ANNALES DE L'I. H. P., SECTION A

R. A. WEDER

Spectral properties of one-body relativistic spin-zero hamiltonians

Annales de l'I. H. P., section A, tome 20, nº 2 (1974), p. 211-220
<http://www.numdam.org/item?id=AIHPA_1974__20_2_211_0>

Ann. Inst. Henri Poincaré,
Vol. XX, n° 2, 1974, p. 211-220.

Section A :
Physique théorique.

Spectral properties of one-body relativistic spin-zero hamiltonians ([1])

by

R. A. WEDER ([2])

Instituut voor Theoretische en Wiskundige Fysica
Katholieke Universiteit Leuven (Belgium) ([3])

ABSTRACT. — We study the spectral properties of the relativistic spin-zero Hamiltonian $H = \sqrt{p^2 + \mu^2} + V$ of a spinless particle, by an extension of the method of Aguilar-Combes, for a class of interactions including $V = -gr^{-\beta}, 0 < \beta < 1$.

Absence of singular-continuous spectrum is proved, together with the existence of an absolutely-continuous spectrum $[\mu, \infty)$. In $R \backslash \{ \mu \}$ the point spectrum consists of finite-dimensional eigenvalues which are bounded. Properties of resonances are investigated.

RÉSUMÉ. — Nous étudions les propriétés de l'Hamiltonien relativiste de spin zero : $H = \sqrt{p^2 + \mu^2} + V$ d'une particule sans spin, grâce à une extension de la méthode d'Aguilar-Combes, pour la classe d'interactions comprenant : $V = -gr^{-\beta}, 0 < \beta < 1$.

L'absence de spectre singulièrement continu est prouvée, en même temps que l'existence d'un spectre absolument continu $[\mu, \infty)$. Dans $R \backslash \{ \mu \}$ le spectre ponctuel est formée de valeurs propres de dimension finie qui sont bornées.

Les propriétés des résonances sont analysées.

([1]) Work supported by: A. G. C. D. (Belgium).

([2]) On leave of absence from Universidad de Buenos Aires (Argentina).

([3]) Postal address: Instituut voor Theoretische en Wiskundige Natuurkunde Celestijnenlaan 200 D, B-3030, Heverlee (Belgium).

INTRODUCTION

Recently [1] [2], spectral properties of Schrödinger operators with interactions satisfying analyticity conditions with respect to the dilatation group were studied.

In this work we investigate the spectral properties of the relativistic spin-zero Hamiltonian $H = \sqrt{p^2 + \mu^2} + V$ of a spinless particle by an extension of the method of [1], for a class of interactions including $V = -gr^{-\beta}$, $0 < \beta < 1$.

In Section I we study the analyticity properties, with respect to the dilatation group, of the free Hamiltonian $H_0 = \sqrt{p^2 + \mu^2}$.

In Section II we define the class of interactions, we are considering, and prove that the interactions $V = -gr^{-\beta}$, $0 < \beta < 1$ are allowed.

In Section III we show that the singular-continuous spectrum is empty together with the existence of an absolutely-continuous spectrum $[\mu, \infty)$. We also show that the point spectrum consists of a bounded set of finite-dimensional eigenvalues different from μ (accumulating, at most, at μ), and possibly an eigenvalue at μ. Properties of resonances are also investigated.

We stress the difference with the non-relativistic case, namely the essential-spectrum of the analytic extention of the Hamiltonian is not a straight-line (see fig. 1) but a part of an hyperbola starting at μ. The basic new technical results for the relativistic case are contained in lemmas 1, 2 and 3. The method of the proof of lemma 4 and theorem 1 were taken from [1] [2] and [6]. For the definitions of vector-valued, and operator-valued analytic functions, and for the classification of the spectrum we refer to T. Kato [3].

In this paper we use the same notation as in [2].

I. THE FREE HAMILTONIAN

Let, the space of wavefunctions of a spinless particle, be the Hilbert space $\mathscr{H} = \mathscr{L}^2(\mathbb{R}^3)$ of square-integrable functions in $\mathbb{R}^3$. Let μ, the particle mass, be a strictly positive constant, and let $\omega(p) = \sqrt{p^2 + \mu^2}$. We define the free Hamiltonian in momentum space by

$$(H_0\Psi)(\vec{p}) = \omega(p)\Psi(\vec{p}) \, ,$$

on the domain, $\mathscr{D}(H_0)$, of all Ψ in $\mathscr{H}$ such that $(\omega(p)\Psi(\vec{p})$ is again in $\mathscr{H}$. We take, of course, the square root with positive sign. H_0 is a positive, and selfadjoint operator [3].

Let $U(\Theta)$, $\Theta \in \mathbb{R}$, be the strongly-continuous unitary representation on $\mathcal{H}$ of the dilatation group defined by

$$(U(\Theta)\Psi)(\vec{p}) = e^{-\frac{3\theta}{2}}\Psi(e^{-\theta}\vec{p}), \qquad \Psi \in \mathcal{H}, \qquad \Theta \in \mathbb{R} .$$

Thus, we have

$$(H_0(\Theta)\Psi)(\vec{p}) = (U(\Theta)H_0U(-\Theta)\Psi)(\vec{p}) = \omega(\Theta, p)\Psi(\vec{p}) ,$$

where

$$\omega(\Theta, p) = \sqrt{e^{-2\Theta}p^2 + \mu^2}, \qquad \Theta \in \mathbb{R} .$$

LEMMA 1. — The family of operators $H_0(\Theta)$, $\Theta \in \mathbb{R}$, can be extended to an analytic family in the strip of the complex-plane

$$S_{\frac{\pi}{2}} = \left\{ \Theta \in \mathbb{C} \mid \ |\operatorname{Im} \Theta| < \frac{\pi}{2} \right\}$$

Proof. — We can write

$$\omega(\Theta, p) = \rho(\Theta, p)e^{i\phi(\Theta,p)} ,$$

with, $\phi(\Theta, p)$, the argument; and a modulus $\rho(\Theta, p) > 0$ for all $p \in [0, \infty)$ and all $\Theta \in S_{\frac{\pi}{2}}$. There exist two real, positive, and bounded functions of $\Theta(M_1(\Theta)$, and $M_2(\Theta))$ such that

$$0 < \frac{\rho(0, p)}{\rho(\Theta, p)} < M_1(\Theta) < \infty ,$$

$$0 < \frac{\rho(\Theta, p)}{\rho(0, p)} < M_2(\Theta) < \infty, \qquad p \in [0, \infty), \qquad \Theta \in S_{\frac{\pi}{2}} .$$

Thus

$$\| H_0\Psi \| \leqslant M_1(\Theta) \| H_0(\Theta)\Psi \| \qquad \Psi \in \mathcal{D}(H_0(\Theta)) ,$$
$$\| H_0(\Theta)\Psi \| \leqslant M_2(\Theta) \| H_0\Psi \| \qquad \Psi \in \mathcal{D}(H_0)$$

That is to say

$$\mathcal{D}(H_0) = \mathcal{D}(H_0(\Theta)) .$$

By a trivial argument, which we omit, we can show that

$$| \omega(\Theta_1, p) + \omega(\Theta_2, p) | \geqslant \rho(\Theta_1, p) (\cos \operatorname{Im} \Theta_1) > 0, \qquad \Theta_1\Theta_2 \in S_{\frac{\pi}{2}}$$

Then, we have the following estimation

$$\left| \frac{\omega(\Theta_2, p) - \omega(\Theta_1, p)}{\Theta_2 - \Theta_1} \right| \leqslant 2\left(\frac{\varepsilon + e^{-2R_e\Theta_1}}{\cos \operatorname{Im} \Theta_1} \right) M_1(\Theta_1)\rho(0, p) ,$$

for Θ_1, $\Theta_2 \in S_{\frac{\pi}{2}}$, $| \Theta_2 - \Theta_1 | < \eta(\Theta_1)$; $\varepsilon > 0$ and $\eta(\Theta_1) > 0$.

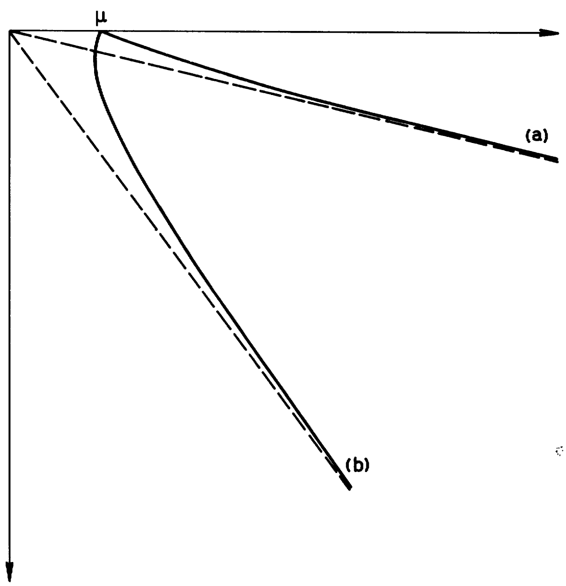
Then, as $\omega(\Theta, p)$ is an analytic function of Θ for all $\Theta \in S_{\frac{\pi}{2}}$, by the Lebesgue's dominated convergence theorem we have

$$\left(\frac{H_0(\Theta_2) - H_0(\Theta_1)}{\Theta_2 - \Theta_1}\Psi\right) \underset{\Theta_2 \to \Theta_1}{\longrightarrow} \left(\frac{d}{d\Theta}\omega(\Theta, p)\right)_{\Theta = \Theta_1} \Psi(\vec{p}), \qquad \Psi \in \mathscr{D}(H_0),$$

in the strong topology in $\mathscr{H}$.

Q. E. D.

LEMMA 2. — The spectrum of $H_0(\Theta)$, denoted $\sigma(H_0(\Theta))$, is a continuous curve, starting at μ, and tending asymptoticaly to the straightline $e^{-i \operatorname{Im} \Theta}\mathbb{R}^+$ (see fig. 1).



a) The spectrum of $H_0(\Theta)$ for $0 < \operatorname{Im}\Theta \leq \frac{\pi}{4}$

b) The spectrum of $H_0(\Theta)$ for $\frac{\pi}{4} < \operatorname{Im}\Theta < \frac{\pi}{2}$

FIG. 1.

Proof. — It is an immediate consequence of the definition of the spectrum that

$$\sigma(H_0(\Theta)) = \{ Z \in \mathbb{C} \mid Z = \sqrt{e^{-2\Theta}p^2 + \mu^2}, \qquad p \in [0, \infty) \}$$

We can write

$$e^{-2\Theta}p^2 + \mu^2 = \rho(\Theta, p)e^{i\phi(\Theta, p)},$$

where $\rho(\Theta, p)$ and $\phi(\Theta, p)$ are respectively the modulus and the argument of $e^{-2\Theta}p^2 + \mu^2$.

For $0 \leqslant \operatorname{Im} \Theta < \frac{\pi}{2}$, $\phi(\Theta, p)$ is a strictly decreasing function of p and

$$- 2 \operatorname{Im} \Theta < \phi(\Theta, p) \leqslant \phi(\Theta, 0) = 0 ,$$

$\rho(\Theta, p)$ is a strictly increasing function of p, bounded below by μ, for $0 \leqslant \operatorname{Im} \Theta \leqslant \frac{\pi}{4}$; and is a convex-function, with a minimum at

$$p = e^{R_e \Theta}\mu \,|\cos 2 \operatorname{Im} \Theta\,|^{1/2}, \qquad \text{for} \qquad \frac{\pi}{4} < \operatorname{Im} \Theta < \frac{\pi}{2}.$$

As $(e^{-2\Theta}p^2 + \mu^2)$ lies in the second-sheet for $p \neq 0$ and $0 < \operatorname{Im} \Theta < \frac{\pi}{2}$, we have

$$(e^{-2\Theta}p^2 + \mu^2)^{1/2} = (\rho(\Theta, p))^{1/2} e^{i\frac{\phi(\Theta, p)}{2}} ,$$

thus, the spectrum of $H_0(\Theta)$ is, for $0 \leqslant \operatorname{Im} \Theta < \frac{\pi}{2}$, a continuous curve starting at μ, and tending asymptoticaly to the straight-line $e^{-i \operatorname{Im} \Theta}\mathbb{R}^+$.

The proof for $-\frac{\pi}{2} < \operatorname{Im} \Theta \leqslant 0$ is similar.

Q. E. D.

Remark. — The spectrum of $H_0(\Theta)$ is independent of $R_e(\Theta)$ because $H_0(\Theta_1)$ and $H_0(\Theta_2)$ are unitary equivalents for $\operatorname{Im} \Theta_1 = \operatorname{Im} \Theta_2$.

II. THE CLASS OF DILATATION ANALYTIC INTERACTIONS

We define a dilatation analytic interaction [*1*] as a symmetric and H_0-compact operator [*3*] V having the following property: the family of operators

$$V(\Theta) = U(\Theta)VU(-\Theta), \qquad \Theta \in \mathbb{R},$$

has an H_0-compact analytic continuation in an open connected domain 0 of the complex-plane (4).

We consider, now, the total Hamiltonian

$$H = H_0 + V,$$

where V is a dilatation analytic interaction in the strip S_a, $0 < a < \frac{\pi}{2}$.

(4) It is clear that the analyticity domain O of $V(\Theta)$ can always be extended to a complex strip $S_a = \{ Z \in \mathbb{C} \mid |\operatorname{Im} Z| < a \}$, $a > 0$ [*1*].

As V is symmetric and H_0-compact, H is selfadjoint, bounded below, and $\mathscr{D}(H) = \mathscr{D}(H_0)$ [3].

By lemma 1, and the definition of dilatation analytic interactions, the family of operators $H(\Theta)$, defined for $\Theta \in \mathbb{R}$ by

$$H(\Theta) = U(\Theta)HU(-\Theta), \qquad \Theta \in \mathbb{R},$$

has an extension to an analytic and selfadjoint family [3], with

$$\mathscr{D}(H(\Theta)) = \mathscr{D}(H_0),$$

in the strip S_b, where $b = \min\left(a, \frac{\pi}{2}\right)$.

LEMMA 3. — The multiplication operator (denoted V) by the function $-gr^{-\beta}$, $0 < \beta < 1$, where g is a constant, is dilatation analytic in the entire complex-plane ([5]).

Proof. — V is a symmetric operator [3]; and

$$U(\Theta)(-gr^{-\beta})U(-\Theta) = -g(\Theta)r^{-\beta};$$

where

$$g(\Theta) = ge^{-\beta\theta}, \qquad \Theta \in \mathbb{R};$$

which has an analytic extension to the entire complex-plane. Let us define the following operator

$$(V_n(\Theta)\Psi)(\vec{r}) = (-g(\Theta)r^{-\beta})_n\Psi(\vec{r}),$$

where

$$(-g(\Theta)r^{-\beta})_n = \begin{cases} -g(\Theta)r^{-\beta}, & r < n \\ 0 \quad, & r > n, \end{cases}$$

(n is a entire positive number) in the domain $\mathscr{D}(V_n(\Theta))$, of all $\Psi \in \mathscr{H}$, such that $[(-g(\Theta)r^{-\beta})_n\Psi(\vec{r})]$ is again in $\mathscr{H}$.

$(-g(\Theta)r^{-\beta})_n \in \mathscr{L}^{\alpha+3}(\mathbb{R}^3)$ ([6]), $\alpha > 0$, if $0 < \beta < 1$. Also

$$(\sqrt{p^2 + \mu^2} + i)^{-1} \in \mathscr{L}^{3+\alpha}[\mathbb{R}^3].$$

Then $V_n(\Theta)$ is H_0-compact by a theorem of [4].

([5]) Clearly, V is defined as a multiplication operator in configuration-space that is to say

$$(V\Psi)(r) = -gr^{-\beta}\Psi(\vec{r}),$$

on the domain, $\mathscr{D}(V)$; of all Ψ in $\mathscr{H}$ such that $(-gr^{-\beta}\Psi(\vec{r}))$ is again in $\mathscr{H}$. $\Psi(\vec{r})$ is the Fourier transform of the wave-function in momentum space $\tilde{\Psi}(\vec{p})$. In momentum space V is an integral operator.

([6]) $\mathscr{L}^{3+\alpha}(\mathbb{R}^3)$ is the Banach space of complex valued functions on $\mathbb{R}^3$, such that

$$\int |\Psi(\vec{r})|^{3+\alpha}d^3\vec{r} < \infty.$$

But as

$$\| (V(\Theta) - V_n(\Theta))(H_0 + i)^{-1}\Psi \| \leqslant \frac{|g(\Theta)|}{n^{\beta/2}} \| (H_0 + i)^{-1} \| \ \| \Psi \| ,$$

$V_n(\Theta)(H_0 + i)^{-1}$ converges in norm to $V(\Theta)(H_0 + i)^{-1}$.

Q. E. D.

We will now study the spectrum of the operators $H(\Theta)$, $\Theta \in S_b/\mathbb{R}$, and then make the transition to real Θ. We note that the remark following lemma 2 is also valid for $H(\Theta)$.

III. SPECTRAL PROPERTIES OF $H = H_0 + V$

LEMMA 4. — The spectrum of $H(\Theta)$ with $0 < |\operatorname{Im} \Theta| < b$ consists of: essential spectrum: $\sigma_e(H(\Theta)) = \sigma(H_0(\Theta))$.

Real bound state energies $(\sigma_d^r(H(\Theta)))$: a bounded set of isolated, finite-dimensional, real-eigenvalues, independent of Θ, with μ as the only possible accumulation point.

Non-real resonance energies $(\sigma_d(H(\Theta))/\sigma_d^r(H(\Theta)))$: a bounded set of non-real isolated, finite-dimensional eigenvalues, contained in the sector of the complex-plane bounded by $[\mu, \infty)$ and $\sigma_e(H(\Theta))$. The only possible accumulation point is μ. A given resonance energy is independent of Θ as long as it belongs to $\sigma_d(H(\Theta))\backslash\sigma_d^r(H(\Theta))$.

For $|\phi| > |\operatorname{Im} \Theta|$ there exist $C(\phi) > 0$ such that for $0 \leqslant \rho < \infty$,

$$\| (H(\Theta) - \lambda_0 + 1 - \rho e^{i\phi})^{-1} \| \leqslant C(\phi)\rho^{-1} ,$$

where λ_0 is the minimum of the spectrum (which is independent of Θ).

Proof. — By the second resolvent equation [3]

$$(H(\Theta) - Z)^{-1} = (H_0(\Theta) - Z)^{-1}(1 + V(\Theta)(H_0(\Theta) - Z)^{-1})^{-1} ,$$

for all $Z \in \mathbb{C}\backslash\sigma(H_0(\Theta))$ such that

$$(1 + V(\Theta)(H_0(\Theta) - Z)^{-1})^{-1}$$

exists.

But, as $V(\Theta)$ is $H_0(\Theta)$-compact and $\| V(\Theta)(H_0(\Theta) + \alpha)^{-1} \| < 1$ for real α and $\alpha > K > 0$, this holds for all $Z \in \mathbb{C}\backslash\sigma(H_0(\Theta))$, except for, at most, a set S of isolated points [3].

Let, for $\lambda \in S$, P_λ be the projection operator defined by [3]

$$P_\lambda = \frac{-1}{2\pi i} \int_\Gamma (H_0(\Theta) - Z)^{-1} dZ + \frac{1}{2\pi i} \int_\Gamma (H(\Theta) - Z)^{-1} V(\Theta)(H_0(\Theta) - Z)^{-1} dZ ,$$

where Γ is a circle separating λ from $\sigma(H(\Theta)) - \{ \lambda \}$.

The first integrand is holomorphic in $Z = \lambda$, and the second compact; hence P_λ is a compact operator.

Then, λ is an isolated finite-dimensional eigenvalue of $H(\Theta)$ [3]. By exchanging the roles of $H_0(\Theta)$ and $H(\Theta)$ we can prove that

$$\sigma_e(H(\Theta)) = \sigma(H_0(\Theta)).$$

Take us λ_0 in the set $\sigma_d(H(\Theta_0))$ of isolated finite-dimensional eigenvalues of $H(\Theta_0)$. As $H(\Theta)$ is an analytic family with spectrum constant for Im Θ constant, $\lambda_0 \in \sigma_d(H(\Theta))$ for Θ in a neighborhood of Θ_0 [3]. Thus the isolated finite-dimensional eigenvalues of $H(\Theta)$ can accumulate only at μ; and the real bound-state energies are independent of Θ, and the non-real resonance energies are independent of Θ as long as they belong to $\sigma_d(H(\Theta))\backslash\sigma_d^r(H(\Theta))$, and are contained in the sector of the complex-plane bounded by $[\mu, \infty)$ and $\sigma_e(H(\Theta))$.

The fact that the set $\sigma_d(H(\Theta))$ is bounded (and the validity of the estimation for the resolvent given above) can be proven in the same lines as in [2], then we will omit the proof here.

Q. E. D.

THEOREM 1. — The point spectrum of H consists of a bounded set of finite-dimensional eigenvalues different from μ (which are precisely the real eigenvalues of $H(\Theta)$ Im $\Theta \neq 0$, different from μ) accumulating at most at μ, and possibly an eigenvalue at μ. The projection operators $P(\Theta, \lambda)$, $\Theta \in S_b$, on the eigenspace of $H(\Theta)$ corresponding to a fixed-real-eigenvalue λ different from μ form a selfadjoint analytic family in S_b.

The eigenvectors Φ of H corresponding to such eigenvalues (λ) are in the dense set $\mathscr{D}_b$ of analytic vectors [5] in S_b, and the analytic extentions $\Phi(\Theta)$ of Φ are eigenvectors of $H(\Theta)$ corresponding to λ. The singular-continuous spectrum is empty, i. e.

$$\mathscr{H} = \mathscr{H}_{a.c.} \oplus \mathscr{H}_p, \quad \text{and} \quad \sigma_{a.c.} = [\mu, \infty).$$

Proof. — Γet Φ and Ψ, be fixed vectors in the dense set $\mathscr{D}_b$; and $\Phi(\Theta)$ and $\Psi(\Theta)$ their analytic extentions.

By lemma 4 the function

$$F_{\Phi,\Psi}(\Theta, Z) = (\Phi(\bar{\Theta}), \quad (H(\Theta) - Z)^{-1}\Psi(\Theta)),$$

is analytic in Θ, for fixed Z such that Im Z > 0 and such that

$$-\arg Z \leqslant \operatorname{Im}\Theta < b.$$

Since

$$F_{\Phi,\Psi}(\Theta, Z) = (\Phi,(H - Z)^{-1}\Psi) \quad \text{for} \quad \Theta \in \mathbb{R},$$

it follows that the equality holds for all Θ with

$$-\arg Z \leqslant \operatorname{Im}\Theta < b \quad \text{and} \quad \operatorname{Im} Z > 0.$$

Now we fix Θ with Im $\Theta > 0$; by lemma 4, $F_{\Phi,\Psi}(\Theta, Z)$ is meromorphic in Z for $Z \notin \sigma_e(H(\Theta))$, then $(\Phi, (H - Z)^{-1}\Psi)$ has a meromorphic continuation from above ($\text{Im } Z > 0$) across the line $[\mu, \infty)$ up to the curve $\sigma(H_e(\Theta))$. Let us denote by $E(\lambda)$ the spectral family of H [3], then we have that

$$(\Phi, (E_\lambda - E_{\lambda-0})\Psi) = \lim_{\substack{Z \to \lambda \\ Z \in C^+_{\lambda,\omega}}} F_{\Phi,\Psi}(\Theta, Z),$$

where

$$C^+_{\lambda\omega} = \left\{ Z \in \mathbb{C} \mid \text{Im } Z > 0, \omega \leqslant \arg(Z - \lambda) \leqslant \pi - \omega, 0 < \omega < \frac{\pi}{2} \right\}.$$

This implies (together with a similar result for $\text{Im } \Theta < 0$ and $\text{Im } Z < 0$) that the eigenvalues of H, different from μ, are precisely the real eigenvalues of $H(\Theta)$, $\text{Im } \Theta \neq 0$, different from μ; and that the real poles of $(H(\Theta) - Z)^{-1}$, $\text{Im } \Theta \neq 0$, different from μ, are simple. Then, the point spectrum of H is bounded and accumulates, at most, at μ.

Let, $P^\pm(\Theta, \lambda)$, be the projection operator [3] on the eigenspace of $H(\Theta)$ corresponding to an eigenvalue, λ, different from μ (+, − corresponds to $\text{Im } \Theta > 0$ and $\text{Im } \Theta < 0$ respectively).

Setting $P(\lambda) = E_\lambda - E_{\lambda-0}$, we obtain as above:

$$(\Phi, P^\pm(\Theta, \lambda)\Psi) = (\Phi(-\overline{\Theta}), P(\lambda)\Psi(-\Theta)),$$

where we have used the fact that λ is a simple pole.

Then, the function $f_{\Phi,\Psi}(\Theta)$ defined for Φ, $\Psi \in \mathcal{D}_b$ by

$$f_{\Phi,\Psi}(\Theta) = \begin{cases} (\Phi, P^\pm(\Theta, \lambda)\Psi), & \text{Im } \Theta \neq 0 \\ (\Phi, P(\Theta, \lambda)\Psi), & \text{Im } \Theta = 0, \end{cases}$$

is analytic in S_b, where $P(\Theta, \lambda) = U(\Theta)P(\lambda)U(-\Theta)$ is the projection operator on the eigenspace of $H(\Theta)$ corresponding to the eigenvalue λ for $\Theta \in \mathbb{R}$.

Now, it is not difficult to show [1] the fact that the family $P(\Theta, \lambda)$, $\Theta \in \mathbb{R}$, has an analytic extention in S_b which equals $P^+(\Theta, \lambda)$ (resp. $P^-(\Theta, \lambda)$ for $\text{Im } \Theta > 0$ (resp. $\text{Im } \Theta < 0$).

This implies that the eigenvalues of H, different from μ, are finite-dimensional.

By standard arguments [1] it is possible to show that the eigenvectors Φ of H corresponding to such eigenvalues, λ, are analytic vectors in S_b, and that their analytic extentions $\Phi(\Theta)$ are eigenvectors of $H(\Theta)$ with the same eigenvalue λ.

Let $\Delta = (a, b)$, be an interval in (μ, ∞) which contains no-eigenvalue of H, then for $\text{Im } \Theta > 0$ and $\Phi \in \mathcal{D}_b$ [3]

$$(\Phi, E_\Delta \Phi) = \frac{1}{2\pi i} \int_a^b \{ (\Phi(\overline{\Theta}), (H(\Theta) - \lambda)^{-1}\Phi(\Theta)) - (\Phi(\Theta), (H(\overline{\Theta}) - \lambda)^{-1}\Phi(\overline{\Theta})) \} \, d\lambda,$$

where $E_\Delta = E_b - E_a$.

The integrand being analytic, the function $(\Phi, E_\lambda \Phi)$ is absolutely continuous on Δ; since $\mathcal{D}_b$ is dense in $\mathcal{H}$ we obtain

$$\mathcal{H}_{\text{s.c.}} = \emptyset ,$$

that is to say

$$\mathcal{H} = \mathcal{H}_{\text{a.c.}} \oplus \mathcal{H}_p ,$$

and also $\sigma_{\text{a.c.}} = [\mu, \infty)$.

ACKNOWLEDGEMENTS

I am very indebted to Professor A. Verbeure, who suggested this work, for discussions and help. It is a pleasure to thank Professors J. P. Antoine and W. Amrein for enlightening discussions and for their encouragement.

We thank the referee for pointing out to us the formulation of Lemma 3 in its most general form.

REFERENCES

[1] J. Aguilar, J. M. Combes, *Comm. Math. Phys.*, t. **22**, 1971, p. 269.
[2] E. Balslev, J. M. Combes, *Comm. Math. Phys.*, t. **22**, 1971, p. 280.
[3] T. Kato, *Perturbation theory for linear operators*. Springer-Verlag, 1966.
[4] W. G. Faris, Quadratic Forms and Essential Self-Adjointness. *Helv. Phys. Acta*, t. **45**, 7, 1973, p. 1074.
[5] E. Nelson, Analytic Vectors. *Ann. Math.*, t. **70**, 1959, p. 3.
[6] E. Balslev, Spectral Theory of Schrödinger Operators of Many-Body Systems. *Lecture notes*, Leuven, 1971.

(Version révisée reçue le 22 octobre 1973)