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Thermodynamic equivalence of spin systems

by

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SUMMARY. — This article is concerned with the thermodynamic equilibrium properties of systems composed of classical spin $\frac{1}{2}$ particles (Ising spins). Given an interaction pattern between the Ising spins the main problem is to calculate the equilibrium state(s) of the system. The point we want to put forward in our paper is the existence of many thermodynamical equivalent spin coordinate systems. As a consequence of this phenomenon the interaction pattern of a system may be very intricate when described with respect to one spin coordinate system whereas it may become simple with respect to another one and *vice versa*.

The content of this article is a systematic investigation of this phenomenon.

In § 2 we introduce the configuration space Γ of an infinite, countable set of Ising spins. The set Γ is provided with a topology and a Borel measure. This mathematical structure enables us to define the equilibrium states of a large class of finite as well as infinite systems of Ising spins. By simple arguments we show that measure preserving homeomorphisms of Γ connect spin coordinate systems that are thermodynamically equivalent as long as one considers finite systems only.

In § 3 we consider infinite systems. The reason for this is that one is interested in the thermodynamic limit rather than in finite systems. We start with the definitions of infinite systems and their equilibrium states. The main result is contained in proposition 4. From it we learn that measure preserving homeomorphisms of Γ satisfying certain additional conditions connect spin coordinate systems that are thermodynamically equivalent both for finite and infinite systems.

In § 4 lattice systems are discussed. That is we consider infinite systems of Ising spins having the translation symmetry of some ν-dimensional

lattice. In this situation the relevant homeomorphisms of Γ are those commuting with the homeomorphisms induced by the lattice translations. In its most explicit form the problem of finding such homeomorphisms appears to be a problem in code theory. That is it appears to be the problem of finding finite codes. For the 1-dimensional lattice this question has already been studied to some extent. For higher dimensional lattices little seems to be known. We conclude with a few explicite examples for 1- and 2-dimensional lattices.

RÉSUMÉ. — Cet article concerne les propriétés thermodynamiques à l'équilibre de systèmes de particules classiques de spin $\frac{1}{2}$ (spins d'Ising). Lorsqu'on se donne un schéma d'interaction entre les spins d'Ising le problème essentiel est de calculer le (ou les) état(s) du système. Le point essentiel de cet article est l'existence de nombreux systèmes de coordonnées de spin thermodynamiquement équivalents. En conséquence de cette propriété le schéma d'interaction d'un système peut être très complexe dans un système de coordonnées de spin mais très simple dans un autre, et *vice versa*. Cet article est une étude systématique de ce phénomène.

Au § 2 nous introduisons l'espace de configuration Γ d'un ensemble infini et comptable de spins d'Ising. L'ensemble Γ est équipé d'une topologie et d'une mesure de Borel. Cette structure mathématique permet de définir les états d'équilibre d'une large classe de systèmes d'un nombre fini ou infini de spins d'Ising. Une argumentation simple permet de démontrer que les homéomorphismes de Γ qui conservent la mesure invariante relient des systèmes de coordonnées de spin qui sont thermodynamiquement équivalents aussi longtemps que seuls des systèmes finis sont considérés.

Au § 3 nous considérons des systèmes infinis. La raison en est que l'on est plus intéressé à la limite thermodynamique qu'aux systèmes finis. Nous définissons les systèmes infinis et leurs états d'équilibre. Le résultat principal est contenu dans la proposition 4 qui nous apprend que lorsqu'ils satisfont certaines conditions supplémentaires, les homéomorphismes de Γ laissant la mesure invariante relient des systèmes de coordonnées de spin thermodynamiquement équivalents, pour le cas fini comme pour le cas infini.

Au § 4 nous discutons des systèmes de réseaux, c'est-à-dire nous considérons des systèmes infinis de spins possédant la symétrie de translation d'un réseau de dimension v. Les homéomorphismes appropriés de Γ sont alors ceux qui commutent avec les homéomorphismes induits par les translations du réseau. Dans sa forme la plus explicite, le problème de trouver ces homéomorphismes se révèle être un problème de la théorie des codes : celui de trouver des codes finis. Dans le cas à une dimension ce problème a déjà été étudié jusqu'à un certain point. Pour les réseaux de dimensions supérieures il semble que peu soit connu. Nous donnons en conclusion quelques exemples explicites de réseaux à une et deux dimensions.

1. INTRODUCTION

This paper is concerned with the equilibrium statistical mechanics of systems composed of classical spin $\frac{1}{2}$ particles. Given an interaction pattern between the particles the main problem is to find out the thermodynamic behaviour of the system.

We investigate configuration space transformations producing thermodynamic relations between spin systems. Let us illustrate the idea by a simple, representative example. Consider a collection of n classical spin $\frac{1}{2}$ particles. Let Γ be the configuration space of the system. So $|\Gamma| = 2^n$.

Let $h: \Gamma \to \mathbb{R}$ be the Hamiltonian of the system. So $h(\gamma)$ equals the energy of the system if it is in configuration γ. The thermodynamic average $\langle g \rangle_h$ of a function g on Γ equals

$$\langle g \rangle_h = \Big[\sum_{\gamma \in \Gamma} g(\gamma).\exp h(\gamma)\Big] \cdot \Big[\sum_{\gamma \in \Gamma} \exp h(\gamma)\Big]^{-1}.$$

Here the factor $-\frac{1}{kT}$ has been absorbed into h. If $A: \Gamma \to \Gamma$ is a permutation of the configurations we have

$$\langle g \rangle_h = \Big[\sum_{\gamma \in \Gamma} g(A\gamma).\exp h(A\gamma)\Big] \cdot \Big[\sum_{\gamma \in \Gamma} \exp h(A\gamma)\Big]^{-1} = \langle g \circ A \rangle_{h \circ A}.$$

Thus configuration space transformations produce thermodynamic relations between systems composed of n spin $\frac{1}{2}$ particles. The object of this paper is to investigate such configuration space transformations in more general situations and to discuss their interest for the equilibrium statistical mechanics of spin systems.

Section 2 is concerned with transformations between finite spin systems. In section 3 we extend the analysis such that infinite systems are also included. In section 4 we specialize the discussion to lattice spin systems.

2. THERMODYNAMIC RELATIONS BETWEEN FINITE SPIN SYSTEMS

2.1. This section is devoted to the construction of a mathematical object that enables us to formulate simultaneously the thermodynamics of all systems of classical spin $\frac{1}{2}$ particles.

Let N be an infinite, countable set. Let Γ be the set of all mappings $\gamma : \mathrm{N} \to \{-1, 1\}$. The set Γ can be identified with the configuration space of an infinite, countable set of classical spin $\frac{1}{2}$ particles, labeled by the elements of N.

We provide Γ with a topology. Suppose K to be a finite subset of N and let α be a mapping of K into $\{-1, 1\}$. To each doublet (K, α) a subset $\{\gamma \in \Gamma \mid \gamma(i) = \alpha(i) \text{ for all } i \in \mathrm{K}\}$ of Γ can be assigned. Such a subset of Γ is called a cylinder. Open sets of Γ are defined as unions of cylinders. As Γ can be considered as the product $\{-1, 1\}^{\mathrm{N}}$ this topology is nothing but the product of the discrete topologies of the sets $\{-1, 1\}$. The topological space Γ is homeomorphic to the Cantor set [*1*]. Hence Γ is a compact, metric, perfect, totally disconnected topological space. As topological space Γ is completely determined by these four properties [*2*].

Next we construct a measure Π_0 on Γ. Let the σ-algebra generated by the cylinders be the collection of measurable subsets of Γ. The measure of a cylinder is defined by $\Pi_0(\{\gamma \in \Gamma \mid \gamma(i) = \alpha(i) \text{ for all } i \in \mathrm{K}\}) = 2^{-|\mathrm{K}|}$. By $\int$ we denote the integral on Γ derived from Π_0. All continuous functions on Γ are integrable.

2.2. In this section we use the space Γ to describe simultaneously all finite systems of classical spin $\frac{1}{2}$ particles.

Consider a system of a finite number of spin $\frac{1}{2}$ particles. The particles of the system can be labeled by the elements of some finite subset K of N. The configuration space of the system can be identified with the set Γ_{K} of all mappings $\alpha : \mathrm{K} \to \{-1, 1\}$. The energy of the system in the various configurations can be described by a function $h_{\mathrm{K}} : \Gamma_{\mathrm{K}} \to \mathbb{R}$ such that $h_{\mathrm{K}}(\alpha)$ equals the energy of the system in configuration α. If g_{K} is some function on Γ_{K} its thermodynamic average equals

$$\langle g_{\mathrm{K}} \rangle = \left[\sum_{\alpha \in \Gamma_{\mathrm{K}}} g_{\mathrm{K}}(\alpha) . \exp h_{\mathrm{K}}(\alpha) \right] . \left[\sum_{\alpha \in \Gamma_{\mathrm{K}}} \exp h_{\mathrm{K}}(\alpha) \right]^{-1} . \quad (2.2.1)$$

Here the factor $-\frac{1}{k\mathrm{T}}$ has been absorbed in h_{K} for convenience.

From an arbitrary function f_{K} on Γ_{K} a function f on Γ can be derived by defining $f(\gamma) = f_{\mathrm{K}}(\gamma|_{\mathrm{K}})$ $(\gamma \in \Gamma)$. One easily sees that

$$\int f = 2^{-|\mathrm{K}|} . \sum_{\alpha \in \Gamma_{\mathrm{K}}} f_{\mathrm{K}}(\alpha) . \quad (2.2.2)$$

Suppose h and g to be functions on Γ derived from h_K and g_K. From 2.2.1 and 2.2.2 it follows that

$$\langle g_K \rangle = \left[\int g.\exp h\right].\left[\int \exp h\right]^{-1}. \tag{2.2.3}$$

A function $f : \Gamma \to \mathbb{R}$ will be called cylindrical if there exists a finite subset K of N such that $f(\gamma)$ only depends on $\gamma\,|_K$ for all $\gamma \in \Gamma$. From the previous considerations it follows that finite spin systems can be identified with cylindrical functions h on Γ. The thermodynamic averages for such a system equal

$$\langle g \rangle_h = \left[\int g.\exp h\right].\left[\int \exp h\right]^{-1}. \tag{2.2.4}$$

Here g is a cylindrical function on Γ corresponding to the quantity one wants to average.

2.3. In this section a class of transformations of Γ is considered producing relations between thermodynamic averages of finite spin systems. Suppose A is a mapping of Γ into itself satisfying the following two conditions.

The function $f \circ A$ on Γ is cylindrical for all cylindrical functions f on Γ. (2.3.1)

$$\int f \circ A = \int f \qquad \text{for all cylindrical functions } f \text{ on } \Gamma. \tag{2.3.2}$$

Let h, g be arbitrary cylindrical functions on Γ. Then

$$\langle g \rangle_h = \frac{\int g.\exp h}{\int \exp h} = \frac{\int (g.\exp h)\circ A}{\int (\exp h)\circ A} = \frac{\int (g\circ A).\exp(h\circ A)}{\int \exp(h\circ A)} = \langle g \circ A \rangle_{h\circ A}. \tag{2.3.3}$$

Hence, for all cylindrical functions h on Γ, the transformation A produces relations between the thermodynamic averages of the finite spin systems described by h and $h \circ A$.

Obviously, if $h \circ A = h$ one has relations between the averages of the system h. Because of these properties the transformations A are of interest for the thermodynamics of finite spin systems.

Conditions (2.3.1) and (2.3.2) can be brought in a more natural form.

PROPOSITION 1. — Conditions (2.3.1) and (2.3.2) are equivalent with:

A is continuous, (2.3.4)

$$\int f \circ A = \int f \qquad \text{for all continuous functions } f \text{ on } \Gamma. \tag{2.3.5}$$

To prove proposition 1 we first give a topological characterization of cylindrical functions.

PROPOSITION 2. — A function $f : \Gamma \to \mathbb{R}$ is cylindrical iff $f^{-1}x$ is an open subset of Γ for all $x \in \mathbb{R}$.

Proof. — Suppose $f : \Gamma \to \mathbb{R}$ is cylindrical. By definition there exists a finite subset K of N such that f is constant on the cylinders belonging to K. So $f^{-1}x$ is the union of a number of cylinders for all $x \in \mathbb{R}$. Since cylinders are open so is $f^{-1}x$. Conversely, suppose f is a function on Γ with $f^{-1}x$ open for all $x \in \mathbb{R}$. Since $f^{-1}x$ is open it is a union of cylinders. Consequently, the space Γ can be covered by cylinders such that f is constant on each cylinder of the covering. Since Γ is compact there exists a finite covering of this type. Now $f(\gamma)$ depends only on $\gamma\,|_{\mathrm{L}}$ where L is the union of all K's in the doublets (K, α) defining the cylinders in the latter covering. So f is a cylindrical function. This completes the proof.

Proof of proposition 1. — Suppose $\mathrm{A} : \Gamma \to \Gamma$ satisfies (2.3.4) and (2.3.5). Using proposition 2 condition (2.3.1) follows from (2.3.4). Since cylindrical functions are continuous condition (2.3.2) follows from (2.3.5). Conversely, suppose A satisfies (2.3.1) and (2.3.2). First we show A to be continuous. It suffices to show that the inverse image of a cylinder (K, α) is open. Let g be the characteristic function of the cylinder (K, α). As g is cylindrical so is $g \circ \mathrm{A}$. Hence $(g \circ \mathrm{A})^{-1}(1)$ is open. However, $(g \circ \mathrm{A})^{-1}(1)$ is the inverse image under A of the cylinder (K, α). So A is continuous. Condition (2.3.5) follows from (2.3.2) by using the argument that a continuous function f can be approximated as closely as one wants by cylindrical functions. That is, for each $\varepsilon > 0$ there exists a cylindrical function f' such that $|f(\gamma) - f'(\gamma)| < \varepsilon$ for all $\gamma \in \Gamma$. The latter statement can be proven by using the compactness of Γ. This completes the proof.

Using the compactness of Γ it follows that transformations A satisfying (2.3.4) and (2.3.5) are surjective. However, A need not to be injective. If, in addition, A is injective the transformation A^{-1} is continuous because Γ is compact. Moreover, $\int f = \int f \circ \mathrm{A}^{-1}$ for all continuous functions f on Γ. Consequently, injective transformations A satisfying (2.3.4) and (2.3.5) constitute a group with respect to composition of transformations. The set of all transformations A satisfying (2.3.4) and (2.3.5) constitute a semi-group.

Let us consider the action $f \rightsquigarrow f \circ \mathrm{A}$ on continuous functions f with A satisfying (2.3.4) and (2.3.5). Because A is surjective this action is injective. In addition the action will be surjective iff A is injective. Consequently, only for injective transformations A satisfying (2.3.4) and (2.3.5) all thermodynamic averages of a system $h \circ \mathrm{A}$ can be expressed in those of a system h. Here h is an arbitrary cylindrical function on Γ.

2.4. In this section we translate the conditions (2.3.1) and (2.3.2) in more explicit terms. That is, in terms of « old » and « new » spin variables.

The vector space $C(\Gamma)$ of all (real) continuous functions on Γ can be provided with an inner product; $(f, f') = \int f.f'$. The spinfunction σ_n belonging to $n \in N$ is defined by $\sigma_n(\gamma) = \gamma(n)$ $(\gamma \in \Gamma)$. For a finite subset K of N we define; $\sigma_K = \prod_{n \in K} \sigma_n$. If K is empty we define; $\sigma_\phi = 1$. Obviously, the functions σ_K are cylindrical. The collection of functions σ_K forms an orthonormal basis in the space of cylindrical functions. That is, each cylindrical function f can uniquely be written as a finite linear combination of σ_K's; $f = \sum_K (f, \sigma_K)\sigma_K$. The constant term in this expansion equals $\int f$.

Let A be an arbitrary mapping of Γ into itself. Obviously, A is completely determined by the functions $\sigma_n \circ A$. Notation; $\sigma_n' = \sigma_n \circ A$.

PROPOSITION 3. — Suppose for each $n \in N$ a function σ_n' on Γ is given. Then: There exists a mapping $A : \Gamma \to \Gamma$ satisfying (2.3.1) and (2.3.2) such that $\sigma_n \circ A = \sigma_n'$ for all $n \in N$ iff the functions σ_n' satisfy the following three conditions:

(*i*) σ_n' can be written as a finite linear combination of σ_K's.

(*ii*) $(\sigma_n')^2 = 1$.

(*iii*) for each non empty, finite subset K of N the expansion in spinfunctions of $\prod_{n \in K} \sigma_n'$ does not contain a constant term.

Proof. — Suppose $A : \Gamma \to \Gamma$ satisfies (2.3.1) and (2.3.2). Since σ_n is a finite function so is $\sigma_n \circ A$. Hence $\sigma_n' = \sigma_n \circ A$ can be written as a finite linear combination of σ_K's. Property (*ii*) is obvious. The constant term in $\prod_{n \in K} \sigma_n'$ equals

$$\int \prod_{n \in K} \sigma_n' = \int \prod_{n \in K} (\sigma_n \circ A) = \int \Big(\prod_{n \in K} \sigma_n\Big) \circ A = \int \prod_{n \in K} \sigma_n = 0.$$

Conversely, suppose A is a mapping of Γ into itself defined by $\sigma_n \circ A = \sigma_n'$ where the functions σ_n' satisfy (*i*), (*ii*), and (*iii*). Condition (2.3.1) follows from (*i*). Let f be a cylindrical function. The function f can be written as a finite linear combination of σ_K's; $f = \sum_K \alpha_K \sigma_K$. So

$$\int f \circ A = \sum_K \alpha_K \int \sigma_K \circ A = \sum_K \alpha_K \int \prod_{n \in K} \sigma_n' = \alpha_\phi = \int f.$$

Hence A satisfies (2.3.2). This completes the proof.

3. THERMODYNAMIC RELATIONS BETWEEN INFINITE SPIN SYSTEMS

3.1. Since in statistical mechanics one is interested in the thermodynamic limit we want to generalize the previous considerations such that infinite spin systems are also included. Mathematically the interaction for finite as well as infinite spin systems can be described as follows. For each finite subset K of N a continuous function $h_K : \Gamma \to \mathbb{R}$ is given. The family of functions $\{ h_K \}$ should satisfy the following condition. For all finite subsets K and K′ of N with $K \subset K'$ one has $(h_K - h_{K'})(\gamma) = (h_K - h_{K'})(\gamma')$ whenever $\gamma |_{N-K} = \gamma' |_{N-K}$. That is, $(h_K - h_{K'})(\gamma)$ is independent of $\gamma |_K$. The physical interpretation of the functions h_K is obvious. The number $h_K(\gamma)$ equals the interaction energy between the spins contained in K (being in configuration $\gamma |_K$) plus their interaction energy with all other spins (being in configuration $\gamma |_{N-K}$).

Instead of thermodynamic averages or correlations functions we will use probability measures to describe the equilibrium properties of the systems. To define equilibrium states we require the following definitions [3]. For a given system $\{ h_K \}$ we construct a family $\{ f^K \}$ (K finite subset of N) of functions $f^K : \Gamma_K \times \Gamma_{N-K} \to \mathbb{R}$ by

$$f^K(\bar{\delta}, \gamma) = [\exp h_K(\bar{\delta}, \gamma)] \cdot \Big[\sum_{\delta \in \Gamma_K} \exp h_K(\delta, \gamma) \Big]^{-1} \qquad (\bar{\delta} \in \Gamma_K, \gamma \in \Gamma_{N-K}). \tag{3.1.1}$$

Here $(\bar{\delta}, \gamma)$ and (δ, γ) on the right hand side of (3.1.1) should be regarded as elements of Γ in the obvious way. For K a subset of N let $r_K : \Gamma \to \Gamma_K$ be the projection defined by $r_K(\gamma) = \gamma|_K$ $(\gamma \in \Gamma)$. Let Π be a measure on Γ (all measures refer to the σ-algebra generated by the cylinders), K a subset of N and μ an element of Γ_{N-K}. Then $r_K\Pi$ and $r_K^\mu\Pi$ are measures on Γ_K defined by

$$r_K\Pi(\Omega) = \Pi(\{ \gamma \in \Gamma \mid r_K(\gamma) \in \Omega \}), \tag{3.1.2}$$

$$r_K^\mu\Pi(\Omega) = \Pi(\{ \gamma \in \Gamma \mid r_K(\gamma) \in \Omega \quad \text{and} \quad r_{N-K}(\gamma) = \mu \}). \tag{3.1.3}$$

Here Ω is a measurable subset of Γ_K. A probability measure Π on Γ is an equilibrium state of the system $\{ h_K \}$ if

$$r_{N-K}^\delta\Pi = f^K(\delta, .) r_{N-K}\Pi \tag{3.1.4}$$

for all finite subsets K of N and configurations δ in Γ_K. It is clear that addition to h_K of a continuous function h_K' independent of $\gamma |_K$ does not influence the definition of equilibrium states of the system $\{ h_K \}$.

From the work of Preston [3] it can be concluded that there exists at least one equilibrium state. Contrary to finite systems infinite systems can have several equilibrium states (phase transition).

3.2. In this section configuration space transformations are considered producing relations between equilibrium states of infinite spin systems. Suppose $A : \Gamma \to \Gamma$ satisfies

A is continuous, (3.2.1)

for each $n \in N$ there exists a finite subset K of N such that $r_{N-K}A\gamma$ is independent of $\gamma(n)$. (3.2.2)

The mapping A induces a transformation (also denoted by A) in the collection of systems $\{ h_K \}$ defined by

$$A \{ h_K \} = \{ h'_K \} \qquad \text{with} \qquad h'_K = h_{K'} \circ A. \qquad (3.2.3)$$

Here K′ should be chosen as follows. Because of property (3.2.2) there exists a finite subset K″ of N such that $r_{N-K''} A\gamma$ is independent of $r_K\gamma$. The subset K′ should be chosen larger or equal to K″. As one sees easily the ambiguity in this definition caused by the infinity of possibilities to choose K′ is harmless because of the ambiguity in the definition of $\{ h_K \}$ for a system (see 3.1). Furthermore one verifies easily that $\{ h'_K \}$ satisfies the conditions to describe a spin system (see 3.1).

Let $A : \Gamma \to \Gamma$ satisfy (3.2.1) and (3.2.2). Let $\{ h_K \}$ describe an arbitrary spin system. One can imagine several relations between the equilibrium states of the systems $\{ h_K \}$ and $A \{ h_K \}$. We are able to prove the following one.

PROPOSITION 4. — Let $A : \Gamma \to \Gamma$ be a homeomorphism satisfying (3.2.2), $\{ h_K \}$ be a family of functions describing a spin system and Π be an equilibrium state of this system. Let Π' be the probability measure on Γ defined by $\Pi'(\Omega) = \Pi(A\Omega)$ with Ω an arbitrary measurable subset of Γ.

Then Π' is an equilibrium state of the spin system described by $A \{ h_K \}$.

Proof. — Suppose $A \{ h_K \} = \{ h'_K \}$. Let $\{ f'^K \}$ be the family of functions associated with $\{ h'_K \}$ (see 3.1.1). Let K be a fixed, finite subset of N. Let $\bar{\delta}$ be a fixed element of Γ_K. We have to show

$$r^{\bar{\delta}}_{N-K}\Pi'(\Omega) = \int_{\Omega} f'^K(\bar{\delta}, \gamma) dr_{N-K}\Pi'(\gamma) \qquad (\gamma \in \Gamma_{N-K}) \qquad (3.2.4)$$

with Ω a measurable subset of Γ_{N-K}. It suffices to prove (3.2.4) for Ω an arbitrary cylinder in Γ_{N-K}. So we may assume

$$\Omega = \{ \gamma \in \Gamma_{N-K} \, | \, \gamma \, |_L = \bar{\nu} \} \qquad (3.2.5)$$

with L a fixed, finite subset of $N - K$ and $\bar{\nu}$ a fixed element of Γ_L. For convenience we introduce the symbols C_1 resp. C_2 for the left resp. right hand side of (3.2.4). By considering suitable partitions of Ω the difference between C_1 and C_2 will be shown to be arbitrary small.

Because A satisfies (3.2.2) there exists a finite subset K_0 of N such that

$$r_{N-K_0}A\gamma \text{ is independent of } r_K\gamma \qquad (\gamma \in \Gamma). \tag{3.2.6}$$

Because f^{K_0} is a positive, continuous function on a compact space it has a positive minimum M. Choose $\varepsilon \in \mathbb{R}$ such that

$$0 < \varepsilon < \frac{1}{2}M. \tag{3.2.7}$$

Because $f'^K(\bar{\delta}, .)$ is a continuous function on the compact space Γ_{N-K} there exists a finite subset K_1 of $N - K$ such that

$$|f'^K(\bar{\delta}, \gamma) - f'^K(\bar{\delta}, \gamma')| < \varepsilon \tag{3.2.8}$$

for all $\gamma, \gamma' \in \Gamma_{N-K}$ with $\gamma|_{K_1} = \gamma'|_{K_1}$.

For the same reasons there exists a finite subset K_2 of $N - K_0$ such that

$$|f^{K_0}(\mu, \gamma) - f^{K_0}(\mu, \gamma')| < \varepsilon$$

for all $\mu \in \Gamma_{K_0}$ and $\gamma, \gamma' \in \Gamma_{N-K_0}$ with $\gamma|_{K_2} = \gamma'|_{K_2}$. (3.2.9)

Because A is continuous $(A\gamma)(i)$ depends on a finite number of $\gamma(j)$'s (see section 2). So there exists a finite subset K_3 of $N - K$ such that $r_{K_0 \cup K_2}A\gamma$ is completely determined by $r_{K_3 \cup K}\gamma$ for all $\gamma \in \Gamma$. (3.2.10)

The subset K_4 of $N - K$ is defined by

$$K_4 = L \cup K_1 \cup K_3. \tag{3.2.11}$$

Now we will show the difference between C_1 and C_2 to be of order ε. First we calculate C_2. Since K_4 is a finite subset of $N - K$ the space Γ_{N-K} can be divided into $2^{|K_4|}$ cylinders belonging to K_4. As $L \subset K_4$ the cylinder Ω in Γ_{N-K} is a union of cylinders $\Omega_1, \ldots, \Omega_r$ belonging to K_4. As $K_1 \subset K_4$ from (3.2.8) it follows that

$$|f'^K(\bar{\delta}, \gamma) - f'^K(\bar{\delta}, \gamma')| < \varepsilon \tag{3.2.12}$$

for all $\gamma, \gamma' \in \Omega$ whenever γ and γ' belong to a same cylinder Ω_i.

This gives the following estimate at C_2

$$C_2 = \left\{ \sum_{i=1}^{r} f'^K(\bar{\delta}, \gamma^i) . r_{N-K}\Pi'(\Omega_i) \right\} + \lambda . r_{N-K}\Pi'(\Omega) \tag{3.2.13}$$

with γ^i an arbitrary, fixed element in Ω_i and $\lambda \in \mathbb{R}$ with $|\lambda| < \epsilon$. Next we calculate C_1. We have

$$r^{\delta}_{N-K}\Pi'(\Omega) = \sum_{i=1}^{r} r^{\bar{\delta}}_{N-K}\Pi'(\Omega_i). \tag{3.2.14}$$

Consider $r^{\delta}_{N-K}\Pi'(\Omega_i)$ for $\delta \in \Gamma_K$ arbitrary. Let us define cylinders $C_{\delta,i}$ in Γ belonging to $K \cup K_4$

$$C_{\delta,i} = \{ \gamma \in \Gamma \mid r_K\gamma = \delta \quad \text{and} \quad r_{N-K}\gamma \in \Omega_i \}. \tag{3.2.15}$$

By definition

$$r^{\delta}_{N-K}\Pi'(\Omega_i) = \Pi(A(C_{\delta,i})). \tag{3.2.16}$$

We can write

$$A(C_{\delta,i}) = \{ \gamma \in \Gamma \mid r_{K_0}\gamma = \mu(\delta, i) \quad \text{and} \quad r_{N-K_0}\gamma \in \tilde{\Omega}_i \}$$

where

$$\mu(\delta, i) = r_{K_0}A\gamma \quad \text{with} \quad \gamma \in C_{\delta,i} \tag{3.2.17}$$

$$\tilde{\Omega}_i = \{ \gamma \in \Gamma_{N-K_0} \mid \text{there exists a } \gamma' \in C_{\delta,i} \text{ such that } r_{N-K_0}A\gamma' = \gamma \}.$$

Notice that $\mu(\delta, i)$ is well defined because of (3.2.10). Because of (3.2.6) the set $\tilde{\Omega}_i$ is independent of δ. Furthermore $\tilde{\Omega}_i$ is measurable in Γ_{N-K_0} (it is the union of a finite number of cylinders in Γ_{N-K_0}). Because Π is an equilibrium state of $\{ h_K \}$ it follows from (3.2.16) and (3.2.17) that

$$r^{\delta}_{N-K}\Pi'(\Omega_i) = r^{\mu(\delta,i)}_{N-K_0}\Pi(\tilde{\Omega}_i) = \int_{\tilde{\Omega}_i} f^{K_0}(\mu(\delta, i), \gamma)dr_{N-K_0}\Pi(\gamma) \qquad (\gamma \in \Gamma_{N-K_0}). \tag{3.2.18}$$

Because of (3.2.10) the configurations of Γ_{N-K_0} belonging to $\tilde{\Omega}_i$ are equal on K_2. Applying (3.2.9) this gives

$$| f^{K_0}(\mu, \gamma) - f^{K_0}(\mu, \gamma') | < \varepsilon$$
$$\text{for all } \gamma, \gamma' \in \tilde{\Omega}_i \text{ and all } \mu \in \Gamma_{K_0}. \tag{3.2.19}$$

This is true for all i. Combination of (3.2.18) and (3.2.19) gives the following estimate

$$r^{\delta}_{N-K}\Pi'(\Omega_i) = \{ f^{K_0}(\mu(\delta, i), r_{N-K_0}A(\delta, \gamma^i)) + \lambda(\delta, i) \} . r_{N-K_0}\Pi(\tilde{\Omega}_i) \tag{3.2.20}$$

with $| \lambda(\delta, i) | < \varepsilon$.

Using (3.2.20) we have

$$r_{N-K}\Pi'(\Omega_i) = \sum_{\delta \in \Gamma_K} r^{\delta}_{N-K}\Pi'(\Omega_i) = \left\{ \left[\sum_{\delta \in \Gamma_K} f^{K_0}(\mu(\delta, i), r_{N-K_0}A(\delta, \gamma^i)) \right] + 2^{|K|} . \lambda_i \right\} . r_{N-K_0}\Pi(\tilde{\Omega}_i) \tag{3.2.21}$$

with $| \lambda_i | < \varepsilon$.

Combination of (3.2.20) and (3.2.21) gives

$$r^{\bar{\delta}}_{N-K}\Pi'(\Omega_i) = \frac{f^{K_0}(\mu(\bar{\delta}, i), r_{N-K_0}A(\bar{\delta}, \gamma^i)) + \lambda(\bar{\delta}, i)}{\left[\sum_{\delta \in \Gamma_K} f^{K_0}(\mu(\delta, i), r_{N-K_0}A(\delta, \gamma^i)) \right] + 2^{|K|} . \lambda_i} . r_{N-K}\Pi'(\Omega_i). \tag{3.2.22}$$

Using (3.2.7), $|\lambda(\bar{\delta}, i)| < \varepsilon$ and $|\lambda_i| < \varepsilon$ this gives

$$r^{\bar{\delta}}_{N-K}\Pi'(\Omega_i) = \left[\frac{f^{K_0}(\mu(\bar{\delta}, i), r_{N-K_0}A(\bar{\delta}, \gamma^i))}{\sum_{\delta \in \Gamma_K} f^{K_0}(\mu(\delta, i), r_{N-K_0}A(\delta, \gamma^i))} + \frac{4\alpha_i}{M}\right] . r_{N-K}\Pi'(\Omega_i) \tag{3.2.23}$$

with $|\alpha_i| < \varepsilon$.

Next we will show

$$f'^K(\bar{\delta}, \gamma^i) = \frac{f^{K_0}(\mu(\bar{\delta}, i), r_{N-K_0}A(\bar{\delta}, \gamma^i))}{\sum_{\delta \in \Gamma_K} f^{K_0}(\mu(\delta, i), r_{N-K_0}A(\delta, \gamma^i))}. \tag{3.2.24}$$

By definition

$$f'^K(\bar{\delta}, \gamma^i) = [\exp h'_{K'}(\bar{\delta}, \gamma^i)] . \left[\sum_{\delta \in \Gamma_K} \exp h'_{K'}(\delta, \gamma^i)\right]^{-1}$$

with K' large enough. By definition $h'_{K'} = h_{K''} \circ A$ for K'' large enough. So

$$f'^K(\bar{\delta}, \gamma^i) = [\exp h_{K''}(A(\bar{\delta}, \gamma^i))] . \left[\sum_{\delta \in \Gamma_K} \exp h_{K''}(A(\delta, \gamma^i))\right]$$

$$= \frac{\exp h_{K''}(r_{K_0}A(\bar{\delta}, \gamma^i), r_{N-K_0}A(\bar{\delta}, \gamma^i))}{\sum_{\delta \in \Gamma_K} \exp h_{K''}(r_{K_0}A(\delta, \gamma^i), r_{N-K_0}A(\delta, \gamma^i))}. \tag{3.2.25}$$

From (3.2.6) it follows

$$r_{N-K_0}A(\delta, \gamma^i) = r_{N-K_0}A(\bar{\delta}, \gamma^i) \qquad \text{for all} \qquad \delta \in \Gamma_K. \tag{3.2.26}$$

Using (3.2.17), (3.2.26) and the definition of f^{K_0} the equality (3.2.25) gives (3.2.24).

Because of (3.2.23) and (3.2.24) we have an estimate at C_1

$$C_1 = \sum_{i=1}^{r} r^{\bar{\delta}}_{N-K}\Pi'(\Omega_i) = \left[\sum_{i=1}^{r} f'^K(\bar{\delta}, \gamma^i) . r_{N-K}\Pi'(\Omega_i)\right] + \frac{4\alpha}{M} . r_{N-K}\Pi'(\Omega) \tag{3.2.27}$$

with $|\alpha| < \varepsilon$.

From (3.2.13) and (3.2.27) it follows $C_1 - C_2 = \left(\frac{4\alpha}{M} - \lambda\right) . r_{N-K}\Pi'(\Omega)$.

So $|C_1 - C_2| < \left(\frac{4}{M} + 1\right) . \varepsilon$. Since ε can be chosen as small as we want this gives $C_1 = C_2$. Thus we have proven proposition 4.

We conclude this section with a few remarks. A homeomorphism $A : \Gamma \to \Gamma$ satisfying (3.2.2) satisfies also (2.3.5). This can be shown by considering the equilibrium state of the system $\{ h_K \}$ with $h_K = 0$ for all K. However, it is clear that the set of transformations satisfying both (2.3.4) and (2.3.5) is larger than the set of homeomorphisms satisfying (3.2.2).

It is clear that a homeomorphism A of Γ satisfying (3.2.2) induces an injection of the set of equilibrium states of system $\{ h_K \}$ into the set of equilibrium states of $A\{ h_K \}$. If A^{-1} also satisfies (3.2.2) this injection is surjective. If A^{-1} does not satisfy (3.2.2) it is not clear whether this injection is surjective.

4. THERMODYNAMIC RELATIONS BETWEEN LATTICE SPIN SYSTEMS

4.1. In this section we discuss spin systems on lattices. So $N = Z^\nu$ with ν some positive integer. Let G_ν be the group of translations of this lattice. A translation $g \in G_\nu$ induces in a natural way a transformation (also denoted by g) in the configuration space Γ_ν; $g(\gamma) = \gamma \circ g^{-1}$ ($\gamma \in \Gamma_\nu$). In this definition the configuration γ should be regarded as a mapping from Z^ν into $\{-1, 1\}$. We consider only systems with translation invariant interaction. The translation invariance of a system $\{ h_K \}$ is reflected by the condition $h_K - h_{gK} \circ g$ is independent of $\gamma |_K$ for all finite subsets K of Z^ν and $g \in G_\nu$. We do not require $h_K = h_{gK} \circ g$ because of the ambiguity in the definition of $\{ h_K \}$ for a system (see 3.1). One easily sees that if $A : \Gamma_\nu \to \Gamma_\nu$ satisfies (3.2.1) and (3.2.2) and commutes with all translations (regarded as transformations of Γ_ν) then the mapping $\{ h_K \} \rightsquigarrow A\{ h_K \}$ conserves the translation invariance. For this reason we restrict ourselves to transformations of Γ_ν that commute with all translations. A homeomorphism A of Γ_ν that commutes with all translations automatically satisfies condition (3.2.2).

So the transformations of interest are the homeomorphisms of Γ_ν commuting with all translations. They constitute a group F_ν with respect to composition of transformations.

Let us summarize the interest of the groups F_ν for the equilibrium statistical mechanics of lattice spin systems. If $A \in F_\nu$, $\{ h_K \}$ a translation invariant system, Π an equilibrium state of $\{ h_K \}$ then Π' defined by $\Pi'(\Omega) = \Pi(A\Omega)$ is an equilibrium state of the translation invariant system $A\{ h_K \}$. In this way A induces a bijection of the set of all equilibrium states of $\{ h_K \}$ into the set of all equilibrium states of $A\{ h_K \}$. In this sense all translation invariant systems belonging to the same orbit with respect to F_ν are equivalent. A special orbit is the one to which the system

$$\{ h_K \} \text{ with } h_K = \sum_{i \in K} \sigma_i \text{ belongs.}$$

All systems in this orbit are equivalent to the system without coupling. Given a translation invariant system $\{ h_K \}$ one can consider the subgroup H of F_ν consisting of all transformations A with $A\{h_K\} = \{h_K\}$.

The group H can be considered as the symmetry group of the system $\{h_K\}$. If Π is an equilibrium state of $\{h_K\}$ then Π' defined by $\Pi'(\Omega) = \Pi(A\Omega)$ is also for all $A \in H$. If $\{h_K\}$ has a unique equilibrium state Π we have $\Pi(A\Omega) = \Pi(\Omega)$. So Π is invariant under the group H. Of course all these statements can be translated into ones about correlation functions. Orbits and symmetry groups are related. Loosely speaking one can say that the larger the orbit to which a system belongs the smaller its symmetry group and *vice versa.*

4.2. The extent to which the group F_ν and its action on the space of translation invariant interactions is known determines the results the foregoing analysis can give. So we need more information about F_ν. Let us consider the 1-dimensional case. Taking into account the results of section 2 we see that $A : \Gamma_1 \to \Gamma_1$ belongs to F_1 iff there exists $n, m \in Z$ with $n \leqslant m$ and a function $f : \{-1, 1\}^{m-n+1} \to \{-1, 1\}$ such that

$$(A\gamma)(i) = f(\gamma(i + n), \gamma(i + n + 1), \ldots, \gamma(i + m))$$

for all $\gamma \in \Gamma_1$ and $i \in Z$, A is bijective.

In the frame-work of code theory A is called a code. Such codes also appear in related work by Ornstein [*4*] on Bernoulli systems. The translations belong to F_1. The inversion C of all spins also belongs to F_1; $(C\gamma)(i) = -\gamma(i)$ $(\gamma \in \Gamma_1, i \in Z)$. Elements of F_1 belonging to the subgroup generated by the translations and C will be called trivial. It is not obvious that non-trivial transformations exist in F_1. However, they do exist. The following results concerning F_1 are taken from a report by O. P. Lossers, J. H. van Lint and W. Nuij [*5*] on the subject.

The smallest $m - n + 1$ for which a non-trivial transformation exists equals 4. Consider the function

$$f(s_1, s_2, s_3, s_4) = s_2 . \left\{ 1 + \frac{1}{4}(s_1 + 1)(s_3 - 1)(s_4 + 1) \right\} \qquad (s_i = \pm 1). \qquad (4.2.1)$$

Let A_f be the transformation described by f. So

$$(A_f\gamma)(i) = \gamma(i+n+1) . \left\{ 1 + \frac{1}{4} [\gamma(i+n) + 1][\gamma(i+n+2) - 1][\gamma(1+n+3) + 1] \right\} . \qquad (4.2.2)$$

By substitution one proves $A_f^2 = T^{2n+2}$ with T the translation defined by $(T\gamma)(i) = \gamma(i + 1)$. Hence A_f is bijective and belongs to F_1.

For $k > 4$ one can define functions $f(s_1, \ldots, s_k)$ similar to (4.2.1)

$$f(s_1, \ldots, s_k) = s_2 . \{ 1 - 2^{-k+2} . (s_1 - 1)(s_3 + 1) \ldots (s_{k-1} + 1)(s_k - 1) \} . \qquad (4.2.3)$$

Let A_f be the transformation described by f. Again one can prove

$A_f^2 = T^{2n+2}$. Hence A_f is bijective and belongs to F_1. The group F_1 also contains transformations A with $A^n \neq T^m$ for all n, $m \in Z - \{0\}$. The product CA_f with A_f defined by (4.2.2) is an example of such a transformation. So beyond the trivial transformations there exist a large variety of non-trivial ones.

With the foregoing transformations we can illustrate the statements in 4.1 about orbits and symmetry groups. Applying transformation (4.2.2) to the trivial system $\{h_K\}$ with $h_K = \sum_{i \in K} \sigma_i$ gives the system with the formal Hamiltonian

$$\sum_{i=-\infty}^{\infty} \frac{3}{4}\sigma_i - \frac{1}{4}\sigma_i\sigma_{i+2} - \frac{1}{4}\sigma_i\sigma_{i+1}\sigma_{i+3} + \frac{1}{2}\sigma_i\sigma_{i+1}\sigma_{i+2} + \frac{1}{4}\sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}.$$

Obviously this system belongs to the orbit of the trivial system and can be decoupled. Consider the system $A_f\{h_K\} + \{h_K\}$ with A_f described by (4.2.3) and $\{h_K\}$ an arbitrary translation invariant system. Because $A_f^2 = T^{2n+2}$ the transformation A_f belongs to the symmetry group of the system $A_f\{h_K\} + \{h_K\}$.

The following example of a non-trivial transformation in F_2 is taken from [5]

$$(A\gamma)(i,j) = \gamma(i,j) \cdot \left\{ 1 - \frac{1}{8}[\gamma(i-1,j)+1][\gamma(i-1,j+1)-1][\gamma(i,j+1)+1][\gamma(i+1,j)-1] \right\}.$$

It is not difficult to prove the equality $A^2 = 1$. So A is bijective and belongs to F_2. Applying this transformation to the trivial system $\{h_K\}$ with $h_K = \sum_{(i,j) \in K} \sigma_{ij}$ gives a system with the formal Hamiltonian

$$\sum_{i,j=-\infty}^{\infty} \left\{ \frac{7}{8}\sigma_0 + \frac{1}{8}\sigma_0\sigma_b - \frac{1}{8}\sigma_0\sigma_c + \frac{1}{4}\sigma_0\sigma_a\sigma_b - \frac{1}{8}\sigma_0\sigma_a\sigma_c + \frac{1}{8}\sigma_0\sigma_a\sigma_d + \frac{1}{8}\sigma_0\sigma_b\sigma_c \right.$$
$$- \frac{1}{8}\sigma_0\sigma_b\sigma_d + \frac{1}{8}\sigma_0\sigma_a\sigma_b\sigma_c - \frac{1}{8}\sigma_0\sigma_a\sigma_b\sigma_d + \frac{1}{8}\sigma_0\sigma_a\sigma_c\sigma_d - \frac{1}{8}\sigma_0\sigma_b\sigma_c\sigma_d$$
$$\left. - \frac{1}{8}\sigma_0\sigma_a\sigma_b\sigma_c\sigma_d \right\}.$$

Here 0, a, b, c, d are shorthand notations for 2-dimensional lattice sites;

$0 = (i,j)$ $\quad a = (i-1,j)$ $\quad b = (i-1, j+1)$ $\quad c = (i, j+1)$ $\quad d = (i+1, j)$.

Since this rather complicated system belongs to the orbit of the trivial system it can be decoupled.

One might conclude from the foregoing considerations that a deeper investigation of the groups F_ν is worth while [6]. As the action of F_ν on the linear space of translation invariant systems $\{ h_K \}$ is crucial the representation theory of F_ν could serve as a mean to classify the different types of ν-dimensional lattice spin systems.

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