $D_i$ as the closure of $\hat{D}_i$. The operator $D_i^*D_i$ is the self-adjoint operator associated with the form $\| D_i\varphi \|^2$. In particular, $Q(D_i^*D_i) = \mathscr{D}(D_i)$. We define the positive self-adjoint operator $H(\alpha)$ by

$$H(\alpha) = -\frac{1}{2}\Sigma_q D_i^*D_i \qquad \text{for} \qquad n = 1 \quad \text{or} \quad N = 2 \tag{4.11}$$

$$H(\alpha) = \frac{1}{2}\Sigma_q D_i^*D_i +_q \alpha^2\left(\left(\frac{1}{2}N - 1\right)V_0 - U\right) \qquad \text{for} \qquad n \geq 2,\ N \geq 3 \tag{4.12}$$

In particular

$$Q(H(\alpha)) = \bigcap_i \mathscr{D}(D_i) \quad \text{for} \quad n=1 \text{ or } N=2 \text{ or } \alpha=0$$

$$Q(H(\alpha)) = \bigcap_i \mathscr{D}(D_i) \cap Q\left(\left(\frac{1}{2}N-1\right)V_0 - U\right) \quad \text{for} \quad n \geq 2,\ N \geq 3 \text{ and } \alpha \neq 0.$$

We are now prepared to introduce the interaction and define the total Hamiltonian. Let the potentials $V_{ij}$ satisfy the condition

(B1) $$V_{ij} \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$$

We define an operator $\tilde{H}$ from $\mathscr{H}$ to $\mathscr{D}'_0$ with domain

$$\mathscr{D}(\tilde{H}) = \{ \varphi : \varphi \in \mathscr{H} \quad \text{and} \quad V\varphi \in L^1_{loc}(\Omega) \}$$

by $\tilde{H}\varphi = H_0\varphi + V\varphi$, in the sense of distributions. We also define an operator $H^+$ from $\mathscr{H}$ to $\mathscr{H}$ as the restriction of $\tilde{H}$ to those $\varphi \in \mathscr{D}(\tilde{H})$ such that $\tilde{H}\varphi \in \mathscr{H}$ (cf. [7]).

Suppose in addition that the $V_{ij}$ satisfy the condition:

(B2) $\quad V_{ij} \geq c r_{ij}^{-2} \quad$ where $\quad c \equiv c(\alpha) \quad$ and $\quad \alpha \geq \alpha_m$.

Define:

$$H = H(\alpha) +_q (V - cV_0) \tag{4.13}$$

In particular:

$$Q(H) = Q(H(\alpha)) \cap Q(V - cV_0)$$

One sees easily that $H \subset H^+$ ([7]).

Let $h$ be the closed positive quadratic form associated with H, and let $\hat{h}$ be the restriction of $h$ to $\mathscr{D}_0 \times \mathscr{D}_0$. $\hat{h}$ is the quadratic form defined in an obvious way by $H_0 + V$ on $\mathscr{D}_0 \times \mathscr{D}_0$. The main result of this section is the following.

THEOREM 4.1. — Let V satisfy conditions (B1, 2) and let $h$ and $\hat{h}$ be defined as above. Then $h$ is the closure of $\hat{h}$.

*Proof.* — We want to prove that $\mathscr{D}_0$ is dense in Q(H) in the sense of the norm $\|\varphi\|_H^2 = \lambda\|\varphi\|^2 + h(\varphi, \varphi)$ where $\lambda$ is some strictly positive number. We choose $\lambda = 1 + N/2$. This property is equivalent to the fact that $\varphi \in Q(H)$ and $\varphi$ orthogonal to $\mathscr{D}_0$ in the sense of the corresponding scalar product imply $\varphi = 0$. Now $\varphi \in Q(H)$ implies $\varphi \in Q(V - cV_0)$, therefore $V\varphi \in L^1_{loc}(\Omega)$ and $\varphi \in \mathscr{D}(\tilde{H})$. The condition that $\varphi \in Q(H)$ be orthogonal to $\mathscr{D}_0$ in the previous sense then becomes

$$(\tilde{H} + \lambda)\varphi = 0 \qquad \text{(weakly on } \mathscr{D}_0) \tag{4.14}$$

From (4.14) and the fact that $V\varphi \in L^1_{loc}(\Omega)$, it follows that also $H_0\varphi \in L^1_{loc}(\Omega)$. Therefore, by Kato's lemma [*6*]:

$$H_0 |\varphi| \leq Re \frac{\bar{\varphi}}{|\varphi|} H_0\varphi \tag{4.15}$$

Therefore:

$$(\mathrm{H}(\alpha) + \lambda) | \varphi | \leq 0 \tag{4.16}$$

where we have used the inequality $\mathrm{V} - c\mathrm{V}_0 \geq 0$.

Let now $\psi$ be defined by (2.5, 6) with $\beta = 0$ and the same $\alpha$ as in condition (B2) and the definition of $\mathrm{H}(\alpha)$. This $\psi$ satisfies condition (A1, 2, 3) stated in the proof of theorem (3.1). In particular (A3) follows from proposition (2.1.1, 2). Define $\mathrm{F}_\varepsilon$ by (3.5). Then $\mathrm{F}_\varepsilon\psi \in \mathscr{D}_0$ and (4.16) implies:

$$\langle \mathrm{F}_\varepsilon\psi, (\mathrm{H}(\alpha) + \lambda) | \varphi | \rangle \leq 0 \tag{4.17}$$

Using the definition of $\mathrm{H}(\alpha)$, we obtain from (4.17):

$$\frac{1}{2}\sum_i \langle \mathrm{D}_i\mathrm{F}_\varepsilon\psi, \mathrm{D}_i | \varphi | \rangle + \left\langle \mathrm{F}_\varepsilon\psi, \left[\lambda + \alpha^2\left(\left(\frac{\mathrm{N}}{2} - 1\right)\mathrm{V}_0 - \mathrm{U}\right)\right] | \varphi | \right\rangle \leq 0$$

or:

$$\frac{1}{2}\left\{ \sum_i \langle \mathrm{F}_\varepsilon\mathrm{D}_i\psi, \mathrm{D}_i | \varphi | \rangle + \sum_i \langle (\nabla_i\mathrm{F}_\varepsilon)\psi, \mathrm{D}_i | \varphi | \rangle \right\} + \left\langle \mathrm{F}_\varepsilon\psi, \left[\lambda + \alpha^2\left(\left(\frac{1}{2}\mathrm{N} - 1\right)\mathrm{V}_0 - \mathrm{U}\right)\right] | \varphi | \right\rangle \leq 0$$

for $n \geq 2$ and $\mathrm{N} \geq 3$. For $n = 1$ or $\mathrm{N} = 2$, the quantity $\alpha^2\left(\left(\frac{1}{2}\mathrm{N} - 1\right)\mathrm{V}_0 - \mathrm{U}\right)$ should be omitted in the L. H. S. of both inequalities. In all cases:

$$2 \langle \mathrm{F}_\varepsilon(\mathrm{H}_0 + c\mathrm{V}_0 + \lambda)\psi, | \varphi | \rangle - \sum_i \langle (\nabla_i\mathrm{F}_\varepsilon).\mathrm{D}_i\psi, | \varphi | \rangle + \sum_i \langle (\nabla_i\mathrm{F}_\varepsilon)\psi, \mathrm{D}_i | \varphi | \rangle \leq 0 \tag{4.18}$$

From (4.18), property (A3) and the identity

$$\mathrm{D}_i\psi = - r^{-1} \sum_{j(\neq i)} x_{ij}\psi$$

it follows that

$$2 \langle \mathrm{F}_\varepsilon\psi, | \varphi | \rangle + \sum_i \left\langle (\nabla_i\mathrm{F}_\varepsilon)\psi, \left(r^{-1} \sum_{j(\neq i)} x_{ij} + \mathrm{D}_i\right) | \varphi | \right\rangle \leq 0 \tag{4.19}$$

From $\varphi \in \mathrm{Q}(\mathrm{H}(\alpha))$, it follows that $\varphi \in \mathscr{D}(\mathrm{D}_i)$ and therefore $\nabla_i\varphi \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$. This implies

$$\mathrm{D}_i | \varphi | = \mathrm{Re} \frac{\bar{\varphi}}{| \varphi |} \mathrm{D}_i\varphi \tag{4.20}$$

in the sense of distributions, i. e. weakly on $\mathscr{D}_0$. The proof of (4.20) is analogous to that of Kato's lemma, but simpler.

We now complete the proof of the theorem, assuming for the moment that $\psi$ satisfies the condition:

(A') $\psi \in \mathscr{H}$ and $(\nabla_i F_\varepsilon)\psi$ tends to zero weakly in $\mathscr{H}$ when $\varepsilon \downarrow 0$.

We let $\varepsilon$ tend to zero in (4.19). It follows from (A'), from (4.20) and the fact that $D_i\varphi \in \mathscr{H}$, and from the inequality

$$\left| \sum_{j(\neq i)} x_{ij} \right| \leq r\mathrm{N}^{1/2}$$

that the second term in the L. H. S. of (4.19) tends to zero while the first term tends to $\langle \psi, | \varphi | \rangle$. Therefore $\langle \psi, | \varphi | \rangle \leq 0$ and therefore $\varphi = 0$ because of (A2).

It remains to be shown that $\psi$ satisfies (A'). One sees easily that $\psi \in \mathscr{H}$, while

$$\nabla_i F_\varepsilon = \sum_{j(\neq i)} [r_{ij}^{-1} x_{ij} f_\varepsilon'(r_{ij}) f_\varepsilon^{-1}(r_{ij})] F_\varepsilon \tag{4.21}$$

where we have again neglected the contribution of the last factor in (3.5). From (2.22) and (3.14), we obtain:

$$| \nabla_i F_\varepsilon | \leq \mathrm{B} \sum_{j(\neq i)} r_{ij}^{-1} g_\varepsilon(r_{ij}) \tag{4.22}$$

(A') follows from (4.22) and proposition 2.4.1, 2.

This completes the proof of theorem 4.1.

REMARK 4.1. — Theorem 4.1 yields the strongest results when the assumption on $V_{ij}$ is the weakest, i. e. when $\alpha = \alpha_m$, $c = c_m$ in assumption (B2). In the special case $\mathrm{N} = 3$, $n \geq 3$, the result of theorem 4.1 can be improved by using a more elaborate $\omega$ with $\beta \neq 0$. The definition of H has to be modified suitably. The net result is to replace $c_m = -(n-2)^2/6$ by $c = -(n-1)(n-4+\sqrt{3})/6$. The proof is obtained by trivial modifications of that of theorem 4.1, using proposition 2.4.3 instead of 2.4.1, 2.

We now exhibit some consequences of theorem 4.1, in order to clarify its meaning. We use the following definition. Let A be a positive symmetric operator with domain $\mathscr{D}(\mathrm{A})$. We define the Friedrichs extension of A as the positive self adjoint operator associated with the closure of the positive closable form $a(\varphi, \psi) = \langle \varphi, \mathrm{A}\psi \rangle$ defined on $\mathscr{D}(\mathrm{A}) \times \mathscr{D}(\mathrm{A})$.

COROLLARY 4.1. — Let $\alpha \geq \alpha_m$ and $c = c(\alpha)$, and let $\hat{\mathrm{H}}(\alpha)$, $\mathrm{H}(\alpha)$ and H be defined as above. Then

(1) $\mathrm{H}(\alpha)$ is the Friedrichs extension of $\hat{\mathrm{H}}(\alpha)$.

(2) $\mathrm{H}(\alpha)$ coincides with the form sum

$$\mathrm{H}(\alpha) = \mathrm{H}(\alpha_m) +_q (c - c_m)\mathrm{V}_0 \tag{4.23}$$

In particular for $\alpha > \alpha_m$, $Q(H(\alpha)) = Q(H(\alpha_m)) \cap Q(V_0)$ is independent of $\alpha$ and contained in $Q(V_0)$.

(3) Let $V_{ij}$ satisfy conditions (B1, 2). Then the form sum

$$H(\alpha') +_q (V - c(\alpha')V_0) \tag{4.24}$$

is independent of $\alpha'$ for $\alpha_m \leq \alpha' \leq \alpha$, and therefore coincides with H.

(4) The form sum $H(\alpha) +_q (-c(\alpha)V_0)$ is independent of $\alpha$ for $\alpha_m \leq \alpha \leq 0$ $(n \geq 3)$ or for $\alpha_m (= 1/2) \leq \alpha \leq 1$ $(n = 1)$. It coincides with $H_0$ in the range $\alpha_m \leq \alpha \leq 0$ if $n \geq 3$, or for $\alpha = \alpha_m = 0$ if $n = 2$.

*Proof.*

(1) Apply theorem 4.1 with $V_{ij} = cr_{ij}^{-2}$.

(2) Take $V_{ij} = cr_{ij}^{-2}$ and apply theorem 4.1 once for $\alpha$ and once for $\alpha_m$: both members of (4.23) coincide with the Friedrichs extension of $\hat{H}(\alpha)$.

(3) By theorem 4.1, the quadratic form of the operator

$$H(\alpha') +_q (V - c(\alpha')V_0)$$

is the closure of its restriction to $\mathscr{D}_0 \times \mathscr{D}_0$, where it coincides with the quadratic form defined by $H_0 + V$ and is therefore independent of $\alpha'$.

(4) Applying (3) with $V = 0$ proves the first point (the first statement is empty for $n = 2$, since the only admissible value of $\alpha$ is zero in this case). For $n \geq 2$, one can prove that $H_0$ coincides with the Friedrichs extension of its restriction to $\mathscr{D}_0$. The proof is almost identical with that of theorem 4.1, with however $\alpha = 0$ and $D_i$ replaced by the operator $\nabla_i$ with its usual domain. The second statement in (4) then follows from equality of the restrictions to $\mathscr{D}_0 \times \mathscr{D}_0$ of the quadratic forms of the operators $H_0$ and $H(\alpha) +_q (-c(\alpha)V_0)$.

REMARK 4.2. — For $n = 1$, the operator $H(\alpha) +_q (-c(\alpha)V_0) = H(1)$ is not $H_0$. It describes instead particles with point hard cores (namely with wave functions vanishing on S).

REMARK 4.3. — If $V_{ij} \geq 0$ and $n \geq 2$, it follows from corollary 4.1.3, 4 that H coincides with the form sum $H_0 +_q V$.

We have seen in theorem 4.1 that H can be defined by extension of the quadratic form $\hat{h}$ with domain $\mathscr{D}_0$. We shall now see that under similar assumptions, H can also be obtained as a suitable restriction of $\tilde{H}$.

THEOREM 4.2. — Let $V_{ij}$ satisfy conditions (B1) and (B2) where now $\alpha > \alpha_m$, $c > c_m$, and let H′ be the restriction of $H^+$ to the domain

$$\mathscr{D}(H') = \mathscr{D}(H^+) \cap Q(V_0)$$

Then $H' = H$ (In particular H′ is self adjoint).

*Proof.* — We already know that $Q(H) \subset Q(V_0)$ because of corollary 4.1.2 and the definition (4.13) of H. Therefore $H \subset H'$. Let now $\theta' \in \mathscr{D}(H')$ and $\lambda = 1 + N/2$. Since H is positive self adjoint, there exists $\theta \in \mathscr{D}(H)$ such that $(\lambda + H')\theta' = (\lambda + H)\theta$. Therefore $(\tilde{H} + \lambda)\varphi = 0$ where $\varphi = \theta' - \theta \in \mathscr{D}(H')$, since both H and H′ are restrictions of $\tilde{H}$. Since $\varphi \in \mathscr{D}(\tilde{H})$, $V\varphi \in L^1_{loc}(\Omega)$. From Kato's lemma [*6*] and assumption (B2), we obtain as in the proof of theorem 4.1

$$(H(\alpha) + \lambda) \mid \varphi \mid \leq 0 \qquad \text{(weakly on } \mathscr{D}_0) \tag{4.25}$$

We introduce the same $\psi$ and $F_\varepsilon$ as in the proof of theorem 4.1 and deduce from (4.25) that

$$\langle F_\varepsilon \psi, (H(\alpha) + \lambda) \mid \varphi \mid \rangle \leq 0 \tag{4.26}$$

From this it follows as in the proof of theorem 3.1 that

$$\langle F_\varepsilon \psi, \mid \varphi \mid \rangle + \langle [H_0, F_\varepsilon]\psi, \mid \varphi \mid \rangle \leq 0 \tag{4.27}$$

From (4.27) and (3.15), we obtain

$$\mid \langle [H_0, F_\varepsilon]\psi, \mid \varphi \mid \rangle \mid \leq C \sum_{i<j} \langle \psi, r_{ij}^{-2} g_\varepsilon(r_{ij}) \mid \varphi \mid \rangle \tag{4.28}$$

$$\leq C \sum_{i<j} \| r_{ij}^{-1} g_\varepsilon(r_{ij})\psi \| \, \| r_{ij}^{-1} g_\varepsilon(r_{ij})\varphi \| \tag{4.29}$$

where C is some constant independent of $\varepsilon$. The last factor in (4.29) tends to zero with $\varepsilon$ because $\varphi \in Q(V_0)$ while the first factor is bounded uniformly in $\varepsilon$ because of proposition 2.4. We then let $\varepsilon$ tend to zero in (4.27) and obtain $\langle \psi, \mid \varphi \mid \rangle \leq 0$ and therefore $\varphi = 0$. Therefore $H' \subset H$. This completes the proof.

A stronger result in the same direction as theorem 4.2 can be obtained if in (B2) one takes for $c$ the values given in theorem 3.1. The corresponding result is the analogue in the present situation of the main theorem in [*7*].

THEOREM 4.3. — Let $V_{ij}$ satisfy conditions (B1) and (B2) where now $c$ takes the values given in theorem 3.1. Then $H = H^+$ (in particular $H^+$ is self adjoint).

*Proof.* — The beginning of the proof is identical with that of theorem 4.2 with $H^+$ replacing H′. One is led to show that if $\varphi \in \mathscr{H}$ and $\varphi$ satisfies (4.25), then $\varphi = 0$. The end of the proof is the same as that of theorem 3.1. Notice however that in order to obtain the values of $c$ given in theorem 3.1 for $N = 3$, one needs to use both in $\psi$ and in the definition of H the improved form of H the improved form of $\omega$ given by (2.6) with $\beta \neq 0$ (Cf. Remark 4.1).

So far, we have assumed that $V_{ij} \in L^1_{loc}(\mathbb{R}^n \setminus \{ 0 \})$. In this case, there is no minimal operator in $\mathscr{H}$ with natural domain associated with $H_0 + V$. Instead we have defined H as a sum of quadratic forms, so that H is self-adjoint by construction. All previous results of this section are of the nature of identifying this H with other possible candidates.

We now come back to the special case where $V_{ij} \in L^2_{loc}(\mathbb{R}^n \setminus \{ 0 \})$. Then there is an operator $\hat{H} = H_0 + V$ with domain $\mathscr{D}_0$ and it is a natural problem to look for self-adjoint extensions of this operator. In the remaining part of this section, we shall obtain some results concerning the uniqueness of self-adjoint extensions of $\hat{H}$ under suitable restrictions. The strongest result in this direction has already been derived in section 3, where we have proved that $\hat{H}$ is essentially self adjoint on $\mathscr{D}_0$ under suitable assumptions on V.

We first state a simple consequence of theorem 4.1 (cf. [*1*]).

COROLLARY 4.2. — Let $V_{ij} \in L^2_{loc}(\mathbb{R}^n \setminus \{ 0 \})$ and $V_{ij} \geq c_m r_{ij}^{-2}$ for all $(i, j)$. Let $\hat{H} = H_0 + V$ with domain $\mathscr{D}_0$. Then $\hat{H}$ has only one self-adjoint extension with domain contained in Q(H). (Equivalently: then H is the Friedrichs extension of $\hat{H}$).

The equivalence of the two statements in corollary 4.2 is a well known property of the Friedrichs extension.

With the same assumptions on V, one can prove the following result, which bears the same relation to theorem 4.2 as does theorem 3.1 to theorem 4.3.

THEOREM 4.4. — Let $V_{ij} \in L^2_{loc}(\mathbb{R}^n \setminus \{ 0 \})$ and $V_{ij} \geq c_m r_{ij}^{-2}$ for all $(i, j)$. Let $\hat{H} = H_0 + V$ with domain $\mathscr{D}_0$. Then $\hat{H}$ has at most one self-adjoint extension with domain contained in $Q(V_0)$.

*Proof.* — Let $H_1$ and $H_2$ be self-adjoint extensions of $\hat{H}$ with domains contained in $Q(V_0)$. It is sufficient to show that this implies $H_2 = H_1$. Let $\theta_2 \in \mathscr{D}(H_2)$ and let $\lambda > 0$ be sufficiently large. Then, since $H_1$ is self-adjoint, there exists $\theta_1 \in \mathscr{D}(H_1)$ such that $(\lambda + i + H_1)\theta_1 = (\lambda + i + H_2)\theta_2$. Therefore $(\lambda + i + \tilde{H})\varphi = 0$ where $\varphi = \theta_1 - \theta_2 \in Q(V_0)$ and $\tilde{H}$ is the operator from $\mathscr{H}$ to $\mathscr{D}'_0$, dual of $\hat{H}$ (cf. the proof of theorem 3.1). From this, one deduces that $\varphi = 0$ by the same argument as in the proof of theorem 4.2. Therefore $H_2 \subset H_1$, therefore $H_2 = H_1$ since both are self adjoint. This completes the proof.

If $\alpha > \alpha_m$, it follows from corollary 4.1.2 and the definition (4.13) of H that $Q(H) \subset Q(V_0)$. Therefore H is a positive self-adjoint extension of $\hat{H}$. This allows one to recover corollary 4.2 from theorem 4.4 in the restricted case $\alpha > \alpha_m$ without using theorem 4.1, provided one uses (4.11, 12) with $\alpha_m$ to define $H(\alpha_m)$ and then (4.23) and (4.13) to define $H(\alpha)$ and H. The statement in corollary 4.2 with $\alpha = \alpha_m$, however, cannot be deduced from theorem 4.4.

A result similar to theorem 4.4 and corollary 4.2 but restricted to the two-body case, has been obtained by Kalf (See theorem (3) in [*14*] and the references contained in that paper).

## 5. CONCLUDING REMARKS

The results obtained in sections 3 and 4 are optimal as regards the behaviour of the potentials near the origin if the latter is governed by a power law $r^{-\mu}$ with $\mu \neq 2$. For $\mu = 2$ however, the result depends on the value of the coupling constant $c$, and the values obtained for $c$ in theorems 3.1 and 4.1 are not expected to be optimal. On the basis of the results for the two-body case, one expects $c = -n(n-4)/4$ for theorem 3.1 and $c = -(n-2)^2/4$ for theorem 4.1, for all N. It is clear from the method of proof that such an improvement depends on the construction of a trial function $\psi$ that correctly reproduces the behaviour of the ground state of H in the neighbourhood of the set S where two or more particles come close together. The trial function used in this paper is sufficiently accurate when no more than two particles come close together, but not otherwise. Possible improvements would consist in introducing $k$-body terms in $\omega$ $(2 \leq k \leq \mathrm{N})$. This introduces many-body potentials in the expression of $\mathrm{H}_0\psi/\psi$. These potentials become more and more difficult to estimate when N increases and there is little hope to reach the expected values of $c$ for arbitrary $n$ and N by this method.

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