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Dirac brackets in geometric dynamics

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ABSTRACT. — Theory of constraints in dynamics is formulated in the framework of symplectic geometry. Geometric significance of secondary constraints and of Dirac brackets is given. Global existence of Dirac brackets is proved.

1. INTRODUCTION

The successes of the canonical quantization of dynamical systems with a finite number of degrees of freedom, the experimental necessity of quantization of electrodynamics, and the hopes that quantization of the gravitational field could resolve difficulties encountered in quantum field theory have given rise to thorough investigation of the canonical structure of field theories. It has been found that the standard Hamiltonian formulation of dynamics is inadequate in the physically most interesting cases of electrodynamics and gravitation due to existence of constraints. Methods of dealing with dynamics with constraints have been developed by several authors and it has been realized that the standard Hamiltonian dynamics can be formulated in terms of constraints (1).

Hamiltonian dynamics has been given a very elegant mathematical formulation in the framework of symplectic geometry (2). The aim of this paper is to give a symplectic formulation of the theory of constraints in dynamics. As a result the geometric significance of the classification of constraints and of Dirac brackets is given. Further, the globalization of

(1) See refs. [*2*], [*3*], [*4*], [*5*], [*6*], [*8*] and the references quoted there.
(2) See eg. refs. [*1*] and [*9*].

the results discussed in the literature in terms of local coordinates is obtained.

As in the case of the Hamiltonian dynamics, the fundamental notion in the geometric analysis of dynamical systems with constraints is that of a symplectic manifold. Basic properties of symplectic manifolds are reviewed in Section 2.

A dynamical system with constraints can be represented by a constraint submanifold of a symplectic manifold. Therefore, constraint dynamics is described by a triplet (P, M, ω), where (P, ω) is a symplectic manifold and M is a submanifold of P, which is called a canonical system [*12*]. Elementary properties of canonical systems, their relations to Lagrangian systems with homogeneous Lagrangians, and the relation between a canonical system and its reduced phase space are discussed in Section 3.

Section 4 contains a discussion of secondary constraints. The generalization of Dirac classification of constraints is given in Section 5. Section 6 is devoted to an analysis of the geometric significance of Dirac brackets and a proof of their global existence.

2. SYMPLECTIC MANIFOLDS

A symplectic manifold is a pair (P, ω) where P is a manifold ([3]) and ω is a symplectic form on P. The most important example of a symplectic manifold in dynamics is furnished by the structure of the phase space of a dynamical system, in this case P represents the phase space and ω is the Lagrange bracket.

Let (P, ω) be a symplectic manifold and f a function on P. There exists a unique vector field v_f, called the Hamiltonian vector field of f, such that $v_f \lrcorner\, \omega = - df$, where $\lrcorner$ denotes the left interior product of a form by a vector field. It v_f and v_g are Hamiltonian vector fields corresponding to functions f and g, respectively, then their Lie brackets $[v_f, v_g]$ is the Hamiltonian vector field corresponding to the Poisson bracket (f, g) of f and g. Further, $(f, g) = v_f(g) = - v_g(f)$, and the Jacobi identity for the Poisson brackets is an immediate consequence of the Jacobi identity for the Lie bracket of vector fields and the assumption that ω is closed.

3. CANONICAL SYSTEMS

DÉFINITION 3.1. — A canonical system is a triplet (P, M, ω) where (P, ω) is a symplectic manifold and M is a submanifold of P.

Canonical systems appear if one passes from the Lagrangian dynamics

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([3]) All manifolds considered in this paper are finite dimensional, paracompact and of class C^∞.

to the canonical dynamics of systems with homogeneous Lagrangians. Let L be a homogeneous function of degree one defined on a conical domain D of the tangent bundle space TX of a configuration space X. We denote by $FL : D \to T^*X$ the Legendre transformation given by the fibre derivative of L and by θ the Liouville form on T^*X defined, for each $\underline{v} \in T_p T^*X$, by $\theta_p(\underline{v}) = p(T\pi(\underline{v}))$ where $\pi : T^*X \to X$ is the cotangent bundle projection. The triplet (T^*X, range FL, $d\theta$) is a canonical system provided range FL is a submanifold of T^*X.

Let (P, M, ω) be a canonical system. There are two subsets K and N of $TP \mid M$ associated to (P, M, ω) as follows:

$$K = \{ \underline{v} \in TP \mid M : (\underline{v} \lrcorner \omega) \mid M = 0 \} \qquad \text{and} \qquad N = K \cap TM.$$

The set N is called the characteristic set of $\omega \mid M$. The dynamical significance of N for a canonical system (T^*X, range FL, $d\theta$) obtained from a Lagrangian system on X with a homogeneous Lagrangian L is given by the following.

PROPOSITION 3.2. — A curve γ in X satisfies the Lagrange-Euler equations corresponding to the Lagrangian L if and only $FL \cdot \dot{\gamma}$, where $\dot{\gamma}$ denotes the prolongation of to TX, is an integral curve of N. Proof of this proposition is given in the reference [*11*] and it will be omitted.

Thus, for a canonical system (P, M, ω), points of M which can be connected by integral curves of N are related in a physically meaningfull manner; they may be related by time evolution as in Proposition 3.2 or by a gauge transformation. Therefore, given a point $p \in M$, the maximal integral manifold of N through p, provided it exists, can be interpreted as the history of p.

DÉFINITION 3.3. — A canonical system (P, M, ω) is regular if N is a subbundle of TM.

Let (P, M, ω) be a regular canonical system. Then N is a subbundle of TM, and since ω is closed N is also involutive. Hence, by Frobenius theorem ([4]), each point $p \in M$ is contained in a unique maximal integral manifold of N. Let P^r denote the quotient set of M by the equivalence relation defined by maximal integral manifolds of N, that is each element in P^r represents a history of the dynamical system described by our canonical system (P, M, ω), and let $\rho : M \to P^r$ denote the canonical projection. If P^r admits a differentiable structure such that ρ is a submersion then there exists a unique symplectic form ω^r on P^r such that $\omega \mid M = \rho^* \omega^r$. The symplectic manifold (P^r, ω^r) is called the reduced phase space of (P, M, ω). Functions on P^r correspond to (gauge invariant) constants of motion, and ω^r gives rise to their Poisson algebra in the manner described in Section 2 ([5]).

([4]) All theorems on differentiable manifolds used in this paper can be found in ref. [*7*].
([5]) This Poisson algebra was first studied in ref. [*2*].

4. SECONDARY CONSTRAINTS

If (P, M, ω) is not regular then dim N_p depends on $p \in M$. In particular the set S^0 of points $p \in M$ on which dim $N_p = 0$ is an open submanifold of M, if it is not empty. Discarding this set S^0 as physically inadmissible have lead Dirac [3] to the notion of secondary constraints generalization of which is given here. Consider the class of all manifolds Y contained in M such that $TY \cap N$ is a subbundle of TY (we allow here dim Y = 0), and let $\leq$ denote the partial order in this class defined as follows $Y \leq Y'$ if and only if Y is a submanifold of Y'. It follows from Zorn's lemma that there exist maximal manifolds in this class. Thus, we are lead to the following.

DÉFINITION 4.1. — A secondary constraint manifold of a canonical system (P, M, ω) is a maximal manifold S contained in M such that $N \cap TS$ is a subbundle of TS (6).

Let S be a secondary constraint manifold of (P, M, ω). If S is not an open submanifold of M then $N \cap TS$ need not be involutive. In this case one cannot use Frobenius theorem to define the reduced phase space. If $N \cap TS$ is involutive its maximal integral manifolds define an equivalence relation, and let P_S^r denote the quotient set of S by this relation. Suppose that there exists a differentiable structure on P_S^r such that the canonical projection $\rho_S : S \to P_S^r$ is a submersion. Since N is the characteristic set of $\omega \mid M$ and $TS \subseteq TM$, then $N \cap TS$ is contained in the characteristic set N_S of $\omega \mid S$ defined by $N_S = \{ \underline{v} \in TS \mid \underline{v} \lrcorner\, \omega \mid S = 0 \}$. Therefore there exists a unique 2-form ω_S^r on P_S^r such that $\omega \mid S = \rho_S^* \omega_S^r$. The form ω_S^r is non-degenerate if and only if $N \cap TS = N_S$. Thus, the reduced phase space P_S^r of the secondary constraint manifold S has a natural structure of a symplectic manifold if and only if $TS \cap N = N_S$. If this condition holds and if S is a submanifold of P, then the structure of S as a secondary constraint manifold of (P, M, ω) is exactly the same as that of a regular canonical system (P, S, ω). This situation appears in all cases of interest in physics, therefore in the following we shall limit our considerations to regular canonical systems.

5. CLASS OF A CANONICAL SYSTEM

Let (P, M, ω) be a canonical system. A function f on P is called a first class function if its Poisson bracket with every function constant on M is identically zero on M. This condition can be reformulated as follows:

(6) This definition is a globalization of the definition given in ref. [10].

f is first class if and only if, for each $v \in K$, $v(f) = 0$. Functions on P which are not first class are called second class functions. Since the constraint submanifold M can be locally described by a system of equations $f_i(p) = 0$, $i = 1, \ldots, \dim P - \dim M$, the notion of a class can be extended to apply a regular canonical system.

DÉFINITION 5.1. — Class of a regular canonical system (P, M, ω) is the pair of integers $(\dim N, \dim K - \dim N)$ [7].

If (P, M, ω) is of class (n, k) then locally M can be described by a system of equations $f_i(p) = 0$, $i = 1, \ldots, n$ and $g_j(p) = 0$, $j = 1, \ldots, k$, where all functions f_i are first class and all functions g_j are second class, and it cannot be described by any system of equations with more than n first class functions. If $k = 0$ we call M a first class submanifold of (P, ω) as it can be given locally by a system of equations $f_i(p) = 0$, $i = 1, \ldots, n$, where all functions f_i are first class. Similarly, if $n = 0$, M is called a second class submanifold of (P, ω). In this case $(M, \omega \mid M)$ is a symplectic manifold. There hold two theorems which we shall need later.

PROPOSITION 5.2. — For any regular canonical system (P, M, ω), $\dim K - \dim N$ is an even number.

Proof. — $\dim K = \dim P - \dim M$ and $\dim P^r = \dim M - \dim N$. Since both P and P^r admit symplectic forms $\dim P$ and $\dim P^r$ are even numbers. Hence, $\dim K - \dim N = \dim P - \dim P^r$ is even.

PROPOSITION 5.3. — Let (P, M, ω) be a regular canonical system of class (n, k) with $k > 0$. Then, there exist two functions f and g such that

$$f \mid M = g \mid M = 0 \qquad \text{and} \qquad (f, g) \mid M = 1.$$

Proof. — For each $p \in M$ there exists a neighbourhood U_p of p in P and two functions f_p and g_p on U_p such that $f_p \mid M \cap U_p = g_p \mid M \cap U_p = 0$ and $(f_p, g_p) \mid M \cap U_p = 1$. Further, we associate to the complement $\bar{M}'$ of the closure of M in P functions f' and g' equal identically to zero. This way we have obtained a covering of P, and since P is paracompact, there exists a locally finite refinement $\{ U_\alpha \}$ of this covering and a partition of unity $\{ h_\alpha \}$ subordinated to the covering $\{ U_\alpha \}$. For each U_α we have two functions f_α and g_α on U_α such that

$$f_\alpha \mid M \cap U_\alpha = g_\alpha \mid M \cap U_\alpha = 0 \qquad \text{and} \qquad (f_\alpha, g_\alpha) \mid M \cap U_\alpha = 1.$$

Let f and g be functions on P defined as follows, for each

$$p \in P, f(p) = \Sigma_\alpha f_\alpha(p) h_\alpha(p) \qquad \text{and} \qquad g(p) = \Sigma_\alpha g_\alpha(p) h_\alpha(p).$$

Then,

$$f \mid M = g \mid M = 0 \qquad \text{and} \qquad (f, g) \mid M = 1.$$

(7) This definition is due to W. M. Tulczyjew (unpublished).

6. DIRAC BRACKETS

Let (P, M, ω) be a regular canonical system of class (n, k). Quantization of (P, M, ω) associates to functions on P Hermitian operators in a Hilbert space H in such a way that the Poisson brackets of functions go into commutators of the corresponding operators divided by $i\hbar$, where $\hbar$ is the Planck constant divided by 2π. Physically admissible states of the system are represented by a subspace H_0 of H such that, for every function $\hat{f}$ on P constant on M, H_0 is contained in the eigenspace of the operator f associated to f corresponding to the eigenvalue equal to the constant value of f on M. If $k > 0$ this condition for H_0 is satisfied only by $H_0 = 0$, since if f and g are functions on P such that $f \mid M = g \mid M = 0$ and $(f, g) \mid M = 1$ then, for each vector $\psi \in H_0$, $\psi = 1\psi = \widehat{(f, g)}\psi = 1/i\hbar(\hat{f}\hat{g} - \hat{g}\hat{f})\psi = 0$.

According to Dirac [3] this implies that the original phase space (P, ω) is too big to be physically interpretable and that one should look for a second class submanifold $\tilde{P}$ of P containing M and such that M is a first class submanifold of $(\tilde{P}, \tilde{\omega})$ where $\tilde{\omega} = \omega \mid \tilde{P}$.

PROPOSITION 6.1. — Let (P, M, ω) be a regular canonical system such that M is a closed submanifold of P. Then there exists a second class submanifold $\tilde{P}$ of (P, ω) such that M is a first class submanifold of $(\tilde{P}, \tilde{\omega})$ where $\tilde{\omega}$ is the restriction of ω to $\tilde{P}$.

Proof. — Since (P, M, ω) is regular dim N is constant on M and hence

$$N' = \{ v \in TP \mid M : (\underline{v} \lrcorner \omega) \mid N = 0 \}$$

is a subbundle of $TP \mid M$. Let $\tilde{M}$ be a submanifold of P such that

$$M \subseteq \tilde{M}, T\tilde{M} \cap N' = TM \qquad \text{and} \qquad T\tilde{M} \mid M + N' = TP \mid M$$

and let $\tilde{N}$ be the characteristic set of $\omega \mid \tilde{M}$, i. e.

$$\tilde{N} = \{ \underline{v} \in T\tilde{M} : \underline{v} \lrcorner \omega \mid \tilde{M} = 0 \}.$$

For each $p \in M$ and each $\underline{v} \in T_p\tilde{M}$ there exists $\underline{u} \in T_p\tilde{M}$ such that $\omega(\underline{v}, \underline{u}) \neq 0$. Thus, $\dim \tilde{N}_p = 0$ for all $p \in \tilde{M}$. Let $\tilde{P}$ be the open submanifold of $\tilde{M}$ defined by $p \in \tilde{P}$ if and only if $\dim \tilde{N}_p = 0$. Then $\tilde{P}$ is a second class submanifold of (P, ω). We denote by $\tilde{\omega}$ the restriction of ω to $\tilde{P}$. Consider now a canonical system $(\tilde{P}, M, \tilde{\omega})$ and let $\tilde{K} = \{ \underline{v} \in TP \mid M : (\underline{v} \lrcorner \tilde{\omega}) \mid M = 0 \}$. The class of $(\tilde{P}, M, \tilde{\omega})$ is $(\dim N, \dim \tilde{K} - \dim N)$, but $\tilde{K} = K \cap T\tilde{P} = N$ where $K = \{ \underline{v} \in TP \mid M : (\underline{v} \lrcorner \omega) \mid M = 0 \}$. Hence $\dim \tilde{K} = \dim N$ and M is a first class submanifold of $(\tilde{P}, \tilde{\omega})$. Therefore, it suffices to prove the existence of a submanifold $\tilde{M}$ satisfying the conditions assumed above.

Since M is closed in P there exists a total tubular neighbourhood of M in P, that is a vector bundle $\zeta : Z \to M$ and a diffeomorphism φ of an open neighbourhood U of M onto Z which maps M onto the zero section 0 of Z. Then, $T\varphi(TP \mid M) = TZ \mid 0$, and, since Z is a vector bundle, there exists a vector bundle isomorphism α: $TZ \mid 0 \to Z \times_M TM$ such that, for each $u \in TZ \mid 0$, the second component of $\alpha(u)$ is $T\zeta(u)$, i. e.

$$\alpha(\underline{u}) = (z, T\zeta(\underline{u})) \qquad \text{for some } z \in Z.$$

Since $TM \subseteq N'$, $\alpha \circ T\varphi(N') = Y \times_M TM$ where Y is a subbundle of Z. There exists a subbundle X of Z complementary to Y, i. e. such that $X \cap Y = 0$ and Z is isomorphic to $X \times_M Y$, in the following we shall identify Z with $X \times_M Y$. Since X is a submanifold of Z and φ is a diffeomorphism,

$$\tilde{M} = \varphi^{-1}(X)$$

is a submanifold of P and M is a submanifold of $\tilde{M}$. Further,

$$T\tilde{M} \mid M = T\varphi^{-1}(TX \mid 0) = T\varphi^{-1} \circ \alpha^{-1}(X \times_M 0 \times_M TM)$$

and

$$N' = T\varphi^{-1} \circ \alpha^{-1}(0 \times_M Y \times_M TM).$$

Hence,

$$N' \cap T\tilde{M} \mid M = T\varphi^{-1} \circ \alpha^{-1}(0 \times_M 0 \times_M TM) = TM,$$

and similarly $TM \mid M + N' = TP \mid M$. Therefore, $\tilde{M} = \varphi^{-1}(X)$ satisfies the required conditions, which completes the proof.

Let (P, M, ω) be a regular canonical system and $\tilde{P}$ a second class submanifold of (P, ω) such that M is a first class submanifold of $(\tilde{P}, \tilde{\omega})$ where $\tilde{\omega}$ is the restriction of ω to P. We want to relate the value at points of P of the Poisson bracket of two functions on P to the Poisson bracket of their restrictions to $\tilde{P}$. Let U be an open set in P such that $\tilde{P} \cap U$ can be characterized by a system of equations $g_i(p) = 0, i = 1, \ldots, k = \dim P - \dim \tilde{P}$, where the functions g_i are independent in U. Then the Hamiltonian vector fields u_i associated to g_i, defined by $u_i \lrcorner\, \omega = -\, dg_i$, are linearly independent in U. Further, since $\tilde{P}$ is second class, the vector fields u_i are transverse to $U \cap \tilde{P}$ and, together with $T\tilde{P} \mid (\tilde{P} \cap U)$, they span $TP \mid (\tilde{P} \cap U)$. Thus, for each $p \in \tilde{P} \cap U$ and each $\underline{v} \in T_pP$ there is a unique decomposition $\underline{v} = \tilde{\underline{v}} + \Sigma_i \alpha_i u_i(p)$, where $\tilde{\underline{v}} \in T_p\tilde{P}$. To interpret the coefficients α_i let us compute $\underline{v}(g_j)$ for $j = 1, \ldots, k$. We have $\underline{v}(g_j) = \Sigma_i \alpha_i u_i(p)(g_j) = \Sigma_i \alpha_i (g_i, g_j)$. Since the matrix function (g_i, g_j) is non-singular on $\tilde{P} \cap U$, there exists the inverse C_{ij} such that $\Sigma_j C_{ij}(g_j, g_l) = \delta_{il}$. Hence, $\alpha_i = \Sigma_j C_{ij} \underline{v}(g_j)$. Let f be a function on P and u_f the Hamiltonian vector field associated to f. For each $p \in \tilde{P} \cap U$ we have $u_f(p) = \tilde{u}_f(p) + \Sigma_{ij} C_{ij}(f, g_j) u_i(p)$ where

$$\tilde{u}_f(p) \in T_p\tilde{P},$$

and

$$df(p) = -\, u_f(p) \lrcorner\, \omega = -\, \tilde{u}_f(p) \lrcorner\, \omega - \Sigma_{ij} C_{ij}(f, g_j) u_i(p) \lrcorner\, \omega.$$

If $\tilde{\underline{v}} \in T_pP$, then $\tilde{\underline{v}}(f) = df(\tilde{v}) = -\, \omega(\tilde{u}_f(p), \tilde{\underline{v}}) = -\, \omega(\tilde{u}_f(p), \tilde{\underline{v}})$, and this implies that $\tilde{u}_f$ is the Hamiltonian vector field on $(\tilde{P}, \tilde{\omega})$ associated to the restriction of f to $\tilde{P}$. If h is any other function on P we have

$$(f, h)(p) = u_f(p)(h) = \tilde{u}_f(p)(h) + \Sigma_{ij}C_{ij}(f, g_j)u_i(p)(h) = (f \mid \tilde{P}, h \mid \tilde{P}) + \Sigma_{ij}C_{ij}(f, g_j)(g_i, h)$$

evaluated at p, where $(f \mid \tilde{P}, h \mid \tilde{P})$ is the Poisson bracket in $(\tilde{P}, \tilde{\omega})$ of the restrictions of f and g to $\tilde{P}$. Therefore, we have on $\tilde{P}$

$$(f \mid \tilde{P}, h \mid \tilde{P}) = (f, h) + \Sigma_{ij}(f, g_i)C_{ij}(g_j, h)$$

which is precisely the Dirac bracket of f and h.

Thus, the Poisson bracket in $(\tilde{P}, \tilde{\omega})$ of the restrictions to $\tilde{P}$ of functions on P gives a global extension of the Dirac bracket.

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