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Green's functions for theories with massless particles (in perturbation theory)

by

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ABSTRACT. — With the method of perturbative renormalization developed by Epstein and Glaser it is shown that Green's functions exist for theories with massless particles such as Q. E. D., and $\lambda : \phi^{2n}$: theories. Growth properties are given in momentum space. In the case of Q. E. D., it is also shown that one can perform the physical mass renormalization.

RÉSUMÉ. — A l'aide de la méthode de renormalisation perturbative développée par H. Epstein et V. Glaser on montre l'existence des fonctions de Green pour des théories comprenant des particules de masse nulle telles que l'électrodynamique quantique et les théories $\lambda : \phi^{2n}$. On donne des propriétés de croissance dans l'espace des moments. Pour l'electrodynamique, on montre qu'il est possible d'effectuer la renormalisation de la masse à sa valeur physique.

I. PRELIMINARIES

1. Introduction

Notations are those of [*1*]. Any change will be explained.

It has been shown in [*1*] that for $g_i(x)$ in $\mathscr{S}(\mathbb{R}^4)$ the various field ope-

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rators $T(\hat{\mathscr{L}}_{i_1}(x_1) \dots \hat{\mathscr{L}}_{i_s}(x_s); g)$ exist as tempered valued distributions on the domain D_1 and possess all the required properties in the sense of formal power series in the g_i. The n^{th} order expansion coefficient of such an operator is of the form

$$
\begin{aligned}
T_n(\hat{\mathscr{L}}_{i_1}(x_1) \dots \hat{\mathscr{L}}_{i_1}(x_s); g) & \\
&= T_n(x_1, \dots, x_s; g) \\
&= \frac{i^n}{n!} \int \quad \int R(y_1, \dots, y_n; x_1, \dots, x_s) g(y_1) \dots g(y_n) dy_1 \dots dy_n \\
&= \frac{i^n}{n!} \int \dots \int \mathscr{L}(y_1) \downarrow \dots \mathscr{L}(y_n) \\
&\qquad \downarrow T(\mathscr{L}_{i_1}(x_1) \dots \mathscr{L}_{i_s}(x_s)) g(y_1) \dots g(y_n(dy_1 \dots dy_n
\end{aligned} \tag{I.1.1}
$$

which we denote, shortening the notation

$$\frac{i^n}{n!} \int Y \downarrow T(X) g(Y) dY$$

We want to show that the « adiabatic limit » when

$$g(y) = (g_0(y), g_1(y), \dots) \to (\lambda, 0, \dots)$$

λ being a constant, of the vacuum expectation value of (I.1.1), always exists in the sense of tempered distribution in the variables X.

In fact, in order to recover after the limiting procedure all the properties of Green's functions we need to study the adiabatic limit of the vacuum expectation value (v. e. v.) of the n^{th} order expansion coefficient of products of T products:

$$
\begin{aligned}
[\hat{T}(X_1) \dots \hat{T}(X_p)]_n &= \frac{i^n}{n!} \int Y \downarrow \{ T(X_1) \dots T(X_p) \} g(Y) dY \\
&= \frac{i^n}{n!} \int \sum_{\substack{Y_1 \cup \dots \cup Y_p = Y \\ Y_r \cap Y_s = \emptyset}} Y_1 \downarrow T(X_1) Y_2 \downarrow T(X_2) \dots Y_p \downarrow T(X_p) g(Y) dY
\end{aligned} \tag{I.1.2}
$$

which will, from now on, be shortened in

$$\int Y \downarrow \mathscr{O}(X) g(Y) dY \tag{I.1.3}$$

In particular, v. e. v. of $\bar{T}$ products, retarded and advanced functions, and Wightman functions can be expressed in terms of such monomials (I.1.2).

2. The starting point

The starting point in the study of the adiabatic limit for operators as (I.1.3) is to study $F^-(X, Y) = (\Omega, Y \downarrow \mathscr{O}(X)\Omega) = \langle Y \downarrow \mathscr{O}(X) \rangle$ and to compare it with $F^+(X, Y) = \langle Y \uparrow \mathcal{O}(X) \rangle$ (Ω is the vacuum state). F^+ and F^- have respectively advanced and retarded support properties relative to the Y variables

$$\text{supp } F^{\pm} \subset C_{\pm} = \{ (X, Y) \in \mathbb{R}^{4(|Y|+|X|)}; [Y] \subset [X] + \bar{V}_{\pm} \} \quad (I.2.1)$$

More precisely

$$C_{\pm} = \{ (x_1, \ldots, x_{|X|}, y_1, \ldots y_{|Y|}) \in \mathbb{R}^{4(|Y|+|X|)} \mid y_i - x_{u(i)} \in \bar{V}_{\pm}, i = 1, \ldots, |Y| \} \quad (I.2.2)$$

for a least a mapping $u: (1, \ldots, |Y|) \to (1, \ldots, |X|)$.

Those two cones are opposite, closed and pointed at the origin.

On the other hand, using the « arrow calculus » it can be shown that the « absorptive part » can be written:

$$F^+(X, Y) - F^-(X, Y) = \langle \Sigma [y_{i_1} \updownarrow \ldots \updownarrow y_{i_n}, y_{i_{n+1}} \updownarrow \ldots \updownarrow y_{i|Y|} \updownarrow \mathcal{O}(X)] \rangle \quad (I.2.3)$$

where the sum extends over a finite number of commutators.

Now, noticing that a monomial $y_1 \updownarrow \ldots \updownarrow y_n \updownarrow \mathcal{O}(X)$ can be expressed as commutators of advanced (or retarded) products with respect to the Y's, of order (in Y) less or equal to n, we see that, knowing $Y \uparrow \mathcal{O}(X)$ [or $Y \downarrow \mathcal{O}(X)$] for $|Y| < n$, the absorptive part (2.3) is known for $|Y| = n$.

Remark. — At this level, Epstein and Glaser [*2*], adding the spectral condition and its consequences in momentum space, are able, in the case of a minimal non-vanishing mass, to deduce the existence of the adiabatic limit.

However, in the case of a particle with a zero mass, troubles appear in the use of the spectral condition. Before going on let us give a more precise meaning to: « adiabatic limit ».

3. The adiabatic limit

DÉFINITION. — *A distribution* $T \in \mathcal{S}'(\mathbb{R}^N)$ *satisfies an adiabatic norm of degree* δ, *if* $0 < \delta$ *and if there exist constants* $C \geqslant 0$, $M \geqslant 0$ *such that for every* $\varphi \in \mathcal{S}(\mathbb{R}^N)$ *one has*

$$|\langle T(x), \varphi(x) \rangle| \leqslant C \sum_{|\alpha| \leqslant M} \sup (1 + \|X\|)^{|\alpha| - \delta} |D^{\alpha}\varphi(x)| \quad (I.3.1)$$

From this definition follows Lemma 1.

LEMMA 1. — *If a distribution* $T \in \mathcal{S}'(\mathbb{R}^N)$ *satisfies an adiabatic norm* (I.3.1), *then for every* $\varphi(x) \in \mathcal{S}(\mathbb{R}^N)$, *the adiabatic limit*

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \langle T(x), \varphi(\varepsilon x) \rangle = L\varphi(0) \quad (I.3.2)$$

exists and is a distribution in φ *of the form given by* (3.2), *where* L *is a constant independent of* φ.

Proof. — In order to prove the existence of the adiabatic limit is suffices in view of

$$\langle T(x), \varphi(\varepsilon x) \rangle = \langle T(x), \varphi(x) \rangle - \int_\varepsilon^1 \left\langle T(x), \frac{d}{d\eta} \varphi(\eta x) \right\rangle d\eta$$

to prove the existence of the integral for $\varepsilon = 0$. In fact, we will show that $\left\langle T(x), \frac{d}{d\eta} \varphi(\eta x) \right\rangle$ is absolutely integrable at $\eta = 0$.

We have

$$\frac{d}{d\eta} \varphi(\eta x) = \sum_\mu x_\mu \left(\frac{\partial}{\partial x_\mu} \varphi \right)(\eta x) = \sum_\mu x_\mu \chi_\mu(\eta x)$$

Now, applying (I.3.1)

$$\left| \left\langle T(x), \frac{d}{d\eta} \varphi(\eta x) \right\rangle \right| \leqslant C \sum_{|\alpha| \leqslant M} \sup\, (1 + \|X\|)^{|\alpha| - \delta} \left| D^\alpha \left(\sum_\mu x_\mu \chi_\mu(\eta x) \right) \right|$$

$$\leqslant C' \sum_\mu \sum_{|\alpha| \leqslant M} \sup\, (1 + \|X\|)^{|\alpha| + 1 - \delta} \eta^{|\alpha|} \,|(D^\alpha \chi_\mu)(\eta x)|$$

$$\leqslant C' \frac{1}{\eta^{1-\delta}} \sum_\mu \sum_{|\alpha| \leqslant M} \sup\, (1 + \|X\|)^{|\alpha| + 1 - \delta} \,|D^\alpha \chi_\mu(x)| \leqslant \text{const.}\ \eta^{\delta - 1}$$

which, therefore, is integrable since $\delta > 0$.

Thus, by a well-known theorem, the adiabatic limit is a tempered distribution S:

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \langle T(x), \varphi(\varepsilon x) \rangle = \langle S, \varphi \rangle$$

Now, we want to estimate

$$\sum_{|\alpha| \leqslant M} \sup\, (1 + \|X\|)^{|\alpha| - \delta} \,|D^\alpha \varphi(\varepsilon x)|$$

For simplicity we treat the case when $\delta < 1$. Then (I.3.3) can be written

$$\sup \frac{|\varphi(\varepsilon x)|}{(1 + \|X\|)^\delta} + \sum_{1 \leqslant |\alpha| \leqslant M} \sup\, (1 + \|X\|)^{|\alpha| - \delta} \,|D^\alpha \varphi(\varepsilon x)|$$

The first term is bounded by

$$|\varphi(0)| + \frac{|\varphi(\varepsilon x) - \varphi(0)|}{(1 + \|X\|)^\delta}$$

and

$$\frac{|\varphi(\varepsilon x) - \varphi(0)|}{(1 + \|X\|)^\delta} \leqslant C_\delta \varepsilon^\delta \sum_{|\alpha| = 1} \sup\, (1 + \|X\|)^{|\alpha| - \delta} \,|D^\alpha \varphi(x)|$$

The second one is bounded by

$$\varepsilon^{\delta} \sum_{1 \leqslant |\alpha| \leqslant M} \sup (1 + \| X \|)^{|\alpha| - \delta} | D^{\alpha}\varphi(x) |$$

Therefore,

$$| \langle T(x), \varphi(\varepsilon x) \rangle | \leqslant C \left\{ | \varphi(0) | + \varepsilon^{\delta} \sum_{1 \leqslant |\alpha| \leqslant M} \sup (1 + \| X \|)^{|\alpha| - \delta} | D^{\alpha}\varphi(x) | \right\}$$

and taking the limit $\varepsilon \to 0$

$$| \langle S, \varphi \rangle | \leqslant C | \varphi(0) |$$

Now, let us choose a fixed element $h(x) \in \mathscr{S}(\mathbb{R}^N)$ such that $h(0) = 1$, and apply this inequality to $\psi(x) = \varphi(x) - \varphi(0)h(x)$. We get

$$\langle S, \varphi \rangle = \langle S, h \rangle \varphi(0)$$

Setting $L = \langle S, h \rangle$, this proves the lemma.

The definition given here for the adiabatic norm in position space coincides with the one given in momentum space in [2]. This equivalence is shown in the mathematical appendix.

However, the existence of a massless particle will force us to work not only with adiabatic norms like (I.3.1), but also with norms like

$$\sum_{|\alpha| \leqslant N'} \sup (1 + \| X \|)^{|\alpha| - D + 1 - \delta} | D^{\alpha}\varphi(x), \quad 0 < \delta < 1 \qquad \text{(I.3.4)}$$

where $D - 1$ measures the lack of convergence towards an adiabatic limit.

Equivalently, to such norms correspond, in momentum space

$$\sum_{|\beta| \leqslant M'} \sup \| q \|^{N + |\beta| + D - 1 + \delta} (1 + \| q \|)^{M} | D^{\beta} \tilde{\varphi}(q) |$$

as can be shown following the techniques developed in the mathematical appendix.

Having given a precise meaning to what we call an adiabatic limit, we can now present the principle of the proof.

4. The principle of the proof

The proof is based on a double inductive procedure acting on the length $|X|$ of X and on the length $|Y|$ of Y. Let us go into details.

Let s and n be two fixed non-negative integers and suppose that for

1) $|X| < s$ and any value of $|Y|$
2) $|X| = s$ and $|Y| \leqslant n - 1$ (I.4.1)

we have proved the existence for $Y \uparrow \mathcal{O}(X)$ [or $Y \downarrow \mathcal{O}(X)$] of an adiabatic norm (in Y), (I.3.1).

The method requires two steps.

First step. — We compute the absorptive part for

$$|Y| = n, \qquad \langle Y \uparrow \mathcal{O}(X) - Y \downarrow \mathcal{O}(X) \rangle$$

and have to show that it satisfies the adiabatic norm.

Let us remark that this difference is expressed (I.2.3) with monomials of only two types:

a) there are only the y's; then the length is less than n;

b) there are y's and x's; then $|X| = s$ and the length of the y's is less than n, thus only a part of the inductive hypothesis is useful.

Second step. — We have to recover $\langle Y \uparrow \mathcal{O}(X) \rangle$ [or $\langle Y \downarrow \mathcal{O}(X) \rangle$] from the absorptive part. This is accomplished as in [*1*] through a cutting procedure (with a suitable $\sqcup$ function), and we have to show that $\langle Y \uparrow \mathcal{O}(X) \rangle$ satisfies the adiabatic norm (in Y).

Let us make another remark. Cases $|X| = 0$ and $|X| \neq 0$ will appear to be quite different. When $|X| = 0$, we have support properties in all the variables and the cutting procedure is exactly the one described in [*1*]. But when $|X| \neq 0$, we only have support properties in the Y's and these properties depend on the X's (as parameters), therefore, the cutting procedure has to be modified. On the other hand, when $|X| = 0$ we can restrict ourselves to connected terms, since at each order (starting from $|Y| = 1$), the vacuum contribution (as intermediate state) can be neglected in the absorptive part. But for $|X| \neq 0$, $\langle \mathcal{O}(X) \rangle$ and $\langle Y \uparrow \mathcal{O}(X) \rangle$ have no reason to be connected; however, in the absorptive parts (2.3), the vacuum contribution, as intermediate state, can be neglected and we have always to deal with connected products.

5. **Outlines**

There are two parts. The first one deals, in a relatively complete way, with the case of Q. E. D. or similar theories. No gauge conditions are used, however, Stora [*3*] and the authors have shown that it is possible to construct T product satisfying such kind of conditions. The second one is related to $\lambda : \phi^{2n}$: theories where $\emptyset(X)$ is a zero mass field, which is treated as an example to show how such methods can be extended to other cases.

II. CASE OF QUANTUM ELECTRODYNAMICS

1. **Introduction**

1.1 NOTATIONS

The notations are nearly the same as in [*1*]. However X being a set of variables $\{ x_1, \ldots, x_{|X|} \}$, $j(X)$ will be the set of indices which numbers the x variables.

We also denote by T(X, Y) any kind of Steinmann monomial of the form $y_1 \updownarrow \ldots \updownarrow y_n \updownarrow \mathcal{O}(X)$ where $\mathcal{O}(X)$ is a product of T products. This notation is due to the fact that any such monomial can be expanded in a sum of T products and what we will say applies to each term.

To specify the theory we define as in [*1*] multi-indices $r = (r_1, \ldots, r_n)$, $r_i = (r_i^1, r_i^2, r_i^3)$ where

1 is associated to $\bar\psi$ } fermion fields
2 is associated to ψ }
3 is associated to A photon field;

and $r_i^j = (a_i^j, \alpha_i^j) a_i^j = 1$ or 0 and α_i^j are the spinor or tensor indices (here $\alpha_i^j = 1, 2, 3$, or 0).

An operator $T_r(X)$ with $a_i^j = 1$ can be understood as « coming » from a Lagrangian at point x_i which is a derivative of the interaction Lagrangian $\mathscr{L}(x_i) = \sum_\mu \bar\psi(x_i)\Gamma_\mu\psi(x_i)A_\mu(x_i)$ with respect to the j^{th} field.

We can also represent graphically the vacuum expectation value $\langle T_r(X) \rangle$ of $T_r(X)$, $|X| = n$, as a « diagram » with n vertices and $\sum_{j \in J(X)} \sum_{i=1}^{3} a_j^i$ external lines: more precisely, with

$$\sum_{j \in J(X)} (a_j^1 + a_j^2) \qquad \text{fermions}$$

$$\sum_{j \in J(X)} a_j^3 \qquad \text{photons}$$

As a tempered distribution in the relative variables $\langle T_r(X) \rangle$ is singular at the origin of order

$$\omega = 4 - \frac{3}{2} \sum_{j \in J(X)} (a_j^1 + a_j^2) - \sum_{j \in J(X)} a_j^3$$

as it was shown in [*1*].

Remark also that in Q. E. D. the only diagrams which are singular at the origin with $\omega \geqslant 0$ are

where ~ stands for photons and — for fermions.

According also to Furry's theorem, $\langle T_r(X) \rangle$ vanishes identically for $\sum_j a_j^1 = \sum_j a_j^2 = 0$ and $\sum_j a_j^3$ odd, and due to charge conservation one has always $\sum_j a_j^1 = \sum_j a_j^2$ (3).

We will now present briefly which kinds of difficulties occur when we are dealing with massless particles.

1.2 DIFFICULTIES

As we have seen in the preliminaries, the method consists in finding properties of the difference

$$F^+(Y,X) - F^-(Y,X) = \langle \Sigma[y_{i_1} \updownarrow \dots \updownarrow y_{i_k}, \; y_{i_{k+1}} \updownarrow \dots \updownarrow y_{i_{|Y|}} \updownarrow \mathcal{O}(X)] \rangle \tag{II.1.1}$$

for $|Y| = n$, $|X| = s$, knowing the properties of $\langle Y \uparrow \mathcal{O}(X) \rangle$ for $|Y| \leqslant n-1$ and $|X| \leqslant s$.

Now consider one of the commutators in (II.1.1). It is a sum of terms of the form

$$\langle T_{r'}(Y')T_{r''}(Y'', X) \rangle$$

with $Y = Y' \cup Y''$, $Y' \cap Y'' = \emptyset$.

According to the Wick's theorem each term is a sum of terms like

$$\langle T_{r'+s'}(Y') \rangle \langle T_{r''+s''}(Y'', X) \rangle \prod_{j=1}^{l} P_j(\partial)\Delta_j^+ \tag{II.1.2}$$

By going into momentum space we will see more easily their structure.

Using the invariance by translation one defines the Fourier transform $t_r(q,p)$ of $\langle T_r(Y,X) \rangle$ by

$$\int \dots \int \langle T_r(Y,X) \rangle e^{-i\sum_{j=1}^{n} q_j . y_j} e^{-i\sum_{j=1}^{s} p_j . x_j} \prod_1^s dx_j \prod_1^n dy_j$$

$$= \delta^{(4)}\left(\sum_1^s p_j + \sum_1^n q_j\right) t(q_1, \dots, q_n, p_1, \dots, p_{s-1})$$

if $|Y| = n$ and $|X| = s$ ($\neq 0$). When $s = 0$, one chooses q_n to be the omitted variable.

(3) From now on, omitting the spinor indices, we will use r_j^i instead of a_j^i.

Then (II.1.2) becomes, up to a $\delta^{(4)}$ function

$$\int \cdots \int t_{r'+s'}(q'_1 + k_1, \ldots, q'_l + k_l, q'_{l+1}, \ldots, q'_{\nu-1})$$
$$t''_{r''+s''}(q''_1 - k_{i_1}, \ldots, q''_u - k^i_\mu, p_1 - k_{i_{\mu+1}}, \ldots, p_{s-1} - k_{i_{\mu+s-1}})$$
$$\delta^{(4)}\left(\sum_1^l k_i - \sum_1^\nu q'_i\right) \prod_{j=1}^l \mathrm{P}_j(k_j)\delta^+(k_j; m_j)dk_j = t_r(q, p) \quad \text{(II.1.3)}$$

with $|\mathrm{Y}'| = \nu, |\mathrm{Y}''| = \mu$, $\mu + \nu = n$ and $(i_1, \ldots, i_{\mu+s-1})$ is a mapping of $(1, \ldots, l)$ into $(1, \ldots, \mu + s - 1)$.

One sees in this formula that, due to the $\delta^{(4)}$ function, $t_r(q, p)$ is a distribution which « vanishes » for $\| q \| < m$ if one of the intermediate states (or particles) has a mass m_j which is non-zero. In this case one can easily be convinced that, tested with $f(p) \in \mathcal{S}$, the distribution $\int t_r(q, p) f(p) dp$ satisfies an adiabatic norm in the q variables.

Therefore, we should distinguish two cases whether there is or not a massive particle in the intermediate states. Going back to position space, we must therefore control the behaviour at infinity in the y's variables of $\langle \mathrm{T}_r(\mathrm{Y}) \rangle$ and $\int \langle \mathrm{T}_r(\mathrm{Y}, \mathrm{X}) \rangle f(\mathrm{X}) d\mathrm{X}$ for $r^j_i = 0, j = 1, 2, j \in \mathrm{J}(\mathrm{Y})$.

1.3 The spinor calculus

All estimates are related to the coefficients in the γ matrices expansion of the different quantities which appear in the theory, and, therefore, except in Section 4, we omit any reference to the spinor indices.

To estimate a given term (spinorial quantity) we can take any of the norms used in matrix calculus. Here, for simplicity, we take an upper bound of the coefficients. For example, in any estimate (*), $\delta^+(p; m)(\not p + m)$ will be replaced by $\mathrm{C}\delta^+(p; m)(1 + \| p \|)$.

2. **Diagrams with photon external lines only**

We first define an index of divergence (infra-red) which measures, in momentum space, the behaviour at the origin (or at infinity in position space) for such diagrams. We then deduce a norm which takes into account this behaviour and show by induction that it is satisfied by the absorptive parts and preserved by the cutting procedure.

2.1 The index

We require from this index to be compatible with *a*) a renormalization

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([4]) $\delta^+(p; m) = \theta(p_0)\delta(p^2 - m^2)$.

at the origin of the photon self-energy and of the photon-photon scattering diagrams; *b*) the internal structure.

Let us explain this last point. Suppose a diagram G is made of two diagrams G_1 and G_2 linked by l intermediate photons

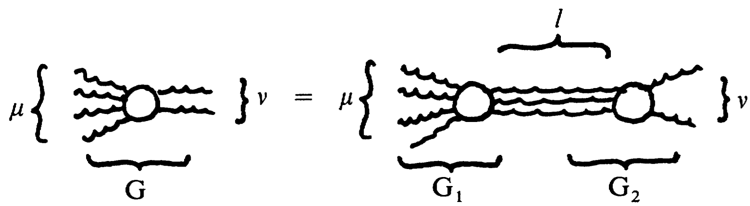

FIG. 1.

Roughly, the behaviour at the origin (in momentum space) will be the product of the behaviour at the origin of G_1, G_2 and of the phase space. This last quantity behaves like λ^{2l-4}. The index will be the worst of the number we get in this way by looking at all internal possible connected structures. Noting D(G) the index of G, we shall get

$$\lambda^{D(G_1)}\lambda^{2l-4}\lambda^{D(G_2)} \leqslant \lambda^{D(G)}$$

therefore, the index D(G) has to satisfy

$$D(G_1) + D(G_2) + 2l - 4 \geqslant D(G)$$

According to this definition $D(G) \leqslant 0$ means that G diverges at the origin.

If now we specify statement *a*) by *a'*) the photon self-energy has to vanish twice at the origin, *a''*) the photon-photon diagrams have to vanish at the origin, we arrive at the result (perhaps not the best)

$$D(G) = -\frac{1}{2}p + 3 \qquad \text{(II.2.1)}$$

where p is the number of external lines (here photon lines). With the example of Figure 1

$$D(G) = -\frac{1}{2}(\mu + \nu) + 3 \qquad D(G_1) = -\frac{1}{2}(\mu + l) + 3$$

$$D(G_2) = -\frac{1}{2}(\nu + l) + 3$$

and

$$D(G_1) + D(G_2) + 2l - 4 = l - 1 + D(G) \geqslant D(G) \qquad \text{(II.2.2)}$$

since $l \geqslant 1$ (we work with connected products!).

2.2 The induction and the norm

We are dealing with vacuum expectation values (v. e. v.) of operators $T_r(Y)$ with $r_j^i = 0$, $i = 1, 2$, $j = 1, \ldots, |Y|$. These v. e. v. are tempered distributions depending on relative variables. We choose the following set $\xi_j = y_1 - y_{|Y|}, j = 1, \ldots, |Y|$ ($\xi_{|Y|} = 0$ and when we will speak about the ξ variables we omit $\xi_{|Y|}$) and note the v. e. v. by $F_r(\xi)$ of $F(\xi)$. The induction will run on the length of $|Y|$.

INDUCTION HYPOTHESIS. — *Let* $|Y|$ *be less than* n, $r_j^i = 0, i = 1, 2, j \in J(Y)$; *then for each distribution* $F_r(\xi)$ *there exist three constants,* $C, \kappa, \varepsilon, \kappa > 0, \varepsilon > 0$ *arbitrarily small such that for any* $\varphi \in \mathcal{S}(\mathbb{R}^{4(|Y|-1)})$

$$\langle F_r(\xi), \varphi(\xi) \rangle \leqslant C \sum_{|\alpha| \geqslant \max(0, D)} \sup_{\xi} \| \xi \|^{\mu_\alpha^+} (1 + \| \xi \|)^{-\mu_\alpha^+ + |\alpha| - D + 1 - \delta} | D^\alpha \varphi | \tag{II.2.3}$$

with $\mu_\alpha^+ = (-\omega + |\alpha| - \varepsilon)^+$, $\delta = 1 - \kappa$

$$\omega = 4 - \sum_{j \in J(Y)} r_j^3 \qquad D = -\frac{1}{2} \sum_{j \in J(Y)} r_j^3 + 3$$

In formula (II.2.3), the fact that $|\alpha| \geqslant \max(0, D)$ results from the hypotheses a') and a'') of 2.1. Indeed, for the self-energy $D = 2$, and $|\alpha| \geqslant 2$, means that the Fourier transform of $F_r(\xi)$ vanishes twice at the origin. In the same way, for the four-photon diagrams $D = 1$ and $|\alpha| \geqslant 1$ means that the Fourier transform of $F_r(\xi)$ vanishes at the origin. This is explained with more details in Appendix B.

One can check formula (II.2.3) for low orders. For example, when $|Y| = 2$, the only photon diagram which will enter in the construction of terms of order three, is the photon self-energy. But, in momentum space, it is analytic in a neighbourhood of the origin; it has, therefore, an adiabatic limit which, after renormalization, can be chosen to be zero; the same can be done for the first derivatives, Therefore, it satisfies, at infinity, in position space, the growth indicated in formula (II.2.3). At the origin, the growth results from [*1*].

Our next step is to show that, starting from the induction hypothesis, the adiabatic parts satisfy (II.2.3) for $|Y| = n$.

2.3 THE ADIABATIC PARTS AND THE NORM

According to Section II.1 we have to find a norm for a distribution of the form

$$\langle T_{r'+s'}(Y') \rangle \langle T_{r''+s''}(Y'') \rangle \prod_{j=1}^{l} P_j(\partial) \Delta_j^+ \tag{II.2.4}$$

where

$$r_j'^{1,2} = 0 \qquad j \in J(Y')$$
$$r_j''^{1,2} = 0 \qquad j \in J(Y'')$$

$$\sum_{j=1}^{J(Y')} \left(s_j'^1 + s_j'^2 + s_j'^3\right) = \sum_{j=1}^{J(Y'')} \left(s_j''^1 + s_j''^2 + s_j''^3\right) = l$$

$$Y' \cup Y'' = Y \qquad Y' \cap Y'' = \emptyset$$

$$l \geqslant 1$$

Then two cases appear: either one of the intermediate « particles » has a non-vanishing mass, or they all have a zero mass.

a) *Zero mass case*

All the $\Delta_j^+ = \Delta_j^+(x\,;m_j)$ have $m_j = 0$. The $P_j(\partial)$ are equal to one. We define, as in Ref. 1), the following set of variables

$$\xi_j' = y_j' - y_{|Y'|}' \qquad j = 1, \ldots, |Y'| \qquad \eta = y_{|Y'|}' - y_{|Y''|}''$$
$$\xi_j'' = y_j'' - y_{|Y''|}'' \qquad j = 1, \ldots, |Y''|$$

and the mappings $j \to u'(j)$ and $j \to u''(j)$.

Then, (II.2.4) becomes in this case

$$F'(\xi')F''(\xi'') \prod_{j=1}^{l} \Delta^+\left(\xi_{u'(j)}' - \xi_{u''(j)}'' + \eta\,; 0\right) = G(\xi', \xi'', \eta) \qquad (II.2.5)$$

where $F(\xi)$ stands for $F_{r+s}(\xi) = \langle T_{r+s}(Y) \rangle$. By the induction hypothesis $F'(\xi')$ and $F''(\xi'')$ satisfy (II.2.3). As in Ref. 1) there are two cases.

i) $\underline{l = 1}$

We apply in this case the second tensor product rule. The indices of F′ being D′ and ω' with

$$D' = -\frac{1}{2}\left(\sum_{j\in J(Y')} r_j'^3 + 1\right) + 3 \qquad \omega' = 4 - \sum_{j\in J(Y')} r_j'^3 - 1$$

the indices of F″ being D″ and ω'', with

$$D'' = -\frac{1}{2}\left(\sum_{j\in J(Y'')} r_j''^3 + 1\right) + 3 \qquad \omega'' = 4 - \sum_{j\in J(Y'')} r_j''^3 - 1$$

the indices of Δ^+ (see Appendix A) being $D''' = -2$ and $\omega''' = -2$, one gets, after applying twice this tensor product rule, that (II.2.5) satisfies (II.2.3) in the variables ξ', ξ'', η with indices D and ω given by

$$D = D' + D'' + D''' = -\frac{1}{2}\sum_{j\in J(Y)} r_j^3 + 3 \qquad \omega = \omega' + \omega'' + \omega''' = 4 - \sum_{j\in J(Y)} r_j^3$$

and, $\varepsilon > \varepsilon' + \varepsilon''$ and $\delta < \delta' + \delta'' - 1$, where ε', δ' and ε'', δ'' are the numbers associated respectively with F′ and F″ in formula (II.2.3).

ii) $l > 1$

As in [1] we set $t_j = \xi'_{u'(j)} - \xi''_{u''(j)}$ and $R(t,\eta) = \prod_{j=1}^{l} \Delta^+(t_j + \eta; 0)$. Then

$$D_t^\alpha R(t,\eta) = \left(\frac{1}{i}\frac{\partial}{\partial\eta_0} + 1\right)\left(\frac{1}{i}\frac{\partial}{\partial\eta_0}\right)^M B_\alpha(t,\eta)$$

with $M = 2l + |\alpha|$. We can replace, since $M > 1$, $B_\alpha(t,\eta)$ by $B_\alpha(t,\eta) - B_\alpha(t,0)$ and we have the following estimate: for any θ, $0 < \theta < 1$, there exists a constant C_θ such that

$$|B_\alpha(t,\eta) - B_\alpha(t,0)| \leqslant C_\theta \|\eta\|^\theta \tag{II.2.6}$$

This results from the fact that for any θ, $0 < \theta < 1$ there exists a constant A_θ such that

$$|e^{ix} - 1| \leqslant A_\theta |x|^\theta \qquad \forall x \in \mathbb{R}$$

Let now $\varphi(\xi', \xi'', \eta) \in \mathcal{S}(\mathbb{R}^{4(|\overline{Y}|-1)})$ and define as in [1]

$$\psi(\xi', \xi'') = \int R(t,\eta)\varphi(\xi',\xi'',\eta)d\eta \Big|_{t_j = \xi'_{u'(j)} - \xi''_{u''(j)}}$$

According to the second tensor product rule

$$I = \int F'(\xi')F''(\xi'')\psi(\xi',\xi'')d\xi'd\xi''$$
$$\leqslant C^t \sum_{|\alpha| \geqslant \max(0, D'+D'')} \sup \|\xi',\xi''\|^{\mu_\alpha^+}(1 + \|\xi',\xi''\|)^{-\mu_\alpha^+ + |\alpha| - D' - D'' + 1 - \delta} |D^\alpha_{\xi',\xi''}\psi(\xi',\xi'')| \tag{II.2.7}$$

with $\mu_\alpha^+ = (-\omega' - \omega'' + |\alpha| - \varepsilon' - \varepsilon'')^+$ and $\delta = \delta' + \delta'' - 1$.

On the other hand

$$|D^\alpha_{\xi',\xi''}\psi(\xi',\xi'')| \leqslant \sum_{|\gamma| \leqslant |\alpha|} C_\theta \int \|\eta\|^\theta \left|\left(i\frac{\partial}{\partial\eta_0} + 1\right)\left(i\frac{\partial}{\partial\eta_0}\right)^{|\alpha| - |\gamma| + 2l} D^\gamma_{\xi',\xi''}\varphi\right| d^4\eta$$
$$\leqslant \sum_{2l+|\alpha| \leqslant |\gamma| \leqslant 2l+|\alpha|+1} \sup_\eta \|\eta\|^{4+\theta-\theta'}(1 + \|\eta\|)^{\theta'+\theta''} |D^\gamma_{\xi',\xi'',\eta}\varphi|, \quad 0 < \theta' < 1, \quad \theta'' > 0$$

Now, we have to estimate

$$\|\xi',\xi'',\eta\|^{(-\omega'-\omega''+|\alpha|-\varepsilon'-\varepsilon'')^+ + 4 + \theta - \theta'}$$
$$(1 + \|\xi',\xi'',\eta\|)^{-(-\omega'-\omega''+|\alpha|-\varepsilon'-\varepsilon'')^+ + |\alpha| - D' - D'' + 1 + \theta' + \theta'' - \delta} |D^\gamma\varphi| \tag{II.2.8}$$

with $|\alpha| \geqslant \max(0, D' + D'')$, $2l + |\alpha| \leqslant |\gamma| \leqslant 2l + |\alpha| + 1$.

First consider $\|\xi',\xi'',\eta\| \leqslant 1$, then (II.2.8) is bounded by

$$C^t \|\xi',\xi'',\eta\|^{(-\omega'-\omega''+|\alpha|-\varepsilon'-\varepsilon'')^+ + 4 - \theta - \theta'} |D^\gamma\varphi| \tag{II.2.9}$$

But

$$(-\omega' - \omega'' + |\alpha| - \varepsilon' - \varepsilon'')^+ + 4 + \theta - \theta'$$
$$\geqslant (-\omega' - \omega'' + |\gamma| - 2l - 1 + 4 + \theta - \theta' - \varepsilon' - \varepsilon'')^+ \geqslant (-\omega + |\gamma| - \varepsilon)^+ = \mu_\gamma^+$$

with $\omega = \omega' + \omega'' + 2l - 4$, $\varepsilon > \varepsilon' + \varepsilon''$ since θ can be chosen close to one and θ' to zero. Then (II.2.9) is bounded by

$$C^t \| \xi', \xi'', \eta \|^{\mu_\gamma^+} | D^\gamma \varphi | \leqslant C^t \| \xi', \xi'', \eta \|^{\mu_\gamma^+} (1 + \| \xi', \xi'', \eta \|)^{-\mu_\gamma^+ + |\gamma| - D + 1 - \delta} | D^\gamma \varphi |$$

since $\| \xi', \xi'', \eta \| \leqslant 1$.

Consider now the case when $\| \xi', \xi'', \eta \| \geqslant 1$, then (II.2.8) is bounded by

$$C^t (1 + \| \xi', \xi'', \eta \|)^{|\gamma| - 2l - D' - D'' + 1 - \delta + 4 + \theta + \theta''} | D^\gamma \varphi |$$

Using formula (II.2.2) the exponent is

$$|\gamma| - D - l + 1 + 1 - \delta + \theta + \theta'' \leqslant |\gamma| - D - \delta + \theta + \theta'' = |\gamma| - D + 1 - \delta$$

since $l \geqslant 2$ and choosing θ'' such that $\theta + \theta'' = 1$.

Then (II.2.8) is bounded by

$$C^t (1 + \| \xi', \xi'', \eta \|)^{|\gamma| - D + 1 - \delta} | D^\gamma \varphi |$$

Now, using $\| \xi', \xi'', \eta \| \geqslant 1$, we get $(1 + \| \xi', \xi'', \eta \|) \leqslant 2 \| \xi', \xi'', \eta \|$ and (II.2.8) is bounded by

$$C^t \| \xi', \xi'', \eta \|^{\mu_\gamma^+} (1 + \| \xi', \xi'', \eta \|)^{-\mu_\gamma^+ + |\gamma| - D + 1 - \delta} | D^\gamma \varphi |$$

Moreover, $\max(2l, 2l + D' + D'') \geqslant \max(0, D)$ because of formula (II.2.2) and we obtain

$$I \leqslant C \sum_{|\gamma| \geqslant \max(0, D)} \sup \| \xi', \xi'', \eta \|^{\mu_\gamma^+} (1 + \| \xi', \xi'', \eta \|)^{-\mu_\gamma^+ + |\gamma| - D + 1 - \delta} | D^\gamma \varphi | \tag{II.2.10}$$

b) *One of the masses is non-zero*

With the same notations as in *a*) and applying the first tensor product [*1*] rule

$$I = \int F'_{r'+s'}(\xi') F''_{r''+s''}(\xi'') \psi(\xi', \xi'') d\xi' d\xi''$$

$$\leqslant C^t \sum_{|\alpha| \leqslant M} \sup \| \xi', \xi'' \|^{(-\omega' - \omega'' + |\alpha| - \varepsilon)^+} (1 + \| \xi', \xi'' \|)^Q | D^\alpha \psi(\xi', \xi'') | \tag{II.2.11}$$

with

$$\omega' = 4 - \frac{3}{2} \sum_{j \in J(Y')} (s_j'^1 + s_j'^2) - \sum_{j \in J(Y')} (s_j'^3 + r_j'^3)$$

$$\omega'' = 4 - \frac{3}{2} \sum_{j \in J(Y'')} (s_j''^1 + s_j''^2) - \sum_{j \in J(Y'')} (s_j''^3 + r_j''^3) \tag{II.2.12}$$

and ψ defined as in *a*).

Since one of the masses at least is different from zero we need better estimates on the derivatives of $R(t, \eta)$. We set

$$D_t^\alpha R(t, \eta) = \left(\frac{1}{i} \frac{\partial}{\partial \eta_0} \right)^{M_\alpha} B_{M_\alpha}(t, \eta)$$

with $M_\alpha = |\alpha| + \omega_0 + 1 + \rho,\ \rho \geqslant 0$ and

$$\omega_0 = \sum_{j\in J(Y')} s_j'^3 + \sum_{j\in J(Y'')} s_j''^3 + \frac{3}{2}\sum_{j\in J(Y')} (s_j'^1 + s_j'^2) + \frac{3}{2}\sum_{j\in J(Y'')} (s_j''^1 + s_j''^2)$$
$$= 2l_3 + 3(l_1 + l_2)$$

l_3 being the number of photons, $l_1 + l_2$ being the number of fermions

$$B_{M_\gamma}(t, \eta) = \int e^{iP.\eta + i\Sigma t_j.p_j} \frac{(ip)^\gamma}{(P_0)^{M_\gamma}} \delta^{(4)}(P - \Sigma p_j)\left(\prod_{j=1}^{l} \delta^+(p_j; m_j)P_j(p_j)d^4p_j\right)d^4P$$

Since one of the particles is a fermion $P_0 \geqslant m$, and

$$|B_{M_\gamma}(t,\eta)| \leqslant \int_{P_0\geqslant m} d^4P\rho_l(P)\frac{(P_0)^{l_1+l_2}}{(P_0)^{M_\gamma - |\gamma|}} \leqslant C' \int_m^\infty \frac{dP_0}{P_0^{2+\rho}} \leqslant C' \quad \text{(II.2.13)}$$

Since $\rho_l(P)$ (see [1]) is bounded by $(P_0)^{2l-4}\theta(P^2 - m^2)\theta(P_0)$. We need another estimate. If one has

$$\left(\frac{1}{i}\frac{\partial}{\partial\eta_0}\right)^{\rho+1} B_{M_\gamma}(t, \eta)$$

one can replace $B_{M_\gamma}(t, \eta)$ by

$$B_{M_\gamma}(t, \eta) - \sum_{j\leqslant\rho} \frac{(\eta_0)^j}{j!}\left[\left(\frac{\partial}{\partial\eta_0}\right)^j B_{M_\gamma}\right](t, \eta)\Bigg|_{\eta_0=0} = \chi_{\gamma,\rho}(t, \eta) \quad \text{(II.2.14)}$$

To estimate $\chi_{\gamma,\rho}(t, \eta)$ one has to get a bound for

$$e^{iP_0\eta_0} - \sum_{j\leqslant\rho} \frac{(iP_0\eta_0)^j}{j!} \quad \text{(II.2.15)}$$

Since $\left| e^{ix} - \sum_{j\leqslant\rho} ((ix)/j!)^j \right|$ is bounded by $C_1|x|^{\rho+1}$ and $C_2(1+|x|)$ there exists, for any value θ, $0 < \theta < 1$, a constant C_θ such that

$$\left| e^{ix} - \sum_{j\leqslant\rho} \frac{(ix)^j}{j!} \right| \leqslant C_\theta |x|^{\rho+\theta}$$

Applying this result to (II.2.15) we obtain

$$|\chi_{\gamma,\rho}(t, \eta)| \leqslant A_\theta |\eta_0|^{\rho+\theta} \int_m^\infty \frac{dP_0}{P_0^{2-\theta}} \leqslant B_\theta ||\eta||^{\rho+\theta} \quad \text{(II.2.16)}$$

Now, we can estimate

$$||\xi', \xi''||^{(-\omega'-\omega''+|\alpha|-\varepsilon)^+}(1 + ||\xi', \xi''||)^Q |D^\alpha\psi(\xi', \xi'')| \quad \text{(II.2.17)}$$

i) $||\xi', \xi''|| < 1$

(II.2.17) is bounded by

$$C' ||\xi', \xi''||^{(-\omega'-\omega''+|\alpha|-\varepsilon)^+} |D^\alpha\psi(\xi', \xi'')|$$

and

$$|D^\alpha \psi| \leqslant C' \sum_{|\gamma| \leqslant |\alpha|} \int D_t^\gamma R(t, \eta) D_{\xi', \xi''}^{\alpha - \gamma} \varphi d^4\eta \Big|_{t_j = \xi'_{u'(j)} - \xi''_{u''(j)}}$$

Consider now

$$J_\gamma(t) = \int D_t^\gamma R(t, \eta) D_{\xi', \xi''}^{\alpha - \gamma} \varphi d^4\eta$$

$$= \int \left(\frac{1}{i}\frac{\partial}{\partial \eta_0}\right)^{\rho+1} B_{M_\gamma}(t, \eta)\left(i \frac{\partial}{\partial \eta_0}\right)^{M_\gamma - \rho - 1} D_{\xi', \xi''}^{\alpha - \gamma} \varphi(\xi', \xi'', \eta) d^4\eta \quad \text{(II.2.18)}$$

ρ will be fixed later. — We introduce now, a function $v(\eta)$, $0 \leqslant v \leqslant 1$, $v \in C^\infty(\mathbb{R}^4)$, $v = 0$, $\|\eta\| \geqslant 1$, $v = 1$, $\|\eta\| \leqslant \frac{1}{2}$ and write

$$J_\gamma^{1,2}(t) = \int \left(\frac{1}{i}\frac{\partial}{\partial \eta_0}\right)^{\rho+1} B_{M_\gamma}(t, \eta)\left(i \frac{\partial}{\partial \eta_0}\right)^{M_\gamma - \rho - 1} D_{\xi' \xi''}^{\alpha - \gamma} \varphi v^{1,2}(\eta) d^4\eta \quad \text{(II.2.19)}$$

with $v^1(\eta) = v(\eta)$, $v^2(\eta) = 1 - v(\eta)$ and $J = J^1 + J^2$.

α) *Estimate with* J_γ^1. — We replace in this case $B_{M_\gamma}(t, \eta)$ by $\chi_{\gamma,\rho}(t, \eta)$ and $J_\gamma^1(t)$ is equal to

$$\sum_{\theta \leqslant \rho+1} \int \chi_{\gamma,\rho}(t, \eta)\left(i \frac{\partial}{\partial \eta_0}\right)^{\theta} v^1(\eta)\left(i \frac{\partial}{\partial \eta_0}\right)^{M_\gamma - \theta} D_{\xi', \xi''}^{\alpha - \gamma} \varphi = \sum_\theta J_{\gamma,\theta}^1(t)$$

1) $\underline{\theta = 0.}$ — Then

$$|J_{\gamma,0}^1(t)| \leqslant C' \sup_{\|\eta\| \leqslant 1} \|\eta\|^{\rho+\theta+4-\varepsilon'} \sum_{|\sigma| = |\alpha| + \omega_0 + \rho + 1} |D^\sigma \varphi|$$

and the corresponding term in (II.2.17) is bounded by

$$C' \sum_{|\sigma| = |\alpha| + \omega_0 + \rho + 1} \sup_{\|\xi', \xi'', \eta\| \leqslant \sqrt{2}} \|\xi', \xi'', \eta\|^{\rho+\theta+4-\varepsilon'+(-\omega'-\omega''+|\alpha|-\varepsilon)^+} |D^\sigma \varphi| \quad \text{(II.2.20)}$$

but

$$\rho + \theta + 4 - \varepsilon' + (-\omega' - \omega'' + |\alpha| - \varepsilon)^+$$
$$\geqslant (-\omega' - \omega'' + |\sigma| - \omega_0 - 1 - \varepsilon + \theta + 4 - \varepsilon')^+ = (-\omega + |\sigma| - \bar{\varepsilon})^+$$

with $\bar{\varepsilon} = 1 - \theta + \varepsilon' + \varepsilon$, and we can choose $\bar{\varepsilon} > \varepsilon$ arbitrarily close to ε. Therefore, (II.2.20) is bounded by

$$C' \sum_{|\sigma| \geqslant \omega_0 + \rho + 1} \sup \|\xi', \xi'', \eta\|^{\mu_\sigma^+} |D^\sigma \varphi|$$

$$\leqslant C' \sum_{|\sigma| \geqslant \omega_0 + \rho + 1} \sup \|\xi', \xi'', \eta\|^{\mu_\sigma^+}(1 + \|\xi', \xi'', \eta\|)^{-\mu_\sigma^+ + |\sigma| - D + 1 - \delta} |D^\sigma \varphi|$$

with $\mu_\sigma^+ = (-\omega + |\sigma| - \bar{\varepsilon})^+$, $\omega = \omega' + \omega'' + \omega_0 - 4$ is given by

$$\omega = 4 - \sum_{j\in J(Y)} r_j^3$$

and

$$D = -\frac{1}{2}\sum_{j\in J(Y)} r_j^3 + 3$$

2) $\underline{\theta \neq 0.}$ — In this case $\frac{1}{2} \leqslant \|\eta\| \leqslant 1$, and

$$|J^1_{\gamma,\theta}(t)| \leqslant C^t \sum_\sigma \sup_{1/2\leqslant\|\eta\|\leqslant 1} |D^\sigma\varphi|$$

with

$$|\sigma| = |\alpha| + \omega_0 + \rho + 1 - \theta \geqslant \omega_0$$

The corresponding term in (II.2.17) is bounded by

$$\sum_{|\sigma|\geqslant\omega_0} \sup \|\xi', \xi'', \eta\|^{\mu_\sigma^+}(1 + \|\xi', \xi'', \eta\|)^{-\mu_\sigma^+ + |\sigma| - D + 1 - \delta} |D^\sigma\varphi| \qquad (II.2.22)$$

since $\frac{1}{2} \leqslant \|\xi', \xi'', \eta\| \leqslant \sqrt{2}$. Moreover, since the external lines are photons and one of the internal line is a fermion, it means that there are at least two fermions as intermediate particles. Then $\omega_0 \geqslant 6 > D$, the maximal value of which is two, and (II.2.22) is bounded by

$$\sum_{|\sigma|\geqslant \max(0,D)} \sup \|\xi', \xi'', \eta\|^{\mu_\sigma^+}(1 + \|\xi', \xi'', \eta\|)^{-\mu_\sigma^+ + |\sigma| - D + 1 - \delta} |D^\sigma\varphi| \qquad (II.2.23)$$

β) *Estimate with* J^2_γ. — We write $J^2_\gamma(t)$ as

$$\sum_{\theta\leqslant\rho+1} \int B_{M_\gamma}(t, \eta)\left(i\frac{\partial}{\partial\eta_0}\right)^\theta v^2(\eta)\left(i\frac{\partial}{\partial\eta_0}\right)^{M_\gamma-\theta} D^{\alpha-\gamma}_{\xi',\xi''}\varphi(\xi', \xi'', \eta)d^4\eta = \sum_\theta J^2_{\gamma,\theta}(t)$$

1) $\underline{\theta = 0.}$ — Then

$$|J^2_{\gamma,0}(t)| \leqslant C^t \sum_{|\sigma|=|\alpha|+\omega_0+\rho+1} \sup_{\|\eta\|\geqslant 1/2} \|\eta\|^{4+\varepsilon'} |D^\sigma\varphi|$$

and the corresponding term in (II.2.17) is bounded by

$$\sup_{\|\xi',\xi'',\eta\|\geqslant\frac{1}{2}} \|\xi', \xi'', \eta\|^{(-\omega'-\omega''+|\alpha|-\varepsilon)^+ + 4 + \varepsilon'} |D^\sigma\varphi|$$
$$\leqslant \sup_{\|\xi',\xi'',\eta\|\geqslant\frac{1}{2}} (1 + \|\xi', \xi'', \eta\|)^{(-\omega'-\omega''+|\alpha|-\varepsilon)^+ + 4 - \varepsilon'} |D^\sigma\varphi|$$

which is

$$\leqslant C^t \sup_{\|\xi',\xi'',\eta\|\geqslant 1/2} (1 + \|\xi', \xi'', \eta\|)^{|\sigma| - D + 1 - \delta} |D^\sigma\varphi|$$
$$\leqslant C^t \sup \|\xi', \xi'', \eta\|^{\mu_\sigma^+}(1 + \|\xi', \xi'', \eta\|)^{-\mu_\sigma^+ + |\sigma| - D + 1 - \delta} |D^\sigma\varphi|$$

if we have chosen ρ such that

$$(-\omega' - \omega'' + |\alpha| - \varepsilon)^+ + 4 + \varepsilon' \leqslant |\sigma| - D + 1 - \delta$$

for example $\rho \geqslant \max(-\omega + 2, 0)$ and $\delta = 1 - \kappa$, $\kappa > 0$ arbitrarily small. On the other hand $|\sigma|$ being greater than ω_0 is greater than max (0, D), one has the estimate (II.2.23).

2) $\underline{\theta \neq 0.}$ — This case is similar to the one with $J^1_{\gamma,\theta}$ since

$$i\frac{\partial}{\partial\eta_0}v^2(\eta) = -i\frac{\partial}{\partial\eta_0}v^1(\eta)$$

ii) $\underline{\|\xi', \xi''\| > 1}$

(II.2.17) is bounded by

$$(1 + \|\xi', \xi''\|)^{Q'} |D^\alpha\psi(\xi', \xi'')|$$

where

$$Q' = \sup_{|\alpha|} [(-\omega' - \omega'' + |\alpha| - \varepsilon)^+ + Q]$$

Using (II.2.13) we have

$$|D^\alpha\psi(\xi', \xi'')| \leqslant C' \sum_{|\sigma| = |\alpha| + \omega_0 + \rho + 1} \sup_\eta (1 + \|\eta\|)^{4+\varepsilon'} |D^\sigma\varphi|$$

and (II.2.17) is bounded by

$$\sum_{|\sigma| \geqslant \omega_0 + \rho + 1} \sup_{\|\xi',\xi'',\eta\| \geqslant 1} (1 + \|\xi', \xi'', \eta\|)^{Q'+4+\varepsilon'} |D^\sigma\varphi(\xi', \xi'', \eta)|$$

which is less than

$$\sum_{|\sigma| \geqslant \omega_0 + \rho + 1} \sup_{\|\xi',\xi'',\eta\| \geqslant 1} (1 + \|\xi', \xi'', \eta\|)^{|\sigma| - D + 1 - \delta} |D^\sigma\varphi|$$

$$\leqslant \sum_{|\sigma| \geqslant \max(0,D)} \sup \|\xi', \xi'', \eta\|^{\mu_\sigma^+}(1 + \|\xi', \xi'', \eta\|)^{-\mu_\sigma^+ + |\sigma| - D + 1 - \delta} |D^\sigma\varphi|$$

if we have chosen ρ such that

$$|\sigma| - D + 1 - \delta \geqslant Q' + 4 + \varepsilon' \qquad \text{or} \qquad \rho \geqslant Q' - \omega_0 + D + 2 + \delta + \varepsilon'$$

this is satisfied if $\rho \geqslant \max(0, Q' - \omega_0 + D + 3)$.

To sum up we have found that if we choose ρ large enough, then

$$I \leqslant C' \sum_{|\sigma| \geqslant \max(0,D)} \sup \|\xi', \xi'', \eta\|^{\mu_\sigma^+}(1 + \|\xi', \xi'', \eta\|)^{-\mu_\sigma^+ + |\sigma| - D + 1 - \delta} |D^\sigma\varphi|$$

with ω and D given by (II.2.21), $\mu_\sigma^+ = (-\omega + |\sigma| - \bar{\varepsilon})^+$, $\delta = 1 - \kappa$, $\kappa > 0$, $\bar{\varepsilon} > 0$, κ being as small as we want.

We have, therefore, proved that the absorptive parts satisfy, at order n, the required norm with D the index of divergence and ω the degree of singularity of the whole diagram. It remains to prove that this norm is preserved by the cutting procedure.

2.4 The cutting procedure and the norm

We apply the procedure described in Chapter V of [1], that is to say, a product by the function $\sqcup$ when $\omega < 0$, a product by the function $\sqcup$ after truncation when $\omega \geqslant 0$ (this last procedure gives a particular solution, the general one being obtained by adding an arbitrary polynomial, in the derivatives of the δ function, of order less or equal to ω). We have, therefore, to estimate for any $\varphi \in \mathscr{S}(\mathbb{R}^N)$.

1)

$$\sum_{|\alpha| \geqslant \max(0,D)} \sup \| \xi \|^{\mu_\alpha^\pm}(1 + \| \xi \|)^{-\mu_\alpha^\pm + |\alpha| - D + 1 - \delta} | D^\alpha(\sqcup(\xi)\varphi(\xi)) | \quad (II.2.24)$$

when $\omega < 0$;

2)

$$\sum_{|\alpha| \geqslant \max(0,D)} \sup \| \xi \|^{\mu_\alpha^\pm}(1 + \| \xi \|)^{-\mu_\alpha^\pm + |\alpha| - D + 1 - \delta} | D^\alpha(\sqcup)(\xi)(W\varphi)(\xi)) | \quad (II.2.25)$$

when $\omega \geqslant 0$.

It is shown in Appendix C that these terms are majorized by

$$\sum_{|\alpha| \geqslant 0} \sup \| \xi \|^{\mu_\alpha^\pm}(1 + \| \xi \|)^{-\mu_\alpha^\pm + |\alpha| - D + 1 - \delta} | D^\alpha \varphi |$$

since ω and D are integers (D is an integer by Furry's theorem). There is, however, a difference with the required result (formula II.2.3) because the sum over $|\alpha|$ begins at zero. Since all diagrams except the photon self-energy and the four-photon diagrams have $D \leqslant 0$, we have to look more carefully at these two cases. We explain in Appendix B how, using the ambiguity on the solution, we recover (II.2.3).

3. Diagrams with photon lines and integrated external lines

Our aim in this Chapter is to develop the second induction hypothesis and to prove at the same time the existence of Green's functions and their growth properties as tempered distributions. More precisely we will be interested in the behaviour at the origin in the position space (or equivalently at infinity in the momentum space).

The proof will follow the same steps as the ones given in Chapter II. We first define a new index which indicates the behaviour of the photon external lines over which we do not integrate. We then define a norm which is supposed to be valid up to a certain order and then prove that it is conserved in constructing the absorptive parts and in applying the cutting procedure.

3.1 The index

In order to treat the case where in constructing the absorptive parts we only have photons as intermediate particles, we need to define an index of divergence for diagrams which are partially integrated and which have only photon lines as non-integrated external lines.

The index must satisfy, in comparison with D, the following two conditions: *a*) it shall reduce to one when there is no external photon; *b*) it has to be compatible with the internal structure in the sense of Chapter II.

From these conditions it results that we can take as index of a diagram G, $D'(G) = -p + 1$, where p is the number of non-integrated external photon lines.

For example, in the case of $\langle T_r(Y; X) \rangle$ where we integrated over the X's.

$$D' = - \sum_{j \in J(Y)} r_j^3 + 1$$

and is only defined if $r_j^i = 0$, $i = 1, 2, j \in J(Y)$.

One can check also the compatibility with the internal structure.

Consider the situation where a diagram G has $\mu + \nu$ external photon lines and is considered as built by linking two diagrams G_1 and G_2 with l

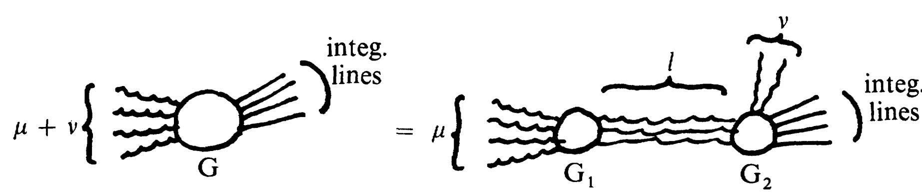

Fig. 2.

photons. Suppose G_1 has $\mu + l$ photons and G_2, the diagram with some integrated external lines, has $\nu + l'$ external photons, $l' \leqslant l$ (we only take into account the photons going away from non-integrated variables). See Figure 2.

Then

$$D'(G) = -(\mu + \nu) + 1 \qquad D'(G_2) = -(\nu + l') + 1$$

$$D(G_1) = -\frac{1}{2}(\mu + l) + 3$$

$$D(G_1) + D'(G_2) + 2l - 4 = -\frac{1}{2}(\mu + l) + 3 - (\nu + l') + 1 + 2l - 4$$

$$= -(\mu + l) + \frac{1}{2}(\mu + l) - (\nu + l') + 2l = D'(G) + \frac{1}{2}(\mu + l) - 1 + l - l'$$

$$\geqslant D'(G)$$

since $l \geqslant l'$ and $\frac{1}{2}(\mu + l) \geqslant 1$ ($l \neq 0$ and G_1 has at least two photons).

3.2 The induction and the norm

The objects under consideration are now the v. e. v. of operators $T_r(Y, X)$ where $r_j^i = 0$, $i = 1, 2$ $j \in J(Y)$. These v. e. v. are tempered distributions in the following set of relative variables

$$\xi_j = y_j - x_{|X|},\ j \in J(Y) \qquad \xi_j = x_j - x_{|X|},\ j \in J(X)$$

We denote them by $F_r(\xi, \zeta)$ or $F(\xi, \zeta)$.

We fix the length of X, $|X| = s$ and the values of r_j^i, $i = 1, 2, 3$, $j \in J(X)$. The induction will run on the length of $|Y|$.

Induction hypothesis. — *Let* $|Y|$ *be less than* n, $r_j^i = 0$, $i = 1, 2$, $j \in J(Y)$; *then for each distribution* $F_r(\xi, \zeta)$ *there exist six positive constants* M, N, C, P, κ, ε, κ *and* ε *positive and arbitrarily small, such that for any* $\varphi(\xi, \zeta) \in \mathscr{S}(\mathbb{R}^{4(|Y|+s-1)})$

$$\langle F_r(\xi, \zeta), \varphi(\xi, \zeta) \rangle \leqslant C \sum_{M \geqslant |\alpha| \geqslant 0} \sum_{N \geqslant |\beta| \geqslant 0} \sup (1 + \|\xi\|)^{|\alpha| - D' + 1 - \delta} \|\zeta\|^{\mu_\beta^+} (1 + \|\zeta\|)^P |D_\xi^\alpha D_\zeta^\beta \varphi(\xi, \zeta)| \quad (II.3.1)$$

with

$$\mu_\beta^+ = (-\omega + |\beta| - \varepsilon)^+ \qquad \omega = 4 - \frac{3}{2} \sum_{j \in J(X)} (r_j^2 + r_j^2) - \sum_{j \in J(X)} r_j^3 \quad (II.3.2)$$

$$D' = -\sum_{j \in J(Y)} r_j^3 + 1, \qquad \delta = 1 - \kappa \quad (II.3.3)$$

and P *an integer.*

We have to make different remarks.

1) If $\varphi(\xi, \zeta) = f(\xi)g(\zeta)$ then the bound in (II.3.1) is a product of norms.

2) When $D' = 1$, $F_r(\xi, \zeta)$ satisfies an adiabatic norm in the ξ variables and has a limit $G(\zeta)$ which, according to the nuclear theorem, is a tempered distribution and satisfies, in view of (II.3.1), for any $\varphi(\zeta) \in \mathscr{S}(\mathbb{R}^{4(s-1)})$,

$$\langle G_r(\zeta), \varphi(\zeta) \rangle \leqslant C \sum_{|\beta| \geqslant 0} \sup \|\zeta\|^{\mu_\beta^+} (1 + \|\zeta\|)^P |D^\beta \varphi(\zeta)| \quad (II.3.4)$$

This gives the existence of Green's functions at order $|Y|$, as tempered distributions with a definite growth at the origin in position space.

3) Moreover if $G_r(\xi)$ has a support property, for example suppose its support is contained in some convex cone Γ defined by $\xi_c > \alpha |\vec{\xi}|$ for some $\alpha > 0$, then one gets pointwise bounds for its Fourier-Laplace transform (in a suitable region).

Starting from

$$|\langle G_r(\zeta), \varphi(\zeta) \rangle| \leqslant C \sum_{0 \leqslant |\beta| \leqslant N} \sup_{\zeta} (1 + \|\zeta\|)^{P+\mu_\beta^+} |D^\beta \varphi(\zeta)|$$

one gets using the Whitney's extension theorem (Ref. 6) that

$$|\langle G_r(\zeta), \varphi(\zeta) \rangle| \leqslant C \sum_{0 \leqslant |\beta| \leqslant N} \sup_{\zeta \in \bar{\Gamma}} (1 + \|\zeta\|)^{P+\mu_\beta^+} |D^\beta \varphi(\zeta)|$$

Thus one can take as test functions $\varphi_k(\zeta) = e^{ik.\zeta}$ where $k = p + iq$ $q \in (\Gamma^*)^0$ (the interior of the dual cone). The Fourier Laplace transform $\tilde{G}_r(k)$ of $G_r(\zeta)$ is an analytic function for $\operatorname{Im} k \in (\Gamma^*)^0$ and one gets

$$\begin{aligned} |\tilde{G}_r(k)| &\leqslant C \sum_{0 \leqslant |\beta| \leqslant N} \sup_{\zeta \in \bar{\Gamma}} (1 + \|\zeta\|)^{P+\mu_\beta^+} |D^\beta e^{ik.\zeta}| \\ &\leqslant C' \sum_{0 \leqslant |\beta| \leqslant N} \frac{|k|^{|\beta|}}{\operatorname{dist}(q, \partial\Gamma^*)^{\mu_\beta^+ + P}} \end{aligned}$$

from which follows also the behaviour of the distribution $\lim_{\substack{q \to 0 \\ q \in (\Gamma^*)^0}} \tilde{G}_r(k)$

4) We have no information on the behaviour at the origin in the ξ variables. It is too hard to control such a behaviour and unnecessary for our purpose. In particular this means that in building absorptive parts we only retain from Chapter 2 that $F_r(\xi)$ satisfies

$$\langle F_r(\xi), \varphi(\xi) \rangle \leqslant C \sum_{M \geqslant |\alpha| \geqslant \max(0,D)} \sup_{\xi} (1 + \|\xi\|)^{|\alpha| - D + 1 - \delta} |D^\alpha \varphi|$$

Now let us start the induction. When $|Y| = 0$ we have to deal with v. e. v. of operators $T_r(X)$. But it has been shown in [*1*] that those v. e. v. are tempered distributions singular at the origin of order ω. Therefore, they satisfy

$$\langle F_r(\zeta), \varphi(\zeta) \rangle \leqslant C \sum_{N \geqslant |\beta| \geqslant 0} \sup \|\zeta\|^{\mu_\beta^+} (1 + \|\zeta\|)^P |D^\beta \varphi(\zeta)|$$

Our next step is to show that the absorptive parts of order $r = |Y|$ satisfy the norm (II.3.1).

3.3 The adiabatic parts and the norm

We have to norm a distribution of the form (II.2.4), with $l \geqslant 1$ (connected terms) and $|Y| = |Y' \cup Y''| = n$. Let us first define a set of variables

$$\xi'_j = y'_j - x_s \qquad j \in J(Y')$$
$$\xi''_j = y''_j - x_s \qquad j \in J(Y'')$$
$$\tau_j = y'_j - y'_{|Y'|} \qquad j \in J(Y')$$

One has the relation $\tau_j = \xi'_j - \xi'_{|Y'|}$ and for commodity we call $\eta = \xi'_{|Y'|}$. Then following the discussion of Section 2.3, we have to distinguish two cases.

a) *Zero mass case*

(II.2.4) becomes

$$F'_{r'+s'}(\tau)F''_{r''+s''}(\xi'',\zeta)\prod_{j=1}^{l'}\Delta^+(\tau_{u'(j)} - \xi''_{u''(j)} + \eta; 0)\prod_{j=1}^{l-l'}\Delta^+(\tau_{v'(j)} - \zeta_{v''(j)} + \eta; 0) \qquad \text{(II.3.5)}$$

where $j \to u'(j)$ and $j \to u''(j)$ are mappings of $(1, \ldots, l')$ respectively, in $(1, \ldots, |Y'| - 1)$ and $(1, \ldots, |Y''|)$. $j \to v'(j)$ and $j \to v''(j)$ are mappings of $(1, \ldots, l-l')$ respectively in $(1, \ldots, |Y'|-1)$ and $(1, \ldots, s-1)$

$$l = \sum_{j \in J(Y')} (s'^1_j + s'^2_j + s'^3_j) = \sum_{j \in J(Y'') \cup J(X)} (s''^1_j + s''^2_j + s''^2_j)$$

$$l' = \sum_{j \in J(Y'')} (s''^1_j + s''^2_j + s''^3_j) \qquad \text{(II.3.6)}$$

Since there is in the case under consideration only photons as intermediate lines, $s'^i_j = s''^i_j = 0$, $i = 1, 2$. The decomposition of l in l' and $l - l'$ is to differentiate the intermediate lines attached to $y's$ from the ones linked to $x's$.

We have now two cases.

i) $\underline{l = 1}$

We apply twice the second tensor product rule in the variables τ, ξ'', η since in the norm of $F''(\xi'', \zeta)$ the ζ variables are disconnected from the ξ'' variables.

Following the case $l = 1$ of Section II.2.3 we find the bound given in (II.3.1) since

$$D_1 + D_2 + D_3 > D' = -\sum_{j \in J(Y)} r^3_j + 1$$

where

$$D_1 = -\frac{1}{2}\sum_{j \in J(Y')} (r'^3_j + s'^3_j) + 3 \qquad \text{is the index of } F'$$

$$D_2 = - \sum_{j \in J(Y'')} (r_j''^3 + s_j''^3) + 1 \qquad \text{is the index of } F''$$

$$D_3 = -2 \qquad \text{is the index of } \Delta^+ \qquad (II.3.7)$$

and here

$$\sum_{j \in J(Y')} s_j'^3 = 1 \geqslant \sum_{j \in J(Y'')} s_j''^3$$

ii) $\underline{l > 1}$

The proof is roughly the same as in Section II.2.3. However, we will give it in detail as an example.

Let

$$\varphi(\xi', \xi'', \zeta) \in \mathscr{S}(\mathbb{R}^{4(n+s-1)})$$

and set

$$\psi(\tau, \xi'', \zeta) = \int R_1(t^1, \eta) R_2(t^2, \eta) \varphi(\xi', \xi'', \eta) d^4\eta \Big|_{\substack{t_j^1 = \tau_{u'(j)} - \xi''_{u''(j)} \\ t_j^2 = \tau_{v'(j)} - \zeta_{v''(j)}}}$$

and

$$R_1(t^1, \eta) = \prod_{j=1}^{l'} \Delta^+(t_j^1 + \eta; 0) \qquad R_2(t^2, \eta) = \prod_{j=1}^{l-l'} \Delta^+(t_j^2 + \eta; 0)$$

We define $R(t, \eta) = R_1 R_2$. Now applying the second tensor product rule to

$$I = \int F'(\tau) F''(\xi'', \zeta) \psi(\tau, \xi'', \zeta) d\tau d\xi'' d\zeta$$

we find that I is bounded by

$$C \sum_{|\alpha| \geqslant 0} \sum_{|\beta| \geqslant 0} \sup (1 + \| \tau, \xi'' \|)^{|\alpha| - D_1 - D_2 + 1 - \delta} \| \zeta \|^{\mu_\beta^+} (1 + \| \zeta \|)^P \, | D^\alpha_{\tau,\xi''} D^\beta_\zeta \psi | \qquad (II.3.8)$$

with D_1 and D_2 defined by (II.3.7)

$$\mu_\beta^+ = (-\omega + |\beta| - \varepsilon)^+ \qquad \omega = 4 - \frac{3}{2} \sum_{j \in J(X)} (r_j''^1 + r_j''^2) - \sum_{j \in J(X)} (r_j''^3 + s_j''^3) \qquad (II.3.9)$$

since $s_j^i = 0$, $i = 1, 2$.

Notice that we have omitted the upper bounds for $|\alpha|$ and $|\beta|$. This will always be the case from now on since the steps we will follow only introduce a finite number of new derivatives and since we only need to know that the constants M and N of (II.3.1) are finite.

Now

$$| D^\alpha_{\tau,\xi''} D^\beta_\zeta \psi(\tau, \xi'', \zeta) | \leqslant C' \sum_{|\gamma| \leqslant |\alpha|} \sum_{|\theta| \leqslant |\beta|} \left| \int D^\gamma_{\tau,\xi''} D^\theta_\zeta R(t, \eta) D^{\alpha-\gamma}_{\tau,\xi''} D^{\beta-\theta}_\zeta \varphi d^4\eta \right| \qquad (II.3.10)$$

Again

$$D_t^\sigma R(t, \eta) = \left(\frac{1}{i} \frac{\partial}{\partial \eta_0} + 1 \right) \left(\frac{1}{i} \frac{\partial}{\partial \eta_0} \right)^{|\sigma| + 2l} B_\sigma(t, \eta)$$

Since $l > 0$ we replace $B_\sigma(t, \eta)$ by

$B_\sigma(t, \eta) - B_\sigma(t, 0)$ and $| B_\sigma(t, \eta) - B_\sigma(t, 0) | \leqslant (1 + \| \eta \|)^{2-\varepsilon'}$, $\varepsilon' < 1$.

Then (II.3.10) is bounded by

$$C^t \sum_{|\gamma| \leqslant |\alpha|} \sum_{|\theta| \leqslant |\beta|} \int (1 + \| \eta \|)^{2-\varepsilon'} \left| \left(i \frac{\partial}{\partial \eta_0} + 1 \right) \left(i \frac{\partial}{\partial \eta_0} \right)^{|\gamma| + |\theta| + 2l} D^{\alpha-\gamma}_{\tau, \xi''} D^{\beta-\theta}_\zeta \varphi \right| d^4\eta$$

$$\leqslant C^t \sum_{2l + |\alpha| + 1 \geqslant |\sigma| \geqslant 2l + |\alpha|} \sum_{|\theta| \leqslant |\beta|} \sup_\eta (1 + \| \eta \|)^{6-\varepsilon'+\varepsilon''} | D^{\sigma+\theta}_{\tau, \xi''} D^{\beta-\theta}_\zeta \varphi |, \quad \varepsilon'' > 0$$

Therefore, from $\| \tau, \xi'', \eta \| \leqslant C \| \xi', \xi'' \|$ it results that

$$I \leqslant C^t \sum_{|\sigma| \geqslant 2l} \sum_{|\theta| \leqslant |\beta|} \sup (1 + \| \xi', \xi'' \|)^{|\sigma| - D_1 - D_2 - 2l + 1 - \delta + 6 - \varepsilon' + \varepsilon''} \| \zeta \|^{\mu^+_\beta} (1 + \| \zeta \|)^P | D^{\sigma+\theta}_{\xi', \xi''} D^{\beta-\theta}_\zeta \varphi | \quad \text{(II.3.11)}$$

By choosing ε' close to one and ε'' close to zero, one gets

$$| \sigma | - D_1 - D_2 - 2l + 1 - \delta + 6 - \varepsilon' + \varepsilon'' \leqslant | \sigma | - D' + 1 - \bar{\delta} \leqslant | \sigma | + | \theta | - D' + 1 - \bar{\delta}$$

with $\bar{\delta} < \delta$ and

$$D' = - \sum_{j \in J(Y)} r^3_j + 1 \quad \text{(II.3.12)}$$

Moreover, since $\mu^+_\beta \geqslant \mu^+_{\beta-\theta}$ and the derivation on ζ is of finite order, there exists an integer P′ and a constant C independant of β and θ such that

$$\| \zeta \|^{\mu^+_\beta} (1 + \| \zeta \|)^P \leqslant C \| \zeta \|^{\mu^+_\beta - \theta} (1 + \| \zeta \|)^{P'} \quad \text{(II.3.13)}$$

and I is, then, bounded by

$$C^t \sum_{|\sigma| \geqslant 0} \sum_{|\rho| \geqslant 0} \sup (1 + \| \xi', \xi'' \|)^{|\sigma| - D' + 1 - \bar{\delta}} \| \zeta \|^{\mu^+_\rho} (1 + \| \zeta \|)^{P'} | D^\sigma_{\xi', \xi''} D^\rho_\zeta \varphi | \quad \text{(II.3.14)}$$

We achieve the proof by remarking that if we define $\bar{\omega}$ by

$$\bar{\omega} = 4 - \frac{3}{2} \sum_{j \in J(X)} (r''^1_j + r''^2_j) - \sum_{j \in J(X)} r''^3_j \quad \text{(II.3.15)}$$

then $\bar{\omega} \geqslant \omega$ and it follows that $(-\omega + | \alpha | - \varepsilon)^+ \geqslant (-\bar{\omega} + | \alpha | - \varepsilon)^+$. Applying an inequality of the form (II.3.13) we conclude that for fixed r's, $| Y | = n$ each term of the absorptive part corresponding to $s^i_j = 0$, $i = 1, 2$ is bounded by the same expression which only depends on the r's,

$$C^t \sum_{|\sigma| \geqslant 0} \sum_{|\rho| \geqslant 0} \sup (1 + \| \xi', \xi'' \|)^{|\sigma| - D' + 1 - \bar{\delta}} \| \zeta \|^{(-\bar{\omega} + |\rho| - \varepsilon)^+} (1 + \| \zeta \|)^P | D^\sigma D^\rho \varphi | \quad \text{(II.3.16)}$$

It remains to consider the following case.

b) *One of the masses is non-zero*

The principle of the proof is the same as in *b*) of Section II.2.3, corrected as we just did to take into account the ζ's. We obtain (II.3.14) as a bound and using a formula of type (II.3.13) we get for each term corresponding to fixed values of the r's the bound (II.3.16).

Therefore, for $|Y| = n$, and r_j^i fixed there exist six constants M, N, C, P, κ, ε such that the absorptive part satisfies for any $\varphi \in \mathcal{S}(\mathbb{R}^{4(n+s-1)})$ the formula (II.3.1), with D′ given by (II.3.12) and ω by (II.3.15).

It remains to define a cutting procedure and to show that this norm is conserved.

3.4 The cutting procedure and the norm

The problem here is quite different from the one treated in Chapter II. We only have support properties on the Y's. More precisely the support of the absorptive parts is contained in the union of the cones C_+ and C_- introduced at the beginning.

From now on, let us call ξ the relative variables ξ' and ξ''. In the ξ and ζ variables, one has, with $|X| = s$ and $|Y| = n$

$$C_\pm = \{ (\xi, \zeta) \in \mathbb{R}^{4(n+s-1)} \mid \xi_j - \zeta_{u(j)} \in \bar{V}^\pm \}$$

for at least a mapping $j \to u(j)$ of $(1, \ldots, n)$ in $(1, \ldots, s-1)$ }. (II.3.17)

We now define a function ш as in [1], except that we replace Γ^+ and $\Gamma^- = -\Gamma^+$ by cones Γ_ρ^+ and $\Gamma_\rho^- = -\Gamma_\rho^+$ where $\Gamma^+ \setminus \{0\}$ is strictly contained in Γ_ρ^+.

For example, one can choose

$$\Gamma_\rho^+ = \left\{ x : x_j - x_N \in \bar{V}_\rho^+,\ 1 \leqslant j \leqslant N-1 \mid \bar{V}_\rho^+ = \left\{ y \in \mathbb{R}^4 ; y_0 \geqslant 0, y_0^2 \geqslant \sum_1^3 \rho y_j^2 \right\} \right\} \tag{II.3.18}$$

with $\rho < 1$. In order to simplify the following we will restrict ourselves to this special choice.

Let us call for convenience $d_r(\xi, \zeta)$ the absorptive part (r fixed). Since it satisfies (II.3.1) it is a tempered distribution of degree of singularity in the ξ variables $\omega_\xi = M$. One has

$$\langle d_r(\xi, \zeta), \varphi(\xi, \zeta) \rangle \leqslant C \sum_{M \geqslant |\alpha| \geqslant 0} \sum_{N \geqslant |\beta| \geqslant 0} \sup \| \xi \|^{(-\omega_\gamma + |\alpha| - \varepsilon')^+}$$

$$(1 + \| \xi \|)^{-(-\omega_\xi + |\alpha| - \varepsilon')^+ + |\alpha| - D' + 1 - \delta} \| \zeta \|^{\mu_\beta^+} (1 + \| \zeta \|)^P | D_\xi^\alpha D_\zeta^\beta \varphi | \tag{II.3.19}$$

because $(-\omega_\xi + |\alpha| - \varepsilon')^+ = 0$.

We can therefore apply the usual cutting procedure when $\omega \geqslant 0$ with the function ш(ξ) we just defined. It is shown in Appendix C that

$\langle d_r(\xi, \zeta),$ш$(\xi)(W\varphi)(\xi, \zeta) \rangle$, where the W-subtraction procedure applies only to the ξ variables, is bounded by a norm of the form (II.3.19) but with $|\alpha| \leqslant M'$, $M' > \omega$. However, we notice that the cut distribution which we denote formally by $(\sqcup d_r)(\xi, \zeta)$ satisfies

$$\langle (\sqcup d_r)(\xi, \zeta), \varphi(\xi, \zeta) \rangle \leqslant C \sum_{M \geqslant |\alpha| \geqslant 0} \sum_{N \geqslant |\beta| \geqslant 0} \sup (1 + \| \xi \|)^{|\alpha| - D' + 1 - \delta} \| \zeta \|^{\mu_\beta^+} (1 + \| \zeta \|)^P | D_\xi^\alpha D_\zeta^\beta \varphi | \quad (II.3.20)$$

The question now is to recover from $\sqcup d$ the advanced or the retarded part in the y's, let us call them $a_r(\xi, \zeta)$ and $r_r(\xi, \zeta)$. By construction one has

$$d_r(\xi, \zeta) = a_r(\xi, \zeta) - r_r(\xi, \zeta) \quad (II.3.21)$$

and we know from [1] that a_r and r_r are tempered distributions singular at the origin of degree ω, with

$$\omega = 4 - \sum_{j \in J(Y)} r_j^3 - \sum_{j \in J(X)} r_j^3 \quad (II.3.22)$$

Hence, we have, for example, the existence of constants L, C, R, ε such that for any $\varphi(\xi, \zeta) \in \mathcal{S}$

$$\langle a_r(\xi, \zeta), \varphi(\xi, \zeta) \rangle \leqslant C \sum_{L \geqslant |\sigma| \geqslant 0} \sup \| \xi, \zeta \|^{(-\omega + |\sigma| - \varepsilon)^+} (1 + \| \xi, \zeta \|)^R | D_{\xi,\zeta}^\sigma \varphi | \quad (II.3.23)$$

Suppose now that we are interested in the advanced part. Then, we write, using (II.3.21)

$$\begin{aligned}(\sqcup d_r)(\xi, \zeta) &= a_r(\xi, \zeta) - (a_r(\xi, \zeta) - (\sqcup a_r)(\xi, \zeta)) - (\sqcup r_r(\xi, \zeta) \\ &= a_r(\xi, \zeta) - b_r(\xi, \zeta)\end{aligned} \quad (II.3.24)$$

We find with (II.3.20) and (II.3.23) that

$$\langle b_r(\xi, \zeta), \varphi(\xi, \zeta) \rangle \leqslant C \Big\{ \sum_{M \geqslant |\alpha| \geqslant 0} \sum_{N \geqslant |\beta| \geqslant 0} \sup (1 + \| \xi \|)^{|\alpha| - D' + 1 - \delta} \| \zeta \|^{(-\bar{\omega} + |\beta| - \bar{\varepsilon})^+} (1 + \| \zeta \|)^P | D_\xi^\alpha D_\zeta^\beta \varphi | + \sum_{L \geqslant |\sigma| \geqslant 0} \sup \| \xi, \zeta \|^{(-\omega + |\sigma| - \varepsilon)^+} (1 + \| \xi, \zeta \|)^R | D_{\xi,\zeta}^\sigma \varphi | \Big\} \quad (II.3.25)$$

Noting $\| \xi \| \leqslant \| \xi, \zeta \|$, $\| \zeta \| \leqslant \| \xi, \zeta \|$ and $1/(1 + \| \xi \|)^\delta \leqslant 1$. One gets, with $\bar{M} = \max(M + N, L)$ and $\bar{P} = P + M - D' + 1$ that (II.3.25) is less than

$$\sum_{\substack{\bar{M} \geqslant |\sigma| \geqslant 0 \\ |\sigma| = |\alpha| + |\beta|}} \{ \sup \| \xi, \zeta \|^{(-\bar{\omega} + |\beta| - \bar{\varepsilon})^+} (1 + \| \xi, \zeta \|)^{\bar{P}} | D_\xi^\alpha D_\zeta^\beta \varphi | + \sup \| \xi, \zeta \|^{(-\omega + |\beta| - \varepsilon)^+} (1 + \| \xi, \zeta \|)^R | D_\xi^\alpha D_\zeta^\beta \varphi | \}$$

Let us choose $\varepsilon = \bar{\varepsilon}$ (this is always possible), according to the definition (II.3.22) of ω and (II.3.27) of $\bar{\omega}$ one has $\bar{\omega} > \omega$ and

$$(-\bar{\omega} + |\beta| - \varepsilon)^+ \leqslant (-\omega + |\beta| - \varepsilon)^+$$

and therefore there exists a constant S such that

$$\langle b_r(\xi,\zeta), \varphi(\xi,\zeta) \rangle \leqslant C \sum_{\substack{\sigma=\alpha+\beta \\ \bar{M} \geqslant |\sigma| \geqslant 0}} \sup \| \xi,\zeta \|^{(-\bar{\omega}+|\beta|-\varepsilon)^+} (1 + \| \xi,\zeta \|)^S | D_\xi^\alpha D_\zeta^\beta \varphi | \quad \text{(II.3.26)}$$

Let us now investigate in more detail the support of $b_r(\xi,\zeta)$.

First, consider the term $a_r - \sqcup a_r$, and fix the ζ's. The ξ's belonging to supp $(a_r - \sqcup a_r)$ are those such that $\xi_j - \zeta_{u(j)} \in \bar{V}^+$ and $\xi_j \notin \bar{V}_\rho^+$. Now, let a and b be two four-vectors such that $a - b \in \bar{V}^+$, $a \notin \bar{V}_\rho^+$ and call $\| a \|$ the Euclidean norm. Then, one can check that

$$\| a \|^2 \leqslant \max\left(\frac{\rho+1}{\rho-1}, 3\right) \| b \|^2$$

Therefore, there exists a constant C_ρ such that

$$\| \xi \| \leqslant C_\rho \| \zeta \| \quad \text{(II.3.27)}$$

in the support of $(a_r - \sqcup a_r)$.

Let us now look at the term $\sqcup r$. Since $\sqcup$ vanishes on Γ_ρ^-, the ξ's belonging to supp $\sqcup r$ are those such that $\xi_j - \zeta_{u(j)} \in \bar{V}^-$ and $\xi_j \notin \bar{V}_\rho^-$. They also satisfy (II.3.27).

Thus, the support of $b_r(\xi,\zeta)$ is contained in the cone

$$K = \{ \xi,\zeta \mid \| \xi \| \leqslant C_\rho \| \zeta \| \}$$

Let $C' < C_\rho$, then the cone $K \setminus \{0\}$ is strictly contained in

$$K' = \{ \xi,\zeta \mid \| \xi \| \leqslant C' \| \zeta \| \}$$

and one can replace the norm in (II.3.26) by

$$\sum_{\substack{\sigma=\alpha+\beta \\ \bar{M} \geqslant |\sigma|}} \sup_{\xi,\zeta \in K'} \| \xi,\zeta \|^{(-\bar{\omega}+|\beta|-\varepsilon)^+} (1 + \| \xi,\zeta \|)^S | D_\xi^\alpha D_\zeta^\beta \varphi | \quad \text{(II.3.28)}$$

and there exists a constant $C_{K'}$ such that (II.3.28) is less than

$$C_{K'} \sum_{\substack{\sigma=\alpha+\beta \\ \bar{M} \geqslant |\sigma|}} \sup_{\zeta,\xi} \| \zeta \|^{(-\bar{\omega}+|\beta|-\varepsilon)^+} \frac{(1 + \| \zeta \|)^{S+\delta}}{(1 + \| \xi \|)^\delta} | D_\xi^\alpha D_\zeta^\beta \varphi | \quad \text{(II.3.29)}$$

Now we come back again to (II.3.24) and write

$$\langle a_r(\xi, \zeta), \varphi(\xi, \zeta) \rangle \leqslant C' \left\{ \sum_{M \geqslant |\alpha|} \sum_{N \geqslant |\beta|} \sup (1+\|\zeta\|)^{|\alpha|-D'+1-\delta} \|\zeta\|^{(-\bar{\omega}+|\beta|-\varepsilon)^+} (1+\|\zeta\|)^{P} | D_\xi^\alpha D_\zeta^\beta \varphi | + \sum_{\substack{\sigma=\alpha+\beta \\ \bar{M} \geqslant |\sigma|}} \sup (1 + \|\xi\|)^{|\alpha|-D'+1-\delta} \|\zeta\|^{(-\bar{\omega}+|\beta|-\varepsilon)^+} (1 + \|\zeta\|)^{S+\delta} | D_\xi^\alpha D_\zeta^\beta \varphi | \right\} \quad \text{(II.3.30)}$$

where we have added in (II.3.29) $(1 + \|\xi\|)^{|\alpha|-D'+1}$ which is larger than one, since $D' \leqslant 1$.

Finally, we see that there exists two constants T and C such that

$$\langle a_r(\xi, \zeta), \varphi(\xi, \zeta) \rangle \leqslant C \sum_{\substack{\bar{M} \geqslant |\sigma| \\ \sigma=\alpha+\beta}} \sup (1 + \|\xi\|)^{|\alpha|-D'+1-\delta} \|\zeta\|^{(-\bar{\omega}+|\beta|-\varepsilon)^+} (1 + \|\zeta\|)^{T} | D_\xi^\alpha D_\zeta^\beta \varphi |$$

which is the required result since from a_r or r_r we can construct the $F_r(\xi, \zeta)$ for $|Y| = n$.

4. The case of the electron self-energies

We will show in this Chapter that the electron self-energy of order n, $n = 0, 1, \dots$ given by

$$\Sigma_{\alpha,\beta}(x_1, x_2) = \int \langle T(\mathscr{L}_{1,\alpha}(x_1)\mathscr{L}_{2,\beta}(x_2)\mathscr{L}(y_1) \dots \mathscr{L}(y_n)) \rangle \, dy_1 \dots dy_n$$

(we have shown in Section 3 that this expression is meaningful as a tempered distribution) can be renormalized in such a way that its Fourier transform vanishes on the mass shell $p^2 = m^2$. Here,

$$\mathscr{L}_{1,\alpha}(x_1) = \sum_{\mu,\beta} \psi_\beta(x_1)\Gamma_\mu^{\alpha,\beta} A_\mu(x_1) \qquad \mathscr{L}_{2,\beta}(x_2) = \sum_{\mu,\alpha} \bar{\psi}_\alpha(x_2)\Gamma_\mu^{\alpha,\beta} A_\mu(x_2)$$

and

$$\mathscr{L}(y) = \sum_{\alpha,\beta,\mu} \bar{\psi}_\alpha(y)\Gamma_\mu^{\alpha,\beta}\psi(y)A_\mu(y)$$

where α and β are the spinor indices.

For convenience, we choose to work in momentum space, but with the equivalence of norms in momentum and position spaces, all the results of Sections 2 and 3 can be transcribed in momentum space (see [5]).

In a more symmetrical way the electron self-energy of order n is related to the following v. e. v. of T product

$$\langle T_r(y_1, \dots, y_{n+2}) \rangle$$

with

$$\sum_{j=1}^{n+2} r_j^1 = \sum_{j=1}^{n+2} r_j^2 = 1 \qquad \sum_{j=1}^{n+2} r_j^3 = 0$$

Let us define now some notations. The Fourier transform $K_r(q_1, \ldots, q_n)$ of $\langle T_r(y_1, \ldots, y_n) \rangle$ is given by

$$K_r(q_1, \ldots, q_n) = \int \langle T_r(y_1, \ldots, y_n) \rangle e^{-i\sum_1^n q_j \cdot y_j} \prod_1^n dy_j \,.$$

Since we have translation invariance in position space, $K_r(q)$ is the product of a δ^4 function by a tempered distribution $t_r(q)$ depending only on $n - 1$ variables: $t_r(q)$ is a reduced kernel. We will say that the omitted variable is q_j, $1 \leqslant j \leqslant n$, if

$$K_r(q_1, \ldots, q_n) = \delta^4\left(\sum_{j=1}^{n} q_j\right) t_r(q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n)$$

The study of the properties of the electron self-energy at order n is therefore equivalent to the study of the adiabatic limit, when the q's go to zero for $K_r(p+q_1, q_2, \ldots, q_{n-1}, -p'+q_{n+2})$ or $K_r(q_1, \ldots, q_{n-1}, p-p'+q_n)$, that that is to say, with the omitted variable q_n, of $t_r(p + q_1, q_2, \ldots, q_{n-1})$ or $t_r(q_1, \ldots, q_{n-1})$.

The idea of the proof is to show by induction on the number of vertices that the self-energies are Hölder continuous functions of p, for $p^2 \leqslant m^2$ (5).

We give in a first section a sketch of the proof. In the second one we present the induction hypothesis. Then we prove, through two technical sections, the Hölder continuity, and finally we show how we can get from this continuity the vanishing on the mass shell.

4.1 The sketch of the proof

We work in momentum space and follow closely the techniques and the results of [2] which have to be familiar to the reader. In fact, we just give outlines of the proof and only insist on what is really new. Let us review them. The general principles are the same as usual, we have to check some properties for $\langle T_r(y_1, \ldots, y_n) \rangle$ knowing them for the v. e. v. of T products of lower orders.

The first step is to construct the absorptive parts of order n, and then to recover the *totally* advanced (or retarded) product from a well-known cutting procedure.

In order to have an idea of what has to be known, let us classify the

(5) It would have been better to obtain, the Hölder continuity around m^2, but this would have needed more complicated technical assumptions.

1) No electron

FIG. 3.1.

2) One electron and photons

FIG. 3.2.

3) More than one electron

FIG. 3.3.

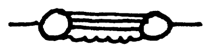

FIG. 3.3′.

4) Only one electron

FIG. 3.4.

terms which appear in the construction of an absorptive part. This classification depends on the structure of the intermediate states.

One sees from those various cases that we need to know the behaviour of the diagrams with only external photon lines, or with external photon lines and two external electron lines. As in Section 3, case 3) presents no difficulty because this absorptive part vanishes for $q_1, \ldots, q_n$ in a neighbourhood of zero and $p^2 < 4m^2$. On the other hand, we see that if we want to avoid a pole of increasing order for the self-energies, we should ask the vanishing on the mass shell for the lower orders. Let us remark that since we need to have a knowledge on diagrams having external photon lines, in Figures 3.1, 3.2, 3.3, 3.3′ and 3.4, one should replace the bubbles by bubbles having external photon lines.

4.2 THE INDUCTION HYPOTHESES

We first remark that being interested only in the behaviour when the q's go to zero, we can restrict ourselves to testing functions having their support in a q-neighbourhood of the origin. Secondly, let us remember in a useful form the results concerning the diagram with photon external lines only. From Section 2, we get:

Let V be any neighbourhood of $\{ q_1 = \ldots = q_{n-1} = 0 \}$, let $r_j^i = 0$, $i = 1, 2, j = 1, \ldots, n$ let R, κ be positive numbers, $R > 0$ and $\kappa > 0$ close to zero, then for any $\varphi \in C^\infty(\mathbb{R}^{4(n-1)})$ with support in

$$V \cap \{ q \mid \| q_j \| \leqslant R, 1 \leqslant j \leqslant n - 1 \}$$

there exist positive constants C and N such that

$$\langle t_r(q_1, \ldots, q_{n-1}), \varphi(q_1, \ldots, q_{n-1}) \rangle \leqslant C \sum_{|\alpha| \leqslant N} \sup \| q \|^{4(n-1)+|\alpha|+D-1+\delta} | D^\alpha \varphi(q) | \quad \text{(II.4.1)}$$

with

$$D = -\frac{1}{2}\sum_{j=1}^{n} r_j^3 + 3 \qquad \delta = 1 - \kappa$$

It remains now to give induction hypotheses for the case when there are two external electron lines. We shall have to distinguish the cases when the external momenta are attached to the same vertices or not (see 4.1). We distinguish also the cases where there is one external photon at least and no external photon. In the first case, according to the form of the reduced kernel « with omitted variable q_n », one has

INDUCTION HYPOTHESIS. A_1. — Let $||Y| < n$ and

$$\sum_{j \in J(Y)} r_j^1 = \sum_{j \in J(Y)} r_j^2 = 1 \qquad \sum_{j \in J(Y)} r_j^3 \geqslant 1 \tag{II.4.2}$$

There exists a neighbourhood V of $\{ q_1 = \ldots = q_{|Y|-1} = 0 \}$ a positive number N, and given any κ, $\kappa > 0$, close to zero, any $R > 0$, any compact convex set $K \in \mathbb{R}^4$, and any $p \in \mathbb{R}^4$, with $p \in K$ and $0 < p^2 \leqslant m^2$, $p_0 > 0$, a positive number B such that if $\varphi \in C^\infty(\mathbb{R}^{4(|Y|-1)})$ with support in

$$V \cap \{ q \mid || q_j || \leqslant R,\ 1 \leqslant j \leqslant ||Y| - 1 \}$$

one has

$$\langle t_r(p + q_1, q_2, \ldots, q_{|Y|-1}), \varphi(q_1, \ldots, q_{|Y|-1}) \rangle \leqslant B \sum_{|\alpha| \leqslant N} \sup || q ||^{4(|Y|-1)+|\alpha|+D_2-1+\delta} | D^\alpha \varphi(q) | \tag{II.4.3}$$

with

$$\delta = 1 - \kappa \qquad D_2 = -\sum_{j \in J(Y)} r_j^3 + 1 \tag{II.4.4}$$

Remark that D_2 is the index introduced in Section 3.

INDUCTION HYPOTHESIS A_2. — *Under the same conditions, except those on p, there exists a positive number* B *such that if* $\varphi \in C^\infty(\mathbb{R}^{4(|Y|-1)})$ *with support in*

$$V \cap \{ q \mid || q_j || \leqslant R,\ 1 \leqslant j \leqslant |Y| - 1 \}$$

one has

$$\langle t_r(q_1, \ldots, q_{|Y|-1}), \varphi(q_1, \ldots, q_{|Y|-1}) \rangle \leqslant B \sum_{|\alpha| \leqslant N} \sup || q ||^{4(|Y|-1)+|\alpha|+D_2-1+\delta} | D^\alpha \varphi(q) | \tag{II.4.5}$$

with δ *and* D_2 *given by* (II.4.4).

Consider now the second case: the self-energies of the electrons; the two external electrons cannot be issued from the same vertex (because of Furry's theorem).

INDUCTION HYPOTHESIS A'_1. — *Let* $|Y| < n$

$$\sum_{j\in J(Y)} r_j^1 = \sum_{j\in J(Y)} r_j^2 = 1 \qquad \sum_{j\in J(Y)} r_j^3 = 0$$

Under the condition of hypothesis A_1, *there exists positive numbers* $A(p)$ *and* B *such that*

$$\langle t_r(p + q_1, \ldots, q_{|Y|-1}), \varphi(q_1, \ldots, q_{|Y|-1}) \rangle$$
$$\leqslant A(p) \left| \int \varphi(q)dq \right| + B \sum_{|\alpha| \leqslant N} \sup \| q \|^{4(|Y|-1)+|\alpha|+\delta} | D^\alpha \varphi(q) | \quad \text{(II.4.7)}$$

and given any θ, $0 < \theta < 1$, *close enough to one, there exists a constant* C_θ *with*

$$A(p) \leqslant C_\theta | p^2 - m^2 |^\theta \quad \text{(II.4.8)}$$

This hypothesis implies the vanishing on the mass shell for the electon self-energies of order less than n. Let us now prove this result at order $|Y| = n$.

4.3 THE CONSTRUCTION OF THE ABSORPTIVE PARTS

We treat the different terms which appear in a absorptive part following the classification of Section 4.1. For simplicity, we will speak about « kernels of type A_1, A_2 or A'_1 » when we refer to reduced kernels of the type which enters in the various induction hypotheses.

1) *No electron*

The proof is the same as the one given in Section 3 (however, written in momentum space) since D_2 is the index D' of Section 3. One gets as a result, the following bound

$$B \sum_{|\alpha| \leqslant N} \sup \| q \|^{4(|Y|-1)+|\alpha|+D_2-1+\delta} | D^\alpha \varphi(q) | \quad \text{(II.4.9)}$$

2) *One electron and at least one photon*

The resulting kernel is of type A_1. Various cases appear which depend on the type of the first and the second kernels. Since we only give outlines of the proof, we just present here the case where the first kernel is of type A_1 and the second of type A_2, and we omit all the parts of the discussion concerning the neighbourhoods.

We have to estimate

$$I = \int t_{r'}(p+q'_1-k_{I_1}, q'_2-k_{I_2}, \ldots, q'_{\nu-1}-k_{I_{\nu-1}})t_{r''}(q''_1+k_{J_1}, \ldots, q''_{\mu-1}+k_{J_{\mu-1}})$$
$$\delta^4\left(p+\sum_1^\nu q'_j - \sum_1^l k_j\right)\delta^+(k_1\,;m)P(k_1)\prod_{j=2}^l \delta^+(k_j\,;0)dk_1, \ldots, dk_l$$
$$\varphi(q'_1, \ldots, q'_\nu, q''_1, \ldots, q''_{\mu-1})\prod_1^\nu dq'_j \prod_1^{\mu-1} dq''_j \qquad \text{(II.4.10)}$$

where $q' = (q'_1, \ldots, q'_\nu)$, $q'' = (q''_1, \ldots, q''_{\mu-1})$ and q''_μ are the conjugate variables associated with Y' and Y''

$$Y' \cap Y'' = \emptyset, \quad Y' \cup Y'' = Y, \quad |Y'| = \nu, \quad |Y''| = \mu, \quad \mu+\nu=n$$
$$r' = \{ r_j \mid j \in J(Y') \} \qquad r'' = \{ r_j \mid j \in J(Y'') \}$$

$I_1, \ldots, I_\nu$ and $J_1, \ldots, J_\mu$ are disjoint sets whose union is $\{1, \ldots, l\}$, l being the number of intermediate states. Since we can restrict ourselves to connected products, $l \geqslant 1$, $P(k_1)$ is a polynomial of degree one.

This is the general form. Restricted to the present case it means that $l \geqslant 2$ and $I_1, \ldots, I_{\nu-1}$ and $J_1, \ldots, J_{\mu-1}$ are disjoint sets whose union is $\{2, \ldots, l\}$, each set having at most one element. Therefore k_{I_j} and k_{J_j} are light-like vectors.

Applying the induction hypotheses we get that I is bounded by

$$B \sum_{|\alpha| \leqslant N'} \sum_{|\beta| \leqslant N''} \sup \| q'_1 - k_{I_1}, \ldots, q'_{\nu-1} - k_{I_{\nu-1}} \|^{4(\nu-1)+|\alpha|+D'_2-1+\delta'}$$
$$\| q''_1 + k_{J_1}, \ldots, q''_{\mu-1} + k_{J_{\mu-1}} \|^{4(\mu-1)+|\beta|+D''_2-1+\delta''}$$
$$| D^\alpha_{q'} D^\beta_{q''} \psi(q'_1 - k_{I_1}, \ldots, q''_{\mu-1} + k_{J_{\mu-1}}) | \qquad \text{(II.4.11)}$$

where

$$\psi(q'_1 - k_{I_1}, \ldots, q''_{\mu-1} + k_{J_{\mu-1}}) = \int \delta^4\left(p + \sum_1^\nu q'_j - \Sigma k_j\right)\delta^+(k_1\,;m)P(k_1)$$
$$\prod_{j=2}^l \delta^+(k_j\,;0)\varphi(q'_1, \ldots, q'_\nu, q''_1, \ldots, q''_{\mu-1})\prod_{j=1}^l dk_j dq'_\nu \qquad \text{(II.4.12)}$$

We now need two technical lemmas:

LEMMA 1. — *Let p, q, k, s be four four-vectors such that*

$$\begin{aligned} &p + q = k + s \\ &k^2 = m^2, \quad k_0 > 0 \quad s^2 \geqslant 0, \quad s_0 \geqslant 0 \qquad \text{(II.4.13)} \\ &0 \leqslant p^2 \leqslant m^2 \end{aligned}$$

Then, there exists a constant $C(p)$ *such that*

$$\| s \| \leqslant C(p) \| q \| \tag{II.4.14}$$

Proof. — Since $s \in \bar{V}^+$, $\| s \| \leqslant \sqrt{2} s_0$.

a) $s_0 \leqslant q_0$

Then $\| s \| \leqslant \sqrt{2} \| q \|$ since $q_0 \leqslant \| q \|$.

b) $s_0 > q_0$

Then $p_0 > k_0$ and

$$p_0 - k_0 = \frac{p_0^2 - k_0^2}{p_0 + k_0} = \frac{|\vec{p}|^2 - |\vec{k}|^2 + p^2 - m^2}{p_0 + k_0}$$

so that

$$p_0 - k_0 \leqslant \frac{(|\vec{p}| - |\vec{k}|)(|\vec{p}| + |\vec{k}|)}{p_0 + k_0}$$

Since $s_0 \leqslant |p_0 - k_0| + |q_0|$ one gets

$$s_0(p_0 + k_0) \leqslant (|\vec{p}| - |\vec{k}|)(|\vec{p}| + |\vec{k}|) + |q_0|(p_0 + k_0)$$

but

$$|\vec{p}| - |\vec{k}| \leqslant |\vec{p} - \vec{k}| = |\vec{q} - \vec{s}| \leqslant |\vec{q}| + |\vec{s}| \leqslant |\vec{q}| + s_0$$

Therefore,

$$s_0(p_0 - |\vec{p}| + k_0 - |\vec{k}|) \leqslant 2|\vec{q}|\,|\vec{p}| + 2|q_0|p_0$$

where we have used $|\vec{p}| > |\vec{k}|$, $p_0 > k_0$. Now

$$p_0 - |\vec{p}| = \frac{p^2}{p_0 + |\vec{p}|} \geqslant \frac{p^2}{\sqrt{2}\| p \|}$$

since $p^2 \geqslant 0$ and

$$k_0 - |\vec{k}| = \frac{m^2}{k_0 + |\vec{k}|} \geqslant \frac{m^2}{\sqrt{2}\| p \|}$$

thus

$$s_0 \leqslant \frac{4\| p \|^2}{m^2 + p^2} \| q \|$$

and

$$\| s \| \leqslant \frac{4\sqrt{2}\| p \|^2}{m^2 + p^2} \| q \|$$

Adding *a*) and *b*) one sees that

$$\| s \| \leqslant C(p) \| q \|$$

where

$$C(p) = \sqrt{2} \max \left(1, \frac{4 \| p \| 2}{m^2 + p^2} \right)$$

We now give a lemma on the phase space

LEMMA 2. — *Define for* $l \geqslant 2$

$$\rho_l(r) = \int \delta^4\Big(r - \sum_{j=1}^{l} k_j\Big) \prod_{j=2}^{l} (\delta^+(k_j\,;\,0)dk_j)\delta^+(k_1\,;\,m)\mathrm{P}(k_1)dk_1 \qquad \text{(II.4.15)}$$

Then, for any $\varepsilon > 0$ *and any compact set* K, *there exists a constant* $\mathrm{C}(\varepsilon, \mathrm{K})$ *such that*

$$|\,\rho_l(p+q)\,| \leqslant \mathrm{C}(\varepsilon, \mathrm{K})\,||\,q\,||^{2l-3-\varepsilon} \qquad \text{(II.4.16)}$$

with $p \in \mathrm{K}$, $q \in \mathrm{K}$, $0 \leqslant p^2 \leqslant m^2$.

Proof. — Define for $l \geqslant 2$

$$\omega_l(r) = \int \delta^4\Big(r - \sum_{j=1}^{l} k_j\Big) \prod_{j=1}^{l} \delta^+(k_j\,;\,0)dk_j$$

Then (see [1]), $\omega_l(r)$ is a continuous function and

$$|\,\omega_l(r)\,| \leqslant \mathrm{C}_l\theta(r_0 - |\,\vec{r}\,|)(r^0)^{2l-4}$$

Now

$$\rho_l(r) = \int \delta^+(r - s, m)\,|\,\mathrm{P}(r-s)\,|\,\omega_{l-1}(s)ds$$

for $l \geqslant 3$. But, with the δ^+ function $r = p + q = k + s$, where $k^2 = m^2$, $k^0 > 0$, we can therefore apply Lemma 1, and

$$|\,\rho_l(p+q)\,| \leqslant \mathrm{C}_{l-1}(\mathrm{C}(p))^{2(l-3)}||\,q\,||^{2(l-3)} \int \delta^+(p+q-s\,;\,m)\,|\,\mathrm{P}(p+q-s)\,|\,\theta(s^0 - |\,\vec{s}\,|)d^4s$$

Let us now estimate

$$\begin{aligned}
&\int \delta^+(p + q - s, m)\,|\,\mathrm{P}(p + q - s)\,|\,\theta(s^0 - |\,\vec{s}\,|)ds \\
&\quad = \int \delta^+(k\,;\,m)\,|\,\mathrm{P}(k)\,|\,\theta((p + q - k)_0 - |\,\vec{p} + \vec{q} - \vec{k}\,|)dk \\
&\quad \leqslant \sup_{\substack{k^2 = m^2 \\ k_0 > 0}} ||\,p + q - k\,||^{3-\varepsilon} \int_{p_0 + q_0 \geqslant \sqrt{\vec{k}^2 + m^2}} \frac{|\,\mathrm{P}(k)\,|}{2\sqrt{|\,\vec{k}\,|^2 + m^2}} \frac{d\vec{k}}{|\,\vec{p} + \vec{q} - \vec{k}\,|^{3-\varepsilon}} \\
&\quad = \mathrm{C}(\varepsilon, q) \sup_{\substack{k^2 = m^2 \\ k_0 > 0 \\ p+q-k \in \overline{\mathrm{V}}^+}} ||\,p + q - k\,||^{3-\varepsilon} \leqslant \mathrm{C}'(\varepsilon, q)\,||\,q\,||^{3-\varepsilon}
\end{aligned}$$

Therefore, for $l \geqslant 3$, there exists a constant $\mathrm{C}(\varepsilon, p, q)$ such that

$$|\,\rho_l(p + q)\,| \leqslant \mathrm{C}(\varepsilon, p, q)\,||\,q\,||^{2l-3-\varepsilon}$$

One can check, by direct calculation, the same result for $l = 2$.

Let us now return to the estimate (II.4.11). Set

$$Q = \sum_{j=1}^{l} q'_j$$

and $s = \sum_{j=2}^{l} k_j$. Now, since the k_{J_i} and $k_{I_j}'s$ are light-like, $\| k_J \| \leqslant \| s \|$ and $\| k_I \| \leqslant \| s \|$. On the other hand, by the δ^4 function of momentum conservation in (II.4.4), $p + Q = k_1 + s$, and using Lemma 1, one gets instead of (II.4.11)

$$B' \sum_{|\alpha| \leqslant N'} \sum_{|\beta| \leqslant N''} \sup \| q'_1, \dots, q'_{\nu-1}, Q, q''_1, \dots, q''_{\mu-1} \|^{4(\mu+\nu-2)+|\alpha|+|\beta|+D'_2+D''_2-2+\delta'+\delta''} | D^{\alpha}_{q'} D^{\beta}_{q''} \psi(q'_1 - k_{I_1}, \dots, q''_{\mu-1} + k_{J_{\mu-1}}) | \qquad \text{(II.4.17)}$$

But

$$| D^{\alpha}_{q'} D^{\beta}_{q''} \psi(q'_1 - k_{I_1}, \dots, q''_{\mu-1} + k_{J_{\mu-1}}) |$$

is bounded by

$$\left| \int \rho_l(p + Q) D^{\alpha}_{q'} D^{\beta}_{q''} \chi(q'_1, \dots, q'_{\nu-1}, Q, q''_1, \dots, q'''_{\mu-1}) dQ \right|$$

Using Lemma 2, this last expression is less than

$$C \int \| Q \|^{2l-3-\varepsilon} | D^{\alpha}_{q'} D^{\beta}_{q''} \chi(q'_1, \dots, q'_{\nu-1}, Q, q''_1, \dots, q''_{\mu-1}) | dQ \leqslant C' \| Q \|^{2l+1-\varepsilon'} | D^{\alpha}_{q'} D^{\beta}_{q''} \chi(q'_1, \dots, q''_{\mu-1}) |$$

with $\varepsilon' > \varepsilon$, and where we have used the fact that the $q's$ are in compact sets.

Adding those results (II.4.17) is bounded by

$$B'' \sum_{|\alpha| \leqslant N'} \sum_{|\beta| \leqslant N''} \sup \| q'_1, \dots, q'_{\nu-1}, Q, q''_1, \dots, q''_{\mu-1} \|^{4(\mu+\nu-1)+|\alpha|+|\beta|+D'_2+D''_2-6+\delta'+\delta''+2l+1-\varepsilon'} | D^{\alpha}_{q'} D^{\beta}_{q''} \chi(q'_1, \dots, q''_{\mu-1}) |$$

Setting $\delta = \delta' + \delta'' - \varepsilon' - 1$, one gets

$$B''' \sum_{|\alpha| \leqslant N''' = N' + N''} \sup \| q', q'' \|^{4(\mu+\nu-1)+|\alpha|+D_2-1+\delta} | D^{\alpha} \varphi(q', q'') | \qquad \text{(II.4.18)}$$

which is the required result. Other cases can be treated in the same way.

3) *More than one electron*

We do not treat this case explicitly. We just remark that in this case, the terms of the absorptive part vanish identically, if, denoting for example by Q the sum of the q's of the first kernel, $\| Q \| < 2m$, with $p^2 \leqslant m^2$. One can check easily that those terms satisfy the required bound.

4) *One electron*

Three cases appear depending on whether or not one of the bubbles is a self-energy.

a) *None of them is a self-energy.* — Let us look for example at the case when the first and the second kernels are of type A_1. One has to estimate an expression of the form (II.4.10) with

$$k_{I_j} = 0 \quad j = 1, \ldots, \nu - 1 \quad k_{J_1} = k \quad k_{J_j} = 0 \quad j = 2, \ldots, \mu$$

and it remains only $\delta^+(k, m)P(k)$.

I is then bounded by

$$B \sum_{|\alpha| \leqslant N'} \sum_{|\beta| \leqslant N''} \sup \| q'_1, \ldots, q'_{\nu-1}, q''_1, \ldots, q''_{\mu-1} \|^{4(\mu+\nu-2)+|\alpha|+|\beta|+D'_2+D''_2-2+\delta'+\delta''} \int \delta^+(p+Q\,; m) \,|\, P(p+Q) \,|\, |\, D^{\alpha}_{q'} D^{\beta}_{q''} \chi(q'_1, \ldots, q'_{\nu-1}, Q, q''_1) \,|\, dQ \tag{II.4.19}$$

Using the fact that on any compact set K

$$\int_{k \in K} \delta^+(k\,; m) \frac{| P(k) |}{\| k - p \|^{3-\varepsilon}} d^4k$$

is bounded, one gets that (II.4.19) is of the form (II.4.18).

b) *One is a self-energy.* — Let us take for example that this is the second. We have to estimate a term of the preceding form. We get a bound in two parts. One is of the form (II.4.19) and can be treated in the same way, the other one is:

$$B \sum_{|\alpha| \leqslant N'} \sup \| q'_1, \ldots, q'_{\nu-1} \|^{4(\nu-1)+|\alpha|+D'_2-1+\delta'} \int \delta^4(p + Q - k) \delta^+(k\,; m) P(k) A(k) dk \left| \int \chi(q'_1, \ldots, q'_{\nu-1}, Q, q''_1, \ldots, q''_{\mu-1}) dq''_1 \ldots dq''_{\mu-1} \right| dQ \tag{II.4.20}$$

and vanishes identically since $A(k)$ vanishes on the mass shell.

If the self-energy had been the first kernel, one would have got, instead of (II.4.20)

$$\mathrm{BA}(p) \sum_{|\beta| \leqslant \mathrm{N}''} \sup \| q''_1, \ldots, q''_{\mu-1} \|^{4(\mu-1)+|\beta|+\mathrm{D}''_2-1+\delta''}$$
$$\int \delta^4(p + \mathrm{Q} - k)\delta^+(k\,;m)\mathrm{P}(k)$$
$$\left| \int \chi(q'_1, \ldots, q'_{\nu-1}, \mathrm{Q}, q''_1, \ldots, q''_{\mu-1}) dq'_1 \ldots dq'_{\nu-1} \right| d\mathrm{Q}dk \qquad (\mathrm{II}.4.21)$$

Here we use the fact that there exists for any $\underline{\theta}$, $0 < \theta < 1$, a constant C_θ such that

$$\mathrm{A}(p) \leqslant \mathrm{C}_\theta \,| p^2 - m^2 |^\theta$$

We get that, on any compact sets K_1 and K_2

$$\int_{\mathrm{Q}\in\mathrm{K}_1} \frac{|\mathrm{P}(p+\mathrm{Q})|}{\|\mathscr{L}\|^{3+\theta-\varepsilon}} | p^2 - m^2 |^\theta \delta^+(p + \mathrm{Q}\,; m) d\mathrm{Q}$$

is, for $p \in \mathrm{K}_2$, bounded. This result comes from the equality

$$p^2 - m^2 = -\mathrm{Q}^2 - 2p.\mathrm{Q} \leqslant \|\mathrm{Q}\|(1 + 2\|p\|)$$

due to the δ^+ function.

We get then the usual estimate (II.4.18), with $\delta = \delta'' - 1 + \theta - \varepsilon - \varepsilon'$, where ε' is due to the estimate on the integration over the q''s.

The other cases can be treated in the same way.

c) *The two are self-energies.* — I is as in case *a*) and is bounded by four terms. There is only one which is different from those previously considered. It is of the form

$$\mathrm{A}_1(p) \left| \int \delta^4(p + \mathrm{Q} - k)\delta^+(k\,; m)\mathrm{P}(k)\mathrm{A}_2(k) \right|$$
$$\int \chi(q'_1, \ldots, q'_{\nu-1}, \mathrm{Q}, q''_1, \ldots, q''_{\mu-1}) dq''_1 \ldots dq''_{\mu-1} \,|\, dk d\mathrm{Q} dq'_1 \ldots dq'_{\nu-1} \,|$$

and vanishes since $\mathrm{A}_2(k)$ vanishes on the mass shell.

Summing up all those results we have therefore shown that an absorptive part of order n is bounded by an expression of the form (II.4.18).

4.4 The cutting procedure

It was described in ref. [2] and has to be completed in position space as it was done in Section 2, in order to take into account the divergence indices D and D_2.

We just give the result. At order n, p satisfying the conditions of the induction hypothesis A_1, one gets for suitable $\varphi(q_1, \ldots, q_{n-1})$ that $t_r(p + q_1, \ldots, q_{n-1})$ obeys

$$\langle t_r(p + q_1, \ldots, q_{n-1}), \varphi(q_1, \ldots, q_{n-1}) \rangle$$
$$\leqslant A \left| \int \varphi(q) dq \right| \delta_{os} + B \sum_{|\alpha| \leqslant N} \sup \| q \|^{4(n-1)+|\alpha|+D_2-1+\delta} | D^\alpha \varphi(q) | \quad \text{(II.4.22)}$$

where δ_{os} is a Kronecker index and $s = \sum_{j \in J(Y)} r_j^3$.

One has a result of the same form for $t_r(q_1, \ldots, q_{n-1})$.

4.5 The vanishing on the mass shell

We point our interest on self-energy: that is for $s = 0$. There is just one case, the reduced kernel being of type A_1. From formula (II.4.22) one sees that $t_r(p + q_1, \ldots, q_{n-1})$ has an adiabatic limit which, on each compact convex set, is bounded by a constant. One gets from Lemma 2 of [*2*] that this limit is a Hölder continuous function of p, of index of continuity δ. More precisely we can have the estimate of Lemma 3 of [*2*]. This Hölder continuity holds for $0 < p^2 \leqslant m^2$, $p_0 > 0$. On the other hand, the degree of singularity of the electron self-energy is $\omega = 1$. We have therefore an arbitrariness in p which allows us to subtract a constant from the limit. In order to do that, let us discuss a little the spinorial character of electron self-energies. The general form of an electron self-energy $\Sigma(p)$ is, in Q. E. D.

$$\Sigma(p) = \not{p}\Sigma_1(p^2) + \Sigma_2(p^2)$$

where $\not{p} = \sum_{\mu=0}^{3} p_\mu \gamma^\mu$ and Σ_1 and Σ_2 are invariant distributions of p. As it is well known, we can recover Σ_1 and Σ_2 from Σ by taking traces over the spinor indices. On the other hand the $t_r(p + q_1, \ldots, q_{n-1})$ depends on the spinor indices and, taking products with $p'_\mu s$ and traces we get linear combinations of t_r whose adiabatic limits are Σ'_1 and Σ'_2. Since $\{ p_\mu \}$ are analytic functions and each limit is Hölder continuous we obtain that Σ'_1 and Σ'_2 are at order n, Hölder continuous functions of p for $0 < p^2 \leqslant m^2$. Depending only on p^2 they are Hölder continuous of the same index in p^2. Therefore, we can subtract from Σ_1 and Σ_2 their values on the mass shell (remark that p plays only the role of a parameter) and define $\Sigma_1(p^2) = \Sigma'_1(p^2) - \Sigma'_1(m^2)$ and $\Sigma_2(p^2) = \Sigma'_2(p^2) - \Sigma'_2(m^2)$. If Σ'_1 and Σ'_2 were Hölder continuous of index θ we would get from this result that for p in a compact set K, there exists a constant $C_\theta(K)$ such that

$$| \Sigma(p) | \leqslant C_\theta(K) | p^2 - m^2 |^\theta$$

which is the required result.

Remark that we know from general analyticity consideration that $\Sigma(p)$ is analytic for $p^2 < m^2$. We have in fact obtained that this function can be Hölder continuously extended to the border of this region $p^2 = m^2$.

III. CASE OF A $\lambda : \phi^{2\nu}(x)$: THEORY

1. Introduction

We treat here as an example the case of $\lambda : \phi^{2\nu}(x)$: theory, $\nu > 1$, and $\phi(x)$ being a zero mass scalar boson field. By such a theory we mean a theory in which $\mathscr{L}^0(x)$ as noted in [*1*], Chapter VI, is equal to $: \phi^{2\nu}(x) :$. The parity of the exponent has been chosen in order to avoid tadpoles. It should also be noticed that, in principle, cases with more complicated interactions can also be treated: in particular, derivative interactions or couplings between massless and massive fields.

The method is exactly the same as the one used to prove the existence of Green's functions in Q. E. D. In some sense it is quite simpler since there are only zero mass particles as intermediate states. We present here only the outlines of the proof.

The notations will be the same, except that here r_j is an index indicating the number of external particles at vertex j, this number being an integer between 0 and 2ν.

2. The indices

We define first an index D_1 which corresponds to the index D of Chapter II.2 in Q. E. D. The choice of such an index is not unique and depends on constraints imposed to the theory (essentially the physical meaning of renormalization). Looking at the norms of adiabatic type in which D enters (see the Q. E. D. part), one sees that roughly D expresses the regularity in momentum space. On the other hand, due to physical considerations some of the Green's functions have to vanish a certain number of times at the origin, and this is only possible if these Green's functions are regular enough, therefore, if their index D is great enough.

We choose here an index D_1 which corresponds to the minimum of constraints:

$$D_1 = - \sum_{j \in J(Y)} r_j + 4 \qquad \text{(III.1)}$$

In the diagrammatic picture it means that only two kinds of diagrams have no divergency at the origin: the vacuum polarization and the self-energy. One can check easily that for $| Y | = 2$ these two diagrams have the right regularity which allows to make the number of subtractions which are necessary (we will prove it later for the self-energy).

One can also check that other indices are possible, for example, one can wake

$$D_1' = - \sum_{j \in J(Y)} r_j + 4(v - 1) \tag{III.2}$$

This index allows more subtractions at the origin and, in particular, for $|Y| = 2$ corresponds to the exact vanishing of the absorptive parts. All results with D_1 can be proved for D_1'. The advantage of D_1 is that the results extend immediately to the cases where the coupling is between massive and zero mass fields.

In the same way we can define an index D_2 (corresponding to D', in Q. E. D.). For our purpose D_2 has only to be one, when there are no external lines which are not integrated. A general choice, whatever D_1 is, can be

$$D_2 = - \sum_{j \in J(Y)} r_j + 1 \tag{III.3}$$

It remains to check that D_1 and D_2 are compatible with the internal structure. Let a diagram G, with no external integrated lines be obtained from two diagrams G′ and G″ linked by l intermediate states (as in Fig. 4).

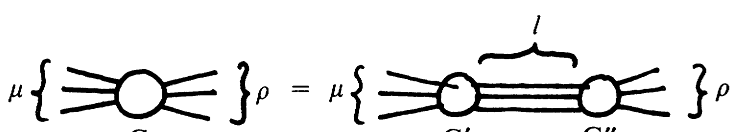

FIG. 4.

The phase space behaviour being $2l - 4$, one has to check that:

$$D_1(G') + D_1(G'') + 2l - 4 \geqslant D_1(G) \tag{III.4}$$

One can check easily from (III.1) that we have, in fact, equality. In the same way, since $v > 1$, one can check that (III.2) satisfies such a relation.

Let now G be a diagram with integrated external lines (as in Fig. 5) and l', $l' \leqslant l$, lines issued from Y's vertices of G″. Then one has

$$D_1(G') + D_2(G'') + 2l - 4 \geqslant D_2(G) \tag{III.5}$$

It remains now to treat the various cases.

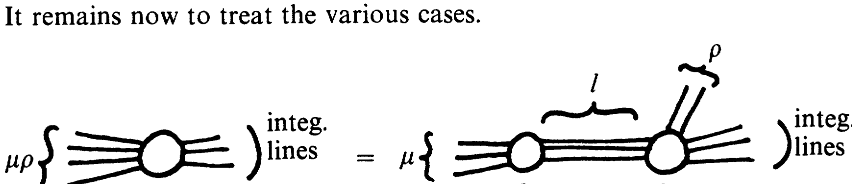

FIG. 5.

3. Diagrams with non-integrated external lines

We define as in Q. E. D., Section 2, the distribution in the relative variables $\xi : F_r(\xi)$.

INDUCTION HYPOTHESIS. — *Let* $|Y| < n$, *then for each distribution* $F_r(\xi)$, *there exist two constants* $C \geqslant 0$, $\kappa > 0$, κ *arbitrarily small, such that for any* $\varphi \in \mathcal{S}(\mathbb{R}^{4(|Y|-1)})$

$$\langle F_r(\xi), \varphi(\xi) \rangle \leqslant C \sum_{|\alpha| \geqslant \max(0,D_1)} \sup_{\xi} (1 + \|\xi\|)^{|\alpha|-D_1+1-\delta} |D^\alpha \varphi| \quad \text{(III.6)}$$

with $\delta = 1 - \kappa$, D_1 *given by* (III.1).

Let us make a remark. Since we have asked for the vanishing of the Fourier transform up to max $(0, D_1)$ [this is the meaning of (III.6) in momentum space] we have to check that this is compatible with the arbitrariness of the cutting procedure, that is to say with the degree ω of the polynomial which can be added. In fact (see [*1*])

$$\omega(G) = |Y|(2\nu - 4) - \sum_{j \in J(Y)} r_j + 4 \quad \text{(III.7)}$$

and one has $D_1 - 1 \leqslant \omega$. Now the proof is exactly the same as in Section II.2.3 *a*) of Q. E. D., however, a little simpler since we have not required an unnecessary know-ledge of the behaviour at the origin in position space.

4. Diagrams with external integrated lines

The notations are as in Section 3.2 of Q. E. D.

INDUCTION HYPOTHESIS. — *Let* $|Y| < n$, $|X| = s$ *fixed, then for each distribution* $F_r(\xi, \zeta)$, *there exist six positive constants* M, N, C, P, κ, ε, $\kappa > 0$ *and* $\varepsilon > 0$ *being arbitrarily small, such that for any* $\varphi \in \mathcal{S}(\mathbb{R}^{4(|Y|+s-1)})$

$$\langle F_r(\xi, \zeta), \varphi(\xi, \zeta) \rangle \leqslant C \sum_{M \geqslant |\alpha| \geqslant 0} \sum_{N \geqslant |\beta| \geqslant 0} \sup (1 + \|\xi\|)^{|\alpha|-D_2+1-\delta} \|\zeta\|^{\mu_\beta^+}(1 + \|\zeta\|)^P |D_\xi^\alpha D_\zeta^\beta \varphi(\xi, \zeta)| \quad \text{(III.8)}$$

with $\mu_\beta^+ = (-\omega + |\beta| - \varepsilon)^+$

$$\omega = (s + |Y|)(2\nu - 4) - \sum_{j \in J(X)} r_j + 4 \quad \text{(III.9)}$$

D_2 *given by* (III.3) *and* $\delta = 1 - \kappa$.

Again the proof of the induction is the same, including also the more elaborated cutting procedure. The only difference is in the value of ω given by (III.9). When $|Y| = 0$, one recognizes the expression (III.7) (with s

instead of $|Y|$), which is the assumed degree of singularity for $\langle T_r(X) \rangle$. In the other steps of the proof one uses the fact that, if $|Y| > |Y'|$, since $\nu \geqslant 2$

$$\omega(|Y|) > \omega(|Y'|)$$

and, therefore, $\mu_\beta^+(|Y|) < \mu_\beta^+(|Y'|)$, which gives the required growth for $a_r(\xi, \zeta)$.

5. **Conclusion**

This shows, in cases of such zero mass theory, the existence of Green's functions and their growth property in momentum space.

IV. CONCLUSION

In Chapters II and III a certain number of consequence have been omitted. In particular, in order to be complete one should have proved for the Green's functions translation invariance, Lorentz invariance, causality and the spectral conditions. All these properties are proved in a paper by Glaser and Epstein [*1*] for the case of massive theories and their proof can be immediately extended to our case.

On the other hand, in the Q. E. D. part, we have only performed the mass renormalization and it is well-known that the wave function renormalization give troubles since it obliges to a better understanding of the physical meaning of soft photons. A treatment, which extends to perturbation theory, the heuristic arguments given by Kulish and Faddeev [*4*], has still to be done.

Finally, one gets as a by-product that for the renormalizable theories: Q. E. D. and $\lambda : \phi^4 :$, the growth properties in momentum space is, at any order of perturbation theory, the same as the free fields case (0^{th} order).

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MATHEMATICAL APPENDIX

I. ÉQUIVALENCE BETWEEN THE DEGREE OF GROWTH IN MOMENTUM SPACE AND THE ORDER OF SINGULARITY AT THE ORIGIN IN POSITION SPACE

This equivalence results from the following two lemmas.

Let $\varphi(x) \in \mathcal{S}(\mathbb{R}^N)$ and $\tilde{\varphi}(p)$ be its Fourier transform, then, one has

LEMMA 1. — *Let* P, ω *and* ε *be three constants,* P *and* ω *being integers with* $P \geqslant 0, \omega \geqslant -N$, *and* $0 < \varepsilon < 1$.

Then there exist four positive constants C, R, M and η, R *and* M *being integers and* $0 < \eta < 1$, *such that, for* $|\alpha| \leqslant P$

$$(1 + \|p\|)^{N+|\alpha|+\omega+\varepsilon} |D^\alpha \tilde{\varphi}(p)| \leqslant C \sum_{|\beta| \leqslant R} \sup_x \|x\|^{(-\omega+|\beta|-\eta)^+}(1 + \|x\|)^M |D^\beta \varphi(x)| \quad (1.1)$$

LEMMA 2. — *Let* R, M, ω *and* η *be four constants,* N, M *and* ω *being integers,* $R \geqslant 0$, $M \geqslant 0$, *and* $0 < \eta < 1$.

Then there exist three positive constants C, P *and* ε, P *being an integers and* $0 < \varepsilon < 1$, *such that, for* $|\alpha| \leqslant R$

$$\|x\|^{(-\omega+|\alpha|-\eta)^+}(1 + \|x\|)^M |D^\alpha \varphi(x)| \leqslant C \sum_{|\beta| \leqslant P} \sup (1 + \|p\|)^{N+|\beta|+\omega+\varepsilon} |D^\beta \tilde{\varphi}(p)| \quad (1.2)$$

1. Proof of lemma 1

We have to estimate

$$(1 + \|p\|)^{N+|\alpha|+\omega+\varepsilon} |D^\alpha \tilde{\varphi}(p)| \quad (1.1.1)$$

We distinguish two cases according to $\|p\| \leqslant 1$ and $\|p\| \geqslant 1$.

a) $\|p\| \leqslant 1$

(1.1.1) is bounded by $C' |D^\alpha \tilde{\varphi}(p)|$, and one has

$$D^\alpha \tilde{\varphi}(p) = \int e^{ip\cdot x}(ix)^\alpha \varphi(x) dx \quad (1.1.2)$$

Again we distinguish two cases according to the values of ω.

i) $\omega \geqslant 0$

One has

$$|D^\alpha \tilde{\varphi}(p)| = \left| \int e^{ip\cdot x}(ix)^\alpha \varphi(x) dx \right| \leqslant \sup_x (1 + \|x\|)^{N+|\alpha|+\delta} |\varphi(x)| \int \frac{dx}{(1 + \|x\|)^{N+\delta}} \leqslant C' \sup_x (1 + \|x\|)^{N+|\alpha|+\delta} |\varphi(x)|$$

with $\delta > 0$.

ii) $\omega < 0$

Then

$$\left| \int e^{ip.x}(ix)^\alpha \varphi(x)dx \right| \leqslant \sup_x \| x \|^{-\omega-\eta}(1 + \| x \|)^{N+|\alpha|+\omega+\eta+\eta'} \, | \varphi(x) | \int \frac{\| x \|^{|\alpha|+\omega+\eta}}{(1 + \| x \|)^{N+|\alpha|+\omega+\eta+\eta'}} dx$$

for $0 < \eta < 1$.

The integral converges for $\eta' > 0$ since $\omega \geqslant - N$. Therefore, one gets, for any value of $\omega \geqslant - N$, that for $\| p \| \leqslant 1$

$$| D^\alpha \tilde{\varphi}(p) | \leqslant C' \sup_x \| x \|^{(-\omega-\eta)^+}(1 + \| x \|)^{N+|\alpha|+\eta'-(-\omega-\eta)^+} | \varphi(x) | \qquad (1.1.3)$$

with $\eta < 1$ and $\eta' > 0$.

b) $\underline{\| p \| \geqslant 1}$

One needs the following estimates and relations

1) $$\frac{1}{\| p \|} \leqslant \frac{2}{1 + \| p \|} \quad \text{when} \quad \| p \| \geqslant 1 \qquad (1.1.4)$$

2) $$\begin{aligned} \int e^{ip.x}\psi(x)dx &= \frac{1}{(ip)^\beta} \int (D_x^\beta e^{ip.x})\psi(x)dx \\ &= \frac{1}{(ip)^\beta} \int e^{ip.x}(-1)^{|\beta|} D_x^\beta \psi(x)dx \end{aligned} \qquad (1.1.5)$$

for any $| \beta | \geqslant 0$ and $\psi \in \mathcal{S}$.

3) $$\begin{aligned} \int e^{ip.x}\psi(x)dx &= \frac{1}{(ip)^\beta} \int (D_x^\beta(e^{ip.x} - 1))\psi(x)dx \\ &= \frac{1}{(ip)^\beta} \int (e^{ip.x} - 1)(-1)^{|\beta|} D_x^\beta \psi(x)dx \end{aligned} \qquad (1.1.6)$$

for any $| \beta | > 0$ and $\psi \in \mathcal{S}$.

4) For any θ, $0 < \theta \leqslant 1$, there exists C_θ such that

$$| e^{ix} - 1 | \leqslant C_\theta | x |^\theta \qquad \forall x \in \mathbb{R} \qquad (1.1.7)$$

Then, with (1.1.6), for $| \beta | > 0$

$$(1 + \| p \|)^{N+|\alpha|+\omega+\varepsilon} | D^\alpha \tilde{\varphi}(p) | \leqslant C \sum_{\beta = |\beta|} (1 + \| p \|)^{N+|\alpha|+\omega+\varepsilon} \frac{1}{\| p \|^{|\beta|}} \int | e^{ip.x} - 1 | \, | D_x^\beta (ix)^\alpha \varphi(x) | dx \qquad (1.1.8)$$

Now, using (1.1.4), (1.1.7) and $| p.x | \leqslant \| p \| \, \| x \|$, (1.1.8) is bounded by

$$C'_\theta(1 + \| p \|)^{N+|\alpha|+\omega+\varepsilon-|\beta|+\theta} \sum_{\substack{\beta_1+\beta_2=\beta \\ |\beta_1| \leqslant |\alpha|}} \int \| x \|^{|\alpha|-|\beta_1|+\theta} | D^{\beta_2}\varphi(x) | dx \qquad (1.1.9)$$

with $0 < \theta \leqslant 1$.

We now choose the values of β and θ.

1) $| \beta | = N + | \alpha | + \omega + 1$
2) $\theta = 1 - \varepsilon$

since $\omega + N \geqslant 0$, $| \beta | \geqslant 1$.

Therefore (1.1.9) becomes

$$C'_\theta \sum_{\substack{\beta_1+\beta_2=\beta \\ |\beta_1|\leqslant|\alpha|}} \int \| x \|^{|\alpha|-|\beta_1|+1-\varepsilon} | D^{\beta_2}\varphi(x) | dx$$

but since $|\beta_1| \leqslant |\alpha|$, $|\beta_2| > \omega$ and one gets that (1.1.10) is bounded by

$$\sum_{\omega+1\leqslant|\beta_2|\leqslant N+\omega+1+|\alpha|} C''_\theta \{ \sup \| x \|^{-\omega+|\beta_2|-\eta} | D^{\beta_2}\varphi(x) | (1+\| x \|) \} \int \frac{\| x \|^{|\alpha|-|\beta_1|-|\beta_2|+\omega+\eta+\theta}}{1+\| x \|} dx$$

and since

$$|\alpha| - |\beta_1| - |\beta_2| + \omega + \eta + \theta = -N + \eta - \varepsilon$$

the integral

$$\int \frac{1}{(1+\| x \|)} \frac{1}{\| x \|^{N+\varepsilon-\eta}} dx$$

converges if $\eta > \varepsilon$. One gets for $\omega \geqslant -N$ and $\| p \| \geqslant 1$

$$(1+\| p \|)^{N+\omega+|\alpha|+\varepsilon} | D^\alpha \tilde{\varphi}(p) | \leqslant C''' \sum_{|\beta|\leqslant N+|\alpha|+\omega+1} \sup \| x \|^{(-\omega+|\beta|-\eta)^+}(1+\| x \|) | D^\beta\varphi(x) |$$

with $\eta > \varepsilon$.

Therefore from (1.1.3) and (1.1.11) one sees that there exist constants C, R, M and η such that (1.1) is valid. Remark that $\eta > \varepsilon$ and $R \geqslant 1$, $M \geqslant 1$.

2. **Proof of lemma 2**

We distinguish two cases according to the values of $\| x \|$.

a) $\underline{\| x \| \leqslant 1}$

Then, we have to estimate

$$\| x \|^{(-\omega+|\alpha|-\eta)^+} | D^\alpha\varphi(x) | \tag{1.2.1}$$

There are also two cases depending on the values of $|\alpha|$.

i) $|\alpha| \leqslant \omega$

Then (1.2.1) reduces to

$$| D^\alpha\varphi(x) | = \left| \int e^{-ip.x}(-ip)^\alpha \tilde{\varphi}(p) dp \right|$$

which is less than

$$\sup (1+\| p \|)^{N+\omega+\varepsilon} | \tilde{\varphi}(p) | \int \frac{\| p \|^{|\alpha|}}{(1+\| p \|)^{N+\omega+\varepsilon}} dp, \qquad \varepsilon > 0 \tag{1.2.2}$$

and the integral converges since $\omega - |\alpha| \geqslant 0$.

ii) $|\alpha| > \omega$

We have to estimate

$$\| x \|^{(-\omega+|\alpha|-\eta)} | D^\alpha\varphi(x) | \tag{1.2.3}$$

Using for the p variables a formula of type (1.1.6) and formula (1.1.7), one has that (1.2.3) is bounded by a sum of terms of the form

$$C_\theta \| x \|^{-\omega+|\alpha|-\eta+\theta-|\beta|} \int \| p \|^{|\alpha|-|\beta_1|+\theta} | D^{\beta_2}\tilde{\varphi}(p) | dp \tag{1.2.4}$$

with $0 < \theta \leqslant 1$, $|\alpha| \geqslant |\beta_1|$.

We set $\theta = \eta$ and $|\beta| = |\alpha| - \omega$ (therefore $|\beta| \geqslant 1$) and (1.2.4) is bounded by

$$C_\theta \int \|p\|^{|\alpha|-|\beta_1|+\eta} |D^{\beta_2}\tilde{\varphi}(p)| dp \leqslant C' \sup (1 + \|p\|)^{N+|\beta_2|+\omega+\varepsilon} |D^{\beta_2}\tilde{\varphi}(p)| \int \frac{\|p\|^{|\alpha|-|\beta_1|+\eta}}{(1+\|p\|)^{N+|\beta_2|+\omega+\varepsilon}} dp$$

and the integral converges if $\varepsilon > \eta$.

To sum up this case

$$\|x\|^{(-\omega+|\alpha|-\eta)^+} |D^\alpha \varphi| \leqslant C' \sum_{|\beta| \leqslant \max(0,|\alpha|-\omega)} \sup (1 + \|p\|)^{N+|\beta|+\omega+\varepsilon} |D^\beta \tilde{\varphi}(p)|$$

b) $\|x\| \geqslant 1$

One has to estimate

$$(1 + \|x\|)^{M'} |D^\alpha \varphi(x)| \tag{1.2.6}$$

where M′ is an integer larger than $M + (-\omega + R - \eta)^+$. Applying the same method (1.2.6) is bounded by a sum of terms of the form

$$(1 + \|x\|)^{M'-|\beta|} \int \|p\|^{|\alpha|-|\beta_1|} |D^{\beta_2}\tilde{\varphi}(p)| dp \tag{1.2.7}$$

with $|\beta_1| \leqslant |\alpha|$, $\beta_1 + \beta_2 = \beta$.

Let us choose $|\beta| = M' + \rho$, $\rho \geqslant 0$ to be fixed, then (1.2.7) is less than

$$\frac{1}{(1+\|x\|)^\rho} \int \|p\|^{|\alpha|-|\beta_1|} |D^{\beta_2}\tilde{\varphi}(p)| dp \leqslant \sup (1 + \|p\|)^{N+|\beta_2|+\omega+\varepsilon} |D^{\beta_2}\tilde{\varphi}(p)| \int \frac{\|p\|^{|\alpha|-|\beta_1|}}{(1+\|p\|)^{N+|\beta_2|+\omega+\varepsilon}} dp$$

and the integral converges if

$$|\beta| + \omega - |\alpha| = M' + \rho + \omega - |\alpha| \geqslant 0$$

hence, we choose $\rho = \max(0, |\alpha| - \omega - M')$, and we get that (1.2.6) is bounded by

$$\sum_{|\beta| \leqslant \max(M', R-\omega)} \sup (1 + \|p\|)^{N+|\beta|+\omega+\varepsilon} |D^\beta \tilde{\varphi}(p)| \tag{1.2.8}$$

Therefore, from (1.2.5) and (1.2.8), one sees that there exist constants C, P and ε such that (1.2) is valid. Remark that $\varepsilon > \eta$.

II. ÉQUIVALENCE BETWEEN THE ADIABATIC NORMS IN MOMENTUM AND POSITION SPACE

This equivalence results from the two following lemmas.

Let $\varphi(x) \in \mathscr{S}(\mathbb{R}^N)$ and $\tilde{\varphi}(p)$ be its Fourier transform.

LEMMA 3. — *Let* P *and* δ *be two constants,* P *an integer,* $P \geqslant 0$ *and* $0 < \delta < 1$.

Then there exist three positive constants C, M *and* δ', M *being an integer and* $0 < \delta' < \delta$, *such that, for* $|\alpha| \leqslant P$

$$(1 + \|x\|)^{|\alpha|-\delta} |D^\alpha \varphi| \leqslant C \left\{ \left| \int \tilde{\varphi}(p) dp \right| + \sum_{|\beta| \leqslant P} \sup \|p\|^{N+|\beta|+\delta'} (1 + \|p\|)^M |D^\beta \tilde{\varphi}(p)| \right\} \tag{2.1}$$

LEMMA 4. — *Let* M, P *and* δ' *be three constants,* M *and* P *integers,* $P > 0$ *and* $0 < \delta' < 1$.

Then, there exist three positive constants C, R *and* δ, R *being an integer and* $0 < \delta < \delta'$, *such that, for* $|\alpha| \leqslant P$

$$\|p\|^{N+|\alpha|+\delta'}(1+\|p\|)^M |D^\alpha\tilde{\varphi}(p)| \leqslant C\sum_{|\beta|\leqslant R} \sup (1+\|x\|)^{|\beta|-\delta}|D^\beta\varphi(x)| \quad (2.2)$$

and

$$\left|\int\tilde{\varphi}(p)dp\right| \leqslant C\sum_{|\beta|\leqslant 1} \sup (1+\|x\|)^{|\beta|-\delta}|D^\beta\varphi(x)| \quad (2.3)$$

1. Proof of lemma 3

We consider two cases according to the values of $\|x\|$.

a) $\|x\| \leqslant 1$

One has to estimate $|D^\alpha\varphi| = \left|\int e^{-ip.x}(-ip)^\alpha\tilde{\varphi}(p)dp\right|$.

i) $|\alpha| = 0$

Then we write

$$\int e^{-ip.x}\tilde{\varphi}(p)dp = \int(e^{-ip.x}-1)\tilde{\varphi}(p)dp + \int\tilde{\varphi}(p)dp \quad (2.1.1)$$

and with (1.1.7)

$$\left|\int(e^{-ip.x}-1)\tilde{\varphi}(p)dp\right| \leqslant C_\theta\int\|p\|^\theta\|x\|^\theta|\tilde{\varphi}(p)|dp$$

$$\leqslant C_\theta\int\|p\|^\theta|\tilde{\varphi}(p)|dp \leqslant C_\theta \sup\|p\|^{N+\delta'}(1+\|p\|)|\tilde{\varphi}(p)|\int\frac{dp}{\|p\|^{N+\delta'-\theta}(1+\|p\|)} \quad (2.1.2)$$

Choosing $\delta' < \theta$, and θ being arbitrary we see that there exists a constant $C(\delta')$, $\delta' < 1$, such that

$$|\varphi(x)| \leqslant C(\delta')\left\{\left|\int\tilde{\varphi}(p)dp\right| + \sup\|p\|^{N+\delta'}(1+\|p\|)|\tilde{\varphi}(p)|\right\} \quad (2.1.3)$$

ii) $|\alpha| \neq 0$

Choose $\delta' < 1$, then

$$|D^\alpha\varphi(x)| \leqslant \int\|p\|^{|\alpha|}|\tilde{\varphi}(p)|dp$$

$$\leqslant \sup\|p\|^{N+\delta'}(1+\|p\|)^{|\alpha|-\delta'+\varepsilon}|\tilde{\varphi}(p)|\int\frac{\|p\|^{|\alpha|}}{\|p\|^{N+\delta'}}\frac{1}{(1+\|p\|)^{|\alpha|-\delta'+\varepsilon}}dp, \qquad \varepsilon > 0$$

and the integral converges since $|\alpha| \geqslant 1 > \delta'$.

Therefore, when $\|x\| \leqslant 1$, for any values $\delta' < 1$, there exists a constant $C_{\delta'}$ such that

$$|D^\alpha\varphi(x)| \leqslant C_{\delta'}\left\{\left|\int\tilde{\varphi}(p)dp\right| + \sup\|p\|^{N+\delta'}(1+\|p\|)^{\max(P,1)}|\tilde{\varphi}(p)|\right\} \quad (2.1.4)$$

b) $\|x\| \geqslant 1$

One has to estimate $(1+\|x\|)^{|\alpha|-\delta}|D^\alpha\varphi|$

i) $|\alpha| = 0$

Then

$$\frac{1}{(1+\|x\|)^\delta}|\varphi| \leqslant \frac{1}{(1+\|x\|)^\delta}\left|\int(e^{-ip.x}-1)\tilde{\varphi}(p)dp\right| + \left|\int\tilde{\varphi}(p)dp\right|$$

and we have to estimate

$$\frac{1}{(1+\|x\|)^\delta}\left|\int(e^{-ip.x}-1)\tilde{\varphi}(p)dp\right| \quad (2.1.5)$$

This is less than

$$\frac{1}{(1+\|x\|)^\delta} C_\theta \int \|p\|^\theta \|x\|^\theta |\tilde{\varphi}(p)| dp \leqslant C_\theta \frac{(1+\|x\|)^\theta}{(1+\|x\|)^\delta} \int \|p\|^\theta |\tilde{\varphi}(p)| dp$$

Choosing $\theta = \delta$, (2.1.5) is bounded by

$$C_\delta \int \|p\|^\delta |\tilde{\varphi}(p)| dp \leqslant C_{\delta'} \sup \|p\|^{N+\delta-\varepsilon}(1+\|p\|)^{\varepsilon+\varepsilon'} |\tilde{\varphi}(p)|$$

ε and ε' being positive and arbitrarily small.

Therefore, there exist $C(\delta')$ and δ', $0 < \delta' < \delta$ such that (2.1.5) is bounded by

$$C(\delta') \sup \|p\|^{N+\delta'}(1+\|p\|)|\tilde{\varphi}(p)| \tag{2.1.6}$$

ii) $|\alpha| \neq 0$

One has to estimate

$$(1+\|x\|)^{|\alpha|-\delta} \left| \int e^{-ip.x}(-ip)^\alpha \tilde{\varphi}(p) dp \right| \tag{2.1.7}$$

Applying a formula of type (1.1.6), (2.1.7) is less than

$$C' \sum_{\substack{\beta_1+\beta_2=\beta \\ |\alpha| \geqslant |\beta_1|}} (1+\|x\|)^{|\alpha|-\delta} \|x\|^{-|\beta|} \int |e^{-ip.x} - 1| \, \|p\|^{|\alpha|-|\beta_1|} |D^{\beta_2}\tilde{\varphi}(p)| dp$$

Using (1.1.7) each term is bounded by

$$C_\theta(1+\|x\|)^{|\alpha|-\delta-|\beta|+\theta} \int \|p\|^{|\alpha|-|\beta_1|+\theta} |D^{\beta_2}\tilde{\varphi}(p)| dp \tag{2.1.8}$$

Now we set $|\alpha| = |\beta|$ ($|\beta| \geqslant 1$) and $\theta = \delta$, and (2.1.8) is bounded by

$$C_\theta \int \|p\|^{|\beta_2|+\delta} |D^{\beta_2}\tilde{\varphi}(p)| dp \leqslant C'_\theta \sup \|p\|^{N+|\beta_2|+\delta'}(1+\|p\|)|D^{\beta_2}\tilde{\varphi}(p)|$$

with $\delta > \delta'$.

One gets that for $\|x\| \geqslant 1$

$$(1+\|x\|)^{|\alpha|-\delta}|D^\alpha\varphi| \leqslant C\left\{ \left| \int \tilde{\varphi}(p)dp \right| + \sum_{|\beta| \leqslant |\alpha|} \sup \|p\|^{N+|\beta|+\delta'}(1+\|p\|)|D^\beta\tilde{\varphi}(p)| \right\} \tag{2.1.9}$$

Therefore, from (2.1.4) and (2.1.9) we see that there exist constants C, R and δ', $\delta' < \delta$, $R \geqslant 1$, such that formula (2.1) is valid.

2. Proof of lemma 4

We give the proof for $M \geqslant 0$, since for $M < 0$, $1/(1+\|p\|)^{|M|} \leqslant 1$, and the proof reduces to $M = 0$.

a) *Estimate on* $\left| \int \tilde{\varphi}(p)dp \right|$

$$\int \tilde{\varphi}(p)dp = \varphi(o)$$

and

$$\varphi(o) = \sup_x \frac{\varphi(o)}{(1 + \| x \|)^\delta} \qquad \delta > 0$$

but

$$\frac{\varphi(o)}{(1 + \| x \|)^\delta} = \frac{\varphi(o) - \varphi(x)}{(1 + \| x \|)^\delta} + \frac{\varphi(x)}{(1 + \| x \|)^\delta} \leqslant \frac{| \varphi(x) |}{(1 + \| x \|)^\delta} + \frac{| \varphi(o) - \varphi(x) |}{(1 + \| x \|)^\delta}$$

But

$$\varphi(x) - \varphi(o) = \int_0^1 \sum_\mu x_\mu \left(\frac{\partial}{\partial x_\mu} \varphi \right)(tx) dt \leqslant \sum_\mu \| x \| \int_0^1 \left| \frac{\partial}{\partial x_\mu} \varphi(tx) \right| dt$$

therefore

$$\frac{| \varphi(o) - \varphi(x) |}{(1 + \| x \|)^\delta} \leqslant \sum_{|\beta| = 1} (1 + \| x \|)^{1-\delta} \int_0^1 | D^\beta \varphi(tx) | dt \tag{2.2.1}$$

Let us choose now $\delta < 1$, then (2.2.1) is bounded by

$$\left\{ \sum_{|\beta| = 1} \sup_{tx} (1 + \| tx \|)^{1-\delta} | D^\beta \varphi(tx) | \right\} \int_0^1 \frac{dt}{t^{1-\delta}}$$

which is less than

$$C_\delta \sum_{|\beta| = 1} \sup (1 + \| x \|)^{1-\delta} | D^\beta \varphi(x) |$$

and finally there exist $C(\delta)$ and δ, $0 < \delta < 1$, such that

$$\left| \int \tilde{\varphi}(p) dp \right| \leqslant C(\delta) \sum_{|\beta| \leqslant 1} \sup (1 + \| x \|)^{|\beta| - \delta} | D^\beta \varphi(x) | \tag{2.2.2}$$

which is formula (2.3) of Lemma 4.

b) *Estimate on* $\| p \|^{N + |\alpha| + \delta'} (1 + \| p \|)^M | D^\alpha \tilde{\varphi}(p) |$.

We distinguish two cases according to the values of $\| p \|$.

i) $\| p \| \leqslant 1$

Then we have to estimate

$$\| p \|^{N + |\alpha| + \delta'} | D^\alpha \tilde{\varphi}(p) | \tag{2.2.3}$$

Applying formula of type (1.1.6) one gets that (2.2.3) is bounded by a sum of terms of the form

$$\| p \|^{N + |\alpha| + \delta' - |\beta| + \theta} \int \| x \|^{\theta + |\alpha| - |\beta_1|} | D^{\beta_2} \varphi(x) | dx \tag{2.2.4}$$

with $|\beta| \geqslant 1$, $\beta_1 + \beta_2 = \beta$, $|\beta_1| \leqslant |\alpha|$, $0 < \theta \leqslant 1$. One chooses $|\beta| = N + |\alpha| + 1$, $\theta = 1 - \delta'$, then (2.2.4) is bounded by

$$\int \| x \|^{|\alpha| - |\beta_1| + 1 - \delta'} | D^{\beta_2} \varphi(x) | dx \leqslant \sup (1 + \| x \|)^{|\beta_2| - \delta} | D^{\beta_2} \varphi(x) | \int \frac{\| x \|^{|\alpha| - |\beta_1| + 1 - \delta'}}{(1 + \| x \|)^{|\beta_2| - \delta}} dx$$

The integral converges if $\delta < \delta'$, since $|\beta_2| \geqslant N + 1 \geqslant 1$. Thus, when $\| p \| \leqslant 1$, (2.2.3) is bounded by

$$C' \sum_{|\beta| \leqslant N + |\alpha| + 1} \sup (1 + \| x \|)^{|\beta| - \delta} | D^\beta \varphi(x) | \tag{2.2.5}$$

with $\delta < \delta'$.

ii) $\| p \| \geqslant 1$.

We have to estimate $(1 + \| p \|)^{N+M+|\alpha|+\delta'} | D^\alpha \tilde{\varphi}(p) |$. The method is the same as in *i*). We choose $| \beta | = M + | \alpha | + 1 + N$ and get

$$(1 + \| p \|)^{N+M+|\alpha|+\delta'} | D^\alpha \tilde{\varphi}(p) | \leqslant C' \sum_{|\beta| \leqslant M+N+|\alpha|+1} \sup (1 + \| x \|)^{|\beta|-\delta} | D^\beta \varphi(x) | \quad (2.2.6)$$

with $\delta < \delta'$.

Therefore, from (2.2.5) and (2.2.6), we see that there exist constants C, R and δ, $\delta < \delta'$, such that formula (2.2) is satisfied.

APPENDIX A

I. THE SECOND TENSOR PRODUCT RULE

This tensor product rule is related to tempered distributions which possess an adiabatic norm. We will prove the following lemma.

LEMMA 1. — Let (ω_1, D_1) and (ω_2, D_2) be two couples of integers and $(\varepsilon_1, \kappa_1)$ and $(\varepsilon_2, \kappa_2)$ be two couples of small enough, positive constants. Let F_1 and F_2 be two tempered distributions respectively in $\mathscr{S}'(\mathbb{R}^{N_1})$ and $\mathscr{S}'(\mathbb{R}^{N_2})$, and suppose that for any $\varphi_1 \in \mathscr{S}(\mathbb{R}^{N_1})$ and $\varphi_2 \in \mathscr{S}(\mathbb{R}^{N_2})$ one has constants (C_1, M_1) and (C_2, M_2) with

$$\langle F_i(\xi_i), \varphi_i(\xi_i) \rangle \leqslant C_i \sum_{M_i \geqslant |\alpha| \geqslant \max(0, D_i)} \sup \| \xi_i \|^{\mu^+_{\alpha,i}} (1 + \| \xi_i \|)^{-\mu^+_{\alpha,i} + |\alpha| - D_i + 1 - \delta_i} | D^\alpha \varphi_i(\xi_i) | , \qquad i = 1, 2 \tag{1.1}$$

where $\xi_i = (\xi_{i_1}, \ldots, \xi_{i_{N_i}})$

$$\mu^+_{\alpha,i} = (-\omega_i + |\alpha| - \varepsilon_i)^+$$

and $\delta_i = 1 - \kappa_i$.

Then there exists a constant C such that for any $\varphi \in \mathscr{S}(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})$ one has

$$\langle F_1(\xi_1) \otimes F_2(\xi_2), \varphi(\xi_1, \xi_2) \rangle \leqslant C \sum_{M \geqslant |\alpha| \geqslant \max(0, D)} \sup \| \xi_1, \xi_2 \|^{\mu^+_\alpha} (1 + \| \xi_1, \xi_2 \|)^{-\mu^+_\alpha + |\alpha| - D + 1 - \delta} | D^\alpha \varphi | \tag{1.2}$$

where $M = M_1 + M_2$, $D = D_1 + D_2$, $\delta = 1 - \kappa$, $\kappa = \kappa_1 + \kappa_2$,

$$\mu^+_\alpha = (-\omega + |\alpha| - \varepsilon)^+ \qquad \text{with} \qquad \omega = \omega_1 + \omega_2, \qquad \varepsilon = \varepsilon_1 + \varepsilon_2 \tag{1.3}$$

Let us comment this lemma.

1) We will use it twice. First with D_1 and D_2 being the divergence indices of photon diagram, and the norm in the induction hypothesis is (1.1). Second with D_2 being the index of divergence when there is some integrated lines, but in this case, in (1.1) the sum over $|\alpha|$ begins always at zero. This is not a difficulty since $D' = 1$ only when there are no floating lines and because we use this tensor product rule to construct connected absorptive parts, one has always $D' < 1$.

2) From the proof we will give, the following lemma also results, which is a weaker form of the previous one.

LEMME 1'. — Under the hypothesis of Lemma 1, but with (1.1), replaced by (1.1')

$$\langle F_i(\xi_i), \varphi_i(\xi_i) \rangle \leqslant C_i \sum_{M_i \geqslant |\alpha| \geqslant \max(0, D_i)} \sup (1 + \| \xi_i \|)^{|\alpha| - D_i + 1 - \delta_i} | D^\alpha \varphi_i | \tag{1.1'}$$

there exists a constant C such that for any $\varphi \in \mathscr{S}(\mathbb{R}^{N_1} \otimes \mathbb{R}^{N_2})$ one has

$$\langle F_1(\xi_1) \otimes F_2(\xi_2), \varphi(\xi_1, \xi_2) \rangle \leqslant C \sum_{M \geqslant |\alpha| \geqslant \max(0, D)} \sup (1 + \| \xi_1, \xi_2 \|)^{|\alpha| - D + 1 - \delta} | D^\alpha \varphi |$$

with M, D and δ given by (1.3).

PROOF OF LEMMA 1

We have to estimate

$$\|\xi_1\|^{\mu^+_{\gamma,1}}\|\xi_2\|^{\mu^+_{\beta,2}}(1+\|\xi_1\|)^{-\mu^+_{\gamma,1}+|\gamma|-D_1+1-\delta_1}(1+\|\xi_2\|)^{-\mu^+_{\beta,2}+|\beta|-D_2+1-\delta_2} |D^\gamma_{\xi_1}D^\beta_{\xi_2}\varphi(\xi_1,\xi_2)| \quad (1.4)$$

for $|\gamma|+|\beta|=|\alpha|$, and $|\alpha|\geqslant \max(0, D_1)+\max(0, D_2)\geqslant \max(0, D_1+D_2)$.
We will distinguish different cases according to the values of $\|\xi_1\|$ and $\|\xi_2\|$.

1) $\|\xi_1\|\leqslant 1,\quad \|\xi_2\|\leqslant 1$

Then (1.4) is bounded by a constant factor time

$$\|\xi_1\|^{\mu^+_{\gamma,1}}\|\xi_2\|^{\mu^+_{\beta,2}}|D^\beta_{\xi_1}D^\beta_{\xi_2}\varphi(\xi_1,\xi_2)| \quad (1.5)$$

which is less than

$$\|\xi_1,\xi_2\|^{\mu^+_{\gamma,1}+\mu^+_{\beta,2}}|D^\gamma_{\xi_1}D^\beta_{\xi_2}\varphi(\xi_1,\xi_2)|$$

and since $\mu^+_{\gamma,1}+\mu^+_{\beta,2}\geqslant(-\omega_1-\omega_2+|\alpha|-\varepsilon_1-\varepsilon_2)^+$ and $\|\xi_1,\xi_2\|\leqslant\sqrt{2}$, there exists a constant C such that (1.5) is bounded by

$$C\sum_{|\sigma|=|\alpha|}\|\xi_1,\xi_2\|^{\mu^+_\sigma}(1+\|\xi_1,\xi_2\|)^{-\mu^+_\sigma+|\sigma|-D+1-\delta}|D^\sigma\varphi| \quad (1.6)$$

with $\mu^+_\sigma=(-\omega_1-\omega_2+|\sigma|-\varepsilon_1-\varepsilon_2)^+$ and δ arbitrary.

2) $\|\xi_1\|\geqslant 1,\quad \|\xi_2\|\leqslant 1$

Then (1.4) is bounded by a constant factor time

$$(1+\|\xi_1\|)^{|\gamma|-D_1+1-\delta_1}\|\xi_2\|^{\mu^+_{\beta,2}}|D^\gamma_{\xi_1}D^\beta_{\xi_2}\varphi| \quad (1.7)$$

Since $\|\xi_2\|\leqslant\|\xi_1,\xi_2\|$ and $\|\xi_1,\xi_2\|\geqslant 1$, this is less than

$$(1+\|\xi_1,\xi_2\|)^{|\alpha|-D_1-D_2+2-\delta_1-\delta_2}|D^\gamma_{\xi_1}D^\beta_{\xi_2}\varphi|$$

where we have used the fact that $|\gamma|\geqslant D_1$ and $|\beta|\geqslant D_2$. Now, from $\|\xi_1,\xi_2\|\geqslant 1$ one gets that (1.7) is bounded by

$$\|\xi_1,\xi_2\|^{\mu^+_\sigma}(1+\|\xi_1,\xi_2\|)^{-\mu^+_\sigma+|\alpha|-D+1-\delta}|D^\gamma_{\xi_1}D^\beta_{\xi_2}\varphi| \quad (1.8)$$

with $|\gamma|+|\beta|=|\alpha|$, $D=D_1+D_2$, $\delta=1-\kappa$, $\kappa=\kappa_1+\kappa_2$.

Now, the two other cases can be treated as 2) and we also get (1.8) as bound. By collecting these various results one obtains formula (1.2).

ESTIMATE AN $\Delta^+(x;0)$

We define $\Delta^+(x;0)$ by

$$\Delta^+(x;0)=\int e^{-ip.x}\theta(p_0)\delta(p^2)d^4p=\left(i\frac{\partial}{\partial x_0}+1\right)\left(i\frac{\partial}{\partial x_0}\right)^2 B(x) \quad (2.1)$$

with

$$B(x)=\int\frac{e^{-ip.x}-1}{p_0^2(1+p_0)}\theta(p_0)\delta(p^2)d^4p \quad (2.2)$$

One has the following estimate on B(x)

1) $$|B(x)|\leqslant C_1 \quad (2.3)$$

2) $$|B(x)|\leqslant C_2(\theta)\|x\|^\theta \qquad 0<\theta<1 \quad (2.4)$$

For the last one we have used the fact that for any constant θ, $0 < \theta \leqslant 1$, there exists a constant $C(\theta)$ such that

$$| e^{ix} - 1 | \leqslant C(\theta) | x |^{\theta} \qquad \forall x \in \mathbb{R}$$

Therefore, for any $\varphi(x) \in \mathscr{S}(\mathbb{R}^4)$ one has

$$| \langle \Delta^+(x), \varphi(x) \rangle | = \left| \int B(x)\left(\frac{1}{i}\frac{\partial}{\partial x_0} + 1\right)\left(\frac{1}{i}\frac{\partial}{\partial x_0}\right)^2 \varphi(x)dx \right|$$
$$\leqslant C_\varepsilon \sum_{2 \leqslant |\alpha| \leqslant 3} \sup | B(x) | \; \| x \|^{4-\varepsilon}(1 + \| x \|) | D^\alpha \varphi(x) |$$

if $1 > \varepsilon > 0$.

Let us estimate

$$| B(x) | \; \| x \|^{4-\varepsilon}(1 + \| x \|) | D^\alpha \varphi(x) | \tag{2.5}$$

1) $\| x \| \leqslant 1$

Then (2.5) is bounded by

$$\| x \|^{4-\varepsilon} | B(x) | \; | D^\alpha \varphi(x) | \leqslant C_2 \| x \|^{4+\theta-\varepsilon} | D^\alpha \varphi(x) | \tag{2.6}$$

after using (2.4).

Let us choose $\theta > \varepsilon$, θ close to one, ε close to zero, then $4 + \theta - \varepsilon \geqslant 2 + | \alpha | - \rho_1$ for $2 \leqslant | \alpha | \leqslant 3$ if $\rho_1 \geqslant 1 - \theta + \varepsilon$, that is to say $\rho_1 > \varepsilon$.

Then (2.5) is bounded by

$$C' \| x \|^{2+|\alpha|-\rho_1} | D^\alpha \varphi(x) | \tag{2.7}$$

2) $\| x \| \geqslant 1$

Then (2.5) is bounded by

$$| B(x) | (1 + \| x \|)^{5-\varepsilon} | D^\alpha \varphi(x) | \leqslant C_1(1 + \| x \|)^{5-\varepsilon} | D^\alpha \varphi |$$

where we have used (2.3). Then, if we choose $\rho_2 \leqslant \varepsilon$,

$$5 - \varepsilon \leqslant 3 + | \alpha | - \rho_2$$

for $2 \leqslant | \alpha | \leqslant 3$ and (2.5) is bounded by

$$C_1(1 + \| x \|)^{3+|\alpha|-\rho_2} | D^\alpha \varphi(x) |$$

and since $\| x \| \geqslant 1$, there exists a constant C_3 such that it is bounded by

$$C_3 \| x \|^{2+|\alpha|-\rho_1}(1 + \| x \|)^{1+\rho_1-\rho_2} | D^\alpha \varphi(x) | \tag{2.8}$$

Finally, with (2.7) and (2.8), one gets

$$\langle \Delta^+(x), \varphi(x) \rangle \leqslant C \sum_{2 \leqslant |\alpha| \leqslant 3} \sup_x \| x \|^{2+|\alpha|-\rho_1}(1 + \| x \|)^{1+\rho_1-\rho_2} | D^\alpha \varphi(x) | \tag{2.9}$$

with $\rho_1 > \varepsilon$ and $\rho_2 < \varepsilon$.

From (2.9) it results that $\Delta^+(x\,;0)$ has indices D and ω given by $D = -2$, $\omega = -2$.

APPENDIX B

1. THE FOUR-PHOTON DIAGRAMS

Let $t(x) \in \mathscr{S}'(\mathbb{R}^N)$ be a distribution singular at the origin of order $\omega = 0$ and satisfying for any $\varphi \in \mathscr{S}(\mathbb{R}^N)$

$$\langle t(x), \varphi(x) \rangle \leqslant C^t \sum_{|\alpha| \geqslant 0} \sup_x (1 + \| x \|)^{|\alpha| - \delta} | D^\alpha \varphi | \tag{1.1}$$

Suppose that $t(x)$ is the result of the cutting procedure. Then any other solution is obtained from $t(x)$ by adding $C\delta^N(x)$ where C is any finite constant.

We want to show that there exists a C such that $\bar{t}(x)$ defined by

$$\bar{t}(x) = t(x) - C\delta^N(x) \tag{1.2}$$

satisfies

$$\langle \bar{t}(x), \varphi(x) \rangle \leqslant C^t \sum_{|\alpha| \geqslant 1} \sup (1 + \| x \|)^{|\alpha| - \delta} | D^\alpha \varphi | \tag{1.3}$$

We remember that (1.1) means that there exists C independant of $\varphi \in \mathscr{S}(\mathbb{R}^N)$ such that

$$\lim_{\varepsilon \to 0} \langle t(x), \varphi(\varepsilon x) \rangle = C\varphi(o)$$

We choose this C to define $\bar{t}$ as in (1.2).

Then

$$\langle \bar{t}(x), \varphi(x) \rangle = \lim_{\varepsilon \to 0} \langle t(x), \psi_\varepsilon(x) \rangle$$

with $\psi_\varepsilon(x) = \varphi(x) - \varphi(\varepsilon x)$.

For $\varepsilon \neq 0$, $\psi_\varepsilon \in \mathscr{S}(\mathbb{R}^N)$, thus

$$| \langle \bar{t}(x), \psi_\varepsilon(x) \rangle | \leqslant C^t \sum_{|\alpha| \geqslant 0} \sup_x (1 + \| x \|)^{|\alpha| - \delta} | D^\alpha \psi_\varepsilon(x) |$$

Since

$$| D^\alpha \psi_\varepsilon(x) | = \left| D^\alpha \sum_\mu x_\mu \int_\varepsilon^1 \left(\frac{\partial}{\partial x_\mu} \varphi \right)(tx) dt \right| \leqslant C \sum_{|\beta| = 1} \sum_{\theta_1 + \theta_2 = \alpha} \| x \|^{|\beta - \theta_1|} \int_\varepsilon^1 t^{|\theta_2|} | D^{\theta_2 + \beta} \varphi(tx) | \, dt$$

and for $\delta < 1$ and $|\varepsilon| < 1$

$$(1 + \| x \|)^{|\alpha| - \delta} | D^\alpha \psi_\varepsilon(x) | \leqslant C' \sum_{\theta_1 + \theta_2 = \alpha} \sup_{tx} (1 + \| tx \|)^{|\theta_2| + |\beta| - \delta}$$
$$| D^{\theta_2 + \beta} \varphi(tx) | \int_{-1}^{+1} \frac{t^{|\theta_2|}}{t^{|\theta_2| + 1 - \delta}} dt \times \{ (1 + \| x \|)^{|\alpha| - \delta - |\theta_2| - 1 + \delta} \| x \|^{|\beta - \theta_1|} \}$$

The term in bracket is less than 1 and the integral converges if $\delta > 0$. Therefore,

$$\lim_{\varepsilon \to 0} | \langle \bar{t}(x), \psi_\varepsilon(x) \rangle | \leqslant C_\delta \sum_{\substack{|\beta| = 1 \\ \theta_1 + \theta_2 = \alpha}} \sup (1 + \| x \|)^{|\theta_2| + 1 - \delta} | D^{\theta_2 + \beta} \varphi(x) |$$
$$= C_\delta \sum_{|\gamma| \geqslant 1} \sup (1 + \| x \|)^{|\gamma| - \delta} | D^\gamma \varphi(x) |$$

which is the required result.

It can be checked that

$$\lim_{\varepsilon \to 0} \langle \bar{t}(x), \varphi(\varepsilon x) \rangle = 0$$

2. THE PHOTON SELF-ENERGIES

The principle is the same than in Section 1. Let $t(x)$ be a distribution singular at the origin of order $\omega = 2$ and satisfying for any $\varphi \in \mathcal{S}(\mathbb{R}^N)$

$$\langle t(x), \varphi(x) \rangle \leqslant C' \sum_{|\alpha| \geqslant 0} \sup (1 + \| x \|)^{|\alpha| - 1 - \delta} | D^\alpha \varphi | \tag{2.4}$$

If $t(x)$ is the result of the cutting procedure another solution is obtained by adding to $t(x)$ any polynomial in the derivatives of $\delta^N(x)$ of order less or equal to 2.

We want to define a new distribution

$$\bar{t}(x) = t(x) - c\delta^N(x) + \sum_{\mu=1}^{N} c_\mu \delta^{N,\mu}(x) \tag{2.5}$$

such that

$$\langle \bar{t}(x), \varphi(x) \rangle \leqslant C' \sum_{|\alpha| \geqslant 2} \sup (1 + \| x \|)^{|\alpha| - 1 - \delta} | D^\alpha \varphi(x) | \tag{2.6}$$

In the above expression

$$\delta^{N,\mu}(x) = \delta(x_1) \ldots \delta(x_{\mu-1})\delta'(x_\mu)\delta(x_{\mu+1}) \ldots \delta(x_N)$$

where $\delta(x_i)$ is the usual one-dimensional δ function and $\delta'(x_i)$ its first derivative. First, (2.4) means that $t(x)$ has an adiabatic limit, and as in Section 1, we define a constant c by

$$\lim_{\varepsilon \to 0} \langle t(x), \varphi(\varepsilon x) \rangle = c\varphi(o)$$

Then we remark also that, according to (2.4), the distributions $x_\mu t(x)$, $\mu = 1, \ldots, N$, [which exists since $x_\mu \in \mathcal{O}(M)$] satisfy

$$\langle x_\mu t(x), \varphi(x) \rangle = \langle t(x), x_\mu \varphi(x) \rangle \leqslant C' \sum_{|\alpha| \geqslant 0} \sup (1 + \| x \|)^{|\alpha| - \delta} | D^\alpha \varphi |$$

The bound in (2.7) means that there exists a family of constants c_μ, $\mu = 1, \ldots, N$ defined by

$$\lim_{\varepsilon \to 0} \langle x_\mu t(x), \varphi(\varepsilon x) \rangle = c_\mu \varphi(o)$$

We set now

$$\bar{t}(x) = t(x) - c\delta^N(x) + \sum_\mu c_\mu \delta^{N,\mu}(x)$$

The new distribution $\bar{t}(x)$ is still a solution of the cutting procedure since it differs from $t(x)$ by a polynomial in the derivatives of δ^N of order one. Let us check now that $\bar{t}(x)$ satisfies formula (2.6)

$$\langle \bar{t}(x), \varphi(x) \rangle = \langle t(x), \varphi(x) \rangle - c\varphi(o) - \sum_\mu c_\mu \left(\frac{\partial}{\partial x_\mu} \varphi \right)(o) = \lim_{\varepsilon \to 0} \langle t(x), \psi_\varepsilon(x) \rangle$$

where

$$\psi_\varepsilon(x) = \varphi(x) - \varphi(\varepsilon x) - \sum_\mu x_\mu \left(\frac{\partial}{\partial x_\mu} \varphi \right)(\varepsilon x)$$

$$= \sum_{|\beta|=2} \frac{2}{\beta !} x^\beta \int_\varepsilon^1 (1 - t) D^\beta \varphi(tx) dt - \varepsilon \sum_\mu x_\mu \left(\frac{\partial}{\partial x_\mu} \varphi \right)(\varepsilon x)$$

Each of these last two terms are in $\mathcal{S}(\mathbb{R}^N)$ for $\varepsilon \neq 0$. We apply for the first one the same analysis as in section 1 and get

$$\left| \left\langle t(x), \sum_{|\beta|=2} \frac{2}{\beta !} x^\beta \int_\varepsilon^1 (1 - t) D^\beta \varphi(tx) dt \right\rangle \right| \leqslant C_\delta \sum_{|\gamma| \geqslant 2} \sup (1 + \| x \|)^{|\gamma| - 1 - \delta} | D^\gamma \varphi(x) |$$

for $0 < \delta < 1$ and $|\varepsilon| < 1$.

For the second one

$$\left|\left\langle t(x), \sum_{\mu} x_{\mu}\left(\frac{\partial}{\partial x_{\mu}} \varphi\right)(\varepsilon x)\right\rangle\right| \leqslant \sum_{|\gamma| \geqslant 1} \sup (1 + \| x \|)^{|\gamma| - 1} | D^{\gamma}\varphi(x) |$$

and thus

$$| \langle t(x), \psi_{\varepsilon}(x) \rangle | \leqslant C_{\delta} \sum_{|\gamma| \geqslant 2} \sup (1 + \| x \|)^{|\gamma| - 1 - \delta} | D^{\gamma}\varphi(x) | + \varepsilon \sum_{|\gamma| \geqslant 1} \sup (1 + \| x \|)^{|\gamma| - 1} | D^{\gamma}\varphi(x) |$$

from which it follows that

$$| \langle \bar{t}(x), \varphi(x) \rangle | = \lim_{\varepsilon \to 0} | \langle t(x), \psi_{\varepsilon}(x) \rangle | \leqslant C_{\delta} \sum_{|\gamma| \geqslant 2} \sup (1 + \| x \|)^{|\gamma| - 1 - \delta} | D^{\gamma}\varphi(x) |$$

which is the required result.

Remark that according to the equivalence of norms between momentum space and position space, formula (1.3) means that the Fourier transform of $\bar{t}(x)$ vanishes at the origin, and formula (2.6) means that the Fourier Transform of $\bar{t}(x)$ vanishes with its first derivatives at the origin.

APPENDIX C

ESTIMATE ON $\| x \|^{-\omega+|\alpha|-\varepsilon}(1 + \| x \|)^{\omega+\varepsilon-D+1-\delta} | D^{\alpha}(\sqcup(x)\varphi(x)) |$

According to [1]

$$| D^{\alpha}(\sqcup(x)\varphi(x)) | \leqslant C' \sum_{|\gamma| \leqslant |\alpha|} | D^{\gamma}\sqcup(x) | \, | D^{\alpha-\gamma}\varphi(x) | \leqslant C' \sum_{|\gamma| \leqslant |\alpha|} \| x \|^{-|\gamma|} | D^{\alpha-\gamma}\varphi(x) |$$

Therefore

$$\| x \|^{-\omega+|\alpha|-\varepsilon}(1 + \| x \|)^{\omega+\varepsilon-D+1-\delta} | D^{\alpha}(\sqcup(x)\varphi(x)) |$$
$$\leqslant C' \sum_{|\gamma| \leqslant |\alpha|} \| x \|^{-\omega+|\alpha|-|\gamma|-\varepsilon}(1 + \| x \|)^{\omega+\varepsilon-D+1-\delta} | D^{\alpha-\gamma}\varphi(x) |$$
$$\leqslant C' \sum_{|\gamma| \leqslant |\alpha|} \| x \|^{-\omega+|\gamma|-\varepsilon}(1 + \| x \|)^{\omega+\varepsilon-D+1-\delta} | D^{\gamma}\varphi(x) | \quad (1'.1)$$

This means that for $\omega < 0$

$$\| x \|^{\mu_{\alpha}^{+}}(1 + \| x \|)^{-\mu_{\alpha}^{+}+|\alpha|-D+1-\delta} | D^{\alpha}(\sqcup(x)\varphi(x)) |$$
$$\leqslant C' \sum_{|\gamma| \leqslant |\alpha|} \| x \|^{\mu_{\gamma}^{+}}(1 + \| x \|)^{-\mu_{\gamma}^{+}+|\gamma|-D+1-\delta} | D^{\gamma}\varphi(x) |$$

where

$$\mu_{\alpha}^{+} = (-\omega + |\alpha| - \varepsilon)^{+} \quad (1.3)$$

ESTIMATE ON $\| x \|^{\mu_{\alpha}^{+}}(1 + \| x \|)^{-\mu_{\alpha}^{+}+|\alpha|-D+1-\delta} | D^{\alpha}(\sqcup(x)(W\varphi)(x)) |$
FOR $|\alpha| \geqslant \max(0, D)$, $\mu_{\alpha}^{+} = (-\omega + |\alpha| - \varepsilon)^{+}$, $\omega \geqslant 0$, ω AND D INTEGERS

Here, as usual $\varepsilon > 0$ small, $0 < \delta < 1$ and δ close to one. $(W\varphi)(x)$ is defined in [1] by

$$(W\varphi)(x) = \varphi(x) - w(x) \sum_{|\alpha|=0}^{\omega} \frac{x^{\alpha}}{\alpha!} D^{\alpha}\varphi(o) = \sum_{|\beta|=\omega+1} x^{\beta}\varphi_{\beta}(x)w(x) + \varphi(x)(1 - w(x)) \quad (2.4)$$

where

$$\varphi_{\beta}(x) = \frac{\omega+1}{\beta!} \int_0^1 dt(1-t)^{\omega}(D^{\beta}\varphi)(tx)$$

and $w \in \mathscr{S}(\mathbb{R}^N)$ with $w(0) = 1$, $D^{\alpha}w(0) = 0$ for $1 \leqslant |\alpha| \leqslant \omega$. From now on let us call I_{α} the expression which has to be estimated.

First, we consider the case when $|\alpha| \leqslant \omega$, then when $|\alpha| > \omega$.

1) $|\alpha| \leqslant \omega$

$$I_{\alpha} = (1 + \| x \|)^{|\alpha|-D+1-\delta} | D^{\alpha}(\sqcup(x)W\varphi(x)) |$$
$$\leqslant C' \sum_{|\gamma| \leqslant |\alpha|} (1 + \| x \|)^{|\alpha|-D+1-\delta} \| x \|^{-|\gamma|} | D^{\alpha-\gamma}W\varphi(x) | = C' \sum_{|\gamma| \leqslant |\alpha|} I_{\alpha,\gamma}$$

Then we write $W\varphi(x) = \psi^1(x) + \psi^2(x)$ with

$$\psi^1(x) = \sum_{|\beta|=\omega+1} x^{\beta}\varphi_{\beta}(x)w(x) \qquad \psi^2(x) = \varphi(x)(1 - w(x)) \quad (2.5)$$

and denote $I_{\alpha,\gamma}^1$ and $I_{\alpha,\gamma}^2$ the corresponding terms in $I_{\alpha,\gamma}$.

i) *Estimate on* $I^1_{\alpha,\gamma}$

We first estimate

$$\left| D^{\alpha-\gamma}\Big(\sum_{|\beta|=\omega+1} x^\beta \varphi_\beta(x) w(x) \Big) \right|$$

It is less than

$$C' \sum_{|\beta|=\omega+1} \sum_{\theta_1+\theta_2+\theta_3=\alpha-\gamma} \| x \|^{|\beta|-|\theta_1|} | D^{\theta_2}\varphi_\beta | \; | D^{\theta_3} w(x) |$$

where

$$| D^{\theta_2}\varphi_\beta | \leqslant C' \int_0^1 dt \, t^{|\theta_2|} \, | D^{\theta_2+\beta}\varphi(tx) |$$

Therefore,

$$I^1_{\alpha,\gamma} \leqslant \sum_{|\beta|=\omega+1} \sum_{\theta_1+\theta_2+\theta_3=\alpha-\gamma} \sup_{tx} \| tx \|^{1+|\theta_2|-\varepsilon} | D^{\beta+\theta_2}\varphi(tx) |$$

$$\times \int_0^1 \frac{dt}{t^{1-\varepsilon}} \times \sup_x \| x \|^{\omega-|\alpha|+|\theta_3|+\varepsilon}(1 + \| x \|)^{|\alpha|-D+1-\delta} | D^{\theta_3} w(x) |$$

The last supremum is bounded by

$$\sup_x (1 + \| x \|)^{\omega+|\theta_3|-D+1-\delta+\varepsilon} | D^{\theta_3} w(x) |$$

since $|\alpha| \leqslant \omega$ and we get

$$I^1_{\alpha,\gamma} \leqslant C' \sum_{|\sigma| \geqslant \omega+1} \sup_x \| x \|^{-\omega+|\sigma|-\varepsilon}(1 + \| x \|)^{\omega+\varepsilon-\min(D,\omega)+1-\bar\delta} | D^\sigma\varphi(x) |$$

since $\omega \geqslant \min(D, \omega)$ and here $\varepsilon > 0$, $1 > \bar\delta > 0$, ε close to zero, $\bar\delta$ close to one.

But since we have asked $|\alpha| \geqslant \max(0, D)$ and $|\alpha| \leqslant \omega$, this means $\omega \geqslant \max(0, D) \geqslant D$ and $\min(D, \omega) = D$. Therefore,

$$I^1_{\alpha,\gamma} \leqslant C' \sum_{|\sigma| \geqslant \omega+1} \sup_x \| x \|^{-\omega+|\sigma|-\varepsilon}(1 + \| x \|)^{\omega+\varepsilon-D+1-\delta} | D^\sigma\varphi | \tag{2.6}$$

ii) *Estimate on* $I^2_{\alpha,\gamma}$

$$| D^{\alpha-\gamma}\varphi(x)(1 - w(x)) | \leqslant C' \sum_{\theta_1+\theta_2=\alpha-\gamma} | D^{\theta_1}\varphi(x) | \; | D^{\theta_2}(1 - w(x)) |$$

First consider the case when $\theta_2 \neq 0$ and let us estimate

$$(1 + \| x \|)^{|\theta_2|+|\gamma|} \| x \|^{-|\gamma|} | D^{\theta_2}(1 - w(x)) |$$

When $\| x \| \leqslant 1$ this is less than

$$C' \| x \|^{-|\gamma|} | D^{\theta_2}(1 - w(x)) |$$

but

$$D^{\theta_2}(1 - w(x)) = \sum_{|\beta|=\omega+1} x^\beta \frac{\omega+1}{\beta!} \int_0^1 (1-t)^\omega t^{\theta_2} D^{\theta_2+\beta} w(tx) dt \leqslant C' \| x \|^{|\beta|} \sup_{tx} | D^{\theta_2+\beta} w(tx) |$$

and

$$\| x \|^{-|\gamma|} | D^{\theta_2}(1 - w(x)) | \leqslant C' \| x \|^{|\beta|-|\gamma|} \leqslant C' \tag{2.7}$$

for any $|\theta_2| \geqslant 0$, since $|\beta| \geqslant |\alpha| + 1 > |\gamma|$.

When $\| x \| \geqslant 1$ this is less than

$$(1 + \| x \|)^{|\theta_2|} | D^{\theta_2} w(x) | \leqslant C'$$

since $|\theta_2| \geqslant 1$. Therefore,

$$(1 + \| x \|)^{|\theta_2|+|\gamma|} \| x \|^{-|\gamma|} | D^{\theta_2}(1 - w(x)) | \leqslant C' \tag{2.8}$$

and

$$(1 + \| x \|)^{|\alpha| - D + 1 - \delta} \| x \|^{-|\gamma|} \, | D^{\theta_1}\varphi(x) | \, | D^{\theta_2}(1 - w(x)) |$$
$$\leqslant (1 + \| x \|)^{|\theta_1| - D + 1 - \delta} \, | D^{\theta_1}\varphi(x) | \, (1 + \| x \|)^{|\theta_2| + |\gamma|} \| x \|^{-|\gamma|} \, | D^{\theta_2}(1 - w(x)) |$$

is bounded by

$$C'(1 + \| x \|)^{|\theta_1| - D + 1 - \delta} \, | D^{\theta_1}\varphi(x) |$$

Let us now consider the case when $\theta_2 = 0$. We have to estimate

$$(1 + \| x \|)^{|\gamma|} \| x \|^{-|\gamma|} \, | 1 - w(x) |$$

When $\| x \| \leqslant 1$ we use estimate (2.7) and when $\| x \| \geqslant 1$ we use $1 + \| x \| \leqslant 2 \| x \|$ and since

$$(1 + \| x \|)^{|\gamma|} \| x \|^{-|\gamma|} \, | 1 - w(x) | \leqslant 2^{|\gamma|} \sup | 1 - w(x) | \leqslant C'$$

we finally get an estimate of the form (2.8). Then

$$(1 + \| x \|)^{|\alpha| - D + 1 - \delta} \| x \|^{-|\gamma|} \, | D^{\alpha - \gamma}\varphi(x) | \, | 1 - w(x) | \leqslant C'(1 + \| x \|)^{|\alpha - \gamma| - D + 1 - \delta} \, | D^{\alpha - \gamma}\varphi(x) |$$

and

$$I^2_{\alpha,\gamma} \leqslant C' \sum_{|\sigma| \leqslant |\alpha|} (1 + \| x \|)^{|\sigma| - D + 1 - \delta} \, | D^{\sigma}\varphi(x) | \tag{2.9}$$

To sum up, when $| \alpha | \leqslant \omega$

$$I_\alpha \leqslant \sum_{|\sigma|} \sup \| x \|^{\mu_{\mathcal{J}}^+} (1 + \| x \|)^{-\mu_{\mathcal{J}}^+ + |\sigma| - D + 1 - \delta} \, | D^{\sigma}\varphi(x) | \tag{2.10}$$

2) $| \alpha | > \omega$

$$I_\alpha = \| x \|^{-\omega + |\alpha| - \varepsilon} (1 + \| x \|)^{\omega + \varepsilon - D + 1 - \delta} \, | D^{\alpha}(\sqcup(x) W \varphi(x)) |$$

As for $| \alpha | \leqslant \omega$, we define $I_{\alpha,\gamma}$, $I^1_{\alpha,\gamma}$ and $I^2_{\alpha,\gamma}$.

i) *Estimate for* $I^1_{\alpha,\gamma}$

If $\omega \geqslant D$ the result is the same as for $| \alpha | \leqslant \omega$, but if $\omega < D$ we have to modify the proof. In fact ω and D are integers and $\omega < D$ means $\omega \leqslant D - 1$, thus $\omega - D + 1 + \varepsilon - \delta < 0$ for ε close to zero and δ close to one. Therefore, in

$$\int_0^1 \| tx \|^{1 - \varepsilon + |\theta_2|} \frac{1}{t^{1-\varepsilon}} (1 + \| x \|)^{\omega + \varepsilon - D + 1 - \delta} \| x \|^{-\omega + |\alpha| - \varepsilon - 1 + \varepsilon - |\theta_2| + |\beta| - |\theta_1| - |\gamma|} \, | D^{\theta_3} w(x) | \, dt$$

we can replace

$$(1 + \| x \|)^{\omega + \varepsilon - D + 1 - \delta} \quad \text{by} \quad (1 + \| tx \|)^{\omega + \varepsilon - D + 1 - \delta}$$

which is larger and we get

$$I^1_{\alpha,\gamma} \leqslant C' \sum_{|\sigma| \geqslant \omega + 1} \sup_x \| x \|^{-\omega + |\sigma| - \varepsilon} (1 + \| x \|)^{\omega + \varepsilon - D + 1 - \delta} \, | D^{\sigma}\varphi |$$

which is again the required result.

ii) *Estimate for* $I^2_{\alpha,\gamma}$

We have to estimate terms of the form

$$(1 + \| x \|)^{\omega + \varepsilon - D + 1 - \delta} \| x \|^{-\omega + |\alpha| - \varepsilon - |\gamma|} \, | D^{\theta_1}\varphi | \, | D^{\theta_2}(1 - w(x)) | \tag{2.12}$$

with $\theta_1 + \theta_2 = \alpha - \gamma$.

First consider the case when $| \theta_1 | \leqslant \omega$. Rewriting the last expression as

$$(1 + \| x \|)^{|\theta_1| - D + 1 - \delta} \, | D^{\theta_1}\varphi | . \{ (1 + \| x \|)^{\omega + \varepsilon - |\theta_1|} \| x \|^{-\omega + |\alpha| - \varepsilon - D - |\gamma|} \, | D^{\theta_2}(1 - w(x)) | \}$$

Let us estimate the term in the brackets.

For $\| x \| \leqslant 1$ it is bounded by

$$C' \| x \|^{-\omega+|\alpha|-\varepsilon-|\gamma|} \, | D^{\theta_2}(1 - w(x)) |$$

which is less than

$$\| x \|^{-|\gamma|} \, | D^{\theta_2}(1 - w(x)) |$$

since $| \alpha | > \omega$, $| \alpha |$ and ω being integers.

According to estimate (2.7), this expression is bounded by a constant. Now, for $\| x \| \geqslant 1$, the expression in the brackets is bounded by

$$(1 + \| x \|)^{|\alpha|-|\gamma|-|\theta_1|} \, | D^{\theta_2}(1 - w(x)) | = (1 + \| x \|)^{|\theta_2|} \, | D^{\theta_2}(1 - w(x)) |$$

which, according to (2.8) and the following is also bounded by a constant.

Therefore (2.12) is for $| \theta_1 | \leqslant \omega$, bounded by

$$C'(1 + \| x \|)^{|\theta_1|-D+1-\delta} \, | D^{\theta_1}\varphi |$$

Consider now the case when $| \theta_1 | > \omega$. In the same way we are led to estimate

$$\| x \|^{|\theta_2|} \, | D^{\theta_2}(1 - w(x)) |$$

which, as we have shown, is bounded by a constant. Then

$$I^2_{\alpha,\gamma} \leqslant C' \sum_{|\sigma| \leqslant |\alpha| - |\gamma|} \| x \|^{-\omega+|\sigma|-\varepsilon}(1 + \| x \|)^{\omega+\varepsilon-D+1-\delta} \, | D^{\sigma}\varphi | \tag{2.13}$$

Adding all the results we obtain

$$\| x \|^{\mu_\alpha^+}(1 + \| x \|)^{-\mu_\alpha^+ + |\alpha| - D + 1 - \delta} \, | D^{\alpha}(\sqcup(x)(W\varphi)(x)) |$$
$$\leqslant C(\varepsilon, \delta) \sum_{|\sigma| \leqslant |\alpha| + \omega + 1} \sup_x \| x \|^{\mu_\sigma^+}(1 + \| x \|)^{-\mu_\sigma^+ + |\sigma| - D + 1 - \delta} \, | D^{\sigma}\varphi(x) | \tag{2.14}$$

with the same ε and δ.

We have got as a by-product the following two results

a) $\omega < 0$

$$(1 + \| x \|)^{|\alpha|-D+1-\delta} \, | D^{\alpha}(\sqcup(x)\varphi(x)) | \leqslant C' \sum_{|\sigma| \leqslant |\alpha|} \sup \, (1 + \| x \|)^{|\sigma|-D+1-\delta} \, | D^{\sigma}\varphi(x) | \tag{2.15}$$

b) $\omega \geqslant 0$, ω and D integers, $| \alpha | \geqslant \max(0, D)$

$$(1 + \| x \|)^{|\alpha|-D+1-\delta} \, | D^{\alpha}(\sqcup(x)W\varphi(x)) |$$
$$\leqslant C' \sum_{|\sigma| \leqslant |\alpha| + \omega + 1} \sup \, (1 + \| x \|)^{|\sigma|-D+1-\delta} \, | D^{\sigma}\varphi(x) | \tag{2.16}$$

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