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Spherically symmetric charged dust distributions in general relativity

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ABSTRACT. — The paper presents a discussion of the dynamics of charged dust distributions with spherical symmetry. The general behaviour is quite complicated and while for $\rho/|\sigma|$ a constant greater than unity there is a collapse of the spatial volume, the circumferential area may show an oscillatory behaviour (ρ and σ indicate the matter and charge densities respectively). Again, for $\rho/|\sigma|$ a constant less than unity, the spatial volume would not in general collapse but the circumferential area may do so. However there cannot be any regular oscillation if ρ/σ be constant. Some static and nonstatic solutions are displayed but excepting a class of static solutions, all these have a singularity at the center of symmetry. A static solution, fitted on one side with a Schwarzschild field and on the other side with the Reissner-Nordstrom metric, shows that a star may have around it a shell of charged dust in equilibrium and the charge mass ratio may have arbitrarily large values for the shell.

I. — INTRODUCTION

The problem of a charged matter distribution in general relativity is interesting for more reasons than one. While on one hand there is the classical problem of the electron, on the other hand one may wonder whether this case, where there is a « repulsive field of force » as also an anisotropic stress, may show some interesting aspects in gravitational collapse. The investigations of Bardeen [*1*], de la Cruz and Israel [*2*] as also of Novikov [*3*] were apparently aimed principally at a study of the temporal behaviour of the Schwarzschild radial coordinate (whose square determines the circumferential area of the spherical shells) and the charge distribution was assumed to embedded in an outside empty space (*i. e.*, the space of Reissner-Nordstrom metric). These authors reached the conclusion that under some circumstances a bounce and re-expansion occurs but into a universe different from that in which the collapse initiated. There might also be regular oscillations between two finite extrema of the Schwarzschild radial coordinate. Again Bekenstein [*4*] was able to show that a bounce cannot occur in the region between the two singularities of the Reissner-Nordstrom metric. However the situation regarding any possible collapse in the radial direction (as distinct from circumferential collapse) seems not to have been properly investigated, although it is known that in the case of anisotropic collapse, even for uncharged dust spheres, the radial and the circumferential dimensions have quite different temporal behaviours and naked singularities of novel type may appear [*5*], [*6*], [*7*]. Recently Misra and Srivastava [*8*] and Vickers [*9*] have independently shown that one can obtain first integrals of the Einstein Maxwell equations for a charged dust sphere, the integrals involving a number of arbitrary functions of the radial coordinate. In the present paper we present these first integrals and then investigate the condition of the regularity of the field at the centre. It turns out that this sets some restrictions on the arbitrary functions and it is found that at the centre one gets an Oppenheimer-Snyder type of collapse to a state of infinite density if $\rho^2/\sigma^2 \geqslant 1$, while if $\rho^2/\sigma^2 < 1$, the collapse is halted and the system after contracting to a minimum expands indefinitely.

However this behaviour at the centre is by no means typical of what happens elsewhere. There the circumferential area may show oscillations between two finite values or expansion from a minimum to arbitrary large values (or the time reversed contraction) as noted by previous authors. However we find that the cases of circumferential oscillation are invariably associated with a radial collapse if σ/ρ be constant and there is a consequent evolution of singularity with infinite values of charge and matter densities. Unlike the investigations of previous authors, these conclusions are obtained independently of any boundary conditions (as, for example, would be imposed by the continuity with an « electrovac » universe outside) and would thus hold also for charged universes of the type as was at one time considered by Bondi and Lyttleton [*10*].

II. — FIELD EQUATIONS AND THEIR FIRST INTEGRALS

With the spherically symmetric line element

$$ds^2 = - e^{\lambda}dr^2 - e^{\omega}d\Omega^2 + e^{\nu}dt^2 \tag{1}$$

one can assume the coordinate system to be comoving. In the above form λ, ω and ν are functions of r and t. The Einstein-Maxwell equations for a charged dust are, in this case

$$- E^2 = - e^{-\omega} + \frac{1}{2} e^{-\lambda}\left(\omega'\nu' + \frac{\omega'^2}{2}\right) + e^{-\nu}\left(- \frac{3}{4}\dot{\omega}^2 + \frac{1}{2}\dot{\omega}\dot{\nu} - \ddot{\omega}\right) \tag{2}$$

$$\begin{aligned} + E^2 = e^{-\lambda}&\left(\frac{1}{2}\omega'' + \frac{1}{4}\omega'^2 + \frac{1}{2}\nu'' + \frac{1}{4}\nu'^2 - \frac{1}{4}\omega'\lambda' + \frac{1}{4}\omega'\nu' - \frac{1}{4}\nu'\lambda'\right) \\ &+ e^{-\nu}\left(- \frac{1}{2}\ddot{\lambda} - \frac{1}{4}\dot{\lambda}^2 - \frac{1}{2}\ddot{\omega} - \frac{1}{4}\dot{\omega}^2 + \frac{1}{4}\dot{\lambda}\dot{\nu} + \frac{1}{4}\dot{\omega}\dot{\nu} - \frac{1}{4}\dot{\omega}\dot{\lambda}\right) \end{aligned} \tag{3}$$

$$- 8\pi\rho - E^2 = - e^{-\omega} + e^{-\lambda}\left(\omega'' + \frac{3}{4}\omega'^2 - \frac{1}{2}\omega'\lambda'\right) - e^{-\nu}\left(\frac{1}{2}\dot{\lambda}\dot{\omega} + \frac{1}{4}\dot{\omega}^2\right) \tag{4}$$

$$0 = - \dot{\omega}' - \frac{1}{2}\dot{\omega}\omega' + \frac{1}{2}\dot{\lambda}\omega' + \frac{1}{2}\dot{\omega}\nu' \tag{5}$$

$$\dot{F}_{10} = \left(\frac{\dot{\lambda} + \dot{\nu}}{2} - \dot{\omega}\right)F_{10} \tag{6}$$

$$F'_{10} + \left(\omega' - \frac{\lambda' + \nu'}{2}\right)F_{10} = 4\pi\sigma v^0 e^{-\lambda+\nu} \tag{7}$$

where

$$E^2 = - F^{10}F_{10}$$

and dots and primes indicate differentiation with respect to t and r respectively.

The following first integrals may be obtained fairly easily (cf. references [8] and [9]).

$$\frac{\partial}{\partial t}(\rho/\sigma) = 0 \tag{8}$$

$$e^{\omega - \frac{\lambda}{2}}\nu' = f \tag{9}$$

$$\omega' = \nu'[g e^{\omega/2} - 1] \tag{10}$$

$$e^{-\nu}\dot{\omega}^2 = A e^{-2\omega} + B e^{-\omega} + C e^{-3\omega/2} \tag{11}$$

where f, g, A, B, and C are functions of r alone and while C is arbitrary, A and B are given by

$$A = f^2\left(1 - \frac{\rho^2}{\sigma^2}\right) \tag{12}$$

$$B = f^2 g^2 - 4. \tag{13}$$

Equation (11) may be formally integrated to give

$$S + K = \int \frac{e^{\omega}\dot{\omega}dt}{\xi} \tag{14}$$

with

$$\xi = \mathrm{A} + \mathrm{B}e^{\omega} + \mathrm{B}e^{\omega/2}$$
$$\mathrm{S} = \int e^{\nu/2}dt$$

and K is an arbitrary function of r alone. The explicit form of the integral on the right of equation (14) will depend on the values of A, B and C and will, in different cases, be as follows

$$\mathrm{S} + \mathrm{K} = \frac{\xi^{1/2}}{\mathrm{B}} - \frac{\mathrm{C}}{2\mathrm{B}^{3/2}} \operatorname{Cosh}^{-1} \frac{e^{\omega/2} + \frac{\mathrm{C}}{2\mathrm{B}}}{\left(\frac{\mathrm{C}^2}{4\mathrm{B}^2} - \frac{\mathrm{A}}{\mathrm{B}}\right)^{\frac{1}{2}}} \quad (\text{if } \mathrm{B} > 0,\ \mathrm{C}^2 > 4\mathrm{AB}) \qquad (15\,a)$$

$$= \frac{\xi^{\frac{1}{2}}}{\mathrm{B}} - \frac{\mathrm{C}}{2\mathrm{B}^{3/2}} \operatorname{Sinh}^{-1} \frac{e^{\omega/2} + \frac{\mathrm{C}}{2\mathrm{B}}}{\left(\frac{\mathrm{A}}{\mathrm{B}} - \frac{\mathrm{C}^2}{4\mathrm{B}^2}\right)^{\frac{1}{2}}} \quad (\text{if } \mathrm{B} > 0,\ \mathrm{C}^2 < 4\mathrm{AB}) \qquad (15\,b)$$

$$= \frac{\xi^{\frac{1}{2}}}{\mathrm{B}} - \frac{\mathrm{C}}{2\mathrm{B}^{3/2}} \log\left(e^{\omega/2} + \frac{\mathrm{C}}{2\mathrm{B}}\right) \quad (\text{if } \mathrm{B} > 0,\ \mathrm{C}^2 = 4\mathrm{AB}) \qquad (15\,c)$$

$$= \frac{\xi^{\frac{1}{2}}}{\mathrm{B}} + \frac{\mathrm{C}}{2(-\mathrm{B})^{3/2}} \operatorname{Sin}^{-1} \frac{e^{\omega/2} + \frac{\mathrm{C}}{2\mathrm{B}}}{\left(\frac{\mathrm{C}^2}{4\mathrm{B}^2} - \frac{\mathrm{A}}{\mathrm{B}}\right)^{\frac{1}{2}}} \quad (\text{if } \mathrm{B} < 0,\ \mathrm{C}^2 > 4\mathrm{AB}) \qquad (15\,d)$$

$$= \frac{4}{3\mathrm{C}^2}(\mathrm{A} + \mathrm{C}e^{\omega/2})^{3/2} - \frac{4\mathrm{A}}{\mathrm{C}^2}(\mathrm{A} + \mathrm{C}e^{\omega/2})^{\frac{1}{2}} \quad (\text{if } \mathrm{B} = 0) \qquad (15\,e)$$

There still remain the field equations (3) and (4) and these give

$$\mathrm{A}'g^2 + \mathrm{C}'g + \mathrm{B}' = 0. \qquad (16)$$

The discussion has brought in six functions of r viz, f, g, A, B, C and K. Amongst these there exist the relations (13) and (16). Thus there remain four independent functions of r. These correspond physically to the distributions of ρ and σ; the initial velocity distribution and the possibility of a transformation of the radial coordinate r [*11*].

III. — REGULARITY OF THE FIELD AT THE CENTRE

The regularity of the field at the centre $r = 0$, requires in particular

i) exp λ, exp ν and their derivatives at least up to second order exist at the origin and further exp λ, exp ν do not vanish.

ii) Considering a circle of infinitesimal radius r at the origin, its radius $\to e^{\lambda/2}r$ and its circumference tends to $2\pi e^{\omega/2}$, so that for the origin to be a regular point, one must have

$$(e^{\lambda/2}r)_{r\to 0} = (e^{\omega/2})_{r\to 0}. \tag{17}$$

iii) For the validity of equation (17) at all times,

$$\dot{\lambda}_{r\to 0} = \dot{\omega}_{r\to 0} \tag{18}$$

iv)

$$\nu'_{r\to 0} = 0. \tag{19}$$

This last condition follows from a critical examination of the field equations. If this were not true σ would be infinite at the origin.

We are thus led to the following forms for exp λ exp ω and exp ν in the neighbourhood of the origin

$$\begin{aligned} e^{\lambda} &= e^{\mu}(1 + \alpha_1 r + \dots) \\ e^{\omega} &= e^{\mu}r^2(1 + \beta_1 r + \dots) \\ e^{\nu} &= 1 + \gamma_1 r^2 + \gamma_2 r^3 + \dots \end{aligned} \tag{20}$$

where μ, the α's, the γ's, the β's are functions of t alone and we have made $[\exp \nu]_{r\to 0} = 1$ by a suitable transformation of the time scale. Substituting these expressions in our equations (9), (10) and (11), we get

$$\begin{aligned} f &= c_0 r^3(1 + c_1 r + \dots) \\ g &= \frac{2}{c_0 r^3}(1 - c_1 r - \dots) \end{aligned} \tag{21}$$

where

$$c_0 = 2e^{\mu/2}\gamma_1 \tag{22}$$

$$c_1 = \frac{3}{2}\gamma_2/\gamma_1. \tag{23}$$

and also

$$4\pi\rho_0 a_0^2 = \frac{3c_0}{2}e^{-3\mu/2} \tag{24}$$

$$4\pi\rho_0(1 - a_0^2) = -\frac{3}{2}\left(\ddot{\mu} + \frac{1}{2}\dot{\mu}^2\right) \tag{25}$$

where ρ_0 and a_0 are the values of ρ and σ/ρ at the origin. In obtaining the above equations it is more convenient to use the following equations which are readily derivable from the field equations and are thus equivalent to the set (9), (10), (11) and (16) [*12*]

$$\frac{4\pi\sigma^2}{\rho} = e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'}{2}\left(\omega' - \frac{\lambda'}{2}\right)\right] + e^{-\lambda}\frac{\sigma}{\rho}\left(\frac{\rho}{\sigma}\right)'\frac{\nu'}{2} \tag{26}$$

$$\begin{aligned} &(4\pi\rho + \mathrm{E}^2)\left(1 - \frac{\sigma^2}{\rho^2}\right) - e^{-\lambda}\frac{\rho}{\sigma}\frac{\nu'}{2}\left(\frac{\sigma}{\rho}\right)' \\ &\qquad = -e^{-\nu}\left(\ddot{\omega} + \frac{\dot{\omega}^2}{2} + \frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\omega}\dot{\nu}}{2} - \frac{\dot{\lambda}\dot{\nu}}{4}\right) \end{aligned} \tag{27 a}$$

which has the equivalent form

$$(4\pi\rho + E^2)\left(1 - \frac{\sigma^2}{\rho^2}\right) = -\frac{\theta^2}{3} - \theta_{,\alpha}v^\alpha - 2q^2 - \left(\frac{\sigma}{\rho}\right)' v' \qquad (27\,b)$$

where θ is the expansion $\equiv v^\mu$; μ and q^2 is the shear term and vanishes at the origin. Equations (24) and (25) together give

$$\frac{d^2}{dt^2}(e^{\mu/2}) = -\frac{c_0}{a_0^2}(1 - a_0^2)e^{-\mu} \qquad (28)$$

which again can be integrated to give

$$\dot{\mu}^2 = \mathrm{P}e^{-\mu} + \frac{4c_0}{a_0^2}(1 - a_0^2)e^{-3\mu/2}. \qquad (29)$$

Thus if $a_0^2 \leqslant 1$, a decreasing exp $(\mu/2)$ would vanish at a finite time in view of equation (28), as from equation (24) c_0 is positive. One would thus have a collapse to a singular state of vanishing spatial volume and infinite density. Again if $a_0^2 > 1$, the right hand side of equation (28) is positive and it is clear that there exists the possibility of a minimum of exp $(\mu/2)$ from which the system may bounce back. However these considerations apply only at the origin and we shall see that elsewhere the situation is essentially different and much more complicated. The difference arises due to the fact that while at the origin the electric intensity and the shear both vanish leading to a situation similar to that obtaining for pure uncharged dust, elsewhere both these quantities exist [*12*] giving rise to an extremely complicated situation. The difference is apparent in the occurrence of the term A exp (-2ω) in equation (11) whereas there is no corresponding term in equation (29). The reason is that in view of equations (12), (20) and (21) A exp (-2ω) vanishes as r^2 as $r \to 0$, whereas the other terms are in general nonvanishing as $r \to 0$. A comparison of equations (11) and (29) shows that C behaves as $4c_0(1 - a_0^2)r^3/a_0^2$ as $r \to 0$, and is positive or negative at the centre according as the central value of $|\sigma|/\rho$ is less or greater than unity.

IV. — THE GENERAL DYNAMICAL BEHAVIOUR AND NONSTATIC SOLUTIONS WITH SINGULARITY AT THE ORIGIN

The behaviour of exp ω which gives the circumferential area may be studied from equation (15). From equation (11), the zeros of $\dot{\omega}$ occur at values of exp ω given by

$$(e^\omega)_{\dot{\omega}=0} = \frac{-\mathrm{C} \pm (\mathrm{C}^2 - 4\mathrm{AB})^{\frac{1}{2}}}{2\mathrm{B}}$$

and one has the following situations:

(a_1) A, B are of the same sign and C is of opposite sign. In this case both the roots are positive and real if $C^2 \geqslant 4AB$. In case A and B be both negative (which corresponds to $\rho > |\sigma|$, and $|fg| < 2$), $\exp \omega$ will execute oscillations between the two roots. This may be compared with the results of de la Cruz and Israel [*2*] who considered the charge distribution in the form of a shell and the field to be continuous with an outside Reissner-Nordstrom metric. The continuity condition requires in our case [*8*], [*9*], [*13*].

$$\mu = \frac{4\pi}{2} \int_0^b \rho \mid fg \mid e^{\lambda/2+\mu} dr$$

$$e = 4\pi \int_0^b \sigma e^{\lambda/2+\mu} dr$$

where b is the value of r at the boundary and μ and e are the mass and charge constants appearing in the Reissner-Nordstrom metric. The de la Cruz-Israel conditions for oscillation of the shell require that at $r = b$, $|e|/\mu$, $|\rho fg/2\sigma|$, $|fg/2|$ are each less than unity.

However with $\rho > |\sigma|$, if σ/ρ be constant, then from equation (27 *b*) there will be a collapse of the volume in finite proper time and hence although $\exp \omega$ oscillates, there will be a radial collapse with a singularity of infinite density ($\exp \lambda \to 0$, $\exp\left(\omega + \frac{\lambda}{2}\right) \to 0$, $\rho \to \infty$). This singularity, in view of equations (9) and (10), would be accompanied by a vanishing of ω' corresponding to the appearance of an extremum of the circumferential radius regarded as a function of the comoving radial coordinate (cf. the shell crossing in case of dust spheres discussed by Yodzis *et al.* [*7*]).

(a_2) If however A and B be both positive with C negative (*i. e.* $|\sigma| > \rho$, $|fg| > 2$), ξ assumes negative values for values of $\exp(\omega/2)$ lying between the two zeros of ξ (*i. e.* the zeros of $\dot{\omega}$) and $\exp \omega$ can either run between a finite positive minimum and infinity (in either direction) or it may run between zero and a finite maximum (again in either direction).

(a_3) If $C^2 = 4AB$, the roots are identical and there is the possibility of a static solution which will be discussed later.

(*b*) A and B are of opposite signs. The roots are real and of opposite signs. In this case there is no oscillation but either a minimum or a maximum as in (a_2) above.

(*c*) A, B, C are all of the same sign and $C^2 > 4AB$. Both the roots are real and negative. Physically admissible cases occur only if all the three are positive and then there is a monotone change of $\exp \omega$ from zero to infinity.

(*d*) $C^2 < 4AB$. The roots are complex. The case A and B are positive is similar to that in (*c*) while if they are negative no real solution exists.

Looking back over the discussion we may say that in no case can there be regular oscillations if σ/ρ be constant-a conclusion also reach by Bailyn [14] for the special case of spheres with $|e| > \mu$.

We have not been able to obtain any time dependent singularity free solution. It may be noted that in view of equations (12), (13) and (21). A vanishes at least as rapidly as r^6 and B as r^2 as r goes to zero. We have noted at the end of the last section that C vanishes at least as rapidly as r^3 as r goes to zero. Thus none of the three functions A, B and C can reduce to a constant other than zero if the field is to be regular at the origin. However equation (16) is trivially satisfied if any two of the three functions vanish and the third is a constant not equal to zero. One can in that case integrate readily the other equations and obtain an explicit solution singular at the origin.

If $A = B = 0$, C a constant other than zero, one gets

$$e^2 = \dot{\omega}^2 e^{3\omega/2} C^{-1}$$
$$e^{5\omega/4}\omega' = \pm 2\alpha e^{\omega/2} - f\alpha$$
$$e^{\lambda/2} = \alpha e^{-\omega/4}$$

where f and α are arbitrary functions of r. Again with $A = C = 0$, $B = a$ constant $\neq 0$, one gets the solution given by Hamoui [13]

$$e^\lambda = 1$$
$$e^\nu = \dot{\omega}^2 e^\omega B^{-1}$$
$$e^\omega \omega' = \pm (B^2 + 4)^{\frac{1}{2}} e^{\omega/2} - f$$

where f again is an arbitrary function of r. In both the above cases $|\sigma| = \rho$. One however gets a case where $|\sigma|/\rho \neq 1$ by taking $A = a$ constant $\neq 0$, $B = C = 0$. One then has

$$e^\nu = e^{2\omega}\dot{\omega}^2 A^{-1}$$
$$e^\lambda = \alpha^2 e^{-\omega}$$
$$e^{3\omega/2}\omega' = \pm 2\alpha e^{\omega/2} - f\alpha$$
$$\frac{\rho^2}{\sigma^2} = 1 - \frac{A}{f^2} < 1.$$

All these solutions may be fitted to an outside Reissner-Nordstrom metric and it turns out that in each of the three cases $\mu > |e|$ where the exterior metric is

$$ds^2 = \left(1 - \frac{2\mu}{r} + \frac{e^2}{r^2}\right)dt^2 - \left(1 - \frac{2\mu}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2.$$

V. — THE STATIC CASE

In view of equation (11) and the conditions for the regularity at the origin, the static nonsingular solutions are characterised by $A = B = C = 0$

or $|\sigma| = \rho$ and $|fg| = 2$. The latter conditions are also sufficient for the solution to be static. For, these conditions give from equations (12) and (13) $A = B = 0$. Equation (16) would then require, in view of the regularity at the origin that C also vanishes and the solution becomes static. Hence nonstatistic regular solutions with $\rho = |\sigma|$ have $|fg| \neq 2$. In general for these cases $\mu \neq |e|$ although $\rho = |\sigma|$. In the static case we let exp $\omega = r^2$ by a transformation of the radial coordinate. If now we choose g in accordance with equation (21) taking care that $g.r$ is everywhere greater than unity in the domain of r where the solution is to apply, then we get a regular static solution with exp λ and exp ν determined by equations (9) and (10) with $f = 2/g$. As a very simple example one may take $g = (2/c_0 r^3)$, so that $f = c_0 r^3$ and

$$e^{-\nu} = e^{-\lambda/2} = \left(1 - \frac{c_0 r^2}{2}\right)$$

$$8\pi\rho = 8\pi\,|\sigma| = 3c_0\left(1 - \frac{c_0 r^2}{2}\right).$$

We restrict the solution to the region $0 \leqslant r \leqslant r_b$ where $C_0 r_b^2 < 2$. The method of generating regular static solutions thus depends solely on choosing a suitable form for the function g. In all these solutions it is easy to see that the parameters μ and $|e|$ in the exterior Reissner-Nordstrom metric must be equal.

One may also have another class of static solutions which however do not satisfy the conditions of regularity at the origin. These correspond to cases (a_3) mentioned in section IV above. With exp $\omega = r^2$, one has in this case

$$C = -\frac{2A}{r}$$
$$B = \frac{A}{r^2}. \tag{30}$$

With the relations (30), equation (16), regarded as a quadratic in g, gives either

$$g = \frac{1}{r} \tag{31 a}$$

or

$$g = \frac{1}{r} - \frac{2}{r^2}\cdot\frac{A}{A'}. \tag{31 b}$$

Equation (10) rules out (31 a), while with (31 b), it yields

$$e^2 A = \text{Const.} \tag{32}$$

Using equations (30), (31 b) and (32), we get from equations (9), (12) and (13)

$$f^2 = \frac{A + 4r^2}{\left(1 - \frac{2}{r}\cdot\frac{A}{A'}\right)^2} \tag{33}$$

$$e^{-\lambda} = \frac{A + 4r^2}{r^2} \cdot \frac{A^2}{(A'r - 2A)^2} \tag{34}$$

$$1 - \frac{\rho^2}{\sigma^2} = \frac{A}{r^2 A'^2} \frac{(A'r - 2A)^2}{A + 4r^2}. \tag{35}$$

Thus the field is completely determined one the function A is specified. The solution has a singulaiity at $r = 0$; one may however exclude the region around the origin from the domain of the solution. Thus if the solution be valid for $r \geqslant a$, one may make it continuous with a Schwarzschild or Reissner-Nordstrom field (but not an euclidean field) in the region $r \leqslant a$. A simple example where it is continuous with the Schwarzschild field may be of some astrophysical interest and is presented below. We take

$$g = \left(1 - \frac{m}{a}\right) \Big/ m \qquad (a > 2m).$$

Then one gets, choosing a constant of integration suitably, for $r \geqslant a > 2m$

$$e^{\nu} = \chi_r^2$$

$$e^{-\lambda} = \frac{m^2 \chi_a + r^2 \chi_r^2}{r^2 \left(1 - \frac{m}{a}\right)^2}$$

$$E^2 = \frac{2m^3}{r^4 \left(1 - \frac{m}{a}\right) \chi_r} \left(\frac{1}{a} - \frac{1}{r}\right)$$

$$4\pi\rho = \frac{m^2}{r^4} \frac{\chi_a}{\left(1 - \frac{m}{a}\right) \chi_r}$$

$$16\pi^2 \sigma^2 = \frac{1}{2} \frac{m^3 a}{r^{10}} \cdot \frac{\chi_a^2 (m^2 \chi_a + r^2 \chi_r^2)}{\left(1 - \frac{m}{a}\right)^3 \chi_r^3 \left(1 - \frac{a}{r}\right)}$$

where we have written

$$\chi_r \equiv 1 - \frac{m}{a} - \frac{m}{r}; \qquad \chi_a = 1 - \frac{2m}{a}.$$

The solution presented above may be fitted with the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

at $r = a$. Although $\sigma \to \alpha$ as $r \to a$, the metric tensor components along with their first derivatives (excepting g'_{rr}) as well as the electric field are all continuous at $r = a$. The infinity of σ arises in the folloving manner. For the continuity with the Schwarzschild field, the electric intensity must vanish at $r = a$ which requires the vanishing of $f\rho/\sigma$ but in view of equation (9), f cannot vanish, hence the vanishing of ρ/σ and consequent infinite value of σ. It is thus apparent that one can get rid of this infinity if the interior field be of the Reissner-Nordstrom type with a nonvanishing electric field; however the interesting feature of our static solution that one can have even for a static distribution $|\sigma| \neq \rho$ if there be « a hole or pocket of alien matter inside » [13]; would not thereby be disturbed.

At any $r = b$, where $b > a$, one may make the solution continuous with the Reissner-Nordstrom metric if

$$\mu = \frac{m\left(1 - \frac{2m}{b}\right)}{\left(1 - \frac{m}{a} - \frac{m}{b}\right)}$$

and

$$|e| = \frac{2^{\frac{1}{2}} m^{3/2}}{\left(1 - \frac{m}{a}\right)^{\frac{1}{2}}\left(1 - \frac{m}{a} - \frac{m}{b}\right)^{\frac{1}{2}}} \left(\frac{1}{a} - \frac{1}{b}\right)^{\frac{1}{2}}$$

Thus e is finite for all values of b and tends to zero as $b \to a$, although σ is infinite at $r \to a$.

One may take the mass of the charged dust cloud to be given by

$$\mathrm{M} \equiv \mu - m = m^2\left(\frac{1}{a} - \frac{1}{b}\right) \Big/ \left(1 - \frac{m}{a} - \frac{m}{b}\right)$$

and the charge-mass ratio for $b \gg a$ becomes $|e|/\mathrm{M} = (2a/m)^{\frac{1}{2}}$ and may have arbitrarily large values for any given m. Thus a star may have around it a shell of charged dust in equilibrium with arbitrarily large charge-mass ratio. However there exists upper bounds to the total charge and mass:

$$\mathrm{M} \leqslant \frac{m^2}{a\left(1 - \frac{m}{a}\right)}$$

$$|e| \leqslant 2^{\frac{1}{2}} m^{3/2} \Big/ \left[\left(1 - \frac{m}{a}\right) a^{\frac{1}{2}}\right].$$

The right hand sides of these inequalities tend towards m and $2m$ respectively as a tends towards the Schwarzschild radius $2m$.

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