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Quasi-local observables and the problem of measurement in quantum mechanics

by

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ABSTRACT. — In the framework of infinite quantum systems, we discuss the possibility that interference terms vanish for $t \to \infty$ for all quasilocal observables. We show that this is possible even if the measuring apparatus that interacts with the object is only a finite subsystem of the infinite (interacting) system. The intensive observables of the infinite system are not altered. Unfortunately, approach to a unique equilibrium state is very likely to happen. We study a particular model for which all interference terms decrease faster than the differences between the expectation values in different states.

RÉSUMÉ. — Dans le cadre des systèmes quantiques infinis, nous discutons la possibilité que les termes d'interférence s'annulent pour $t \to \infty$ pour toutes les observables quasi locales. Nous démontrons que cela est possible même si l'appareil de mesurage n'est qu'un sous-système fini du système infini (avec interactions). Les observables intensives du système infini ne sont pas altérées. Malheureusement, il est très probable qu'on aboutisse à un unique état d'équilibre. Nous étudions un modèle particulier pour lequel tous les termes d'interférence décroissent plus rapidement que les différences entre les espérances mathématiques en états différents.

1. INTRODUCTION

We call « problem of measurement » the problem of justifying the transition

$$|\psi\rangle\langle\psi| \to \sum_n |\langle\alpha_n|\psi\rangle|^2 |\alpha_n\rangle\langle\alpha_n|$$

(which takes place when an observable $A = \sum_n \alpha_n \mid \alpha_n \rangle \langle \alpha_n \mid$ is measured on a system in the state ψ) in terms of an interaction between the observed system and a measuring apparatus, and, if possible, without any consideration of the conscious ego of the observer.

After the first treatment by von Neumann [1], many authors dealt with this problem, looking for the possibility that in some sense a pure state can evolve into a mixture by a Hamiltonian time evolution. Certainly, vector states can evolve only into vector states. But if not every selfadjoint operator corresponds to an observable (for the apparatus), then it is possible that a vector state is equivalent to a mixture.

Let us consider a system and an apparatus, with Hilbert spaces h and $\mathscr{H}$ respectively. It is possible, following [1], to construct a unitary operator U acting on $h \otimes \mathscr{H}$ such that

$$
\begin{aligned}
&U(\mid \psi \rangle \langle \psi \mid \otimes \mid \phi_0 \rangle \langle \phi_0 \mid)U^* \\
&\qquad = U\Big(\sum_{mn} \langle \alpha_m \mid \psi \rangle \langle \psi \mid \alpha_n \rangle \mid \alpha_m \rangle \langle \alpha_n \mid \otimes \mid \phi_0 \rangle \langle \phi_0 \mid\Big)U^* \\
&\qquad = \sum_{mn} \langle \alpha_m \mid \psi \rangle \langle \psi \mid \alpha_n \rangle \mid \alpha_m \rangle \langle \alpha_n \mid \otimes \mid \phi_m \rangle \langle \phi_n \mid
\end{aligned}
$$

where $\{ \phi_n \}$ in a suitably chosen orthonormal system for $\mathscr{H}$. If all « interference terms » $\mathrm{Tr}\,(\mid \alpha_m \rangle\langle\alpha_n \mid \otimes \mid \phi_m \rangle\langle\phi_n \mid A_1 \otimes A_2)$ $(m \neq n)$ are zero for every observable, the vector state so obtained is equivalent to the mixture

$$\sum_n \mid \langle \alpha_n \mid \psi \rangle \mid^2 \mid \alpha_n \rangle \langle \alpha_n \mid \otimes \mid \phi_n \rangle \langle \phi_n \mid$$

No exact vanishing of all interference terms can be found, as long as one deals with finite systems, even with superselection rules (see [2]).

On the contrary, as K. Hepp [3] has shown, the problem of measurement can be solved in the framework of the algebraic approach to quantum mechanics of infinite systems.

In his approach, the infinite system is intended to be the apparatus, *i. e.* the apparatus is assumed to be a macroscopic system and is studied in the thermodynamic limit. The object on which the measurement is performed interacts with the whole infinite system.

Our work is manifestly inspired by Hepp's paper. But our purpose is to obtain the vanishing of all interference terms considering the infinite system as a large, but finite, apparatus coupled to an infinite reservoir (or « to the rest of the world »). So we want to overcome the objection that every actual apparatus interacting directly with an object is a finite system.

We use standard algebraic methods of Statistical Mechanics (see for instance [4], [6]) and study a particular model, in which the well known X-Y model (for an essential treatment, see [5]) is made to represent the apparatus plus reservoir. We find that the interference terms between suitably chosen vector states vanish at least as fast as $t^{-N/2}$ for $t \to \infty$ (N is the number of degrees of freedom of the apparatus), while the states approach the same limit, but at a much slower rate (t^{-1}).

In Section 2, we recall the notations and the results of the algebraic approach to Statistical Mechanics, that are useful in connection with the problem of measurement. In Section 3, we recall the X-Y model. In Section 4, we study our model for a measuring process and show the results we have announced. A short discussion follows.

2. GENERAL CRITERIA FOR INFINITE SYSTEMS

In the algebraic approach to quantum statistical mechanics:

— the observables are structured in the quasi-local C*-algebra $\mathscr{A}$,

— the states are positive linear functionals on $\mathscr{A}$.

For every state ω, the GNS representation π_ω of $\mathscr{A}$ as a subalgebra of $\mathscr{B}(\mathscr{H}_\omega)$ can be constructed. The concept of « vector state » depends on the representation (for example, every state is a vector state in its own GNS representation) and is completely independent from the concept of « pure state » (a state that cannot be decomposed into the convex combination of two different states).

In connection with the problem of measurement, we will look for pairs of states ω_1 and ω_2 such that in a representation π they are vector states with vectors Ω_1 and Ω_2, such that a coherent superposition

$$\langle c_1\Omega_1 + c_2\Omega_2 \mid \pi(.) \mid c_1\Omega_1 + c_2\Omega_2 \rangle$$

and an incoherent mixture

$$| c_1 |^2\omega_1(.) + | c_2 |^2\omega_2(.)$$

give the same expectation values for every observable $A \in \mathscr{A}$, or, that is the same, such that the interference terms

$$\langle \Omega_1 \mid \pi(A)\Omega_2 \rangle \qquad \text{vanish for every} \qquad A \in \mathscr{A}.$$

As in general not every selfadjoint operator in $\mathscr{B}(\mathscr{H}_\pi)$ corresponds to an observable, this condition may be satisfied.

We give as a first example a spin system, which is the essential of the Coleman model in [3]. Be $\mathscr{A} = \bigotimes_{n=1}^{\infty} \mathscr{B}(\mathscr{H}_n)$; $\mathscr{H}_n \cong C^2$. Choose for states $\Omega^\pm = \bigotimes_{n=1}^{\infty} \psi_n^\pm$ with $\sigma_n^3\psi_n^\pm = \pm\,\psi_n^\pm$. As a local observable concerns only a finite number of degrees of freedom, and, for each n, $\langle \psi_n^+ \mid \psi_n^- \rangle = 0$, it is easily seen that $\langle \Omega^+ \mid A\Omega^- \rangle = 0$, $\forall A \in \mathscr{A}$.

A general criterion is found in [3]:

$$\text{I)} \qquad \langle \Omega_1 \mid \pi(A)\Omega_2 \rangle = 0 \qquad \forall A \in \mathscr{A}$$

if the two states $\omega_i = \langle \Omega_i \mid \pi(.)\Omega_i \rangle$ $(i = 1,2)$ are disjoint, *i.e.* no subrepresentation of the GNS representation π_{ω_1} is equivalent to a subrepresentation of π_{ω_2}. If we want this relation to hold for every representation π for which ω_1 and ω_2 are vector states, the condition is also necessary.

In view of the fact that, if the time evolution of the system is given by a one-parameter group of automorphisms τ_t of $\mathscr{A}$, two non-disjoint states cannot become disjoint in a finite time interval [3], some sufficient criteria on the vanishing of interference terms in the limit $t \to \infty$ are more useful.

$$\text{II)} \qquad \langle \Omega_1 \mid U(t)^*\pi(A)U(t) \mid \Omega_2 \rangle \to 0 \qquad \text{for} \qquad t \to \infty$$

if the states $\omega_{i,t} = \langle \Omega_i \mid U(t)^*\pi(.)U(t)\Omega_i \rangle$ $(i = 1,2)$ tend weakly towards disjoint states for $t \to \infty$ [3].

The relevance of this criterion is due to the fact that, if two factor states give different expectation values for an intensive observable (of the kind $\tilde{A} = w\text{-}\lim_{N \to \infty} N^{-1} \sum_{n=1}^{N} \pi(A_n)$, where $A_n \in \mathscr{A}_{\Lambda_n}$ and $\{\Lambda_n\}$ is a sequence of bounded regions such that $\cup_n \Lambda_n$ covers the entire space), then they are disjoint [3]. This is the key to Hepp's results.

III) If $(\mathscr{A}, \tau_t)$ is such that $\| [\tau_t A, B] \| \underset{t \to \infty}{\to} 0$ $\forall A, B \in \mathscr{A}$ (asymptotic abelianness), and there exists $T \in \mathscr{A}$ such that $\pi(T^*)\Omega_1 = 0$, $\pi(T)\Omega_2 = \Omega_2$, then one finds:

$$\langle \Omega_1 \mid \pi(\tau_t A)\Omega_2 \rangle = \langle \Omega_1 \mid \pi([\tau_t A, T])\Omega_2 \rangle \underset{t \to \infty}{\longrightarrow} 0 \qquad \forall A \in \mathscr{A}$$

IV) If the time evolution is given by a group of automorphisms of $\mathscr{A}$, $\omega_2(A) = \omega_1(B^*AB) \forall A \in \mathscr{A}$, with $\omega_1(B) = 0$, $\omega_1(B^*B) = 1$ and ω_1 is strongly clustering (*i. e.*, $\mid \omega_1(\tau_t A \cdot B) - \omega_1(\tau_t A).\omega_1(B) \mid \to 0$ for $t \to \infty$), then:

$$\langle \Omega_1 \mid \pi(\tau_t A)\Omega_2 \rangle = \langle \Omega_1 \mid \pi(\tau_t A \cdot B)\Omega_1 \rangle = \omega_1(\tau_t A \cdot B)$$
$$\underset{t \to \infty}{\sim} \omega_1(\tau_t A)\omega_1(B) = 0 \qquad \forall A \in \mathscr{A}$$

We recall that the properties of asymptotic abelianness and of strongly clustering are usually considered in connection with the problem of return to equilibrium (see *e. g.* [6]). In our case, they can give weak convergence of $\omega_1 \circ \tau_t$ and $\omega_2 \circ \tau_t$ to the same limit. For instance, if both are present, then:

$$\omega_2(\tau_t A) = \omega_1(B^* \tau_t A \cdot B) \underset{t \to \infty}{\sim} \omega_1(\tau_t A)\omega_1(B^*B) = \omega_1(\tau_t A)$$

Moreover, if B is unitary, the clustering property is not even necessary.

In our approach, we are going to assume that only a finite part of the infinite system interacts with the object. We call it the « Apparatus » and we assume that it is taken into orthogonal states in correspondence to orthogonal states of the object. The apparatus interacts with the « rest of the world » (we may suppose that the universe is an actually infinite system), or at least with an infinite part of it (an infinite reservoir), and the time evolution of the compound system apparatus plus reservoir cancels the interference terms. We must not alter the macroscopic quantities of the universe, looking for disjoint states, but we may use the criterion III) or IV). It is probable that the expectation values of an arbitrary observable in the different states of the apparatus tend to the same limit for $t \to \infty$. So we lack a permanently recording apparatus, but we can still call such a process a « measurement » if all interference terms vanish considerably faster than the difference between the expectation values of one significant observable, which plays the role of « pointer position ».

3. THE X-Y MODEL [5]

The X-Y model is a one-dimensional spin-one-half system. To each $j \in Z$, a Hilbert space $\mathscr{H}_j \cong C^2$ and a C*-algebra $\mathscr{B}(\mathscr{H}_j)$ are attached. For a finite interval $[m, n]$, the local algebra is $\mathscr{A}_{m,n} = \mathscr{B}(\bigotimes_{j=m}^{n} \mathscr{H}_j)$, and the quasi-local algebra is:

$$\mathscr{A} = \overline{\bigcup_{-\infty < m \leqslant n < +\infty} \mathscr{A}_{m,n}}$$

The time evolution is given by a group of automorphisms τ_t, $t \in \mathbb{R}$

$$\tau_t A = \lim_{\substack{m \to -\infty \\ n \to +\infty}} \exp(iH_{m,n}t) A \exp(-iH_{m,n}t) \tag{3.1}$$

with the local Hamiltonian given by:

$$H_{m,n} = \frac{J}{2} \sum_{j=m}^{n-1} a_j^* a_{j+1} + a_{j+1}^* a_j + h \sum_{j=m}^{n} a_j^* a_j \tag{3.2}$$

with

$$a_j = \frac{1}{2}(\sigma_j^1 + i\sigma_j^2)$$

$$\{ a_j, a_j \} = 0 \; ; \quad 1 = \{ a_j, a_j^* \} \; ; \quad [a_i, a_j] = 0 \; ; \quad [a_i, a_j^*] = \delta_{ij}\sigma_j^3 \tag{3.3}$$

The algebra $\mathscr{A}^e$ generated by the even polynomials in the a^*, a's can be equally generated by the even polynomials in Fermi operators b^*, b, satisfying:

(3.4) $$\{ b_i, b_j \} = 0 \,; \qquad \{ b_i, b_j^* \} = \delta_{ij}$$

and such that, for $p \leqslant q$:

(3.5) $$\begin{aligned} b_p^* b_q &= \quad a_p^* \prod_{r=p+1}^{q-1} (1 - 2a_r^* a_r) a_q \\ b_p b_q &= - a_p \prod_{r=p+1}^{q-1} (1 - 2a_r^* a_r) a_q \end{aligned}$$

and analogous expressions.

Now the local Hamiltonian (3.2) is re-expressed by:

(3.2′) $$\mathrm{H}_{m,n} = \frac{\mathrm{J}}{2} \sum_{j=m}^{n-1} b_j^* b_{j+1} + b_{j+1}^* b_j + h \sum_{j=m}^{n} b_j^* b_j$$

The time evolution of the fundamental $b's$ is computed to be:

(3.6) $$\tau_t b_p = \sum_q \int_0^{2\pi} \frac{d\vartheta}{2\pi} e^{i(p-q)\vartheta} e^{it(\mathrm{J}\cos\vartheta + h)} b_q$$

So we find:

(3.7) $$\begin{aligned} \{ \tau_t b_p, b_q^* \} &= \int_0^{2\pi} \frac{d\vartheta}{2\pi} e^{i(p-q)\vartheta} e^{it(\mathrm{J}\cos\vartheta + h)} \to 0 \qquad \text{for} \quad t \to \infty \\ \{ \tau_t b_p^*, b_q \} &= \int_0^{2\pi} \frac{d\vartheta}{2\pi} e^{-i(p-q)\vartheta} e^{-it(\mathrm{J}\cos\vartheta + h)} \to 0 \qquad \text{for} \quad t \to \infty \\ \{ \tau_t b_p, b_q \} &= 0 = \{ \tau_t b_p^*, b_q^* \} \qquad \forall t \end{aligned}$$

This proves that $(\mathscr{A}^e, \tau_t)$ is asymptotically Abelian (see [5] [6]). The asymptotic rate of decrease of the anticommutators is $t^{-1/2}$ (they are Bessel functions $\mathrm{J}_{p-q}(t)$, multiplied by a phase factor).

4. A MODEL FOR THE PROCESS OF MEASUREMENT

Now we apply the X-Y model to the problem of measurement. We call « apparatus » the part $[1,2n]$ of the spin chain, and « reservoir » the rest. We write:

$$\mathscr{A} = \mathscr{A}_{1,2n} \otimes \tilde{\mathscr{A}}$$

We take for Hilbert space

$$\mathscr{H} = \left(\bigotimes_{j=1}^{2n} \mathscr{H}_j \right) \otimes \mathscr{H}_{\tilde{\omega}}$$

where $\tilde{\omega}$ is a state on $\tilde{\mathscr{A}}$, and $\pi_{\tilde{\omega}}$ its GNS representation.

We choose the representation π defined by:

$$\pi(A_1 \otimes A_2) = A_1 \otimes \pi_{\tilde{\omega}}(A_2); \qquad \text{if} \qquad A_1 \in \mathscr{A}_{1,2n}, A_2 \in \tilde{\mathscr{A}}$$

In order to obtain interesting results, it is necessary to restrict our consideration to even states (*i. e.* states giving zero for the expectation values of every odd polynomial in the a^*, $a's-$ or b^*, $b's-$), so that we need only to consider the asymptotically Abelian even algebra. Furthermore, the initial state of the apparatus is to be chosen in an appropriate way, as either an eigenstate of $\bar{\sigma}^3 = (2n)^{-1} \sum_{1 \leqslant j \leqslant 2n} \sigma_j^3$, or an incoherent mixture of eigenstates of $\bar{\sigma}^3$, with an expectation value $\omega(\bar{\sigma}^3) \neq 0$.

We set, for brevity:

(4.1) $$\phi_\alpha^\pm = \bigotimes_{j=1}^{2n} \psi_j^{\pm\varepsilon(j,\alpha)} \qquad \omega_\alpha^\pm = \langle \phi_\alpha^\pm \mid (.) \phi_\alpha^\pm \rangle$$

where α labels a set of indices $\{ j_1 \dots j_k \} \subset [1,2n]$, $\varepsilon(j, \alpha) = -$ for $j \in \alpha$, $+$ for $j \notin \alpha$

When $\alpha = \emptyset$, we write simply $\phi^\pm$.

So we have

$$\phi_\alpha^+ = \phi_{[1,2n]\setminus\alpha}^-$$

$$\bar{\sigma}^3 \phi_\alpha^\pm = \pm \left(1 - \frac{N(\alpha)}{n}\right) \phi_\alpha^\pm$$

where $N(\alpha)$ is the number of indices belonging to α.

i. e. the $\phi_\alpha^\pm$'s are the eigenvectors of $\bar{\sigma}^3$.

The Gibbs state ω_G for the apparatus in a magnetic field $h \gg J$ is given approximately by a mixture of the ω_α^+'s (or ω_α^-'s)

(4.2) $$\omega_G = \frac{\sum_\alpha \omega_\alpha^+ e^{-\beta \frac{h}{2}\left(1 - \omega_\alpha^+(\bar{\sigma}^3)\right)}}{\sum_\alpha e^{-\beta \frac{h}{2}\left(1 - \omega_\alpha^+(\bar{\sigma}^3)\right)}}$$

If we suppose to prepare the state of the apparatus by applying a strong magnetic field h in the region $[1,2n]$ from $t = -\infty$ to $t = 0_-$, we can think that the apparatus is a system with Hamiltonian given approximately by $(h/2) \sum_{j=1}^{2n} (1 - \sigma_j^3)$, weakly coupled to a reservoir. So it is reasonable to suppose that the state of the apparatus will be ω_G for $t = 0$, at least if the state $\tilde{\omega}$ of the reservoir is KMS.

We want to use this apparatus to measure the spin of a particle. Let

$P^{\pm}$ be the projections on the eigenvectors of σ^3 for the particle. We set

$$H_{\text{int.}} \div \sum_{j=1}^{2n} \sigma_j^1 P^- ;$$

in order to obtain, in a very short time, such that we may neglect the « unperturbed » evolution of the X-Y model during it, the unitary time evolution operator

$$W = P^+ + P^- U \qquad U = \exp\left(-i \sum_{j=1}^{2n} \sigma_j^1\right) \tag{4.3}$$

(U can also be obtained as in the Coleman model of ref [3]).

U turns all the spins of the apparatus upside down:

$$\langle \phi_\alpha^+ \mid U^*(.)U\phi_\alpha^- \rangle = \langle \phi_\alpha^- \mid (.)\phi_\alpha^- \rangle \tag{4.4}$$

or

$$\omega_\alpha^{+U} = \omega_\alpha^-$$

After the interaction object-apparatus ($t = 0_+$) we have the states:

$$\begin{aligned} &\omega_G \text{ in correspondence to } P^+ \\ &\omega_G^U \text{ in correspondence to } P^- \end{aligned} \tag{4.5}$$

As in ω_G there is a prevalence of spins up, in ω_G^U there is a prevalence of spins down. The possibility of reading the measurement is given by the difference

$$\omega_G(\tau_t \bar{\sigma}^3) - \omega_G^U(\tau_t \bar{\sigma}^3) \tag{4.6}$$

For $0_+ < t < +\infty$, the compound system apparatus plus reservoir evolves according to the time automorphism of the X-Y model (without magnetic field).

The $\omega_\alpha^{\pm}$'s (and so ω_G and ω_G^U) are even states on $\mathscr{A}_{1,2n}$. If $\tilde{\omega}$ is an even state on $\tilde{\mathscr{A}}$ (which is certain if $\tilde{\omega}$ is KMS, see [7]), then $\omega_G^{(U)} \otimes \tilde{\omega}$ are even states on $\mathscr{A}$ and we may restrict our consideration to $\mathscr{A}^e$, which is asymptotically Abelian.

We express $\mathscr{A}^e$ in terms of the Fermi operators b_i^*, b_j. Using Equations (3.6) and (3.7) and the assumptions above we can show that:

a) All interference terms between these states decrease at least as fast as t^{-n} for $t \to \infty$ (notice that all interference terms are zero for the elements of the odd algebra, since they equal $\omega_G \otimes \tilde{\omega}(AU)$ and AU is odd if A is odd).

b) Let $\bar{\omega}_G^{(U)} = \omega_G^{(U)} \otimes \tilde{\omega}$. The difference $\bar{\omega}_G(\tau_t \bar{\sigma}^3) - \bar{\omega}_G^U(\tau_t \bar{\sigma}^3)$ decreases like t^{-1} for $t \to \infty$.

So, if n is large enough, all the interference terms have practically disappeared when the difference $\overline{\omega}_G(\tau_t\sigma^3) - \overline{\omega}_G^U(\tau_t\sigma^3)$ is still appreciable, allowing to read the measurement.

Proof of a). — Consider

$$T_j = T_j^* = b_j^* b_j, \qquad j \in [1,2n], \qquad T_j \in \mathscr{A}^e$$

$$(4.7) \qquad T_j\phi_\alpha^+ = \begin{cases} 0 & \text{for } j \notin \alpha \\ \phi_\alpha^+ & \text{for } j \in \alpha \end{cases} \qquad T_j\phi_\alpha^- = \begin{cases} \phi_\alpha^- & \text{for } j \notin \alpha \\ 0 & \text{for } j \in \alpha \end{cases}$$

Let us treat first ϕ^+ and ϕ^-, and then generalize to $\phi_\alpha^\pm$. A typical interference term is:

(4.8)

$$\langle \tilde\Omega | \otimes \langle \phi^+ | \pi(\tau_t A) | \phi^- \rangle \otimes | \tilde\Omega \rangle = \langle \tilde\Omega | \otimes \langle \phi^+ | \pi([\tau_t A, T_j]) | \phi^- \rangle \otimes | \tilde\Omega \rangle \to 0$$
$$\text{for} \quad t \to \infty \quad , \quad A \in \mathscr{A}^e, j \in [1,2n].$$

Let $A = \prod_{k\in\Lambda} b_k^\#$, $b^\# = b$ or b^*, Λ finite. Then:

(4.9)

$$\langle \tilde\Omega | \otimes \langle \phi^+ | \pi\Big(\prod_{k\in\Lambda} \tau_t b_k^\#\Big) | \phi^- \rangle \otimes | \tilde\Omega \rangle$$
$$= \langle \tilde\Omega | \otimes \langle \phi^+ | \pi\Big(\Big[\prod_{k\in\Lambda} \tau_t b_k^\#, b_1^* b_1\Big]\Big) | \phi^- \rangle \otimes | \tilde\Omega \rangle$$
$$= \sum_j \langle \tilde\Omega | \otimes \langle \phi^+ | \pi\big(\tau_t b_{k_1}^\# \dots \tau_t b_{k_{j-1}}^\# [\tau_t b_{k_j}^\#, b_1^* b_1] \tau_t b_{k_{j+1}}^\# \dots \tau_t b_{k_{N(\Lambda)}}^\#\big) | \phi^- \rangle \otimes | \tilde\Omega \rangle$$
$$= \sum_j \pm \langle \tilde\Omega | \otimes \langle \phi^+ | \pi\big(\tau_t b_{k_1}^\# \dots \tau_t b_{k_{j-1}}^\# b_1^\# \tau_t b_{k_{j+1}}^\# \dots \tau_t b_{k_{N(\Lambda)}}^\#\big) | \phi^- \rangle \otimes | \tilde\Omega \rangle$$
$$\times \int_0^{2\pi} \frac{d\vartheta}{2\pi} \exp(\pm i\vartheta(k_j - 1) \pm it J \cos\vartheta)$$

($+$ for b, $-$ for b^*; $b_1^\#$ starred if $b_{k_j}^\#$ is starred; we used

$$[A, BC] = \{A, B\}C - B\{A, C\})$$

For each summand we can repeat the operation, commuting with $b_2^* b_2$ (which commutes with $b_1^\#$), and so on, up to $b_{2n}^* b_{2n}$. So we obtain a sum on many indices of scalar products (which are bounded for $t \to \infty$) multiplied by $2n$ factors, each with asymptotical falloff $t^{-1/2}$. So the typical interference term decreases at least as fast as t^{-n}, Q. E. D.

For $\phi_\alpha^\pm$, we can still commute with the $b_r^* b_r$'s, $r = 1, \dots, 2n$. The null term is that with $b_r^* b_r$ on the left for $r \notin \alpha$, on the right for $r \in \alpha$; the other term equals the original scalar product. So we find the same results again, apart from a minus sign, if an odd number of r's belongs to α.

Proof of b). — Let $\overline{\omega}_\alpha^\pm = \omega_\alpha^\pm \otimes \tilde{\omega}$. It is easy to realize that, for $t \to \infty$:

$$\overline{\omega}_\alpha^+(\tau_t A) - \overline{\omega}_\alpha^-(\tau_t A) \to 0 \qquad \forall A \in \mathscr{A}^e \tag{4.10}$$

In fact, we can immediately verify that:

$$(4.11) \quad \begin{aligned} |\phi^+\rangle\langle\phi^-| &= \prod_{j=1}^{2n} a_j = (-1)^n \prod_{j=1}^{2n} b_j \\ |\phi_\alpha^+\rangle\langle\phi_\alpha^-| &= \prod_{j=1}^{2n} a_j^{(*)} = \pm \prod_{j=1}^{2n} b_j^{(*)} \quad \text{(the * if } j \in \alpha) \end{aligned}$$

Then the difference (4.10) is:

$$(4.10') \quad \langle\tilde{\Omega}| \otimes \langle\phi_\alpha^+|\pi([\tau_t A, |\phi_\alpha^+\rangle\langle\phi_\alpha^-|])|\phi_\alpha^-\rangle \otimes |\tilde{\Omega}\rangle \to 0 \text{ for } t \to \infty,$$

because A and $|\phi_\alpha^+\rangle\langle\phi_\alpha^-|$ belong to the asymptotically Abelian even algebra $\mathscr{A}^e$.

Now we consider the particular observable $b_r^* b_r = (1 - \sigma_r^3)/2$ which is the simplest « component » of the observable $\overline{\sigma}^3$.

$$\begin{aligned} &\overline{\omega}^+(\tau_t(b_r^* b_r)) - \overline{\omega}^-(\tau_t(b_r^* b_r)) \\ &= (-1)^n \langle\tilde{\Omega}| \otimes \langle\phi^+|\pi\Big(\Big[\tau_t b_r^* \tau_t b_r, \prod_{j=1}^{2n} b_j\Big]\Big)|\phi^-\rangle \otimes |\tilde{\Omega}\rangle \\ &= (-1)^{n+1} \sum_{k=1}^{2n} \langle\tilde{\Omega}| \otimes \langle\phi^+|\pi(b_1 \dots b_{k-1}\{\tau_t b_r^*, b_k\}\tau_t b_r b_{k+1} \dots b_{2n})|\phi^-\rangle \otimes |\tilde{\Omega}\rangle \\ &= (-1)^{n+1} \sum_{k=1}^{2n} \sum_q \langle\tilde{\Omega}| \otimes \langle\phi^+|\pi(b_1 \dots b_{k-1} b_q b_{k+1} \dots b_{2n})|\phi^-\rangle \otimes |\tilde{\Omega}\rangle \\ &\times \int_0^{2\pi} \frac{d\vartheta}{2\pi} \exp(-i(r-k)\vartheta - itJ\cos\vartheta) \int_0^{2\pi} \frac{d\vartheta'}{2\pi} \exp(i(r-q)\vartheta' + itJ\cos\vartheta') \end{aligned}$$

The scalar product is zero for $q \neq k$ and $(-1)^n$ for $q = K$. So we find

$$- \sum_{k=1}^{2n} \left| \int_0^{2\pi} \frac{d\vartheta}{2\pi} \exp(i(r-k)\vartheta + itJ\cos\vartheta) \right|^2 \tag{4.12}$$

which tends to zero not faster than t^{-1}.

For $\overline{\omega}_\alpha^\pm$, the treatment is quite analogous and the result is the same, apart a plus sign in the q-th summand for every starred b_q^* in the product expressing $| \phi_\alpha^+ \rangle\langle \phi_\alpha^- |$, *i. e.* for every $q \in \alpha$. As the states ω_α^+ with large α appear in ω_G with very small weights, we can conclude that the falloff of $\bar{\omega}_G(\tau_t(b_r^* b_r)) - \bar{\omega}_G^U(\tau_t(b_r^* b_r))$ is t^{-1}.

5. DISCUSSION

While the vanishing of both interference terms and differences between expectation values is a general property of τ_t-asymptotically Abelian systems (which are not so many, by the way), there is no reason to believe that the property of having different and well separed decay times is more than a peculiar property of the model studied, with the choice of the initial states made in an appropriate way. In other words, not every asymptotically Abelian system provides a measuring apparatus, and only a direct study of the system can decide whether it is the case.

In our approach, no *a priori* correlation was supposed between the problems of

— justifying the macroscopic behaviour of macroscopic bodies,
— cancelling the interference terms;

which are usually related in the literature (see *e. g.* [*3*] [*8*] and references *ibi quoted*). However we found, within our model, that a large apparatus provides a faster decrease of the interference terms, and so a better separation between characteristic times.

Our results resemble for some respect those found by Prosperi ([*8*] and previous papers *ibi quoted*) with the use of the master equation; but there a generally valid result is derived with approximations and restricting to the macroscopic observables of the apparatus, here for a single oversimplified model everything is derived rigorously, apart from the choice of the initial states, which is only heuristically justified.

A possible progress in our approach might be available with the study of inhomogeneous infinite systems, treating the apparatus as an inhomogeneity of the infinite system. The hope is to find non asymptotically Abelian systems, for which $\| [\tau_t A, B] \|$ tends to zero only for some particular choice of B. Then it would be possible to find vector states ϕ^+ and ϕ^- such that $| \phi^+ \rangle\langle \phi^- |$ does not commute asymptotically with every observable, but a certain $T = T^*$, such that $T\phi^+ = 0$, $T\phi^- = \phi^-$, does. So all interference terms would vanish and there would be a permanent recording.

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