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Axiomatic field theory and quantum electrodynamics : the massive case

by

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ABSTRACT. — Massive quantum electrodynamics of the electron is formulated as an LSZ theory of the electromagnetic field $F_{\mu\nu}$ and the electron-positron fields ψ, $\bar{\psi}$. The interaction is introduced with the help of mathematically well defined subsidiary conditions. These are: 1) gauge invariance of the first kind, assumed to be generated by a conserved current j_μ; 2) the homogeneous Maxwell equations and a massive version of the inhomogeneous Maxwell equations; 3) a minimality condition concerning the high momentum behaviour of the theory. The « inhomogeneous Maxwell equation » is a linear differential equation connecting $F_{\mu\nu}$ with the current j_μ. No Lagrangian, no non-linear field equations, and no explicit expression of j_μ in terms of ψ, $\bar{\psi}$ are needed. It is shown in perturbation theory that the proposed conditions fix the physically relevant (i. e. observable) quantities of the theory uniquely.

1. INTRODUCTION

Any attempt to fit quantum electrodynamics (henceforth called QED) into the framework of axiomatic field theory encounters two types of problems, namely

1) the general problem of characterizing particular models without using such mathematically dubious notions as Lagrangians, non-linear equations of motion, or equal time commutators of interacting fields,

2) problems specific to QED, which are connected with the vanishing photon mass. The best known of these are the infrared divergences of the S-matrix, which prevent the application of the LSZ formalism in its esta-

blished form. It is known, furthermore, that QED does not even fit into the more general Wightman framework [*1*]: in formulating QED as a field theory one must violate one or several of Wightman's axioms.

In the present work we shall deal only with the first type of problems. The difficulties mentioned under point 2) will be avoided by giving the photon a small non-vanishing mass Λ. We hope to make the limit $\Lambda \to 0$ the subject of a subsequent paper.

We consider the QED of a charged spin $\frac{1}{2}$ particle, called « electron », with mass $m > 0$. This theory we want to formulate as an LSZ theory of the electromagnetic field $F_{\mu\nu}$ and the electron-positron fields ψ, $\bar{\psi}$. The interaction shall be specified with the help of mathematically well-defined subsidiary conditions.

In Section 2 we consider the LSZ formulation of the theory in question. In particular we discuss the GLZ theorem, which permits the complete characterization of the theory by its retarded functions.

In Section 3 we propose a set of conditions singling out QED from all possible $F_{\mu\nu}$-ψ-$\bar{\psi}$-theories. We do not use Lagrangians or non-linear field equations. The conditions are formulated directly for the field $F_{\mu\nu}$. The introduction of a vector potential is not necessary, though later on it will turn out to be useful as a means of algebraic simplification. Hence no mention will be made of gauge transformations of the second kind. They cannot be fitted easily into the axiomatic frame, because in general they do not respect translation invariance. Since they leave the observables completely untouched, their physical significance is anyway not clear, so that their absence should not be considered a flaw of our formalism (1).

Our conditions are of the following kind. $F_{\mu\nu}$ shall satisfy the homogeneous and a massive version of the inhomogeneous Maxwell equations. The latter are linear equations between $F_{\mu\nu}$ and the electromagnetic current j_μ, so that no distributionistic difficulties arise. An explicit expression of j_μ in terms of ψ, $\bar{\psi}$ is not needed. The coupling to ψ, $\bar{\psi}$ is achieved *via* the commutation relations of these fields with the space integral over j_0, the charge Q. In addition, we postulate a minimality condition for the behaviour of certain physically important quantities at large 4-momenta.

We have no general proof that these conditions do indeed specify a theory. In Section 4 we show, however, that they can be satisfied in all finite orders of perturbation theory and determine there the physically relevant quantities uniquely (2). For this we follow the methods developed

(1) The author is aware that this attitude towards the gauge group is not in accordance with the fashion of the day.

(2) Perturbative QED has already been discussed in a similar vein, but on a lower level of rigour, some time ago by Nishijima [*3*]. Nishijima considered directly the case $\Lambda = 0$, disregarding the infrared problems.

for the simpler case of a single scalar field in Ref. [2], henceforth quoted as B. The results of B will be used freely. Their generalizations to the present case are mostly straightforward and will not be explicitly proved. Equation (n. m.) and Theorem n. m. of B will be quoted as Eq. (B. n. m.) and Theorem B. n. m. respectively.

Section 5 is devoted to a brief discussion of observable fields other than $F_{\mu\nu}$ and j_μ.

As was to be expected we do not gain any fundamental new insights, but reproduce merely in a mathematically clean way some well-known results of the canonical formalism. Also, the expert reader will easily perceive that our methods of proof have often been inspired by the corresponding canonical considerations.

2. THE LSZ FORMALISM

The generalization to massive QED of the formalism discussed in B for a scalar field does not present any fundamental difficulties. Therefore we only collect here without proofs the facts and notions which we shall need later on.

We consider a theory of two four-component spinor fields ψ, $\bar{\psi}$ and a real antisymmetric tensor field $F_{\mu\nu}$ which satisfy all the Wightman axioms [4] [5]. In particular we assume invariance under the orthochronous Lorentz group $L^\uparrow$, including the parity component. Our theory will also be C-invariant, hence, due to the CTP theorem, also T-invariant, so that the latter invariance need not be postulated explicitly.

We use the following notations: arguments of ψ-fields appearing in retarded products and their vacuum expectation values are denoted by small latin letters: $x, y, \ldots, p, q, \ldots$, arguments of $\bar{\psi}$-fields by barred small latin letters: $\bar{x}, \bar{y}, \ldots, \bar{p}, \bar{q}, \ldots$, arguments of $F_{\mu\nu}$-fields by small script letters: $\mathscr{x}, \mathscr{y}, \ldots, \mathscr{p}, \mathscr{q}, \ldots$. The bar over a variable is not part of the variable, but signifies only the occurrence of this variable as a $\bar{\psi}$-argument. The same variable x may occur in a given mathematical expression once with a bar, once without a bar, e. g. in different x-dependent factors of a product. Small greek letters: $\xi, \eta, \ldots, \rho, \omega, \ldots$ stand for variables which may be arguments of any type of field. As in B we use capitals to denote sets of small letters of the same character, e. g. $\bar{X} = \{ \bar{x}_1, \ldots, \bar{x}_n \}$. $|X|$ stands for the number of elements in the set X (note that this convention differs from B, where $|X|$ had another meaning).

We use Dirac matrices $\gamma_0, \ldots, \gamma_3$ with

$$\{ \gamma_\mu, \gamma_\nu \} = 2g_{\mu\nu}, \tag{2.1}$$

$g_{\mu\nu}$ the Minkowski tensor defined with signature $(+ - - -)$. γ_0 is hermitian, $\gamma_{1,2,3}$ are anti-hermitian. Vector indices are raised and lowered with the metric tensor $g_{\mu\nu}$. Indices occurring once as subscript, once as superscript are summed over. For V^{μ} an arbitrary 4-vector we define

$$\not{V} := V^{\mu}\gamma_{\mu}\,. \tag{2.2}$$

The fields ψ, $\bar{\psi}$ are connected by

$$\bar{\psi}(x) = \psi^{*}(x)\gamma_{0}\,. \tag{2.3}$$

The field $F_{\mu\nu}$ shall satisfy the homogeneous Maxwell equation (HM equation)

$$\partial_{\alpha}F_{\beta\gamma}(x) + \text{cycl.} = 0\,, \tag{2.4}$$

with $\partial_{\alpha} = \dfrac{\partial}{\partial x^{\alpha}}$.

For large positive or negative times ψ, $\bar{\psi}$, F are assumed to converge in the LSZ sense [6] to free fields ψ^{ex}, $\bar{\psi}^{\text{ex}}$, F^{ex}, where *ex* stands for *in* in the case $t \to -\infty$, for *out* in the case $t \to +\infty$. The asymptotic fields shall satisfy the Dirac equations

$$(i\not{\partial} - m)\psi^{\text{ex}}(x) = 0, \qquad \bar{\psi}^{\text{ex}}(x)(i\overleftarrow{\not{\partial}} + m) = 0\,, \tag{2.5}$$

and the Klein-Gordon equation

$$(\Box + \Lambda^{2})F^{\text{ex}}_{\mu\nu}(x) = 0 \tag{2.6}$$

respectively, with

$$0 < \Lambda \ll m\,. \tag{2.7}$$

The Fourier transform of any field $\varphi(\xi)$ is defined as

$$\tilde{\varphi}(\rho) = (2\pi)^{-\frac{5}{2}}\int d\xi e^{i\rho\xi}\varphi(\xi)\,. \tag{2.8}$$

Since we shall mostly work in p-space we shall henceforth omit the tilde in $\tilde{\varphi}$. It will usually be clear from the argument of φ and from the context whether we are in x-space or p-space. In case of doubt this will be explicitly specified.

The free fields ψ^{ex}, $\bar{\psi}^{\text{ex}}$ are in p-space of the form, dropping the superscript *ex* for the moment:

$$\begin{aligned}\psi(p) &= \delta_{+}(p)\hat{\psi}_{+}(\boldsymbol{p}) + \delta_{-}(p)\hat{\psi}_{-}(-\boldsymbol{p})\,,\\ \bar{\psi}(p) &= \delta_{+}(p)\hat{\bar{\psi}}_{+}(\boldsymbol{p}) + \delta_{-}(p)\hat{\bar{\psi}}_{-}(-\boldsymbol{p})\,,\end{aligned} \tag{2.9}$$

with $\delta_{\pm}(p) = \theta(\pm p_{0})\delta(p^{2} - m^{2})$ and

$$\hat{\bar{\psi}}_{\pm}(\boldsymbol{p}) = [\hat{\psi}_{\mp}(\boldsymbol{p})]^{*}\gamma_{0}\,. \tag{2.10}$$

They satisfy the anticommutation relations

$$\begin{aligned}\{\,\psi(p), \psi(q)\,\} &= \{\,\bar{\psi}(p), \bar{\psi}(q)\,\} = 0\,,\\ \{\,\psi(p), \bar{\psi}(q)\,\} &= \delta^{4}(p + q)(\not{p} + m)[\delta_{+}(p) - \delta_{-}(p)]\,,\end{aligned} \tag{2.11}$$

from which the anticommutators of the creation operators $\hat{\psi}_{-}$, $\hat{\bar{\psi}}_{-}$ and the destruction operators $\hat{\psi}_+$, $\hat{\bar{\psi}}_+$ are easily obtained. For the free electromagnetic field F^{ex} we have

$$F_{\mu\nu}(k) = \delta_+(p)\hat{F}_{\mu\nu}(\boldsymbol{k}) + \delta_-(k)[\hat{F}_{\mu\nu}(-\boldsymbol{k})]^* \tag{2.12}$$

with (3) $\delta_\pm(k) = \theta(\pm k_0)\delta(k^2 - \Lambda^2)$ and

$$[F_{\alpha\beta}(k), F_{\gamma\delta}(q)] = \delta^4(k + q)M_{\alpha\beta\gamma\delta}(q)[\delta_-(q) - \delta_+(q)], \tag{2.13}$$

$$M_{\alpha\beta\gamma\delta}(q) := -g_{\alpha\gamma}q_\beta q_\delta + g_{\alpha\delta}q_\beta q_\gamma + g_{\beta\gamma}q_\alpha q_\delta - g_{\beta\delta}q_\alpha q_\gamma . \tag{2.14}$$

We assume asymptotic completeness:

$$\mathfrak{H}^{in} = \mathfrak{H}^{out} = \mathfrak{H}, \tag{2.15}$$

where $\mathfrak{H}$ is the total Hilbert space of the theory, $\mathfrak{H}^{ex}$ the Fock space of the fields ψ^{ex}, $\bar{\psi}^{ex}$, F^{ex}. The identity operator in $\mathfrak{H}^{in}$ can be written

$$\mathbb{1} = \sum_{\alpha,\beta=0}^{\infty} E_{\alpha\beta} \sum_{\gamma=0}^{\infty} E'_\gamma, \tag{2.16}$$

with $E_{\alpha\beta}$ the orthogonal projection onto the α-electron β-positron subspace of the ψ^{in}-$\bar{\psi}^{in}$ Fock space, E'_γ the projection onto the γ-photon space in the F^{in} Fock space. These projections have the representations

$$E_{\alpha\beta} = \frac{(-1)^\alpha}{\alpha!\beta!(2m)^{\alpha+\beta}} \sum_{\sigma_i,\tau_j} \int \prod_1^\alpha \frac{d^3p_i}{2\omega(p_i)} \prod_1^\beta \frac{d^3q_j}{2\omega(q_j)}$$

$$\times \overrightarrow{\prod_1^\alpha} \hat{\psi}^-_{\sigma_i}(p_i) \overrightarrow{\prod_1^\beta} \hat{\bar{\psi}}^-_{\tau_j}(q_j) \,|\,0\,\rangle\langle\,0\,|\, \overleftarrow{\prod_1^\beta} \hat{\bar{\psi}}^+_{\tau_j}(q_j) \overleftarrow{\prod_1^\alpha} \hat{\psi}^+_{\sigma_i}(p_i), \tag{2.17}$$

$$E'_\gamma = \frac{(-1)^\gamma}{\gamma!(2\Lambda^2)^\gamma} \int \prod_1^\gamma \frac{d^3\boldsymbol{k}_i}{2\omega(\boldsymbol{k}_i)} \prod_1^\gamma [\hat{F}^{\mu_i\nu_i}(\boldsymbol{k}_i)]^* \,|\,0\,\rangle\langle\,0\,|\, \prod_1^\gamma \hat{F}_{\mu_i\nu_i}(\boldsymbol{k}_i). \tag{2.18}$$

Here $\omega(\boldsymbol{p}) = (\boldsymbol{p}^2 + m^2)^{1/2}$, $\omega(\boldsymbol{k}) = (\boldsymbol{k}^2 + \Lambda^2)^{1/2}$. $|\,0\,\rangle\langle\,0\,|$ is the projection onto the vacuum $|\,0\,\rangle$. The arrows over the products in (2.17) mean that in $\overrightarrow{\Pi}$ the factors stand in order of ascending indices i, in $\overleftarrow{\Pi}$ in order of descending i.

The GLZ theorem [7] tells us that we can characterize the theory by specifying its retarded functions [8]. Due to the anticommutativity of spinor fields there are some trivial changes of sign, relative to the scalar case, in the definition of retarded products. In Ruelle's formal definition [9] of a retarded product (or a generalized retarded product) as a sum over permuted products of fields multiplied with appropriate step functions,

(3) Note that the mass in the definition of $\delta_\pm$ is m or Λ according to the character of its argument. We hope that this illegitimate notation will not lead to confusion.

we have an additional minus sign for the terms in which the ordering of the spinor variables differs from a standard ordering (defined as the ordering exhibited in the argument of R(...)) by an odd permutation. In the axiomatic definition of B we replace Eq. (B. 2.22) by

$$\mathrm{R}(\xi, \eta, \Xi) \pm \mathrm{R}(\eta, \xi, \Xi) = -i \sum_{\mathrm{L}} \varepsilon_{\mathrm{L}}[\mathrm{R}(\xi, \Xi_{\mathrm{L}}), \mathrm{R}(\eta, \Xi_{\mathrm{R}})]_{\pm} . \quad (2.19)$$

Here $\Xi = \{ \xi_1, \ldots, \xi_n \}$ is a set of variables of arbitrary type. The tensor indices of F and the spinor indices of ψ, $\bar{\psi}$ are to be considered part of the corresponding variables ξ_i. This convention will be used throughout this paper. In cases where explicit exhibition of these indices is desirable, they will be shown as subscripts standing in front of the variable. For instance, the expression ${}_{\mu\nu}x$ in the argument of a retarded product or function means that x is the argument of a field $\mathrm{F}_{\mu\nu}(x)$, while x_μ denotes the μ-component of the 4-vector x.

The sum in the right-hand side of (2.19) extends over all partitions of Ξ into two complementary subsets Ξ_{L} and Ξ_{R}. An anticommutator occurs if both factors $\mathrm{R}(\xi, \Xi_{\mathrm{L}})$ and $\mathrm{R}(\eta, \Xi_{\mathrm{R}})$ contain an odd number of spinors, a commutator otherwise. $\varepsilon_{\mathrm{L}} = \pm 1$ is the parity of the spinor variables in the ordering ξ, Ξ_{L}, η, Ξ_{R} as compared to their ordering in ξ, η, Ξ. Within Ξ_{L}, Ξ_{R} the variables stand in the same order as in Ξ. The alternative sign in the left-hand side is positive if both ξ and η are spinor variables, negative otherwise.

The *retarded function* $r(\Xi)$ is the vacuum expectation value of $\mathrm{R}(\Xi)$.

Amputation of a retarded function with respect to a photon variable $\not{k}$ in p-space means multiplication with $(\not{k}^2 - \Lambda^2)$. For the electron momenta p, q we use two different amputation prescriptions:

$$r^n(\Omega\,; \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) := \prod_{\mathscr{K}} (\not{k}_i^2 - \Lambda^2) \prod_{\mathrm{P}} (\not{p}_i - m) r(\Omega, \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) \prod_{\mathrm{Q}} (\not{q}_i + m) ,$$

$$r^a(\Omega\,; \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) := \prod_{\mathscr{K}} (\not{k}_i^2 - \Lambda^2) \prod_{\mathrm{P}} (p_i^2 - m^2) \prod_{\mathrm{Q}} (q_i^2 - m^2) r(\Omega, \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$$

$$= \prod_{\mathrm{P}} (\not{p}_i + m) r^n(\Omega\,; \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) \prod_{\mathrm{Q}} (\not{q}_i - m) . \quad (2.20)$$

The amputated variables are separated from the non-amputated ones by a semi-colon. We shall never have occasion to use the completely unamputated r-functions. Therefore we can again drop the index n in r^n, with the understanding that henceforth $r(\ldots)$ will stand for $r^n(\ldots)$.

The reduction formulæ for matrix elements of fields and retarded products between *in*-states look exactly like in the scalar case (see (B.2.32)), with the r^a-functions used.

The GLZ equations are obtained from the relations (2.19) by forming the vacuum expectation value, inserting the identity representation (2.16)-(2.18) on the right-hand side and expressing the resulting *in* matrix elements with the help of the reduction formulæ. We obtain the following equations, written in p-space, for the totally n-amputated r-functions:

$$
\begin{aligned}
& r(\rho_1, \rho_2, \Omega) \pm r(\rho_2, \rho_1, \Omega) \\
& = -i\left[\sum_{\mathrm{L}} \varepsilon_{\mathrm{L}} \sum_{\substack{\alpha,\beta,\gamma=0 \\ \alpha+\beta+\gamma>0}}^{\infty} \frac{(2\pi)^{2(\alpha+\beta+\gamma)}\varepsilon_{\mathrm{L}}'^{\alpha+\beta}}{\alpha!\beta!\gamma!(-2\Lambda^2)^{\gamma}} \int \prod_{1}^{\alpha} ds_i \delta_+(s_i)\right. \\
& \times \prod_{1}^{\beta} dt_j \delta_+(t_j) \prod_{1}^{\gamma} d\ell_h \delta_+(\ell_h) r(\rho_1, \Omega_{\mathrm{L}}, -\mathrm{S}, -\bar{\mathrm{T}}, -\mathscr{L}) \prod_{1}^{\alpha} (\not{s}_i^{\mathrm{T}} - m) \prod_{1}^{\beta} (\not{t}_j + m) \\
& \left. \times\, r(\rho_2, \Omega_{\mathrm{R}}, \mathrm{T}, \bar{\mathrm{S}}, \mathscr{L}) - \varepsilon_{\mathrm{L}}' \{ \rho_1 \leftrightarrow \rho_2 \} \right]. \qquad (2.21)
\end{aligned}
$$

ε_{L} and the sign on the left-hand side are as in (2.19). ρ_1, ρ_2, Ω, are called external variables, S, T, $\mathscr{L}$, internal variables. Let $\mathrm{N_L}$ be the number of external spinor variables in the left-hand r-factor of the integrand. Then [4] $\varepsilon_{\mathrm{L}}' = (-1)^{\mathrm{N_L}}$. The tensor indices of the internal F-variables are, according to (2.18), upper indices in the left r, lower indices in the right r. With respect to the spinor indices matrix multiplication is implied. For the variable s_1, e. g., this looks, written out explicitly:

$$\sum_{\sigma,\sigma'} r(\ldots, {}_{\sigma}(-s_1), \ldots)(\not{s}_1^{\mathrm{T}} + m)_{\sigma\sigma'} r(\ldots, {}_{\sigma'}\bar{s}_1, \ldots)$$

with $\not{s}^{\mathrm{T}}_{\sigma\sigma'} = \not{s}_{\sigma'\sigma}$.

By a simple generalization of Theorems B.2.1 and B.2.2 a solution $\{ r(\Omega) \}$ of the GLZ equations (2.21) defines a field theory of the desired type, provided that the distributions $r(\Omega)$ satisfy the following conditions:

a) *Reality.* — The relation (2.3) leads to

$$\left[r^a(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) \prod_{\mathrm{Q}} \gamma^0_{q_i} \right]^* = (-1)^{|\mathrm{Q}|} r(-\mathscr{K}, -\bar{\mathrm{P}}, -\mathrm{Q}) \prod_{\mathrm{P}} \gamma^0_{p_i}. \qquad (2.22)$$

Here $\gamma^0_{q_i}$ acts on the spinor index belonging to q_i and analogously for $\gamma^0_{p_i}$. The ordering of the variables in the argument of r^a need not be as shown here. It can be arbitrary, but it must be the same on both sides [5]. In (2.22)

(4) Because of the well-known fermion superselection rule only the r with an even number of spinors can be non-zero. Hence $\varepsilon_{\mathrm{L}}' = \varepsilon_{\mathrm{R}}'$ except in the trivial case of an odd number of external spinor variables, in which case both sides of (2.21) vanish identically.

(5) Unless noted otherwise, this remark will also apply to similar expressions in the future.

we have assumed $|P| = |Q|$, as will be the case in QED, due to charge conservation.

b) *Covariance.* — Under orthochronous Poincaré transformations r transforms like the corresponding product of classical fields F, ψ, $\bar{\psi}$.

c) *Symmetries.* — We have

$$r(\ldots, \rho_i, \rho_{i+1}, \ldots) = \pm\, r(\ldots, \rho_{i+1}, \rho_i, \ldots), \tag{2.23}$$

where the minus sign applies if the exchanged variables are both spinor variables, the plus sign in all other cases. (2.23) does not hold if one of the exchanged variables is the foremost standing one. This exceptional first variable will henceforth be called the « distinguished variable » of $r(\rho_1, \ldots)$. We shall occasionally take the liberty of not putting it at the front of the argument.

Note that the two r in (2.23) represent different functions if ρ_i and ρ_{i+1} are not of the same type, e. g. $r(x, \bar{y}) \neq r(\bar{x}, y)$. We apologize for this possibly confusing notation which has been introduced to avoid an even more confusing proliferation of indices.

d) *Support.* — In x-space $r(\xi, \xi_1, \ldots, \xi_n)$ vanishes outside the set $(\xi - \xi_i) \in \bar{V}_+$, $\forall i$.

e) *Mass shell restriction.* — The restriction of r to the mass shell in several or all of its variables exists and satisfies the smoothness property (B.2.43), which we will not repeat here. It guarantees the local existence of the integrals in (2.21). Since this condition is automatically satisfied by the perturbative construction of B we will not consider it any further.

f) *Normalization of the 2-point functions.* — For $p^2 < (m + \Lambda)^2$ and $\not{k}^2 < 4\Lambda^2$ respectively we have

$$\left.\begin{aligned} r(p, \bar{q}) &= -\frac{1}{2\pi}\delta^4(p+q)(\not{p} - m)[1 + F_1(p)(\not{p} - m)] \\ r(\bar{p}, q) &= -\frac{1}{2\pi}\delta^4(p+q)[1 + (\not{p}^T + m)F_2(p)](\not{p}^T + m) \end{aligned}\right\}, \tag{2.24}$$

$$r(\not{k}, \not{q}) = \frac{1}{2\pi}\delta^4(\not{k} + \not{q})(\not{k}^2 - \Lambda^2)[M(\not{k}) + (\not{k}^2 - \Lambda^2)F_3(\not{k})], \tag{2.25}$$

with F_i analytic. The other 2-point functions vanish identically.

As a further condition we have the HM equation (2.4) which has no equivalent in the scalar case of B. Before formulating it as a condition for r we must make a preparatory remark. The fields defined by a solution of (2.21) with all the necessary properties are explicitly given by their

Haag expansion. Let $\varphi(\xi)$ be any local field of our theory, i. e. $F_{\mu\nu}$, ψ, $\bar{\psi}$, or any of the fields to be introduced later on. Then we have in p-space

$$\varphi(\rho) = \sum_{\alpha,\beta,\gamma=0}^{\infty} \frac{(-2\pi)^{\alpha+\beta+\gamma}}{\alpha!\beta!\gamma!(2\Lambda^2)^\gamma}$$
$$\times \int \prod_1^\alpha du_i \prod_1^\beta dv_j \prod_1^\gamma d\ell_h r(\rho\,;\, -\bar{u}_1, \ldots, -\bar{u}_\alpha, -v_1, \ldots, -v_\beta, -\ell_1, \ldots, -\ell_\gamma)$$
$$\times :\bar{\psi}^{\text{in}}(v_\beta) \ldots \bar{\psi}^{\text{in}}(v_1)\psi^{\text{in}}(u_\alpha) \ldots \psi^{\text{in}}(u_1)F^{\text{in}}(\ell_1) \ldots F^{\text{in}}(\ell_\gamma): . \tag{2.26}$$

Summation over corresponding spinor and tensor indices in r and the *in* fields is understood. Because of the relations

$$2m\psi^{\text{in}}(p) = (\not p + m)\psi^{\text{in}}(p), \qquad 2m\bar{\psi}^{\text{in}}(p) = \bar{\psi}^{\text{in}}(p)(\not p - m) \tag{2.27}$$

following from the Dirac equations (2.5) we can replace r by r^a in (2.26). The combinatorial coefficient in front of the integral acquires then the additional factor $(2m)^\alpha(-2m)^\beta$.

Since the fields ψ, $\bar{\psi}$, F, determine the theory completely, the r-functions are physically relevant only in so far as they contribute to the Haag expansions of these fields. More exactly: two sets of retarded functions $\{ r_1 \}$ and $\{ r_2 \}$ for which

$$r_1^a(\rho\,;\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) = r_2^a(\rho\,;\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) \qquad \text{for} \qquad k_i^2 = \Lambda^2,\ p_j^2 = q_h^2 = m^2$$

are physically equivalent.

After this side remark we return to the HM equations.

g) *Homogeneous Maxwell equations.* — We must have

$$\varepsilon^{\mu\alpha\beta\gamma} k_\alpha r^a(\Omega\,;\, {}_{\beta\gamma}k) = 0 \qquad \text{for} \qquad k^2 = \Lambda^2 \tag{2.28}$$

and

$$\varepsilon^{\mu\alpha\beta\gamma} k_\alpha r^a({}_{\beta\gamma}k\,;\, \mathscr{L}, \mathrm{P}, \bar{\mathrm{Q}}) = 0 \qquad \text{for} \qquad \ell_i^2 = \Lambda^2,\ p_j^2 = q_h^2 = m^2. \tag{2.29}$$

Condition (2.28) reflects that the r^a occurring in it is, for $k_0 > 0$, apart from a numerical factor the matrix element $\langle 0 \,|\, F_{\beta\gamma}^{\text{in}}(k)R(\Omega) \,|\, 0 \rangle$ and for $k_0 < 0$ a similar matrix element with the one-photon state on the right. (2.29) stipulates that every term in the Haag expansion of F satisfies the HM equation. (2.28) is not a consequence of (2.29) and the asymptotic condition, because the Haag expansion presupposes validity of (2.28).

We end this section with a remark concerning the normalization conditions (2.24). In anticipation of the limit $\Lambda \to 0$ to be performed at a later occasion it would be desirable to generalize these conditions to

$$\begin{aligned} r(p, \bar{q}) &= -\frac{Z^2}{2\pi}\delta^4(p+q)(\not p - m)[1 + \ldots], \\ r(\bar{p}, q) &= -\frac{Z^2}{2\pi}\delta^4(p+q)[1 + \ldots](\not p^{\mathrm{T}} + m), \end{aligned} \tag{2.30}$$

with $Z^2 > 0$ an arbitrary function of Λ. An additional factor $Z^{-2\gamma}$ must then be inserted in the GLZ equations (2.21) and the Haag expansion (2.26). We can, however, at once find a solution $\{ r_Z \}$ of the generalized case from a solution $\{ r \}$ with $Z = 1$, to wit:

$$r_Z(\mathscr{K}, P, \bar{Q}) = Z^{|P|+|Q|} r(\mathscr{K}, P, \bar{Q}). \tag{2.31}$$

Hence we can put $Z = 1$ without restricting generality.

3. FIXING THE INTERACTION

From the collection of theories covered by the general formalism of Section 2 we want to single out QED by appropriate subsidiary conditions.

Firstly we demand gauge invariance of the first kind. A gauge transformation of the first kind is a substitution

$$\begin{aligned} \psi(x) &\to \psi(x)e^{i\alpha} \\ \bar{\psi}(x) &\to \bar{\psi}(x)e^{-i\alpha} \\ F(x) &\to F(x) \end{aligned} \tag{3.1}$$

with α a real number. These transformations form an Abelian group. Invariance of the theory under this group means existence of a continuous unitary representation $U(\alpha)$ with

$$U(\alpha + \beta) = U(\alpha)U(\beta) = U(\beta)U(\alpha), \tag{3.2}$$

$$U(\alpha) | 0 \rangle = | 0 \rangle, \tag{3.3}$$

$$\left.\begin{aligned} &U(\alpha)\psi(x)U^*(\alpha) = \psi(x)e^{i\alpha}, \qquad U(\alpha)\bar{\psi}(x)U^*(\alpha) = \bar{\psi}(x)e^{-i\alpha} \\ &U(\alpha)F_{\mu\nu}(x)U^*(\alpha) = F_{\mu\nu}(x). \end{aligned}\right\}, \tag{3.4}$$

By Stone's theorem there exists a self-adjoint operator Q' such that

$$U(\alpha) = e^{i\alpha Q'}, \tag{3.5}$$

$$\left.\begin{aligned} &[Q', \psi(x)] = \psi(x), \qquad [Q', \bar{\psi}(x)] = -\bar{\psi}(x) \\ &[Q', F(x)] = 0 \end{aligned}\right\}, \tag{3.6}$$

$$Q' | 0 \rangle = 0. \tag{3.7}$$

Gauge symmetry shall be generated by a conserved current, the electromagnetic current. This means that in $\mathfrak{H}$ there exists a vector field $j^\mu(\xi)$ satisfying Wightman's axioms, which is local relative to F, ψ, $\bar{\psi}$, and is conserved:

$$\partial_\mu j^\mu(x) = 0, \tag{3.8}$$

such that

$$Q := \int_{x^0 = t} d^3\boldsymbol{x}\, j^0(x) = eQ' \tag{3.9}$$

with e a real number which will serve as coupling constant. For the exact mathematical sense in which the integral (3.9) must be understood we refer to the review article by Orzalesi [10] and the original papers quoted there.

Q is assumed to commute with j^μ.

The current j^μ shall be coupled to the field $F_{\mu\nu}$ through the « inhomogeneous Maxwell equation » (IM equation)

$$(\square + \Lambda^2)F_{\mu\nu}(x) = -\partial_\mu j_\nu(x) + \partial_\nu j_\mu(x). \qquad (3.10)$$

Note that this equation is linear in the distributions F and j, hence mathematically meaningful. We need no explicit expression for j in terms of ψ and $\bar{\psi}$.

A final condition concerning the high momentum behaviour of the theory will yet have to be introduced. But first we want to transcribe the conditions formulated up to now into properties of the retarded functions. We consider now also retarded products containing j^μ-fields. Their arguments will be denoted by barred script letters: $\bar{x}, \ldots, \bar{p}, \ldots$. j-variables are never amputated.

It is well known that gauge invariance implies vanishing of the Wightman functions, and hence the retarded functions, with unequal numbers of ψ and $\bar{\psi}$ variables:

$$r(\mathscr{X}, \bar{\mathscr{Y}}, \mathrm{U}, \bar{\mathrm{V}}) = 0 \qquad \text{if} \qquad |\mathrm{U}| \neq |\mathrm{V}|. \qquad (3.11)$$

The conservation equation (3.8) and the IM equation (3.10) can be translated by analogy to (2.29) into conditions on the Haag coefficients of j and F.

For fixing the numerical value of e we need a normalization condition. This we obtain by inserting the definition (3.9) of Q into

$$\langle 0 | \psi^{\mathrm{in}}(p)\mathrm{Q}\bar{\psi}^{\mathrm{in}}(q) | 0 \rangle = -\,\mathrm{e}\langle 0 | \psi^{\mathrm{in}}(p)\bar{\psi}^{\mathrm{in}}(q) | 0 \rangle \qquad (3.12)$$

and expressing the resulting matrix element of j_0 with the reduction formula. The result is

$$(\not{p} - m)r({}_0\bar{\ell}\,; p, \bar{q})(\not{q} + m)\,|_{\ell=0, p_0=\omega(\boldsymbol{p})}$$
$$= 2(2\pi)^{-\frac{7}{2}}p_0 e(\not{p} - m)\delta^4(p + q)\,|_{p_0=\omega(\boldsymbol{p})} \qquad (3.13)$$

which generalizes by covariance to

$$(\not{p} - m)r({}_\mu\bar{\ell}\,; p, \bar{q})(\not{q} + m)\,|_{\ell=0, p_0=\omega(\boldsymbol{p})}$$
$$= 2(2\pi)^{-\frac{7}{2}}p_\mu e(\not{p} - m)\delta^4(p + q)\,|_{p_0=\omega(\boldsymbol{p})}. \qquad (3.14)$$

Conversely, let us assume (3.11), (3.14), and the validity of the IM equation and the conservation equation for the Haag coefficients of F and j respectively. From the latter two assumptions we find at once that the fields F and j themselves satisfy the IM equation (3.10) and the divergence condition (3.8) respectively. But (3.8) implies that the operator Q defined by (3.9) annihilates the vacuum [*10*]. Making use of results due to Kraus and Landau [*11*] we find furthermore

$$[Q, \psi^{in}(x)] = e\psi^{in}(x) + f\psi^{in^C}(x), \tag{3.15}$$

$$[Q, F^{in}(x)] = hF^{in}(x), \tag{3.16}$$

with e, f, h as yet undetermined complex numbers and ψ^{in^C} the charge conjugate of ψ^{in}. In deriving these equations we have used that $[Q, \psi^{in}(x)]$ must satisfy the Dirac equation and that Q is a Lorentz scalar. As a consequence of (3.11) the functions $\langle 0 | \psi^{in} j \psi^{in} | 0 \rangle$ and $\langle 0 | \psi^{in}\psi^{in} | 0 \rangle$ vanish, hence

$$f \langle 0 | \psi^{in}(x)\psi^{in^C}(y) | 0 \rangle = \langle 0 | \psi^{in}(x) Q \psi^{in}(y) | 0 \rangle = 0 .$$

This implies $f = 0$, because $\langle 0 | \psi^{in}\psi^{in^C} | 0 \rangle$ does not vanish identically. Insertion of (3.15) into $\langle 0 | [Q, \psi^{in}(p)]\bar{\psi}^{in}(q) | 0 \rangle$ gives (3.12), and the normalization condition (3.13) shows that e has the desired real value. We have then also

$$[Q, \bar{\psi}^{in}(x)] = - e\bar{\psi}^{in}(x). \tag{3.17}$$

Commuting Q once to the right, once to the left in $\langle 0 | F^{in}(x) Q F^{in}(y) | 0 \rangle$ we obtain

$$h \langle 0 | F^{in}(x)F^{in}(y) | 0 \rangle = - h \langle 0 | F^{in}(x)F^{in}(y) | 0 \rangle ,$$

hence $h = 0$. From their respective Haag expansions we easily find the commutators of Q with the interacting fields:

$$[Q, \psi(x)] = e\psi(x), \qquad [Q, \bar{\psi}(x)] = - e\bar{\psi}(x),$$
$$[Q, F(x)] = [Q, j(x)] = 0, \tag{3.18}$$

i. e. we recover the relations (3.6). According to Kraus and Landau [*11*] the operator Q is self-adjoint. This is then also true for $Q' = Q/e$, and equation (3.5) defines the desired unitary representation of the gauge group.

As a result of these considerations we find that our conditions on the r-functions are equivalent to the operator conditions formulated at the beginning of this section.

We come now to the minimality requirement at high momenta already alluded to. The conditions discussed until now do not determine the theory uniquely.

We need yet a condition corresponding to the small distance condition of *B*. A direct generalization of this condition to our case, translated into a p-space form, reads as follows. We define the *asymptotic degree* of the p-space distribution $r(\Omega)$, abbreviated AD(r), as the real number β, for which

$$\begin{aligned} &\lim_{\lambda \to \infty} \lambda^{-\beta-\varepsilon} r(\lambda\omega_1, \ldots, \lambda\omega_n) = 0 , \\ &\lim_{\lambda \to \infty} \lambda^{-\beta+\varepsilon} r(\lambda\omega_1, \ldots, \lambda\omega_n) \quad \text{does not exist} \end{aligned} \tag{3.19}$$

for all $\varepsilon > 0$. Here $r(\lambda\Omega)$, $0 < \lambda < \infty$, is a distribution in Ω and the limits $\lambda \to \infty$ must be taken in $\mathscr{D}'$. The asymptotic degree is connected with the x-space scaling degree of B by

$$\mathrm{AD}(r(\Omega)) = - \mathrm{SD}(r(\Xi)) - 4 \mid \Xi \mid , \tag{3.20}$$

where $r(\Omega)$ and $r(\Xi)$ are Fourier transforms of each other. The smoothness condition of B becomes then: *the distributions $r(\Omega)$ shall have the minimal* AD *that is compatible with the conditions already enumerated.* We call this condition the *first minimality condition.*

Unfortunately it will turn out in Section 4 that at least in perturbation theory first minimality still does not determine the theory uniquely: the theory is unrenormalizable in the terminology of B, Chapter VIII. In order to escape this predicament we examine more closely which objects of the theory are physically relevant, meaning that they enter into measurable quantities. Obviously these are the matrix elements of observables between physical states. We do not wish to discuss here what are the most general observables of the theory. For the moment we note only that the fields $F_{\mu\nu}$ and j_μ are observables (after integration over real test functions), but not the fields ψ and $\bar{\psi}$. More general observable fields, e. g. local polynomials of ψ, $\bar{\psi}$, will be discussed in Section 5.

$\mathfrak{H}^{\mathrm{in}}$ constitutes a complete set of states. Hence it suffices to consider the matrix elements of F, j, between *in*-states [6], and these matrix elements are, according to the reduction formulae, determined by the restrictions to the mass shell $k_i^2 = \Lambda^2$, $p_i^2 = q_i^2 = m^2$ of $r^a(k; \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$ and $r^a(\bar{k}; \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$ respectively. More generally we include retarded products of F and j fields among the prospective observables and define

$$r^{\mathrm{MS}}(\mathscr{K}, \mathscr{L}, \mathrm{P}, \bar{\mathrm{Q}}) := r^a(\mathscr{K}, \mathscr{L}, \mathrm{P}, \bar{\mathrm{Q}}) \prod_{\mathrm{P}} \delta(p_i^2 - m^2) \prod_{\mathrm{Q}} \delta(q_j^2 - m^2) . \tag{3.21}$$

We demand now minimality of $\mathrm{AD}(r^{\mathrm{MS}})$ *within the class of r-functions admitted by the earlier conditions.* This condition we call the *second minimality condition.* Excluded from the second minimality condition are the 2-point functions $r(p, \bar{q})$ and $r(\bar{p}, q)$, for which r^{MS} does not exist. The functions $r(\mathrm{P}, \bar{\mathrm{Q}})$ with $\mid \mathrm{P} \mid = \mid \mathrm{Q} \mid > 1$ must, however, be included, because they will play an essential role in our later discussion of the consequences of second minimality, and because they are closely related to certain S-matrix elements, which are also observable quantities.

In the following section we shall show in perturbation theory that second minimality determines the r^{MS} uniquely, whilst the off-mass-shell continuation of r^a remains strongly ambiguous. These ambiguities do not, however, influence the physical content of the theory.

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[6] Note that these matrix elements also determine the matrix elements of polynomials, and even more complicated functions, of the smeared fields.

Let us briefly recapitulate our conditions: the theory is determined by a set $\{ r(\mathscr{K}, \mathscr{P}, \mathrm{P}, \bar{\mathrm{Q}}) \}$ of distributions which

i) satisfy the GLZ equations (2.21) and conditions *a*)-*g*) of Section 2,

ii) satisfy the gauge condition (3.11),

iii) taken as Haag coefficients of F or j satisfy, in analogy to (2.29), the IM equation (3.10) and the conservation equation (3.8),

iv) satisfy the normalization condition (3.14),

v) satisfy the second minimality condition.

In order to keep the unphysical parts of r as smooth as possible we can also demand first minimality. For the sake of simplicity we shall do this in the present paper, even though it is not necessary.

For the purposes of the following section it turns out to be convenient to reformulate the theory somewhat by the introduction of a vector potential. We define

$$\mathrm{A}_\mu(p) := -\frac{1}{\Lambda^2}[ip^\nu \mathrm{F}_{\mu\nu}(p) + j_\mu(p)]. \tag{3.22}$$

From the conservation of j and the Maxwell equations (2.4) and (2.10) we derive

$$\mathrm{F}_{\mu\nu}(p) = -ip_\mu \mathrm{A}_\nu(p) + ip_\nu \mathrm{A}_\mu(p), \tag{3.23}$$

$$j_\mu(p) = (p^2 - \Lambda^2)\mathrm{A}_\mu(p), \tag{3.24}$$

$$p^\mu \mathrm{A}_\mu(p) = 0. \tag{3.25}$$

Conversely, if A_μ is a conserved local field, then (3.23) and (3.24) define local fields $\mathrm{F}_{\mu\nu}$ and j_μ satisfying the Maxwell equations and current conservation. Hence knowledge of A is equivalent to knowledge of F and j.

For large times A_μ converges in the LSZ sense towards free vector fields $\mathrm{A}_\mu^{\mathrm{ex}}$ with

$$\begin{aligned} \mathrm{A}_\mu^{\mathrm{ex}}(p) &= -\frac{i}{\Lambda^2} p^\nu \mathrm{F}_{\mu\nu}^{\mathrm{ex}}(p), \\ \mathrm{F}_{\mu\nu}^{\mathrm{ex}}(p) &= -ip_\mu \mathrm{A}_\nu^{\mathrm{ex}}(p) + ip_\nu \mathrm{A}_\mu^{\mathrm{ex}}(p). \end{aligned} \tag{3.26}$$

A^{in} can take over the part of F^{in} in asymptotic completeness: the Fock space of A^{in}, ψ^{in}, $\bar{\psi}^{\mathrm{in}}$, is the $\mathfrak{H}^{\mathrm{in}}$ of our theory.

We can therefore reformulate massive QED as an LSZ theory of the fields A, ψ, $\bar{\psi}$, instead of F, j, ψ, $\bar{\psi}$. This theory is again characterized by its n-amputated functions $r(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$. Unless noted otherwise, script letters: $x, \ldots, p, \ldots, k, \ldots$ in the argument of r-functions will henceforth denote A-variables. Amputation with respect to an A-momentum p means multiplication with $(p^2 - \Lambda^2)$.

The GLZ equations of the new formulation look almost exactly like the former GLZ equations (2.21). The only difference, aside from the reinterpretation of the photon variables as A-variables, is the replacement of the factor $(-2\Lambda^2)^{-\gamma}$ by $(-1)^{-\gamma}$ in the combinatorial coefficient on the right-hand side. Similarly we obtain the new Haag expansions from (2.26) by dropping the factor $(2\Lambda^2)^{-\gamma}$ and replacing $F^{in}(\ell_i)$ by $A^{in}(\ell_i)$. Conditions *a*)-*e*) and the normalizations (2.24) are taken over unchanged, while (2.25) is replaced by

$$r(k, q) = -\frac{1}{2\pi}\delta^4(k + q)(k^2 - \Lambda^2)[N(k) + (k^2 - \Lambda^2)F_3(k)], \quad (3.27)$$

with N the second-rank tensor

$$N_{\mu\nu}(k) = g_{\mu\nu} - \frac{k_\mu k_\nu}{\Lambda^2}. \quad (3.28)$$

The new version of condition *g*) expresses conservation of A:

$$k^\mu r(\Omega\,;\,{}_\mu k) = 0 \qquad \text{for} \qquad k^2 = \Lambda^2, \quad (3.29)$$

$$k^\mu r^a({}_\mu k\,;\,P, \bar{Q}, \mathscr{L}) = 0 \qquad \text{for} \qquad p_i^2 = q_j^2 = m^2,\ \ell_h^2 = \Lambda^2. \quad (3.30)$$

The normalization condition (3.14) becomes in view of (3.24)

$$(\not p - m)r({}_\mu \ell, p, \bar{q})(\not q + m)\,|_{\ell = 0, p_0 = \omega(\boldsymbol{p})}$$
$$= 2(2\pi)^{-\frac{7}{2}} e p_\mu(\not p - m)\delta^4(p + q)\,|_{p\ = \omega(\boldsymbol{p})}. \quad (3.31)$$

The gauge condition (3.11) and the two minimality conditions are taken over unchanged.

In connection with condition (3.29) it is convenient to write the GLZ equations in a slightly more complicated way. By (3.29) we have

$$N_\mu{}^\nu(k)r(\Omega\,;\,{}_\nu k) = r(\Omega\,;\,{}_\mu k) \qquad \text{for} \qquad k^2 = \Lambda^2. \quad (3.32)$$

Therefore we can introduce factors $N(\ell_i)$ into the integrals of the GLZ equations without changing anything. The equations read then (see (2.21) for definitions of the symbols)

$$r(\rho_1, \rho_2, \Omega) \pm r(\rho_2, \rho_1, \Omega)$$
$$= -i\Bigg[\sum_L \varepsilon_L \sum_{\alpha,\beta,\gamma=0}^{\infty} \frac{(2\pi)^{2(\alpha+\beta+\gamma)}\varepsilon_L'^{\alpha+\beta}(-1)^\gamma}{\alpha!\beta!\gamma!} \prod_1^\alpha ds_i\delta_+(s_i)$$
$$\times \prod_1^\beta dt_j\delta_+(t_j) \prod_1^\gamma d\ell_h\delta_+(\ell_h) r(\rho_1, \Omega_L, -S, -\bar{T}, -\mathscr{L}) \prod_1^\alpha (\not s_i^T - m)$$
$$\times \prod_1^\beta (\not t_j + m) \prod_1^\gamma N(\ell_h)\, r(\rho_2, \Omega_R, T, \bar{S}, \mathscr{L}) + \varepsilon_L' \{\rho_1 \leftrightarrow \rho_2\}\Bigg]. \quad (3.33)$$

The reverse of (3.32) holds as follows: let $a_\mu(k) = N_\mu{}^\nu(k)b_\nu(k)$, then $k^\mu a_\mu(k) = 0$ for $k^2 = \Lambda^2$. Using this we can prove

THEOREM 3.1. — *Let* $\{\bar{r}(\Omega)\}$ *be a solution of the* GLZ *equations* (3.33) *satisfying all subsidiary conditions stated before with the possible exception of* (3.29). *Then*

$$r(\mathrm{P}, \bar{\mathrm{Q}}, \mathscr{K}) = \prod_{\mathscr{K}} \mathrm{N}(\not{k}_i)\bar{r}(\mathrm{P}, \bar{\mathrm{Q}}, \mathscr{K}) \tag{3.34}$$

solves (3.33) *and satisfies all subsidiary conditions including* (3.29).

The proof is trivial and will not be given here.

In considering only solutions of the form (3.34) we are not restricting generality: any admissible solution r' is physically equivalent to, i. e. yields the same Haag coefficients for A, ψ, $\bar{\psi}$, as, a solution of the form (3.34), where we can set $\bar{r} = r'$. This is so because these Haag coefficients are not changed by multiplication with $\mathrm{N}(\not{k}_i)$, $\not{k}_i$ any A-variable, due to (3.29) and (3.30). Hence, if using the GLZ equations in the form (3.33), we can ignore condition (3.29), since it can eventually be satisfied with the help of Theorem 3.1. In Section 4 we shall therefore not take (3.29) into account.

4. PERTURBATION THEORY

In this section we solve the model defined in Section 3 in perturbation theory. We expand the quantities of the theory, in particular the retarded functions, into formal power series in the coupling constant e, e. g.

$$r(\Omega) = \sum_{\sigma=0}^{\infty} e^{\sigma} r_{\sigma}(\Omega). \tag{4.1}$$

We will not discuss convergence of this series, but only derive the coefficients r_σ in every finite order.

In zeroth order the fields are free, i. e.

$$\begin{aligned} r_0(p, \bar{q}) &= -\frac{1}{2\pi}\delta^4(p+q)(\not{p} - m), \\ r_0(\bar{p}, q) &= -\frac{1}{2\pi}\delta^4(p+q)(\not{p}^{\mathrm{T}} + m), \\ r_0(\not{k}, \not{q}) &= -\frac{1}{2\pi}\delta^4(\not{k} + \not{q})(\not{k}^2 - \Lambda^2)\mathrm{N}(\not{k}). \end{aligned} \tag{4.2}$$

All other r_0 vanish.

In orders $\sigma \geqslant 1$ we follow exactly the procedure established in *B* for the scalar case, ignoring for the moment the divergence condition (3.8) and the second minimality condition, which have no equivalents in the scalar case. The generalization of the procedure to our model being straightforward we give here only the merest outline. The only point whose generalization is not obvious: the proof that covariance of r can be satisfied, will be dealt with in an appendix.

The expansions (4.1) are substituted into the GLZ equations (3.33) and the terms of a given order σ collected on both sides. This yields equations

$$r_\sigma(\rho_1, \rho_2, \Omega) \pm r_\sigma(\rho_2, \rho_1, \Omega) = \mathrm{I}_\sigma(\rho_1, \rho_2, \Omega), \tag{4.3}$$

where I_σ is, in very abbreviated notation, of the form

$$\mathrm{I}_\sigma(\rho_1, \rho_2, \ldots) = -i\sum_{\tau=1}^{\sigma-1}\left[\ldots\int\ldots r_\tau(\rho_1, \ldots)\ldots r_{\sigma-\tau}(\rho_2, \ldots)\ldots\right]. \tag{4.4}$$

For the meaning of the dotted parts see (3.33). The important feature of this expression is that it does not contain r_σ: I_σ can be computed from the r_τ, $r < \sigma$, by summation. This allows a recursive determination of r_σ. Let r_τ, $\tau < \sigma$, be known and satisfy the subsidiary conditions, possibly with the two exceptions noted above. Then I_σ can be calculated, and r_σ is obtained as solution of the linear functional equation (4.3), where in the course of the solution full account must be taken of the subsidiary conditions. The basic idea behind the solution is that, due to the postulated x-space support of r, we must have

$$r_\sigma(\xi, \eta, \ldots) = \mathrm{I}_\sigma(\xi, \eta, \ldots) \tag{4.5}$$

if $(\xi - \eta) \notin \bar{\mathrm{V}}_-$, $\bar{\mathrm{V}}_-$ the closed backward cone, and

$$r_\sigma(\xi, \eta, \ldots) = 0 \tag{4.6}$$

if $(\xi - \eta) \notin \bar{\mathrm{V}}_+$. This fixes r_σ outside the manifold $\xi = \eta$. The result must be continued onto this manifold in such a way that our conditions are satisfied. That such a continuation exists has been shown in *B*. There we have also discussed to what extent first minimality restricts the possible solutions. The solution is unique if $\mathrm{AD}(\mathrm{I}_\sigma) < -4$, otherwise it is ambiguous, the number of ambiguities increasing with increasing $\mathrm{AD}(\mathrm{I}_\sigma)$. The ambiguous parts of r_σ are solutions of the homogeneous equation

$$h_\sigma(\rho_1, \rho_2, \ldots) \pm h_\sigma(\rho_2, \rho_1, \ldots) = 0 \tag{4.7}$$

and are of the form

$$h_\sigma(\Omega) = \mathfrak{P}(\Omega)\delta^4(\Sigma\omega_i), \tag{4.8}$$

with $\mathfrak{P}$ a polynomial with appropriate symmetry and covariance properties. For this h_σ we have

$$\mathrm{AD}(h_\sigma) = -4 + \deg \mathfrak{P} \tag{4.9}$$

provided that $\mathfrak{P} \not\equiv 0$. deg $\mathfrak{P}$ is the degree of the polynomial $\mathfrak{P}$.

In first order we find $\mathrm{I}_1(\Omega) = 0$ for all Ω, and first minimality implies the vanishing of all r_1 except the 3-point function $r_1(\not{k}, p, \bar{q})$—and the functions connected to it by permutations of the arguments—for which the normalization condition (3.31) must be satisfied. This is achieved by choosing

$$r_1({}_\mu k, p, \bar{q}) = -(2\pi)^{-\frac{7}{2}}\gamma_\mu\delta_4(k + p + q). \tag{4.10}$$

Namely, for $p_0 = \omega(\boldsymbol{p})$ the relation

$$(\not p - m)\gamma_\mu(\not p - m) = -(\not p - m)(\not p + m)\gamma_\mu + (\not p - m)\{\gamma_\mu, \not p\} = 2p_\mu(\not p - m)$$

holds because $(\not p - m)(\not p + m) = p^2 - m^2$ vanishes on the mass shell.

r_1 as given by (4.10) has the asymptotic degree -4, which is the minimal possible value for an expression of the form (4.8). (4.10) satisfies also the as yet ignored divergence condition (3.30), since for $k = -p - q$, $p^2 = q^2 = m^2$ we find

$$k^\mu(\not p + m)\gamma_\mu(-\not q + m) = -(p^2 - m^2)(-\not q + m) - (\not p + m)(q^2 - m^2) = 0.$$

(4.10) is the only ansatz of the form (4.8), with asymptotic degree -4, satisfying both these conditions.

Note that r_1 as given by (4.10) saturates condition (3.31). Hence we have for $\sigma > 1$ the requirement

$$(\not p - m)r_\sigma(k, p, \bar{q})(\not q + m)\,|_{k=0, p_0=\omega(\boldsymbol{p})} = 0. \tag{4.11}$$

We must now turn to the remaining conditions in orders $\sigma > 1$, i. e. conservation of A and second minimality, which are not covered by the results of *B*. In the remainder of this section we prove

THEOREM 4.1. — *In all finite orders σ of perturbation theory there exist distributions $r_\sigma(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$ satisfying all the conditions enumerated in Section* 3, *including condition* (3.30) *and second minimality. The corresponding mass shell restrictions r_σ^{MS} are uniquely determined.*

The proof of this theorem is attained in several steps.

First we introduce, as an auxiliary mathematical construct, a one-parameter family of LSZ theories of vector fields ${}^\delta\mathrm{A}_\mu$ and spinor fields ${}^\delta\psi$, ${}^\delta\bar{\psi}$. δ is a parameter characterizing a particular theory of the family. It lies in the interval $0 \leqslant \delta \leqslant 1$. The couples ${}^\delta\psi$, ${}^\delta\bar{\psi}$ are related by (2.3). The GLZ equations of the δ-theory look exactly like (3.33), except that the kernels $\mathrm{N}(\ell)$ are replaced by

$${}^\delta\mathrm{N}_{\mu\nu}(\ell) := g_{\mu\nu} - \delta\frac{\ell_\mu\ell_\nu}{\Lambda^2}. \tag{4.12}$$

The distributions ${}^\delta r$ satisfy the same reality, symmetry, covariance, and x-space support conditions as the physical r. The 2-point normalizations (2.24) hold for general δ, while the 2-A-function satisfies

$${}^\delta r(k, q) = -\frac{1}{2\pi}\delta^4(k + q)(k^2 - \Lambda^2)[{}^\delta\mathrm{N}(k) + (k^2 - \Lambda^2)\mathrm{F}_3(k)], \tag{4.13}$$

F_3 regular in $k^2 < 4\Lambda^2$.

The free δ-theory, which is also the 0^{th} order in the perturbation expansion, is defined by the δ-independent 2-spinor functions of (4.2) and

$$^{\delta}r_0(\not{p}, \not{q}) = -\frac{1}{2\pi}\delta^4(\not{p} + \not{q})^{\delta}\mathrm{N}(\not{p}). \tag{4.14}$$

The Wightman function corresponding to (4.14) is

$$\langle 0 \,|\, {}^{\delta}\mathrm{A}_{\mu}(\not{p})^{\delta}\mathrm{A}_{\nu}(\not{q}) \,|\, 0 \rangle = -{}^{\delta}\mathrm{N}(\not{p})\delta^4(\not{p} + \not{q})\delta_{+}(\not{p}) \tag{4.15}$$

and is not positive definite for $\delta \neq 1$. This means that the auxiliary theories with $0 \leqslant \delta < 1$ belong to « Hilbert spaces » with indefinite metric. This does not bother us for two reasons. First, these theories have no direct physical relevance, they occur only as convenient tools in the proof of Theorem 4.1, whose explicit formulation makes no reference to them. Second, the perturbation methods of B apply to these theories without any trouble, because they never use the definiteness of the metric. This definiteness is anyway violated in finite orders of perturbation theory.

The normalization condition (3.31) shall hold for all δ.

We solve the auxiliary models again in perturbation theory with the recursive method of B, starting from the δ-independent first-order expression (4.10). In orders $\sigma > 1$ we demand first minimality. Furthermore, we demand that the Ward-Takahashi identities (WT identities)

$$\begin{aligned}
&\not{k}_j{}^{\delta}r_{\sigma}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) \\
&\quad = (2\pi)^{-\frac{5}{2}}\sum_{p_i \in \mathrm{P}}(\not{p}_i - m)(\not{p}_i + \not{k}_j - m)^{-1}\, {}^{\delta}r_{\sigma-1}(\mathscr{K}^j, \mathrm{P}^i, p_i + \not{k}_j, \bar{\mathrm{Q}}) \\
&\quad - (2\pi)^{-\frac{5}{2}}\sum_{q_i \in \mathrm{Q}}{}^{\delta}r_{\sigma-1}(\mathscr{K}^j, \mathrm{P}, \bar{\mathrm{Q}}^i, \overline{q_i + \not{k}_j})(\not{q}_i + \not{k}_j + m)^{-1}(\not{q}_i + m)
\end{aligned} \tag{4.16}$$

be satisfied. Here $\not{k}_j$ is an arbitrary element of $\mathscr{K}$, $\mathscr{K}^j$ is the set obtained from $\mathscr{K}$ by omitting $\not{k}_j$, P^i and Q^i are defined analogously. $p_i + \not{k}_j$ is a ψ-variable taking the place of p_i in the argument and analogously for $q_i + \not{k}_j$. If $\not{k}_j$ is the distinguished variable in the $^{\delta}r_{\sigma}$ on the left-hand side, then $p_i + \not{k}_j$ and $q_i + \not{k}_j$ must in the terms of the right-hand side be permuted through to the distinguished foremost position. This leads possibly to a change of sign according to (2.23). $(\not{p} - m)^{-1}$ stands for

$$(\not{p} + m)(p^2 - m^2 \pm i\varepsilon p_0)^{-1},$$

the upper sign applying if p is the distinguished variable, the lower sign otherwise. The same goes for $(\not{q} + m)^{-1}$. Multiplication of $r(\ldots, p, \ldots, \bar{q}, \ldots)$ with $(\not{p} - m)^{-1}$ or $(\not{q} + m)^{-1}$ de-amputates r in the variables p or q respectively. The product $\not{k}_j r(\ldots)$ is the scalar product $\not{k}_j^{\mu} r(\ldots, {}_{\mu}\not{k}_j, \ldots)$.

$(p_i + k_j)^{-1}\, {}^\delta r_\tau(\ldots, p_i + k_j, \ldots)$ is a distribution in the variables $\mathscr{K}$, P, Q, with obvious symmetry and covariance properties. Moreover, it has the same x-space retardedness as ${}^\delta r_\tau(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$, and its restriction to the mass shell in some or all the variables (p_i and k_j taken as separate variables!) exists in the same way as for the latter function. The proof of this proceeds, mutatis mutandis, like the proof of Theorem B.7.1, which is the theorem establishing existence of the mass shell restrictions of $r_\tau(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$. Note that the de-amputation factor $[(p + k)^2 - m^2]^{-1}$ is regular on the intersection of the mass shells $p^2 = m^2$, $k^2 = \Lambda^2$.

The WT identities imply conservation of ${}^\delta\mathrm{A}_\sigma$ for $\sigma \geqslant 1$. By (4.16) the divergences of the Haag coefficients ${}^\delta r^a_\sigma(k\,; \ldots)$ are sums of terms of the form

$$(p_i^2 - m^2)\,\frac{{}^\delta r_\sigma(p_i + k_j \ldots)}{k^2 - \Lambda^2}.$$

But the quotient in this expression exists on the mass shell $p_i^2 = m^2$, according to the foregoing consideration, hence the complete expression vanishes at $p_i^2 = m^2$.

In 0^{th} order, however, ${}^\delta\mathrm{A}_0$ is not conserved for $\delta \neq 1$, as is seen from (4.15), hence the total field ${}^\delta\mathrm{A}$ is not conserved. But for $\delta = 1$, A_0, and thus $\mathrm{A} = \Sigma e^\sigma \mathrm{A}_\sigma$, are conserved. The $\delta = 1$ case is obviously the physical theory in which we are interested, still ignoring second minimality.

Disregarding the WT identities, we can find ${}^\delta r_\sigma$ with the methods of *B*. We prove now that the WT identities can be satisfied in addition to the other conditions.

LEMMA 4.2. — *There exist solutions ${}^\delta r_\sigma$ of the auxiliary models $0 \leqslant \delta \leqslant 1$ in all orders $\sigma \geqslant 1$ which satisfy the* WT *identities. The corresponding minimal asymptotic degrees* $\mathrm{AD}({}^\delta r_\sigma)$ *are the same as those obtained without taking the* WT *identities into account.*

It is easy to see that the first order ansatz (4.10) satisfies the WT identities. That they can be satisfied in higher orders is proved, as usual, by induction with respect to σ.

We consider a model with a fixed δ, dropping the index δ for convenience. Assume that the WT identities hold in all orders $1 \leqslant \tau < \sigma$. Let $\hat{r}_\sigma$ be a solution with all the desired properties except possibly the WT identities. Define $\Delta^j_\sigma(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$ as the difference of the two sides of equation (4.16), the left-hand side formed with $\hat{r}_\sigma$. Let $\mathrm{I}_\sigma(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$ be the right-hand side of the GLZ equation (4.3). We wish to repeat here our former notational remark, namely that the arguments of I_σ and the other functions considered may stand in any order, not necessarily the one shown here explicitly. In particular, the two distinguished variables of the GLZ equation may be of any type.

With our definitions we obtain

$$\begin{aligned}\Delta^j_\sigma(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) \pm \Delta^j_\sigma(\leftrightarrow \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) &= \not{k}_j \mathrm{I}_\sigma(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) \\ &- (2\pi)^{-\frac{5}{2}} \sum_{\mathrm{P}} (\not{p}_i - m)(\not{p}_i + \not{k}_j - m)^{-1} \mathrm{I}_{\sigma-1}(\mathscr{K}^j, \mathrm{P}^i, p_i + k_j, \bar{\mathrm{Q}}) \\ &+ (2\pi)^{-\frac{5}{2}} \sum_{\mathrm{Q}} \mathrm{I}_{\sigma-1}(\mathscr{K}^j, \mathrm{P}, \bar{\mathrm{Q}}^i, \overline{q_i + k_j})(\not{q}_i + \not{k}_j + m)^{-1}(\not{q}_i + m) \\ &=: \mathrm{D}^j_\sigma(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) . \end{aligned} \tag{4.17}$$

The arrow in the second Δ^j_σ means that the first two variables (the variables ρ_1, ρ_2 of (4.3)) have been exchanged. The variables $p_i + k_j$, $q_i + k_j$ stand in the argument of $\mathrm{I}_{\sigma-1}$ in the same places as p_i and q_i in the I_σ argument. $\mathrm{I}_{\sigma-1}(\ldots, p_i + k_j, \ldots)$ is zero by definition if p_i and k_j happen to be the two distinguished variables ρ_1, ρ_2 of the GLZ equation. $p_i + k_j$ is one of the two distinguished variables in $\mathrm{I}_{\sigma-1}$ if this is true for either p_i or k_j in I_σ.

The induction hypothesis: validity of the WT identities in orders $\tau < \sigma$, implies $\mathrm{D}^j_\sigma = 0$. Namely, consider a typical integral occurring in the definition of I_σ:

$$\int \Pi ds_i \delta_+(s_i) \ldots r_\tau(\mathscr{K}^j_{\mathrm{L}}, k_j, \mathrm{P}_{\mathrm{L}}, \bar{\mathrm{Q}}_{\mathrm{L}}, -\mathrm{S}, -\bar{\mathrm{T}}, -\mathscr{L}) \\ \Pi(\not{s}^{\mathrm{T}}_i - m) \ldots r_{\sigma-\tau}(\ldots) . \tag{4.18}$$

We contract this with k_j and use (4.16) in order τ. This gives, on the one hand, terms of the form

$$r_{\tau-1}(\ldots, -s_i + k_j, \ldots)(\not{s}^{\mathrm{T}}_i - \not{k}^{\mathrm{T}}_j + m)^{-1}(\not{s}^{\mathrm{T}}_i + m) .$$

These vanish on the mass shell after multiplication with $(\not{s}^{\mathrm{T}}_i - m)$, hence do not contribute to (4.18). On the other hand we get terms in which k_j is added to external spinor variables, and these terms just cancel the corresponding terms in the $\mathrm{I}_{\sigma-1}$ parts of D^j_σ.

Thus Δ^j_σ is a solution of the homogeneous equation (4.7) and is of the form (4.8):

$$\Delta^j_\sigma(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) = \delta^4(\Sigma k_i + \ldots)\mathfrak{P}^j_\sigma(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) , \tag{4.19}$$

$\mathfrak{P}^j_\sigma$ being a polynomial with suitable symmetry and covariance properties.

Computing the minimal asymptotic degree of the various terms in Δ^j_σ with the methods of *B*, Chapter VIII (later on we shall do this explicitly for the case $\delta - 0$), one finds that they are all equal. Let G_σ be this common degree. $\mathfrak{P}^j_\sigma$ is then a polynomial of degree $\leqslant \mathrm{G}_\sigma + 4$. This already implies vanishing of Δ^j_σ, i. e. validity of the WT identities, if $\mathrm{G}_\sigma < -4$.

For $\mathrm{G}_\sigma \geqslant -4$ we show that there is a solution $h_\sigma(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$ of the homogeneous GLZ equation (4.7) with asymptotic degree

$$\leqslant \mathrm{AD}(\hat{r}_\sigma) = \mathrm{G}_\sigma - 1 ,$$

such that $k_j h_\sigma = \delta^4(\ldots)\mathfrak{P}^j_\sigma$. $r_\sigma = \hat{r}_\sigma - h_\sigma$ is then the solution whose existence is claimed in Lemma 4.2.

A necessary condition for the existence of h_σ is the vanishing of Δ^j_σ at $k_j = 0$. We show that this is true for both sides of (4.16), hence also for their difference Δ^j_σ.

Since Δ^j_σ is a solution of (4.7) it does not depend on whether k_j is the distinguished variable in r or not. We can therefore assume that k_j is not distinguished. But then the restriction of $\hat{r}_\sigma(\mathscr{K}, \mathrm{P}, \mathrm{Q})$ to the manifold $k_j = 0$ exists as a distribution in the remaining variables. This follows from the analyticity properties of r, as explained e. g. in Epstein's lectures [12]. Let $\Omega = \{ \omega_0, \ldots, \omega_n)$, ω_0 the distinguished variable. Then

$$r(\Omega) = \delta^4(\Sigma\omega_i)r'(\omega_1, \ldots, \omega_n),$$

and r' is, as a consequence of its x-space support, the boundary value of a function analytic in $\mathrm{Im}\ \omega_i \in \mathrm{V}_-$. Moreover, if we keep all the ω_i except ω_n real, the corresponding analytic function of ω_n can be analytically continued into an open neighbourhood of the real space-like points. This neighbourhood contains in particular the origin $\omega_n = 0$, which proves our contention about the existence of $\hat{r}_\sigma$ at this point. But then $k_j \hat{r}_\sigma(\ldots, k_j, \ldots) = 0$ at $k_j = 0$, as desired.

For the contribution of the right-hand side of (4.16) to Δ^j_σ we note that

$$(\not{p}_i - m)(\not{p}_i + \not{k}_j - m)^{-1} r(\ldots, p_i + k_j, \ldots)|_{k_j=0} = r(\ldots, p_i, \ldots),$$

which leads immediately to a total cancellation of the terms in this right-hand side at $k_j = 0$, because of $|\mathrm{P}| = |\mathrm{Q}|$.

This shows that $\mathfrak{P}^j_\sigma = 0$ at $k_j = 0$, i. e. $\mathfrak{P}^j_\sigma$ is of the form $\mathfrak{P}^j_\sigma = k^\mu_j \mathfrak{P}^j_{\sigma,\mu}$, $\mathfrak{P}^j_{\sigma,\mu}$ polynomials of degree (deg $\mathfrak{P}^j_\sigma - 1$). It is not hard to prove by simple algebra that the $\mathfrak{P}^j_{\sigma,\mu}$ can be chosen such that $h_\sigma(\ldots, {}_\mu k_j, \ldots) = \delta^4(\ldots)\mathfrak{P}^j_{\sigma,\mu}$ has the correct symmetries and covariance. For $|\mathscr{K}| > 1$ one can also show that $k_i \mathfrak{P}^j_\sigma = k_j \mathfrak{P}^i_\sigma$, hence $\mathfrak{P}^j_{\sigma,\mu}$ can be chosen to be independent of j, so that h_σ is independent of j, as it must be. The necessary symmetries can be achieved without raising the degree of $\mathfrak{P}^j_{\sigma,\mu}$, so that r_σ will not have a higher asymptotic degree than $\hat{r}_\sigma$.

Note that the 3-point normalization (4.11) and the 2-point normalizations (2.24) fit consistently into the WT identity for $r_\sigma(k, p, \bar{q})$, hence do not prevent its fulfilment. This ends the proof of Lemma 4.2.

Before we can proceed we must introduce a new type of objects: the « modified » retarded products and functions. In a modified retarded product (*mrp* for short) two spinor variables, called the modified variables, are set apart from the others. This is indicated by capping them with inverted carets: $\check{p}$, $\check{\bar{q}}$, ... The definition of the *mrp* is similar to the axiomatic definition of the ordinary retarded products given in *B*, Chapter II. The *mrp* have the same x-space support and the same transformation law under the Poincaré group as the corresponding unmodified products. They also have the same symmetries, with the modified variables forming a class of their own, i. e. the *mrp* is not invariant under exchange of a modified and an unmodified variable. The perturbative version of the identities (2.19) holds for the *mrp* with the following definition of a « commutator » of two retarded products with one modified variable each:

$$
\begin{aligned}
&[\mathrm{R}_\sigma(\,.\,.\,, \hat{p}, \,.\,.\,), \mathrm{R}_\tau(\,.\,.\,, \hat{\bar{q}}, \,.\,.\,)]_\pm \\
&:= \frac{(2\pi)^{-3}}{\Lambda^2} \int d\ell\, \delta_+(\ell) \{ (\not{p} - \not{\ell} - m)^{-1} \mathrm{R}_{\sigma-1}(\,.\,.\,, p - \ell, \,.\,.\,) \\
&\times \mathrm{R}_{\tau-1}(\,.\,.\,, \overline{q+\ell}, \,.\,.\,)(\not{q} + \not{\ell} + m)^{-1} \pm \mathrm{R}_{\tau-1}(\,.\,.\,, \overline{q-\ell}, \,.\,.\,)(\not{q} - \not{\ell} + m)^{-1} \\
&\times (\not{p} + \not{\ell} - m)^{-1} \mathrm{R}_{\sigma-1}(\,.\,.\,, p + \ell, \,.\,.\,) \} . \qquad (4.20)
\end{aligned}
$$

Here $p \pm \ell$ are ψ-variables, $\overline{q \pm \ell}$ $\bar{\psi}$-variables. If the modified variables are both of the same type (both ψ or both $\bar{\psi}$) then the right-hand side of (4.20) is multiplied with -1, and the de-amputation factors $(\ldots)^{-1}$ are, of course, changed accordingly. Note that (4.20) is not a true commutator: the factors $\mathrm{R}_\sigma(\,.\,.\,, \hat{p}, \,.\,.\,)$ and $\mathrm{R}_\tau(\,.\,.\,, \hat{\bar{q}}, \,.\,.\,)$ are not even defined separately. However, the expression (4.20) has all the properties of a (anti)-commutator which are relevant for our purposes.

The modified retarded function (= *mrf*) $r_\sigma(\,.\,.\,, \hat{p}, \,.\,.\,, \hat{\bar{q}}, \,.\,.\,)$ is the vacuum expectation value of $\mathrm{R}_\sigma(\,.\,.\,, \hat{p}, \,.\,.\,, \hat{\bar{q}}, \,.\,.\,)$. The mass shell restrictions of the *mrf* shall exist in the same sense as those of the ordinary r and the reality condition (2.22) shall hold. The WT identities (4.16) shall hold for the *mrf*, with the factors $(\not{p}_i - m)(\not{p}_i + \not{k}_j - m)^{-1}$ and $(\not{q}_i + \not{k}_j + m)^{-1}(\not{q}_i + m)$ replaced by 1 for the modified variables. Finally, the 0^{th} and 1^{st} order contributions to the *mrf* shall vanish:

$$
r_0(\ldots, \hat{p}, \hat{\bar{q}}, \ldots) = r_1(\ldots, \hat{p}, \hat{\bar{q}}, \ldots) = 0 . \qquad (4.21)
$$

The LSZ reduction formulæ and the Haag expansion (2.26) hold for the *mrp*, with *mrf* with the same modified variables as coefficients. Moreover, we can derive GLZ-like equations for the *mrf* from the modified identities (2.19), using the definition (4.20) for the commutators of products with one modified variable each. It is important to note that on the right-hand side of these modified GLZ equations the terms coming from expressions of the type (4.20) contain only ordinary r-functions, no modified ones.

The *mrf* are determined by solving the modified GLZ equations in perturbation theory, with the subsidiary conditions mentioned above, demanding first minimality. The solution follows exactly the by now familiar pattern used already for the determination of the ordinary r. A close step-by-step examination of this procedure, as explained in *B*, shows that it applies with some obvious changes to the modified case. It is clear how the definition (4.20) extends to the generalized retarded products used in

Chapter VI (7) of B. In the adaption of the existence proof of B, Chapter VII, we use that the restriction of $(\not p + \not\ell - m)^{-1} r(\ldots, p + \ell, \ldots)$ to the mass shell $p^2 = m^2, \ell^2 = \Lambda^2$, exists, as has already been mentioned in connection with equation (4.16).

We have noted earlier that the right-hand side of the modified GLZ equation contains terms depending only on the ordinary r-functions. This occurs for the first time in order $\sigma = 2$, hence the modified r_2 do not all vanish in spite of (4.21).

We return to the main stream of the argument of this section. Up to now we have considered models with a fixed δ. Now we proceed to considering δ-dependent families ${}^{\delta}r_{\sigma}$ of solutions.

In a finite order σ we can select a unique solution $\{ {}^{\delta}r_{\sigma} \}$ (including the *mrf*) for fixed δ among the minimal solutions by prescribing a certain finite number of normalization constants. These constants we can, e. g., choose to be the values at the p-space origin of some specified derivatives of certain ${}^{\delta}r_{\sigma}$, after discarding the momentum-conservation δ^4 factor (see B, Chapter VIII, for examples). For $0 < \delta \leqslant 1$ the minimal necessary number of such constants can be shown to be independent of δ. If we choose them as differentiable functions of δ, then the corresponding ${}^{\delta}r_{\sigma}$ will also be differentiable in δ. This fact one can ascertain by examining the explicit construction of ${}^{\delta}r_{\sigma}$ as described in B. For $\delta = 0$ minimality fixes the solution uniquely, as we shall show later on. Hence no additional conditions are necessary in this case. In order to obtain a differentiable minimal family of the type just described in the whole interval $0 \leqslant \delta \leqslant 1$ we choose the normalization constants such that they tend for $\delta \to 0$ towards the values of the corresponding quantities in the minimal ${}^{0}r_{\sigma}$.

We prove:

LEMMA 4.3. — *Any solution ${}^{0}r_{\sigma}$ of the $\delta = 0$ model can be imbedded into a differentiable family ${}^{\delta}r_{\sigma}$ of solutions for $0 \leqslant \delta \leqslant 1$, such that ${}^{\delta}r_{\sigma}^{\mathrm{MS}}$ is independent of δ.*

For the definition of r^{MS} see (3.21).

Lemma 4.3 is again proved by induction with respect to σ. The lemma is true in order $\sigma = 1$, since the only non-vanishing function in this order is the δ-independent 3-point function (4.10).

Assume that the lemma holds in orders $\tau < \sigma$. Let ${}^{\delta}\hat{r}_{\sigma}$ be a differentiable family of solutions such that ${}^{0}\hat{r}_{\sigma} = {}^{0}r_{\sigma}$. Define

$$
\begin{aligned}
{}^{\delta}\mathrm{D}_{\tau}(\ldots) &= \frac{\partial}{\partial\delta}\, {}^{\delta}r_{\tau}(\ldots), \\
{}^{\delta}\hat{\mathrm{D}}_{\sigma}(\ldots) &= \frac{\partial}{\partial\delta}\, {}^{\delta}\hat{r}_{\sigma}(\ldots),
\end{aligned}
\tag{4.22}
$$

(7) The CTP considerations of this chapter get rather involved due to the complicated CTP behaviour of the spinors ψ, $\bar{\psi}$, but this does not create any essential problems.

$\hat{\mathrm{D}}$ has the same symmetry and covariance properties and the same x-space support as $\hat{r}$, and its restriction to the mass shell exists in the same way as for $\hat{r}$. ${}^{\delta}\hat{\mathrm{D}}_\sigma$ satisfies GLZ-like equations obtained from the GLZ equations for ${}^{\delta}\hat{r}_\sigma$ by differentiation with respect to δ. This gives on the left-hand side the usual difference of two $\hat{\mathrm{D}}$ with exchanged variables. On the right-hand side we obtain terms which differ from the original GLZ terms by the substitution of a D_τ for one of the r_τ factors. In addition there are terms coming from the δ-dependence of the kernel factors ${}^{\delta}\mathrm{N}$. Consider the integral shown in (3.33) in order σ, with N replaced by ${}^{\delta}\mathrm{N}$, and see what happens on differentiating ${}^{\delta}\mathrm{N}(\ell_h)$ with respect to δ. Because of the symmetry of the integrand in $\mathscr{L}$, differentiation of each ${}^{\delta}\mathrm{N}(\ell_h)$ gives the same contribution. We can therefore differentiate ${}^{\delta}\mathrm{N}(\ell_\gamma)$ only and multiply the result with γ. This gives

$$-\frac{\gamma}{\Lambda^2}\int \cdots \prod_1^{\gamma} \{ d\ell_h \delta_+(\ell_h) \} \, r_\tau(\ldots, {}_\mu(-\ell_\gamma)) \ldots \prod_1^{\gamma-1} {}^{\delta}\mathrm{N}(\ell_h) \ell_\gamma^\mu \ell_\gamma^\nu r_{\sigma-\tau}(\ldots, {}_\nu\ell_\gamma) \, .$$

For the two factors $\ell_\gamma r_\tau$ and $\ell_\gamma r_{\sigma-\tau}$ we substitute the WT expressions (4.16). The terms in which ℓ_γ is added to an internal variable do not contribute, since they vanish on the corresponding mass shell after multiplication with $(\not{s}_i^{\mathrm{T}} - m)$ or $(\not{t}_j + m)$ respectively. This was already noted after equation (4.19). There remain terms of the form

$$\begin{aligned}
&(2\pi)^{-\frac{5}{2}} \frac{\gamma}{\Lambda^2} (\not{p}_i - m) \\
&\times \int \cdots \prod_1^{\gamma} \{ d\ell_h \delta_+(\ell_h) \} (\not{p}_i - \not{\ell}_\gamma - m)^{-1} \, {}^{\delta}r_{\tau-1}(\ldots, p_i - \ell_\gamma, \ldots) \prod_1^{\gamma-1} {}^{\delta}\mathrm{N}(\ell_h) \\
&\times r_{\sigma-\tau-1}(\ldots, \overline{q_j + \ell_\gamma}, \ldots)(\not{q}_j + \not{\ell}_\gamma + m)^{-1}(\not{q}_j + m)
\end{aligned}$$

and similar terms in which two ψ-variables or two $\bar{\psi}$-variables are concerned. Comparing this with the modified GLZ equations one finds that

$$\begin{aligned}
{}^{\delta}\mathrm{D}_\sigma(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) = & \sum_{p_i, p_j \in \mathrm{P}} (\not{p}_i - m)(\not{p}_j - m) \, {}^{\delta}\hat{r}_\sigma(\mathscr{K}, \mathrm{P}^{\hat{i}\hat{j}}, \bar{\mathrm{Q}}) \\
& + \sum_{q_i, q_j \in \mathrm{Q}} {}^{\delta}\hat{r}_\sigma(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}^{\hat{i}\hat{j}})(\not{q}_i + m)(\not{q}_j + m) \qquad (4.23) \\
& + \sum_{p_i \in \mathrm{P}, q_j \in \mathrm{Q}} (\not{p}_i - m) \, {}^{\delta}\hat{r}_\sigma(\mathscr{K}, \mathrm{P}^{\hat{i}}, \bar{\mathrm{Q}}^{\hat{j}})(\not{q}_j + m)
\end{aligned}$$

solves the same GLZ-like equation as ${}^{\delta}\hat{\mathrm{D}}_\sigma$ and has all its desired linear properties. The notation $\mathrm{P}^{\hat{i}}$, $\mathrm{P}^{\hat{i}\hat{j}}$ means that the variables p_i, or p_i, p_j respectively, are modified.

Since ${}^{\delta}\mathrm{D}_{\sigma}$ and ${}^{\delta}\hat{\mathrm{D}}_{\sigma}$ both solve the same GLZ-like equation, their difference is of the form

$$ {}^{\delta}\mathrm{D}_{\sigma}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) - {}^{\delta}\hat{\mathrm{D}}_{\sigma}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) = \delta^{4}(\ldots)\, {}^{\delta}\mathfrak{P}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}), \tag{4.24} $$

${}^{\delta}\mathfrak{P}$ a polynomial with suitable symmetry and covariance properties. The polynomial

$$ {}^{\delta}\mathfrak{Q}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) = \int_{0}^{\delta} d\varepsilon\, {}^{\varepsilon}\mathfrak{P}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) \tag{4.25} $$

will then also have these properties, and

$$ {}^{\delta}r_{\sigma}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) = {}^{\delta}\hat{r}_{\sigma}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) + \delta^{4}(\ldots)\, {}^{\delta}\mathfrak{Q}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) \tag{4.26} $$

is again a differentiable family of solutions coinciding with the given solution for $\delta = 0$. The δ-derivative of ${}^{\delta}r_{\sigma}$ is ${}^{\delta}\mathrm{D}_{\sigma}$. But from (4.23) we see that ${}^{\delta}\mathrm{D}_{\sigma}$, multiplied with $\Pi(\not{p}_i + m)\Pi(\not{q}_j - m)$ vanishes on the mass shell $p_i^2 = q_j^2 = m^2$, hence ${}^{\delta}r_{\sigma}^{\mathrm{MS}}$ is independent of δ, which proves Lemma 4.3.

We learn from Lemma 4.3 that for each solution of the auxiliary model $\delta = 0$ there is a solution of the physical model $\delta = 1$ such that the physically relevant quantities r_{σ}^{MS} coincide. Conversely, to each solution for $\delta = 1$ there is a $\delta = 0$ solution with the same r^{MS}, as can be proved in the same way. We can therefore use the $\delta = 0$ model for the determination of the physical r^{MS} and can study the consequences of second minimality in this simpler model ([8]). In what follows, r_{σ} stands for ${}^{0}r_{\sigma}$, unless noted otherwise.

Let us determine the asymptotic degree

$$ d(\,|\mathscr{K}|, |\mathrm{P}| = |\mathrm{Q}|\,;\sigma) := \mathrm{AD}(r_{\sigma}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})) \tag{4.27} $$

given by first minimality. We use the results of B, Chapter VIII. From (4.10) we obtain

$$ \begin{aligned} &d(1, 1\,; 1) = -4, \\ &d(\,|\mathscr{K}|, |\mathrm{P}|\,; 1) = -\infty \qquad \text{for} \qquad \{\,|\mathscr{K}|, |\mathrm{P}|\,\} \neq \{1, 1\}. \end{aligned} \tag{4.28} $$

Let r_{σ}^{λ} be constructed in the same way as r_{σ}, starting from the same $r_1^{\lambda} = r_1$, but in a theory with masses m/λ, Λ/λ. Lemma B.8.1 becomes

$$ r_{\sigma}(\lambda\mathscr{K}, \lambda\mathrm{P}, \overline{\lambda\mathrm{Q}}) = \lambda^{-|\mathscr{K}| - 3|\mathrm{P}|} r_{\sigma}^{\lambda}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}). \tag{4.29} $$

By Theorem B.8.2 r_{σ}^{λ} increases for $\lambda \to \infty$ slower than any positive power λ^{ε}, $\varepsilon > 0$ arbitrarily small. In analogy to (B.8.3) we obtain

$$ d(\,|\mathscr{K}|, |\mathrm{P}|\,;\sigma) \leqslant -|\mathscr{K}| - 3|\mathrm{P}|. \tag{4.30} $$

This estimate is relevant only for the r_{σ} that do not vanish identically because $\mathrm{I}_{\sigma} \equiv 0$. For these vanishing functions, and they comprise in any finite order all but finitely many, we have $d = -\infty$.

([8]) As a side remark we note that in the $\delta = 0$ theory the Λ^{-2} terms in N are not present. This is important in view of the fact that eventually we want to go to the limit $\Lambda = 0$.

The bounds (4.30) do not depend on σ: our auxiliary $\delta = 0$ theory is renormalizable in the sense of *B*. $r_\sigma(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$ is uniquely determined if

$$d(|\mathscr{K}|, |\mathrm{P}|; \sigma) < -4. \tag{4.31}$$

This is satisfied, independently of σ, in all cases except

$$\{ |\mathscr{K}|, |\mathrm{P}| = |\mathrm{Q}| \} = \{4, 0\}, \{3, 0\}, \{2, 0\}, \{1, 1\}, \{0, 1\}.$$

The 3-point function $\{3, 0\}$ is one of those vanishing identically in all orders (9), so that (4.30) is not relevant in this case. In the case $\{4, 0\}$ we are at the borderline of onsetting ambiguity. The only possible ambiguity is of the form $h_\sigma(k_1, \ldots, k_4) = c_{\mu_1,\ldots,\mu_4}\delta^4(k_1 + \ldots + k_4)$, with c a totally symmetric constant tensor. The only such tensor at our disposal is const. $(g_{\mu_1\mu_2}g_{\mu_3\mu_4}$ + cyclic in 234). But the resulting h_σ violates the conservation condition (3.30) and is therefore inadmissible. (Note that $r^{\mathrm{MS}}(k_1, \ldots, k_4) = r(k_1, \ldots, k_4)$, so that the 4-point functions of the $\delta = 0$ and the $\delta = 1$ models coincide.) In the case $\{1, 1\}$ we obtain also $d = -4$, which at first implies an ambiguity of the form const. $\delta^4(\ldots)$. But this ambiguity is removed by using conditions (4.11) and, again (3.30). The ambiguities in the 2-point functions $\{2, 0\}$ and $\{0, 1\}$ are removed by conditions (4.13) and (2.24). As a result, we have proved:

LEMMA 4.4. — *The retarded functions r_σ of the $\delta = 0$ model are uniquely determined.*

We turn now to the second minimality condition. First we prove

LEMMA 4.5. — *For the unique first-minimal solution of the $\delta = 0$ model the estimates*

$$\mathrm{AD}(r_\sigma^{\mathrm{MS}}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})) \leqslant -|\mathscr{K}| - 5|\mathrm{P}| \tag{4.32}$$

hold in all orders σ of perturbation theory.

Proof. — From (4.29) we obtain

$$r^{\mathrm{MS}}(\lambda\mathscr{K}, \lambda\mathrm{P}, \overline{\lambda\mathrm{Q}}) = \lambda^{-|\mathscr{K}|-5|\mathrm{P}|} r^{\mathrm{MS}\lambda}(\mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}). \tag{4.33}$$

The mass shell δ-functions in the definition of $r^{\mathrm{MS}\lambda}$ must of course be taken for mass m/λ. Again, $r^{\mathrm{MS}\lambda}$ diverges for $\lambda \to \infty$ slower than any power λ^ε, $\varepsilon > 0$. This is a special case of the generalization of Theorem B.8.2 to our model. Namely, let $\mathrm{P}_\pm$, $\mathrm{Q}_\pm$ be the variables to be restricted to the positive and negative mass shell respectively (10). Let

$$\varphi(\mathscr{K}, \mathrm{P}, \mathrm{Q}) \in \mathscr{S} \subset \mathrm{H}_\varepsilon(\mathscr{K}, \mathrm{P}, \mathrm{Q}),$$

(9) This is the result of a non-trivial cancellation in $\mathrm{I}_\sigma(k_1, k_2, k_3)$ which we will not prove here.

(10) P_- is a set of 4-vectors and should not be confused with the P^- of *B*, which is a set of scalar variables. Apart from that and some obvious renaming of variables the notation is as in *B*.

H_ε the space of Hölder continuous functions defined in B. We first integrate the product $r_\sigma^{MS\lambda}\varphi$ over $\mathscr{K}$ and $\mathbf{P}_+$, $\mathbf{Q}_+$. According to Theorem B.8.2 this yields a function $f_\lambda(\mathrm{P}^-_{+\lambda}, \mathrm{Q}^-_{+\lambda}, \mathrm{P}_-, \mathrm{Q}_-) \in \mathrm{H}_{\varepsilon'}$, $\varepsilon' < \varepsilon$. f_λ diverges for $\lambda \to \infty$ at most like $\lambda^{c_\sigma \varepsilon}$, c_σ some fixed constant. This property is not destroyed by the integration over the remaining variables. The lemma follows then immediately.

Next we want to show that the first-minimal solution is the only solution satisfying (4.32).

LEMMA 4.6. — *Let*

$$h^a(\mathscr{K}, \mathrm{P}, \mathrm{Q}) = \delta^4(\ldots)\mathfrak{P}(\mathscr{K}, \mathrm{P}, \mathrm{Q}), \tag{4.34}$$

$\mathfrak{P}$ *a polynomial,* $\{ |\mathscr{K}|, |\mathrm{P}| = |\mathrm{Q}| \} \neq \{0, 1\}$, *be a solution of the homogeneous* GLZ *equation. Let* $h^{\mathrm{MS}} \not\equiv 0$, i. e. $\mathfrak{P}$ *shall not vanish identically on the electron mass shell* $p_i^2 = q_j^2 = m^2$. *Then*

$$\mathrm{AD}(h^{\mathrm{MS}}) \geqslant -4 - 2|\mathrm{P}|. \tag{4.35}$$

Proof. — Define

$$\bar{\delta}(\mathscr{K}, \mathrm{P}, \mathrm{Q}) := \delta^4(\Sigma k_i + \Sigma p_i + \Sigma q_i)\prod_{\mathrm{P}}\delta(p_i^2 - m^2)\prod_{\mathrm{Q}}(q_i^2 - m^2). \tag{4.36}$$

This is a tempered distribution. Obviously

$$\begin{aligned}\lim_{\lambda \to \infty} \lambda^{4+4|\mathrm{P}|}\bar{\delta}(\lambda\mathscr{K}, \lambda\mathrm{P}, \lambda\mathrm{Q}) &= \delta^4(\Sigma k_i + \ldots)\prod_{\mathrm{P}}\delta(p_i^2)\prod_{\mathrm{Q}}\delta(q_i^2)\\ &=: \bar{\delta}_\infty(\mathscr{K}, \mathrm{P}, \mathrm{Q}).\end{aligned} \tag{4.37}$$

$\bar{\delta}_\infty$ is again a tempered distribution, as can be shown by a slight adaption of the $\sigma = 1$ part of the proof of Theorem B.7.1. The exceptional case $|\mathscr{K}| = 0$, $|\mathrm{P}| = |\mathrm{Q}| = 1$, is of no interest in the present context. We have

$$h^a = \mathfrak{P}\bar{\delta}. \tag{4.38}$$

We write $\bar{\delta}$ as a sum over $2^{|\mathrm{P}|+|\mathrm{Q}|}$ terms by decomposing the mass shell δ's as $\delta(p^2 - m^2) = \delta_+(p) + \delta_-(p)$. Let $\bar{\mathfrak{P}}$ be the restriction of $\mathfrak{P}$ to the support of one of these $\bar{\delta}$-contributions. For $\bar{\mathfrak{P}}$ we can find a suitable representation as follows. First, we substitute

$$(p_{i,0})^{2n} = (\boldsymbol{p}_i^2 + m^2)^n, \qquad (p_{i,0})^{2n+1} = \pm\,\omega(\boldsymbol{p}_i)(\boldsymbol{p}_i^2 + m^2)^n, \tag{4.39}$$

and analogously for powers of $q_{i,0}$. The sign in the second equation accords with the sign in the $\delta_\pm(p_i)$ factor in the $\bar{\delta}$-contribution under consideration. Next, if $|\mathscr{K}| \neq 0$, we substitute for k_1 the sum

$$k_1 = -\sum_2^{|\mathscr{K}|} k_i - \sum_{\mathrm{P}} p_i - \sum_{\mathrm{Q}} q_i, \tag{4.40}$$

obtaining for $\mathfrak{P}$ the representation

$$\mathfrak{P} = \sum_{P',Q'} \prod_{P'} \omega(\boldsymbol{p}_i) \prod_{Q'} \omega(\boldsymbol{q}_j) \mathfrak{P}_{P'Q'}(\mathcal{K}^1, \mathbf{P}, \mathbf{Q}). \tag{4.41}$$

The sum extends over all subsets P′, Q′, of P, Q, the empty and full sets included. The $\mathfrak{P}_{P'Q'}$ are polynomials. $\mathcal{K}^1$ is the set $\{ \boldsymbol{k}_2, \ldots, \boldsymbol{k}_{|\mathcal{K}|} \}$.

In the case $|\mathcal{K}| = 0$, $|P| = |Q| > 1$ we substitute

$$\begin{aligned} \boldsymbol{p}_1 &= -\sum_{2}^{|P|} \boldsymbol{p}_i - \sum_{Q} \boldsymbol{q}_j, \\ \pm\, \omega(\boldsymbol{p}_1) &= -\sum_{2}^{|P|} \pm\, \omega(\boldsymbol{p}_i) - \sum_{Q} \pm\, \omega(\boldsymbol{q}_j). \end{aligned} \tag{4.42}$$

The second of these equations does not yet exhaust energy conservation, because of the elimination of $\omega(\boldsymbol{p}_i)^2$ in favour of $\boldsymbol{p}_i^2$ used in (4.39). We take account of this by solving the equation

$$\boldsymbol{p}_1^2 + m^2 = \left[\sum_{2}^{|P|} \pm\, \omega(\boldsymbol{p}_i) + \sum_{Q} \pm\, \omega(\boldsymbol{q}_j) \right]^2 \tag{4.43}$$

for the product $\omega(\boldsymbol{q}_1)\omega(\boldsymbol{q}_2)$ and substituting the resulting expression wherever the said product occurs in $\bar{\mathfrak{P}}$. This yields

$$\bar{\mathfrak{P}} = \sum_{P'',Q''} \prod_{P''} \omega(\boldsymbol{p}_i) \prod_{Q''} \omega(\boldsymbol{q}_j) \mathfrak{P}_{P''Q''}(\mathbf{P}^1, \mathbf{Q}). \tag{4.44}$$

The sum extends over all subsets $P'' \subset P^1 = \{ p_2, \ldots, p_{|P|} \}$ and all subsets $Q'' \subset Q$ which do not contain both q_1 and q_2. $\mathfrak{P}_{P''Q''}$ are polynomials.

The representation (4.41) and (4.44) respectively are unique: for any $\bar{\mathfrak{P}}$ there exists exactly one such representation, and $\bar{\mathfrak{P}} \equiv 0$ if and only if all the $\mathfrak{P}_{P'Q'}$ or $\mathfrak{P}_{P''Q''}$ vanish.

Let us, for the moment, treat the ω's as independent variables, i. e. consider $\bar{\mathfrak{P}}$—written in the form (4.41) or (4.44)—as polynomial in the variables $\mathcal{K}$, $\mathbf{P}$, $\mathbf{Q}$, $\omega(\boldsymbol{p}_i)$, $\omega(\boldsymbol{q}_j)$. We decompose this polynomial into its homogeneous parts. Let $\mathfrak{P}^N$ be such a part, i. e. a form of degree N with a representation (4.41) or (4.44). Taking the $\boldsymbol{p}$-dependence of $\omega(\boldsymbol{p})$ again into account, we obtain for $|\mathcal{K}| \neq 0$

$$\begin{aligned} \lim_{\lambda \to \infty} \lambda^{-N} \mathfrak{P}^N(\lambda\mathcal{K}, \lambda P, \lambda Q) &= \left[\sum_{P',Q'} \prod_{P'} |\boldsymbol{p}_i| \prod_{Q'} |\boldsymbol{q}_j|\, \mathfrak{P}_{P'Q'}(\mathcal{K}^1, \mathbf{P}, \mathbf{Q}) \right]_N \\ &=: \mathfrak{P}^N_\infty(\mathcal{K}, P, Q) \end{aligned} \tag{4.45}$$

and an analogous result for $|\mathscr{K}| = 0$. Here $[\ldots]_N$ denotes the homogeneous component of degree N in $[\ldots]$. The right-hand side of (4.45) is the special case $m = 0$ of the representation (4.41), so that $\mathfrak{P}^N_\infty \neq 0$ if and only if at least one $\mathfrak{P}_{P'Q'} \neq 0$.

If N is the maximal degree occurring in $\mathfrak{P}$, we obtain

$$\lim_{\lambda\to\infty} \lambda^{4+4|P|-N} h^{MS}(\lambda\mathscr{K}, \lambda P, \lambda Q) = \mathfrak{P}^N_\infty(\mathscr{K}, P, Q)\bar{\delta}_\infty(\mathscr{K}, P, Q). \quad (4.46)$$

Under our assumption $h^{MS} \not\equiv 0$ at least one $\mathfrak{P}_{P'Q'}$ in at least one $\bar{\delta}$-component does not vanish identically, so that the right-hand side of (4.46) is different from zero. We note furthermore that r^{MS} is formed with the amputated functions r^a of (2.20), so that any admissible homogeneous addition h^{MS}_σ must contain the factor $\Pi_P(\not{p}_i + m)\Pi_Q(\not{q}_i - m)$, i. e. satisfy the Dirac equations $(\not{p}_i - m)h^{MS}_\sigma = h^{MS}_\sigma(\not{q}_j + m) = 0$ for all i, j. This implies that the limit polynomial $\mathfrak{P}^N_\infty$ satisfies the corresponding massless Dirac equations, and this in its turn implies the presence of the factors $\Pi_P \not{p}_i$ and $\Pi_Q \not{q}_j$ in $\mathfrak{P}^N_\infty$, so that $|N| \geqslant |P| + |Q| = 2|P|$. (4.46) gives then the result claimed in Lemma 4.6.

Lemma 4.5 and Lemma 4.6 permit us now to conclude that the unique solution with first minimality of Lemma 4.4 also satisfies second minimality, and that any other solution with second minimality yields the same r^{MS}_σ. For, according to (4.35) and (4.32), the addition of a homogeneous solution to the r_σ of Lemma 4.4 increases $AD(r^{MS}_\sigma)$ unless $|\mathscr{K}| + 3|P| \leqslant 4$. This leaves exactly the same candidates for ambiguities as in the discussion of first minimality (see the paragraph following (4.31)), and the possible ambiguities are discarded by the same arguments as used there.

Because of Lemma 4.3 these results carry over immediately to the $\delta = 1$ model: the second-minimal r^{MS}_σ for $\delta = 1$ are the same as those for $\delta = 0$, hence uniquely determined. But the $\delta = 1$ model is exactly the physical model defined in Section 3, except that in Section 3 we did not demand the WT identities. However, the WT identities have only been used to obtain the estimate (4.32) for the second-minimal r^{MS}_σ. Lemma 4.6 and the unicity proof for r^{MS}_σ derived from it do not use the WT identities, i. e. our minimal solution $\{ r^{MS}_\sigma \}$ is unique within the wider conditions of Section 3, without assuming the WT identities. This completes the proof of Theorem 4.1.

5. LOCAL OBSERVABLE FIELDS

The only observables considered until now are the fields $F_{\mu\nu}$ and j_μ. In this section we want to discuss briefly some more general observables. We understand the term « observable » in the general sense of including the local operations of Haag-Kastler [13]. Of course, we are dealing not with an abstract algebra of observables, but with an algebra of operators in our Hilbert space $\mathfrak{H}$. Also, we do not insist on these operators being bounded.

We assume that all observables are functions in some sense or other of « observable fields », or O-fields for short. An O-field is a Wightman field $B(\xi)$ in $\mathfrak{H}$ which is local relative to the fundamental fields F, ψ, $\bar{\psi}$:

$$[B(\xi), F(x)] = [B(\xi), \psi(x)] = [B(\xi), \bar{\psi}(x)] = 0 \qquad \text{for} \qquad (x-\xi)^2 < 0\,, \tag{5.1}$$

and which commutes with the charge Q:

$$[Q, B(\xi)] = 0\,. \tag{5.2}$$

Under the Lorentz group B shall transform according to a (one-valued) tensor representation.

Because of the general notion of « observable » that we are using B need not be self-adjoint. However, $B^*(\xi)$ is an O-field if $B(\xi)$ is one.

According to a result of Borchers's [*14*] (5.1) implies that all O-fields are relatively local among themselves. In particular, any O-field commutes at space-like distances with the special O-fields j_μ and A_μ, so that we can use the formulation of QED in terms of A instead of F that we have introduced in Section 3.

The O-field B is completely fixed by its Haag coefficients, hence by the retarded functions of one B-variable and any number of ψ, $\bar{\psi}$, and A-variables. B-variables in the argument of a retarded function will be denoted by a barred greek letter: $\bar{\xi}, \ldots, \bar{\rho}, \ldots$. In the Haag expansion of B enter only the amputated functions r^a with all A, ψ, $\bar{\psi}$, variables on the mass shell. r^a and r^n are defined as in (2.20). The B-variable is never amputated and need therefore not be separated from the rest by a semi-colon. The index n in r^n will again be dropped, r henceforth standing for r^n.

The r-functions with one B-variable $\bar{\rho}$ satisfy GLZ-equations of the form (3.33), $\bar{\rho}$ being one of the external variables. On the right only one of the r-factors contains $\bar{\rho}$, so that the GLZ-equations are linear in the as yet undetermined functions $r(\ldots, \bar{\rho}, \ldots)$. These functions also satisfy all the familiar linear subsidiary conditions. Of course, no restriction to a mass shell in ρ need exist, since B is in general not associated with any particle, so that a mass shell is not defined. Because of (5.2) the $r(\ldots, \bar{\rho}, \ldots)$ with unequal numbers of ψ and $\bar{\psi}$ variables vanish.

We develop $r(\ldots, \bar{\rho}, \ldots)$ into a perturbative expansion (4.1) and determine its coefficients r_σ recursively by solving the GLZ equations, using the $r_\tau(\mathscr{K}, P, \bar{Q})$ from Section 4. The right-hand side of these GLZ equations vanishes in 0^{th} order, hence $r_0(\bar{\rho}, \mathscr{K}, P, \bar{Q})$ is of the form (4.8). Inserting this into the Haag expansion and transforming into x-space we find that $B_0(\xi)$ is a sum of local Wick products of the free fields A_0, ψ_0, $\bar{\psi}_0$, and their derivatives, i. e. of terms of the form

$$:D_1\psi_0(\xi) \ldots D_\alpha\psi_0(\xi)D_{\alpha+1}\bar{\psi}_0(\xi) \ldots D_\beta\bar{\psi}_0(\xi)D_{\beta+1}A_0(\xi) \ldots D_\gamma A_0(\xi):, \tag{5.3}$$

with $\beta = 2\alpha$. The D_i are derivatives of arbitrary orders. If we demand that our fields are tempered, i. e. that their Haag coefficients r^{MS} are tempered distributions, then B_0 is a sum of at most finitely many terms (5.3) [15]. Because of the linearity of the problem we can assume without restricting generality that B_0 is the monomial (5.3).

In higher orders $\sigma > 0$ we demand second minimality, i. e. minimality of $AD(r^{MS}(\bar{p}, \mathscr{K}, P, \bar{Q}))$. The resulting expression for the O-field B can be considered a generalization to interacting fields of the Wick product (5.3) and is called the normal product $N(D_1\psi(\xi) \dots D_\gamma A(\xi))$. Our procedure of determining this normal product is a transposition of Zimmermann's canonical method [16] to our formalism.

The vacuum expectation value of an O-field need not vanish, i. e. we can have $\gamma = 0$ in (5.3). This means $B_0(\xi) = c$, a constant multiple of the identity. Since the 1-point function $r(\bar{p})$ does not enter the GLZ equations, our minimality requirement implies then vanishing of all $r_\sigma(\bar{p}, \dots)$ with $\sigma > 0$, so that $B(\xi) = B_0(\xi) = c$. In what follows we consider only the non-trivial case $\gamma > 0$.

The determination of $r_\sigma(\dots, p, \dots)$ follows exactly the recursive method used for $r_\sigma(\mathscr{K}, P, \bar{Q})$. The constructions and existence proofs of *B* are immediately applicable. The methods used in Section 4 for the discussion of second minimality are also easily extended to the present case. We introduce again the auxiliary theories with $0 \leqslant \delta \leqslant 1$ and consider solutions satisfying WT identities obtained from (4.16) by inserting an additional variable $\bar{p}$ into all the terms. That the WT identities can be satisfied in all orders is proved in the same way as in Section 4. This induction proof holds, provided that the WT identities hold at the start of the induction. This is here the order $\sigma = 0$, not $\sigma = 1$ like in Section 4.

We start from the form (5.3) that B_0 takes in the physical theory $\delta = 1$. The retarded functions $r_0(\bar{p}, \dots)$ of this field can be calculated explicitly, taking due account of the ambiguities inherent in the definition of r. Only the $r_0(\bar{p}, \mathscr{K}, P, \bar{Q})$ with $|P| = |Q| = \alpha = \beta - \alpha$, $|\mathscr{K}| = \gamma - \beta$ is different from zero. A possible form of this non-trivial r_0 is

$$r_0(\bar{p}, \mathscr{K}, P, \bar{Q})$$
$$= \delta^4(p + \dots)(-2\pi)^{-\gamma} \prod_{1}^{\alpha} \tilde{D}_i(p_i) \prod_{\alpha+1}^{\beta} \tilde{D}_i(q_i) \prod_{\beta+1}^{\gamma} \tilde{D}_i(k_i) \sum_{P(\sigma_1,\dots,\sigma_\alpha)} \prod_{i=1}^{\alpha} \delta_{\sigma_{j_i}\tau_i}$$
$$\times \sum_{P(\sigma'_{\alpha+1},\dots,\sigma'_\beta)} \prod_{i=\alpha+1}^{\beta} \delta_{\sigma'_{j_i}\tau'_i} \sum_{P(\mu_{\beta+1},\dots,\mu_\gamma)} \prod_{i=\beta+1}^{\gamma} \bar{N}_{\mu_{j_i}\nu_i}(k_i), \tag{5.4}$$

with

$$\bar{N}_{\mu\nu}(k) := \Lambda^{-2}(k^2 g_{\mu\nu} - k_\mu k_\nu). \tag{5.5}$$

The $\tilde{D}_i$ are polynomials obtained as Fourier transforms of the derivations D_i. The variables are $\mathscr{K} = \{ {}_{\nu_i}k_i \}$, $P = \{ {}_{\tau_i}p_i \}$, $Q = \{ {}_{\tau'_i}q_i \}$. The indices μ_i, σ_i, σ'_i, belong to the fields in (5.3), i. e. they are associated with ρ. The sums run over all permutations of $(\sigma_1, \ldots, \sigma_\alpha)$, etc.

It is easy to verify that (5.4) has all the desired properties and, if substituted into the Haag expansion, reproduces (5.3): this r_0 is physically equivalent to any solution of the $\sigma = 0$ GLZ equation leading to (5.3). But, because of $\ell^\nu \bar{N}_{\mu\nu}(\ell) = 0$, (5.4) satisfies the WT identity

$$\ell_j r_0(\bar{\rho}, \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}}) = 0\,, \tag{5.6}$$

when we define $r_{-1} := 0$. The ${}^\delta r_0$ of the auxiliary theories are set equal to r_0 by definition, so that they, too, satisfy the WT identity.

With the methods of Section 4 we find then that the minimal Haag coefficients $r_\sigma^{\mathrm{MS}}(\bar{\rho}, \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$ of the physical theory $\delta = 1$ are the same as those of the unphysical theory $\delta = 0$. Therefore we can use this simpler theory for the discussion of the ambiguities in r_σ^{MS}.

For the $\delta = 0$ theory second minimality is again a consequence of first minimality, hence we use this simpler condition. Let the polynomial multiplying $\delta^4(\ldots)$ in (5.4) be homogeneous of degree N. Then

$$r_0(\overline{\lambda\rho}, \lambda\mathscr{K}, \lambda\mathrm{P}, \overline{\lambda\mathrm{Q}}) = \lambda^{\mathrm{N}-4} r_0(\bar{\rho}, \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})\,. \tag{5.7}$$

Let a be the number of A-factors, b the number of ψ-factors in (5.3). We introduce again the functions r_σ^λ belonging to a theory with masses m/λ, Λ/λ, and starting with $r_0^\lambda = r_0$. The factor Λ^{-2} of (5.5) is *not* scaled. We find in analogy to (4.29)

$${}^0 r_\sigma(\overline{\lambda\rho}, \lambda\mathscr{K}, \lambda\mathrm{P}, \overline{\lambda\mathrm{Q}}) = \lambda^{-|\mathscr{K}|-3|\mathrm{P}|+\mathrm{N}+a+3b-4}\, {}^0 r_\sigma^\lambda(\bar{\rho}, \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})\,, \tag{5.8}$$

hence

$$\mathrm{AD}({}^0 r_\sigma(\bar{\rho}, \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})) \leqslant -|\mathscr{K}| - 3|\mathrm{P}| + \mathrm{N} + a + 3b - 4\,. \tag{5.9}$$

As was the case for (4.30), this bound is independent of σ. This means that the possible ambiguities in B_σ are of the same form in all orders. Moreover, the number of ambiguities is finite, because the unicity condition

$$\mathrm{AD}({}^0 r_\sigma) < -4 \tag{5.10}$$

is satisfied for all but a finite number of $\{|\mathscr{K}|, |\mathrm{P}|\}$-combinations. The larger the number γ of factors in (5.3) and the higher the order of the derivatives D_i, the larger is the number of ambiguities in the definition of B.

We conclude the section with two important remarks.

1) Consider the O-field characterized by the 0^{th} order

$$\mathrm{B}_0^\mu(x) = :\bar{\psi}(x)\gamma^\mu\psi(x):\,. \tag{5.11}$$

We have $\mathrm{N} = 0$, $a = 0$, $b = 1$, and from (5.9) and (5.10) we obtain

$$|\mathscr{K}| + 3|\mathrm{P}| > 3 \tag{5.12}$$

as unicity condition for $r_\sigma(\bar{\rho}, \mathscr{K}, \mathrm{P}, \bar{\mathrm{Q}})$. It is easy to see by comparing the respective lowest order Haag coefficients that

$$\mathrm{B}_0^\mu(x) = j_1^\mu(x)\,, \tag{5.13}$$

j^μ the electromagnetic current. A close scrutiny of our construction in higher orders shows that

$$\mathrm{B}_\sigma(x) = j_{\sigma+1}(x) \tag{5.14}$$

is a solution for B_σ in all orders. This particular solution is singled out from the set of all possible solutions by demanding:

a)
$$r_\sigma(\bar{\rho}, \not{k}) = 0 \qquad \text{at} \qquad \not{k}^2 = \Lambda^2, \tag{5.15}$$

b) $r_\sigma(\bar{\rho}, p, \bar{q})$ satisfies for $\sigma > 0$ the normalization condition (4.11) with ρ replacing $\not{k}$,

c) $r_\sigma(\bar{\rho}, \not{k}_1, \not{k}_2, \not{k}_3)$ and $r_\sigma(\bar{\rho}, p, \bar{q})$ satisfy the conservation conditions

$$\begin{aligned} \rho^\mu r_\sigma({}_\mu\bar{\rho}, \not{k}_1, \not{k}_2, \not{k}_3) &= 0 \qquad \text{at} \qquad \not{k}_i^2 = \Lambda^2 \\ \rho^\mu r_\sigma({}_\mu\bar{\rho}, p, \bar{q}) &= 0 \qquad \text{at} \qquad p^2 = q^2 = m^2. \end{aligned} \tag{5.16}$$

Under these additional conditions we find $j^\mu = e\mathrm{B}^\mu$, and in this sense the canonical expression

$$j_\mu(x) = e\mathrm{N}(\bar{\psi}(x)\gamma_\mu\psi(x)) \tag{5.17}$$

is true in our formalism.

2) We have demanded validity of the WT identities for $r(\bar{\rho}, \ldots)$. In contrast to the situation of Section 4 this is here a genuine restriction of ambiguity. Therefore the question arises how the WT identities can be motivated. In dealing with this question it is important to remember condition (3.29), which has been ignored up to now in view of Theorem 3.1. The $r_\sigma(\bar{\rho}, \ldots)$ as calculated above do not yet give the correct Haag coefficients r^a, they must first be multiplied with the factors $\mathrm{N}(\not{k}_i)$ in all A-variables. This has been done in the zero-order ansatz (5.4) but not in higher orders. Let r^c be the correct r-function obtained by multiplication of r with $\Pi\mathrm{N}$. Now we take into account that eventually we want to pass to the limit $\Lambda = 0$. In any attempt to attain this limit the Λ^{-2} terms in the factors $\mathrm{N}(\not{k}_i)$ of r^c are bound to create problems unless they can be shown not to contribute to the physically relevant quantities. This is the case if $\not{k}_j r_\sigma(\bar{\rho}, \ldots, \not{k}_j, \ldots)$ vanishes on the mass shell, and this is true if the WT identities are satisfied for r_σ. Hence the restricted set of O-fields singled out by the WT identities has a much better chance of existing in the limit $\Lambda = 0$ than the other O-fields.

We have already mentioned that the necessity of going from r to r^c has been allowed for in the zero-order expression (5.4). As a result we find there the factors $\bar{\mathrm{N}}$ containing the denominators Λ^{-2}, so that again we can expect serious problems if we wish to go to the limit $\Lambda = 0$. These problems are avoided by narrowing the set of possible O-fields by admitting for B_0

only expressions of the form (5.3) in which fields $F_{\alpha\beta,0}$ replace the potentials $A_{\mu,0}$. The factors $\bar{N}_{\mu\nu}(k)$ of (5.4) are then replaced by

$$\bar{M}_{\alpha\beta\nu}(k) = k_\alpha g_{\beta\nu} - k_\beta g_{\alpha\nu}, \tag{5.18}$$

and this exists in the limit $\Lambda = 0$. Only for this restricted class of O-fields can we hope to prove existence, in some appropriate sense, of the limit $\Lambda \to 0$. In canonical language, the O-fields singled out by the foregoing considerations are those that are invariant under gauge transformations of the second kind.

APPENDIX

COVARIANCE

The methods used in *B*, Chapter IV, to prove the existence of invariant solutions of the GLZ equations do not generalize easily to the more complicated, covariant instead of invariant, situation met with in QED. These methods were also unnecessarily complicated. We shall therefore describe here an alternative method with a larger scope.

We consider the GLZ equation

$$r(\Omega) \pm r(\leftrightarrow \Omega) = \mathrm{I}(\Omega) \tag{A.1}$$

where I is known. The double arrow in the second r signifies exchange of the first two variables. We assume that I transforms under the proper Lorentz group (11) $L_+^\uparrow$ according to a certain finite dimensional representation, the product of the representations associated with the fields occurring in I. We wish to prove existence of a solution r of (A.1) transforming under $L_+^\uparrow$ with the same representation as I. This representation, being a direct product, is in general not irreducible. It can be decomposed, however, into irreducible components, and we look for an r containing the same irreducible representations. The number of components in I is, of course, finite.

Let $r'(\Omega)$ be a solution of (A.1) satisfying all the subsidiary conditions except possibly the covariance condition just discussed. According to a result by Bros *et al.* [17] we can decompose r' into a finite sum of terms, each of which transforms under some irreducible representation. Moreover, each of these components has the same x-space support as r' itself: it is retarded.

We can find the irreducible components of r' explicitly as follows. Let

$$M_\mu^\nu = \sum_{\omega_i \in \Omega} \left(\omega_{i,\mu}\partial_i^\nu - \omega_i^\nu \partial_{i,\mu}\right), \tag{A.2}$$

$\partial_{i,\mu} = \dfrac{\partial}{\partial \omega_i^\mu}$, be the infinitesimal generators of $L_+^\uparrow$. Then

$$C_1 = M_\alpha^\beta M_\beta^\alpha, \qquad C_2 = M_\alpha^\beta M_\beta^\gamma M_\gamma^\alpha \tag{A.3}$$

are two independent Casimir operators. An irreducible representation is completely specified by the corresponding eigenvalues $c_{1,2}$ of $C_{1,2}$.

The part of r' transforming under the representation (c_1, c_2) is given by

$$r^{(c_1,c_2)}(\Omega) = \prod_{c_1^i \neq c_1} \frac{C_1 - c_1^i}{c_1 - c_1^i} \prod_{c_2^i \neq c_2} \frac{C_2 - c_2^i}{c_2 - c_2^i} r'(\Omega). \tag{A.4}$$

Here the two products extend over all those eigenvalues $c_\nu^i \neq c_\nu$ which belong to at least one of the finitely many representations in r'.

The C_ν are totally symmetric under permutations of the ω_i. In x-space M_μ^ν takes exactly the same form (A.2) as in p-space, hence the C_ν are local operators. This means that $r^{(c_1,c_2)}$ has the same symmetries and the same x-space support as r'.

The components $r^{(c_1,c_2)}$ belonging to representations not present in I solve the homogeneous equation $r \pm r_\leftrightarrow = 0$, as is seen by applying the projection (A.4) to (A.1). Sub-

(11) The discrete symmetry P can easily be dealt with by symmetrization of the solution and is therefore not considered here.

tracting them from r' results then in a new solution of (A. 1) which now satisfies all the requirements, including covariance.

Application of M_{α}^{β}, and therefore of C_{ν}, to the distribution r' does not increase its asymptotic degree, because multiplication by $\omega_{i,\mu}$ raises the asymptotic degree by at most 1, while differentiation with respect to $\omega_{j,\nu}$ lowers it by at least 1. Hence the new solution r has at most the same asymptotic degree as r': it is minimal if r' is.

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