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Singular operations on algebras (generalized contractions)

by

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ABSTRACT. — Let $\mathscr{L}$ be an algebra structure of underlying finite dimensional vector space E; we propose to define a semi-group Σ of operations that generalize the Saletan contractions. Many categories of algebras are preserved by the operations.

The existence of Σ implies the one of a lattice of subalgebras of the initial algebra and every operation determines a lattice of nilpotents ideals in the transformed algebra.

INTRODUCTION

First recall the Saletan contraction and its principal properties [7].

Let $\mathscr{L}$ be a real Lie algebra, of underlying finite dimensional vector space E, with the law:

$$(x, y) \to [x, y] \qquad x, y \in \mathrm{E}.$$

Let $\Phi_t \in \mathrm{End}\ \mathrm{E}$ of the form:

$$\Phi_t = u + t(1 - u) \qquad u \in \mathrm{End}\ \mathrm{E}$$

such that Φ_t is regular for $t \neq 0$ but singular for $t = 0$.

The Saletan contracted algebra is defined by:

$$(i) \qquad [x, y]_{(1)} = \lim_{t \to 0} \Phi_t^{-1}[\Phi_t x, \Phi_t y]$$

Using the Fitting decomposition of E respect to u;

$$\mathrm{E} = \mathrm{E_R} \oplus \mathrm{E_N}, \qquad \mathrm{E_R} = \bigcap_{k \geq 1} \mathrm{Im}\ u^k, \qquad \mathrm{E_N} = \sum_{k \geq 1} \mathrm{Ker}\ u^k$$

the author shows that the limit (*i*) exists if and only if:

(*ii*) $$[ux, uy]_N = u([ux, y]_N + [x, uy]_N - u[x, y]_N)$$

Under this condition one has a Lie algebra $\mathscr{L}_{(1)}$:

(*iii*) $$[x, y]_{(1)} = u_R^{-1}[ux, uy]_R + [ux, y]_N + [x, uy]_N - u[x, y]_N$$

The properties given by Saletan are the following ones:

Further contraction by u leads to a sequence of algebras $\mathscr{L}_{(i)}$, where $\mathscr{L}_{(i+1)}$ is obtained by contracting $\mathscr{L}_{(i)}$ by u. Any algebra in the sequence can be obtained from any previous one by contraction by a power of u; i. e. $\mathscr{L}_{(i+j)}$ can be obtained by contracting $\mathscr{L}_{(i)}$ by u. This sequence terminates at least with $\mathscr{L}_{(m)}$, where m is the least integer for which $u^m E = E_R$.

In every algebra of the sequence the $u^i E$ form a chain of subalgebras. The subalgebras formed by E_R in each of the $\mathscr{L}_{(i)}$ are isomorphic. If $E_i = \text{Ker } u^i$ for some i, then it forms an ideal in $\mathscr{L}_{(i+j)}$; $j = 0, 1, \ldots$.

In this paper we define in an algebrical way, an operation that we shall call « singularization », on certain classes of algebras which are not necessarily Lie algebras.

This operation generalizes the Saletan contraction in the meaning that it generally needs the extension of the base field and that it leads with the help of a condition analogous to (*i*), to a distributive lattice of subalgebras, that may be peculiarly boolean, and not only to a chain.

As for the « singularizations » themselves, we shall show that they form a semi-group. We shall finally give three examples in the Lie algebra sl (3, $\mathbb{C}$): two of them are Saletan contractions, the third one is of boolean type so the construction is meaning-full.

(See also final remarks).

0. PRELIMINARY DEFINITIONS

1) Set:

$\mathbb{K}$ a commutative field of characteristic zero. $\mathbb{K}[x]$ the ring of polynomials in X; $\mathbb{K}(X)$ the quotient field, $\mathbb{K}^0[x]$ the distributive lattice of polynomials of leading coefficient 1, with the relation order:

$$a \geqslant b \quad \text{when a divide } b, \text{ with } a, b \in \mathbb{K}^0[x].$$

$$a \wedge b = g.l.b(a, b) = l.c.m(a, b) \quad \text{and} \quad a \vee b = l.u.b(a, b) = g.c.d(a, b)$$

In the next we shall use a sublattice of $\mathbb{K}^0[x]$ written [q, 1], which is the set of divisors of $q \in \mathbb{K}^0[X]$.

2) Let the set be:

$$\mathbb{K}_q(X) = \left\{ \frac{a}{b}; a \in \mathbb{K}[X]; b \in \mathbb{K}^0[X]; a \vee b = 1; b \vee q = 1 \right\}$$

It is obvious that $\mathbb{K}_q(X)$ is a subring of $\mathbb{K}(X)$ and that:

$$p \geqslant q \Rightarrow \mathbb{K}_p(X) \supseteq \mathbb{K}_q(X)$$

3) Set α belonging to a finite algebraic extension of $\mathbb{K}$, that is to say such that $q(\alpha) = 0$ for $q \in \mathbb{K}^0[X]$; let X_α be the homomorphism of $\mathbb{K}_q(X)$ on the field $\mathbb{K}[\alpha]$ defined by: $X_\alpha(X) = \alpha$.

4) If now E is a $\mathbb{K}$-vector space, one may associate to X_α the homomorphism $\omega_\alpha = I \otimes X_\alpha$ of the $\mathbb{K}_q(X)$-module: $E_q(X) \equiv E \otimes_{\mathbb{K}} \mathbb{K}_q(X)$ on the $\mathbb{K}[\alpha]$-vector space: $E_{(\alpha)} \equiv E \otimes_{\mathbb{K}} \mathbb{K}[\alpha]$.

(I designates the identity on E).

5) At last, let us consider the extension of E: $E(X) = E \otimes_{\mathbb{K}} \mathbb{K}(X)$. An endomorphism u in E (of finite dimension) extends to E(X). Every element of the ring $\mathscr{A}_u \equiv \mathbb{K}[u] \otimes_{\mathbb{K}} \mathbb{K}[X]$ of the polynomials in u with coefficients in $\mathbb{K}(X)$, operate in E(X):

$$(u \otimes I)(x \otimes I) = ux \otimes I; \qquad x \in E$$

In order to shorten the notation one will write $\bar{f} = f(u)$ when $f(X) \in \mathbb{K}[X]$.

I. ASSOCIATED SINGULAR STRUCTURES

Let $\mathscr{L}$ be an algebra of underlying space E on $\mathbb{K}$ and $\mathscr{L}(X)$ an extension of $\mathscr{L}$ of underlying space E(X).

In the next $\mathscr{L}$ will designate the algebra as well as the law. One may define in E(X) a new law $\mathscr{L}^\Phi$:

$$(1) \qquad \mathscr{L}^\Phi(x, y) = \Phi^{-1}\mathscr{L}(\Phi x, \Phi y) \qquad x, y \in E \quad \text{or} \quad E(X)$$

Φ being an invertible element of $\mathscr{A}_u$.

The algebra $\mathscr{L}^\Phi(X)$ is isomorphic to $\mathscr{L}(X)$ by construction.

Let us suppose that u may be such that for $q \in \mathbb{K}^0[X]$ one may have:

$$(2) \qquad \mathscr{L}^\Phi(x, y) \in E_q(X) \quad \text{for} \quad x, y \in E_q(X)$$

that is to say: $\mathscr{L}^\Phi$ operate in $E_q(X)$.

Let α be such that $q(\alpha) = 0$; If $\alpha \notin \mathbb{K}$ one has a finite algebraic extension of $\mathbb{K}$. If the condition (2) is verified one may apply ω_α to $\mathscr{L}^\Phi(x, y)$ that which defines a law $\mathscr{L}_{(\alpha)}$ on the space $E_{(\alpha)}$:

$$(3) \qquad \mathscr{L}_{(\alpha)}(x, y) = \omega_\alpha \mathscr{L}^\Phi(x, y) \quad \text{for} \quad x, y \in E_{(\alpha)}$$

For instance if $\mathscr{L}$ is real, the operation implies in general the complexification of $\mathscr{L}$.

If $\alpha \in \mathbb{K}$, one may have $\mathscr{L}_{(\alpha)}$ not isomorphic to $\mathscr{L}$ although $E_{(\alpha)} = E$.

DEFINITION. — By means of the condition $\mathscr{L}^\Phi(x, y) \in E_q(X)$ for $x, y \in E_q(X)$ we shall call « singular algebra associated to $\mathscr{L}$ » an algebra $\mathscr{L}(\alpha)$ of underlying space $E_{(\alpha)}$ the law of which is:

$$\mathscr{L}_\alpha(x, y) = \omega_\alpha \mathscr{L}^\Phi(x, y) \quad \text{for} \quad x, y \in E_{(\alpha)}$$

The application $\mathscr{L} \rightarrow \mathscr{L}_{(\alpha)}$ will be called « singularization ».

II. STABILITY OF THE ALGEBRA CATEGORY

Let H an algebra category $\{ \mathscr{L}_i \}_{i \in I}$ on a space E of finite dimension on a characteristic zero field $\mathbb{K}$ such that the following homogeneous relations are satisfied:

$$\mathscr{K}_{(n)}(x_1, \ldots, x_n) = \sum_{\sigma \in G_n} \nu_\sigma x^{L}_{\sigma(1)} \ldots x^{L}_{\sigma(n-1)}(x_{\sigma(n)}) + \mu_\sigma x^{R}_{\sigma(n)} \ldots x^{R}_{\sigma(2)}(x_{\sigma(1)}) = 0$$

for $\qquad x_1, \ldots, x_n \in E \quad \text{et} \quad n = 1, 2, \ldots, N.$

where:

$$x^{L}(y) = \mathscr{L}(x, y) = y^{R}(x)$$
$$\lambda_\sigma, \mu_\sigma \in \mathbb{K}$$

G_n is the permutation group of n elements.

(i. e.: Associative algebras, Lie algebras, Jordan algebras, ...).

If $\qquad f \in \mathrm{Hom}_{\mathbb{K}}(\mathscr{L}_1, \mathscr{L}_2) \qquad \mathscr{L}_1, \mathscr{L}_2 \in H,$

that is to say:

$$f x^{L} = (f x)^{L} f \qquad \text{or} \qquad f y^{R} = (f y)^{R} f$$

we have, for arbitrary ν_σ and μ_σ:

$$f \mathscr{K}_n(x_1, \ldots, x_n) = \mathscr{K}_n(f x_1, \ldots, f x_n).$$

Now consider the extended category H(X) obtained by extension of $\mathbb{K}$ to $\mathbb{K}(X)$:

Let: Φ as in § I;

$$\Phi(x^{L}) = \Phi^{-1}(\Phi x)^{L}\Phi, \qquad \Phi(y^{R}) = \Phi^{-1}(\Phi y)^{R}\Phi \qquad x, y \in E.$$

(contravarient transformation).

Then:

$$\Phi(x^{L})(y) = \mathscr{L}^{\Phi}(x, y) = \Phi(y^{R})(x)$$

$$(4) \qquad \mathscr{K}^{\Phi}_{(n)}(x_1, \ldots, x_n) \equiv \sum_{\sigma \in G_n} \nu_\sigma \Phi(x^{L}_{\sigma(1)}) \ldots \Phi(x^{L}_{\sigma(n-1)})(x_{\sigma(n)}) + \mu_\sigma \Phi(x^{R}_{\sigma(n)}) \ldots \Phi(x^{R}_{\sigma(2)})(x_{\sigma(1)}) = \Phi^{-1} \mathscr{K}_{(n)}(\Phi x_1, \ldots, \Phi x_n)$$

(the $\mathscr{K}^{\Phi}_{(n)}$ are related to $\mathscr{L}^{\Phi}$ as the $\mathscr{K}_{(n)}$ are related to $\mathscr{L}$).

By the condition (2), if $x_i \in E_q(X)$, all the terms in the second member of (4) are also in $E_q(X)$.

Therefore, one may apply ω_α to the terms in (4) and $\mathscr{L}_{(\alpha)}$ is in the category H, or $H_{(\alpha)}$ eventually (Space $E_{(\alpha)}$).

The same fact is true for the subcategory H^0 of H of the algebras such that:

$$\mathscr{L}(y, x) = \nu(x, y)\mathscr{L}(x, y) \qquad \nu(x, y) \in \mathbb{K}.$$

We may conclude:

PROPOSITION 1. — The categories $H_{(\alpha)}$ and $H^0_{(\alpha)}$ are stable under the operation $\mathscr{L} \to \mathscr{L}_{(\alpha)}$.

III. FORMULATION OF THE LAWS AND EXISTENCE CONDITION FOR THE CHARACTERISTIC ENDOMORPHISM OF $u \in \operatorname{End} E$

First recall that if $u \in \operatorname{End} E$, then the characteristic endomorphism $X - u$ is an unit of $\mathscr{A}_u$; in fact:

$$(5) \qquad (X - u)^{-1} = \frac{1}{m(X)} \sum_{k=1}^{\deg m} \frac{(X-u)^{k-1}}{k!} m^{(k)}(u)$$

where $m(X)$ is the minimum polynomial of u (or a multiple).

Now let q be a complemented element of $[m, 1]$ that is to say $p = \frac{m}{q}$ be prime with q.

Polynomials a and $b \in \mathbb{K}^0[X]$ exist such that:

$$aq + bp = 1.$$

The endomorphisms $Q = \bar{a}\bar{q}$ and $P = \bar{b}\bar{p}$ are complementary projections since $\bar{p}\bar{q} = \bar{m} = 0$.

On the other hand, one has:

$$(6) \qquad QE = \operatorname{Im} \bar{q} = \operatorname{Ker} \bar{p}, \qquad PE = \operatorname{Im} \bar{p} = \operatorname{Ker} \bar{q}$$

The minimum polynomial of the restriction from u to QE, being p, we have, in setting $u_Q = u \mid QE$ that operates in QE, the inverse of $(X - u_Q)$:

$$(7) \qquad (X - u_Q)^{-1} = \frac{1}{p(X)} \sum_{k \geqslant 1} \frac{(X - u_Q)^{k-1}}{k!} p^{(k)}(u_Q)$$

Likewise on PE:

$$(7)' \qquad (X - u_P)^{-1} = \frac{1}{q(X)} \sum_{k \geqslant 1} \frac{(X - u_P)^{k-1}}{k!} q^{(k)}(u_P)$$

Let us set now, in E(X), for $x, y \in E$, v operating in E:

$$(8) \quad F_{x,y}(X, v, 1) \equiv X^2 \mathscr{L}(x, y) - X\mathscr{L}(vx, y) - X\mathscr{L}(x, vy) + \mathscr{L}(vx, vy)$$

therefore, with $\Phi = u - X$:

$$(9) \quad \begin{aligned} \mathscr{L}^{u-X}(x, y) &= (u - X)^{-1} F_{x,y}(X, u, 1) \\ &= (u_Q - X)^{-1} Q F_{x,y}(X, u, 1) \\ &+ (u_P - X)^{-1} P \left\{ F_{x,y}(u, u, 1) - (u - X) \frac{\partial}{\partial X} F_{x,y}(u, u, 1) \right. \\ &\qquad \left. + \frac{(u - X)^2}{2} \frac{\partial^2}{\partial X^2} F_{x,y}(u, u, 1) \right\} \end{aligned}$$

Now the condition (2), that allows to apply ω_α, from (7)′ and in developping (9), is equivalent to the condition:

(10) $\mathrm{PF}_{x,y}(u, u, 1) \equiv \mathrm{P}\{ u^2\mathscr{L}(x, y) - u\mathscr{L}(ux, y) - u\mathscr{L}(x, uy) + \mathscr{L}(ux, uy) \} = 0$

Under the condition (10) one has therefore:

(9)′ $$\mathscr{L}^{u-\mathrm{X}}(x, y) = (u_{\mathrm{Q}} - \mathrm{X})^{-1}\mathrm{QF}_{x,y}(u, u, 1) + \mathrm{P}\{ \mathscr{L}(ux, y) + \mathscr{L}(x, uy) - (u + \mathrm{X})\mathscr{L}(x, y) \}$$

Therefore, the expression of the new law is:

(11) $$\mathscr{L}_{(\alpha)}(x, y) = \Omega(u, \alpha)\mathrm{Q}\{ \mathscr{L}(ux, uy) - \alpha\mathscr{L}(ux, y) - \alpha\mathscr{L}(x, uy) + \alpha^2\mathscr{L}(x, y) \} + \mathrm{P}\{ \mathscr{L}(ux, y) + \mathscr{L}(x, uy) - u\mathscr{L}(x, y) - \alpha\mathscr{L}(x, y) \}$$

with $\Omega(u, \alpha) = \mathrm{X}_\alpha(u_{\mathrm{Q}} - \mathrm{X})^{-1}$ which is a polynomial in u and in α since $\mathbb{K}[\alpha]$ is a field.

Remark. — If $\alpha = 0$ one finds again a law of the type introduced by Saletan [7], about Lie algebras.

IV. CONDITION EQUIVALENT TO (10)

Let us set, for $x, y \in \mathrm{E}$:

$$\mathscr{L}(x)y = \mathscr{L}(x, y)$$
$$(\mathrm{A}\mathscr{L})(x) = \mathrm{A}\mathscr{L}(x) - \mathscr{L}(\mathrm{A}x); \qquad \mathrm{A} \in \mathbb{K}[u].$$

The relation (10) can be written:

$$\mathrm{P}[(u\mathscr{L})(x), u] = 0$$

Therefore, by obvious recurrence:

(12)′ $$\mathrm{P}[(u^m\mathscr{L})(x), u] = 0 \qquad m, m \in \mathbb{N}.$$

and

(12)″ $$\mathrm{P}[(\mathrm{A}\mathscr{L})(x), \mathrm{B}] = 0 \qquad \mathrm{A}, \mathrm{B} \in \mathbb{K}[u]$$

(If u is regular, $u^{-1} \in \mathbb{K}[u]$ and (12)′ is valid for $m, n \in \mathbb{Z}$).

This last relation takes also the form:

(12) $$\mathrm{P}\mathscr{L}(\mathrm{A}x, \mathrm{B}y) = \mathrm{P}\{ \mathrm{A}\mathscr{L}(x, \mathrm{B}y) + \mathrm{B}\mathscr{L}(\mathrm{A}x, y) - \mathrm{AB}\mathscr{L}(x, y) \}$$

On the other hand, in setting this time:

$$(u^{n+1}.\mathscr{L})(x) = u(u^n.\mathscr{L})(x) - (u^n.\mathscr{L})(ux)$$

where:

$$(u.\mathscr{L})(x) = (u\mathscr{L})(x) \quad \text{and} \quad (1.\mathscr{L})(x) = \mathscr{L}(x)$$

we obtain:

$$(u^n.\mathscr{L})(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u^{n-k} \mathscr{L}(u^k x)$$

Therefore, the condition (10) is also equivalent to:

(12‴) $P[(\bar{a}.\mathscr{L})(x), \bar{b}] = 0$ for all $a \in X\mathbb{K}[X]$, $b \in \mathbb{K}[X]$, $x \in E$.

V. THE OTHER SINGULAR LAWS ASSOCIATED TO $\mathscr{L}$ FOR u VERIFYING (10)

Henceforth, we shall use an extension of $\mathbb{K}$ that decompose q.
Let $\gamma \in \mathbb{K}^0[X]$ be such that:

(13) $$m \leqslant q \leqslant \gamma \leqslant 1$$

Let us consider now $\Phi = \bar{\gamma} - X \in \mathscr{A}_u$.

$\bar{\gamma}$ is invertible on E, if and only if $\gamma \vee m = 1$ (see (6), appendix) but also if and only if its minimum polynomial $\mu_{\bar{\gamma}}(X)$ is not divisible by X.

Therefore, if $\gamma \neq 1$, one has from (13): $\mu_{\bar{\gamma}}(X)$ divisible by X.

On the other hand, since $\gamma \vee p = 1$, $\bar{\gamma}$ is invertible on Ker $\bar{p}$ = QE (cf. (6) appendix); therefore the minimum polynomial V(X) of $\bar{\gamma}_Q = \bar{\gamma} \mid QE$ is not divisible by X; therefore:

$$(X - \bar{\gamma}_Q)^{-1} = \frac{1}{V(X)} \sum_{k \geqslant 1} \frac{(X - \bar{\gamma}_Q)^{k-1}}{k!} V^{(k)}(\bar{\gamma}_Q)$$

Therefore, one may apply ω_0 (that is $X \to 0$) in this expression that figures in the analogous of (9) where $\bar{\gamma}$ substitutes u. The condition analogous to (10) can be written: $PF_{x,y}(\bar{\gamma}, \bar{\gamma}, 1) = 0$ and is verified from (12).

Therefore the law on E is: (cf. (9)′):

(14) $$\mathscr{L}_{\bar{\gamma}}(x, y) = \bar{\gamma}_Q^{-1} Q \mathscr{L}(\bar{\gamma}x, \bar{\gamma}y) + P\{\mathscr{L}(\bar{\gamma}x, y) + \mathscr{L}(x, \bar{\gamma}y) - \bar{\gamma}\mathscr{L}(x, y)\}$$

One may see that (11) is not different from (14) with $\gamma = X - \alpha$ which satisfies (13): $\mathscr{L}_{(\alpha)} = \mathscr{L}_{u-\alpha}$.

More, one has the property: $\mathscr{L}$ is homomorphic to $\mathscr{L}_{\bar{\gamma}}$:

(15) $$\bar{\gamma}\mathscr{L}_{\bar{\gamma}}(x, y) = \mathscr{L}(\bar{\gamma}x, \bar{\gamma}y)$$

which results from (14) and (12) for $\bar{a} = \bar{b} = \bar{\gamma}$.

GENERALIZATION. — One may see from the expression (14) that one may define a singularization $\mathscr{L}_{\bar{\gamma}}$ for all $\gamma \in \mathbb{K}^0[X]$ provided γ be prime with p which is the necessary and sufficient condition in order that $\bar{\gamma}$ may be invertible on QE.

Remarks.

1) If $\bar{\gamma}$ is invertible on $E(\gamma \vee m = 1)\mathscr{L}_{\bar{\gamma}}$ is isomorphic to $\mathscr{L}$ from (15).

2) Let us consider $\mathscr{L}_{\bar{q}}$ and the application invertible on E: $A = \bar{a}Q + P$ (where a has been defined in (6), is invertible on QE since $a \vee p = 1$). One has, since: $A\bar{q} = \bar{a}(\bar{a}\bar{q})\bar{q} = Q^2 = Q$.

$$\begin{aligned}\mathscr{L}_q(Ax, Ay) &= (\bar{q}_Q)^{-1}Q\mathscr{L}(Qx, Qy) + P\{\mathscr{L}(Qx, Ay) + \mathscr{L}(Ax, Qy)\}\\ &= (\bar{q}_Q)^{-1}Q\mathscr{L}(Qx, Qy) + P\{\mathscr{L}(Qx, Py) + \mathscr{L}(Px, Qy)\}\end{aligned}$$

as one may see it in VI, QE is a $\mathscr{L}$-subalgebra.

Therefore:

$$(16)\quad A^{-1}\mathscr{L}_{\bar{q}}(Ax, Ay) = \mathscr{L}(Qx, Qy) + P\{\mathscr{L}(Qx, Py) + \mathscr{L}(Px, Qy)\}$$

which is not different from the Inönü-Wigner contraction with respect to the subalgebra QE of $\mathscr{L}$ [6].

VI. STRUCTURAL PROPERTIES

1) The condition (10), and therefore (12) is valid for $\mathscr{L}_{\bar{\gamma}}$.

Let us set:

$$F_{x,y}(X, v, \bar{\gamma}) = X^2\{\mathscr{L}_{\bar{\gamma}}(x, y) - X\mathscr{L}_{\bar{\gamma}}(vx, y) - X\mathscr{L}_{\bar{\gamma}}(x, vy) + \mathscr{L}_{\bar{\gamma}}(vx, vy)\}$$

Taking (14) in account:

$$PF_{x,y}(u, u, \bar{\gamma}) = PF_{\bar{\gamma}x,y}(u, u, 1) + PF_{x,\bar{\gamma}y}(u, u, 1) - P\bar{\gamma}F_{x,y}(u, u, 1)$$

This expression is void from (10), therefore:

$$(17)\qquad PF_{x,y}(u, u, \bar{\gamma}) = 0 \quad \text{for} \quad x, y \in E$$

2) LATTICE OF SUBALGEBRAS (¹).

The relation (17) implies (as in IV) for $\delta \in \mathbb{K}^0[X]$:

$$(18)\quad P\mathscr{L}_{\bar{\gamma}}(\bar{\delta}x, \bar{\delta}y) = P\bar{\delta}\{\mathscr{L}_{\bar{\gamma}}(\bar{\delta}x, y) + \mathscr{L}_{\bar{\gamma}}(x, \bar{\delta}y) - \bar{\delta}\mathscr{L}_{\bar{\gamma}}(x, y)\} \in \operatorname{Im}\bar{\delta}$$

On the other hand if, $\operatorname{Im}\bar{\delta} \supseteq QE$ that is if and only if $\delta \vee m \geqslant q$ on has peculiarly:

$$Q\mathscr{L}_{\bar{\gamma}}(\bar{\delta}x, \bar{\delta}y) \in \operatorname{Im}\bar{\delta}$$

Therefore $\operatorname{Im}\bar{\delta}$ is a sub-algebra of $\mathscr{L}_{\bar{\gamma}}$.

From (10) (Appendix, formulas (1) and (2))

$$\operatorname{Im}\overline{\delta_1 \wedge \delta_2} = \operatorname{Im}\bar{\delta}_1 \cap \operatorname{Im}\bar{\delta}_2, \qquad \operatorname{Im}\overline{\delta_1 \vee \delta_2} = \operatorname{Im}\bar{\delta}_1 + \operatorname{Im}\bar{\delta}_2$$

for

$$\delta_1, \delta_2 \in [q, 1]$$

Therefore one has:

(¹) For the definitions see (10) Appendix.

PROPOSITION 2. — The set of the subspaces Im $\bar{\delta}$ of E for δ divisor of q, is a lattice of subalgebras of $\mathscr{L}_{\bar{\gamma}}$ what ever may be $\gamma \in \mathbb{K}^0[X]$ prime with p, lattice isomorphic to $[q, 1]$.

This is true peculiarly for the initial algebra $\mathscr{L}(\gamma = 1)$, what is a necessary structure condition of $\mathscr{L}$.

3) LATTICE OF (LEFT) IDEALS.

Let $\gamma \in \mathbb{K}^0[X]$ be such $q \leqslant \gamma \leqslant \delta \leqslant 1$ and let it be: $\xi \in \operatorname{Ker} \bar{\delta} \subseteq \operatorname{Ker} \bar{\gamma}$ therefore $\bar{\gamma}\xi = 0$ and from (14):

$$\mathscr{L}_\gamma(\xi, y) = \mathrm{P}\{\mathscr{L}(\xi, \bar{\gamma}y) - \bar{\gamma}\mathscr{L}(\xi, y)\}$$

Therefore, taking (12) in account.

$$\begin{aligned}\bar{\delta}\mathscr{L}_\gamma(\xi, y) &= \bar{\delta}\mathrm{P}\{\mathscr{L}(\xi, \bar{\gamma}y) - \bar{\gamma}\mathscr{L}(\xi, y)\} \\ &= \mathrm{P}\{\mathscr{L}(\bar{\delta}\xi, \bar{\gamma}y) - \bar{\gamma}\mathscr{L}(\bar{\delta}\xi, y)\} = 0.\end{aligned}$$

That is to say: Ker $\bar{\delta}$ is an ideal of $\mathscr{L}\bar{\gamma}$.

Then, let them be $\xi_1, \xi_2, \xi_k, \xi_{k+1}, \ldots, \eta \in \operatorname{Ker} \bar{\delta}$.

Let us show that:

(19) $$\mathscr{L}_{\bar{\gamma}}(\xi_k) \ldots \mathscr{L}_{\bar{\gamma}}(\xi_1)\eta = (-\mathrm{P}\bar{\gamma})^k \mathscr{L}(\xi_k) \ldots \mathscr{L}(\xi_1)\eta$$

where one has set $\mathscr{L}(\xi)\eta \equiv \mathscr{L}(\xi, \eta)$ and analogous.

Let us set:

$$y = \mathscr{L}(\xi_k) \ldots \mathscr{L}(\xi_1)\eta \quad \text{and} \quad \xi = (-\mathrm{P}\bar{\gamma})^k y.$$

The result is true for $k = 1$ what proceeds immediately from (14). On the other hand:

$$\begin{aligned}\mathscr{L}_{\bar{\gamma}}(\xi_{k+1})\mathscr{L}_{\bar{\gamma}}(\xi_k) \ldots \mathscr{L}_{\bar{\gamma}}(\xi_1)\eta &= \mathrm{P}\{\mathscr{L}(\bar{\gamma}\xi_{k+1}, y) - \bar{\gamma}\mathscr{L}(\xi_{k+1}, y)\} \\ &= \mathrm{P}(-\mathrm{P}\bar{\gamma})^k\{\mathscr{L}(\bar{\gamma}\xi_{k+1}, y) - \bar{\gamma}\mathscr{L}(\xi_{k+1}, y)\} \\ &= (-\mathrm{P}\bar{\gamma})^{k+1}\mathscr{L}(\xi_{k+1})\mathscr{L}(\xi_k) \ldots \mathscr{L}(\xi_1)\eta\end{aligned}$$

(One took (12) and $\bar{\gamma}\xi_{k+1} = 0$ in account). Now Let $\gamma_{i_1} \ldots \gamma_{i_l}$ be the prime factors of γ.

On may write $q = q_1 q_2$ with $q_1 = \gamma_{i_1}^{\nu_{i_1}} \ldots \gamma_{i_l}^{\nu_{i_l}}$ and $q_1 \vee q_2 = 1$.

Therefore:

$$\delta \geqslant \gamma \geqslant q_1 \geqslant pq_1$$

implies:

$$\operatorname{Ker} \bar{\delta} \subseteq \operatorname{Ker} \bar{p}\bar{q}_1 = \operatorname{Im} q_2 \quad \text{since} \quad q_2 \quad \text{and} \quad pq_1$$

are complemented each other with respect to m.

On the other hand $-\mathrm{P}\bar{\gamma}$ is nilpotent on the subspace Ker $\bar{p}\bar{q}_1$ since δ exists such that: $p\gamma^s \leqslant pq_1$.

Now Ker $\bar{p}\bar{q}_1 = \operatorname{Im} \bar{q}_2$ is a subalgebra of $\mathscr{L}$ from 2°; therefore $y \in \operatorname{Im} \bar{q}_2$ and from (12) on sees that Ker $\bar{\delta}$ is a nilpotent ideal of $\mathscr{L}_{\bar{\gamma}}$.

At last, from 10° (appendix, formulas (2) and (3)

$$\operatorname{Ker} \overline{\delta_1 \vee \delta_2} = \operatorname{Ker} \bar{\delta}_1 \cap \operatorname{Ker} \bar{\delta}_2, \qquad \operatorname{Ker} \delta_1 \wedge \delta_2 = \operatorname{Ker} \bar{\delta}_1 + \operatorname{Ker} \bar{\delta}_2$$

for

$$\delta_1, \delta_2 \in [\gamma, 1]$$

Therefore one has:

PROPOSITION 3. — The set of the subspaces Ker $\bar{\delta}$ for $q \leqslant \gamma \leqslant \delta \leqslant 1$ is a lattice of left (or right) nilpotents ideals of $\mathscr{L}_{\bar{\gamma}}$, lattice isomorphic to $[\gamma, 1]$.

Let us remark that Ker $\bar{\delta}$ is a trivial ideal of $\mathscr{L}_{\bar{q}}$ and in this case, the lattice of trivial ideals is isomorphic to $[q, 1]$.

VII. COMPOSITION OF SINGULARIZATIONS

One may define from (18), the singularization:

$$(\mathscr{L}_{\bar{\gamma}})_{\bar{\delta}}(x, y) = (\bar{\delta}_Q)^{-1} Q \mathscr{L}_{\bar{\gamma}}(\bar{\delta}x, \bar{\delta}y) + P \{ \mathscr{L}_{\bar{\gamma}}(\bar{\delta}x, y) + \mathscr{L}_{\bar{\gamma}}(x, \bar{\delta}y) - \bar{\delta}\mathscr{L}_{\bar{\gamma}}(x, y) \}$$

provided one may have δ prime with P.

From (14) one may write:

$$\begin{aligned}(\mathscr{L}_{\bar{\gamma}})_{\bar{\delta}}(x, y) = (\bar{\gamma}_Q \bar{\delta}_Q)^{-1} Q \mathscr{L}(\bar{\gamma}\bar{\delta}x, \bar{\gamma}\bar{\delta}y) & \\ + P \{ \mathscr{L}(\bar{\gamma}\bar{\delta}x, y) + \mathscr{L}(x, \bar{\gamma}\bar{\delta}y) - \bar{\gamma}\bar{\delta}\mathscr{L}(x, y) \} & \\ + P \{ \mathscr{L}(\bar{\gamma}x, \bar{\delta}y) - \bar{\gamma}\mathscr{L}(x, \bar{\delta}y) - \bar{\delta}\mathscr{L}(x, \bar{\gamma}y) + \bar{\gamma}\bar{\delta}\mathscr{L}(x, y) \} & \\ + P \{ \mathscr{L}(\bar{\delta}x, \bar{\gamma}y) - \bar{\delta}\mathscr{L}(x, \bar{\gamma}y) - \bar{\gamma}\mathscr{L}(\delta\bar{x}, y) + \bar{\gamma}\bar{\delta}\mathscr{L}(x, y) \} &\end{aligned}$$

The two last lines are void from (12), therefore:

$$(\mathscr{L}_{\bar{\gamma}})_{\bar{\delta}}(x, y) = \mathscr{L}_{\bar{\gamma}\bar{\delta}}(x, y)$$

or, by setting $\Gamma_\gamma \mathscr{L} = \mathscr{L}_{\bar{\gamma}}$, one has:

$$(20) \qquad \Gamma_\delta \Gamma_\gamma = \Gamma_{\delta\gamma} \qquad \text{with} \qquad \gamma \vee p = \delta \vee p = 1$$

Peculiarly, since $\mathbb{K}$ splits q:

$$\begin{aligned} q(X) &= (X - \alpha_1)^{s_1} \dots (X - \alpha_r)^{s_r} \quad \text{with} \quad \alpha_i \in \mathbb{K}; \quad i = 1, 2, \dots, r. \\ \gamma(X) &= (X - \alpha_2)^{v_1} \dots (X - \alpha_r)^{v_r} \quad \text{with} \quad 0 \leqslant v_1 \leqslant s_i \\ \Gamma_\gamma &= \Gamma_{X-\alpha_1}^{v_1} \dots \Gamma_{X-\alpha_r}^{v_r} \end{aligned}$$

Let us also remark that $\bar{\gamma}$ is a homomorphism of $\mathscr{L}_{\bar{\gamma}\bar{\delta}}$ in $\mathscr{L}_{\bar{\delta}}$ from (15), (18) and (20).

We shall show now:

PROPOSITION 4. — If $\sigma \in \mathbb{K}^0[X]$ is prime with p, then $\Gamma_\sigma \mathscr{L}$ is isomorphic to $\Gamma_{\sigma \vee q} \mathscr{L}$.

First, let us consider should this happen: $\sigma = q_0 \gamma$ with: $q \leqslant q_0 \dot{<} 1$, γ having the same prime factors as q_0, and q_0 complemented in $[m, 1]$.

Let P_0 be the projection associated to q_0.

$$(P_0 E = \text{Ker } \bar{q}_0 \quad \text{and} \quad (1 - P_0)E = \text{Im } \bar{q}_0).$$

One has $PP_0 = P_0P = P_0$ and the application: $S = P_0 + \bar{\gamma}(1 - P_0)$ is invertible on E. Indeed, $\bar{\gamma}$ is invertible on $(1 - P_0)E$ since one has, in setting $q = q_0 r$:

$$\gamma \vee \left(\frac{m}{q_0}\right) = \gamma \vee (pr) = \gamma \vee r = 1 \quad \text{for} \quad q_0 \vee r = 1.$$

We shall show that S is an algebra isomorphism of $\mathscr{L}_{\bar{q}_0}$ on $\mathscr{L}_{\bar{\sigma}}$. Indeed, since $P_0 \bar{\sigma} = P_0 \bar{q}_0 \bar{\gamma} = 0$ one has:

(*i*) $x, y \in P_0E$

$$\mathscr{L}_{\bar{\sigma}}(x, y) = - P\bar{\sigma}\mathscr{L}(x, y) = P(P - P_0)\mathscr{L}(x, \bar{\sigma}y) - P(P - P_0)\bar{\sigma}\mathscr{L}(x, y)$$
$$= P\{ \mathscr{L}((P - P_0)x, \bar{\sigma}y) - \bar{\sigma}\mathscr{L}((P - P_0)x, y)\} = 0 \quad \text{(cf. (12))}$$

On the other hand:

$$\mathscr{L}_{\bar{q}_0}(Sx, Sy) = \mathscr{L}_{\bar{q}_0}(x, y) = - P_{\bar{q}_0}\mathscr{L}(x, y) = 0.$$

(*ii*) $x \in P_0E, \quad y \in (1 - P_0)E$

$$\mathscr{L}_{\bar{\sigma}}(x, y) = P\{ \mathscr{L}(x, \bar{\sigma}y) - \bar{\sigma}\mathscr{L}(x, y)\}$$
$$S\mathscr{L}_{\sigma}(x, y) = P_0\mathscr{L}(x, \bar{\sigma}y) + \bar{\gamma}(P - P_0)\{ \mathscr{L}(x, \bar{\sigma}y) - \bar{\sigma}\mathscr{L}(x, y)\}$$
$$= P_0\mathscr{L}(x, \bar{\sigma}y).$$

The second term is void for the same reason as (*i*)

$$\mathscr{L}_{\bar{q}_0}(Sx, Sy) = \mathscr{L}_{\bar{q}_0}(x, \bar{\gamma}y) = P\{ \mathscr{L}(x, \bar{q}_0\bar{\gamma}y) - \bar{q}_0\mathscr{L}(x, \bar{\gamma}y)\}$$
$$= P_0\mathscr{L}(x, \bar{\sigma}y) + (P - P_0)\{ \mathscr{L}(x, \bar{q}_0\bar{\gamma}y) - \bar{q}_0\mathscr{L}(x, \bar{\gamma}y)\}$$
$$= P_0\mathscr{L}(x, \bar{\sigma}y)$$

At last:

(*iii*) $x, y \in (1 - P_0)E.$

$$\mathscr{L}_{\bar{\sigma}}(x, y) = (\bar{\sigma}_Q)^{-1}Q\mathscr{L}(\bar{\sigma}x, \bar{\sigma}y) + P\mathscr{N}_{\bar{\sigma}}(x, y)$$

where:

$$\mathscr{N}_{\bar{\sigma}}(x, y) \equiv \mathscr{L}(\bar{\sigma}x, y) + \mathscr{L}(x, \bar{\sigma}y) - \bar{\sigma}\mathscr{L}(x, y)$$
$$S\mathscr{L}_{\bar{\sigma}}(x, y) = \bar{\gamma}(1 - P_0(\bar{q}_0\bar{\gamma})_Q^{-1}\mathscr{L}(\bar{\sigma}x, \bar{\sigma}y) + \bar{\gamma}(P - P_0)\mathscr{N}_{\sigma}(x, y)$$
$$= (\bar{q}_0)_Q^{-1}Q\mathscr{L}(\bar{q}_0\bar{\gamma}x, \bar{q}_0\bar{\gamma}y) + P\mathscr{N}_{\bar{q}_0\bar{\gamma}}(x, y)$$

(Recall that $P_0Q = 0$ and $(1 - P_0)E = \operatorname{Im} \bar{q}_0$ is a sub-algebra of $\mathscr{L}$ from prop. 2 and $P_0\mathscr{N}_{\bar{\sigma}}(x, y) = 0$).

On the other hand:

$$\mathscr{L}_{\bar{q}_0}(Sx, Sy) = \mathscr{L}_{\bar{q}_0}(\bar{\gamma}x, \bar{\gamma}y)$$
$$= (\bar{q}_0)_Q^{-1}Q\mathscr{L}(\bar{q}_0\bar{\gamma}x, \bar{q}_0\bar{\gamma}y) + P\mathscr{N}_{\bar{q}_0}(\bar{\gamma}x, \bar{\gamma}y)$$

From (12) we have:

$$P\mathscr{L}(\bar{q}_0\bar{\gamma}x, \bar{\gamma}y) = P\{ \bar{q}_0\bar{\gamma}\mathscr{L}(x, \bar{\gamma}y) + \bar{\gamma}\mathscr{L}(\bar{q}_0\bar{\gamma}x, y) - \bar{q}_0\bar{\gamma}^2\mathscr{L}(x, y)\}$$
$$P\mathscr{L}(\bar{\gamma}x, \bar{q}_0\bar{\gamma}y) = P\{ \bar{\gamma}\mathscr{L}(x, \bar{q}_0\bar{\gamma}y) + \bar{q}_0\bar{\gamma}\mathscr{L}(\bar{\gamma}x, y) - \bar{q}_0\bar{\gamma}^2\mathscr{L}(x, y)\}$$
$$P\bar{q}_0\mathscr{L}(\bar{\gamma}x, \bar{\gamma}y) = P\{ \bar{q}_0\bar{\gamma}\mathscr{L}(x, \bar{\gamma}y) + \bar{q}_0\bar{\gamma}\mathscr{L}(\bar{\gamma}x, y) - \bar{q}_0\bar{\gamma}^2\mathscr{L}(x, y)\}$$

Then:

$$P\mathcal{N}_{\bar{q}_0}(\bar{\gamma}x, \bar{\gamma}y) = P\bar{\gamma}\mathcal{N}_{\bar{q}_0}\bar{\gamma}(x, y)$$

Therefore, from (*i*), (*ii*) and (*iii*) one has at last:

$$S\mathcal{L}_{\bar{\sigma}}(x, y) = \mathcal{L}\bar{q}_0(Sx, Sy) \quad \text{for} \quad x, y \in E$$

Secondly, for any σ, but prime with p, one may write:

$$\sigma = (q_1\gamma_1) \dots (q_r\gamma_r)\gamma_0\zeta$$

q_α are complemented in $[m, 1]$.
$q \leqslant q_\alpha \leqslant 1$; γ_α having the same prime factors as q_α, $\alpha = 1, 2 \dots r$.
$q \leqslant \gamma_0 \leqslant 1$; ζ prime with q.
Γ_ζ being an isomorphism and $\sigma \vee q = q_1 \dots q_r\gamma_0$ one has:

$$\mathcal{L}_{\bar{\gamma}} \quad \text{isomorphic to} \quad \mathcal{L}_{\overline{\sigma \vee q}}. \tag{22}$$

So, the number of distinct singularizations up to an isomorphism, is the one of the elements of $[q, 1]$.

On the other hand, $[q, 1]$ provided with the law: $\gamma.\delta = (\gamma q) \vee q$ for γ and δ in $[q, 1]$ is a commutative semi-group, with unit, verifying:

$$(\alpha.\gamma) \vee (\beta.\gamma) = (\alpha \vee \beta).\gamma$$
$$(\alpha.\gamma) \wedge (\beta.\gamma) = (\alpha \wedge \beta).\gamma$$

Therefore one may state:

THEOREM. — Let $\mathcal{L}$ be an algebra on a finite dimensional vector space E on a characteristic zero field $\mathbb{K}$, u an endomorphism of E, P a projection commuting with u, such that one may have:

(*i*) $uP\{\mathcal{L}(ux, y) + \mathcal{L}(x, uy) - u\mathcal{L}(x, y)\} = P\mathcal{L}(ux, uy)$ for $x, y \in m$.

(*ii*) The minimum polynomial of u_p splittable on $\mathbb{K}$.

Then, a finite polynomial semi-group Σ, with unit, exists, which is a distributive lattice $\hat{\Sigma}$, that depends of u and P so that the application Γ of Σ on the set of the singularizations defined by:

$$\Gamma_\gamma\mathcal{L}(x, y) = \gamma(u_{1-P})^{-1}(1 - P)\mathcal{L}(\gamma(u)x, \gamma(u)y) + P\{\mathcal{L}(\gamma(u)x, y) + \mathcal{L}(x, \gamma(u)y) - \gamma(u)\mathcal{L}(x, y)\}$$

for $x, y \in E$ and $\gamma \in \Sigma$

as the following properties:

(*j*) Γ is an epimorphism.

(*jj*) $\gamma(u)$ is a homomorphism of $\Gamma_{\gamma\delta}\mathcal{L}$ in $\Gamma_\delta\mathcal{L}$.

(*jjj*) The set: $\{\operatorname{Im}\gamma(u);\ \gamma \in \Sigma\}$ is a distributive lattice isomorphic to $\hat{\Sigma}$, of subspaces of E which are subalgebras of $\Gamma_\delta\mathcal{L}$ for any $\delta \in \Sigma$.

(*jv*) The set: $\{\operatorname{Ker}\delta(u);\ \delta \in \Sigma;\ \delta \text{ divides } \gamma \text{ fixed};\ \gamma \in \Sigma\}$ is a distributive lattice of subspaces of E, which are left (or right) nilpotents ideals of $\Gamma_\gamma\mathcal{L}$.

VIII. EXAMPLES

Let us consider the Lie algebra $\mathscr{L} = \mathrm{sl}(3, \mathbb{C})$; H a Cartan subalgebra of basis $\{ h_1, h_2 \}$; $\{ e_1, e_2 \}$ and $\{ \bar{e}_1, \bar{e}_2 \}$ the eigenvectors corresponding respectively to positives and negatives roots of a simple system of roots relative to H; then:

$$e_{12} = [e_1, e_2] \quad \text{and} \quad \bar{e}_{12} = [\bar{e}_1, \bar{e}_2]$$

that achieve a basis of sl 3$\mathbb{C}$ (See column I).

We give firstly two examples of Saletan contractions ($\hat{\Sigma}$ is a chain).

EXEMPLE 1. — Set:

$$E_1 = 2h_2 + 3e_2 + \bar{e}_2$$
$$E_2 = - h_2 - 2e_2$$

$R = \mathbb{C}\{ e_1, e_{12} \}$ abelian subalgebra of $\mathscr{L}$.
$T_1 = \mathbb{C}\{ E_1, E_2 \}$ solvable subalgebra of $\mathscr{L}([E_1, E_2] = 2(E_1 + E_2))$.
$T_2 = \mathbb{C}\{ h_1, h_2, \bar{e}_1, \bar{e}_{12} \}$ solvable subalgebra of $\mathscr{L}$.
If E designates the vector space underlying to $\mathscr{L}$, one has:

$$E = R \oplus T_1 \oplus T_2.$$

Then let Q, P_1, P_2 be the system of projections to the three subspaces R, T_1, T_2 and the application:

$$t: T_2 \to T_1 \quad \text{defined by:} \quad th_1 = th_2 = 0; \quad t\bar{e}_1 = E_1; \quad t\bar{e}_{12} = E_2$$

One easily verifies that:

1) $$[t\bar{e}_1, t\bar{e}_{12}] = t\{ [t\bar{e}_1, \bar{e}_{12}] + [e_1, t\bar{e}_{12}] \}.$$

2) $N = tP_2$ is nilpotent: $N^2 = 0$.

3) $$\mathscr{U} = Q - N \quad \text{and} \quad P = P_1 + P_2$$

are so as:

$$P[\mathscr{U}x, \mathscr{U}y] = P\mathscr{U}\{ [\mathscr{U}x, y] + [x, \mathscr{U}y] - \mathscr{U}[x, y] \} \quad \text{for all} \quad x, y \in E$$

One has therefore, from (10), a Lie algebra $\mathscr{L}'$, of underlying space E, and the law of which is: (cf. (14)).

$$[x, y]' = Q[\mathscr{U}x, \mathscr{U}y] + P\{ [\mathscr{U}x, y] + [x, \mathscr{U}y] - \mathscr{U}[x, y] \}$$

(See App. 11°, column II).

One may notice that:

$\mathrm{Im}\, \mathscr{U} = \mathscr{R} \oplus T_1$ is a subalgebra of $\mathscr{L}$ and $\mathscr{L}'$.
$\mathrm{Ker}\, \mathscr{U} = \mathrm{Ker}\, N = T_1 \oplus \mathrm{Ker}\, t = \mathbb{C}\{ h_1, h_2, E_1, E_2 \}$
$= \mathbb{C}\{ h_1, h_2, e_2, \bar{e}_2 \}$ is an abelian ideal of $\mathscr{L}'$.

Here, one has:

$m(X) = X^2(X-1)$: minimum polynomial of $\mathcal{U}$.
$q(X) = X^2$ minimum polynomial of $\mathcal{U}_p$.
$\gamma(X) = X$ for the explicited $\mathcal{L}_{\bar{\gamma}}$ singularization.
$\hat{\Sigma} = \{ X^2, X, 1 \}$ which the subalgebras chains corresponds to:

$$\{ R = QE, R \oplus P_1E, E \}$$

EXEMPLE 2. — Set now:

$$E = h_2 + 2e_2$$
$$\Sigma_1 = \mathbb{C}E$$
$$\Sigma_2 = \mathbb{C}\{ h_1, h_2, \bar{e}_1, \bar{e}_2, \bar{e}_{12} \} \text{ maximal solvable subalgebra of } \mathcal{L}.$$

One has still: $E = R \oplus \Sigma_1 \oplus \Sigma_2$ and let Q, Π_1, Π_2 be the system of projections relative to these subspaces and the application:

$$\sigma : \Sigma_2 \to \Sigma_1 \quad \text{defined by:} \quad \sigma h_1 = \sigma h_2 = \sigma \bar{e}_2 = 0; \sigma \bar{e}_1 = -\sigma \bar{e}_{12} = E$$

As previously one verifies that:

1) $$[\sigma \bar{e}_1, \sigma \bar{e}_{12}] = \sigma \{ [\sigma \bar{e}_1, \bar{e}_{12}] + [\bar{e}_1, \sigma \bar{e}_{12}] \}$$

2) $M = \sigma \Pi_2$ is nilpotent: $M^2 = 0$.

3) $v = Q - M$ and $P = \Pi_1 + \Pi_2$ lead, with formulas analogous to exemple 1, to a Lie algebra $\mathcal{L}''$ (see App. 11° column III) such that:

$\operatorname{Im} v = R \oplus \Sigma_1$ is a subalgebra of $\mathcal{L}$ and $\mathcal{L}''$.
$\operatorname{Ker} v = \operatorname{Ker} M = \Sigma_1 \oplus \operatorname{Ker} \sigma = \mathbb{C}\{ h_1, h_2, \bar{e}_2, E, \bar{e}_1 + \bar{e}_{12} \}$
$= \mathbb{C}\{ h_1, h_2, e_2, \bar{e}_2, \bar{e}_1 + \bar{e}_{12} \}$ is a nilpotent (non abelian) ideal of $\mathcal{L}''$.

The minimum polynomials of v, v_p, etc., are the same as in example 1. In column IV, we gave as comparison the Inonu-Wigner contraction with respect to $\mathcal{R}$.

We give now an example, which is not a Saletan Contraction:

EXEMPLE 3. — Set:

$$S_1 = \mathbb{C}\{ e_1, e_2, e_{12} \} \quad \text{and} \quad S_2 = \mathbb{C}\{ \bar{e}_1, \bar{e}_2, \bar{e}_{12} \}$$

which are nilpotent subalgebras of $\mathcal{L}$.

$$R_1 = H \oplus S_1 \quad \text{and} \quad R_2 = H \oplus S_2$$

which are maximal solvable subalgebras of $\mathcal{L}$.

Then let q, p_1, p_2 be the system of projections relative to H, S_1 and S_2 and:

$$\mathcal{U}_1 = q + ap_2 \qquad (a \neq 1) \quad p = p_1 + p_2.$$

One has still a relation:

$$p[\mathcal{U}_1 x, \mathcal{U}_1 y] = p\mathcal{U}_1 \{ [\mathcal{U}_1 x, y] + [x, \mathcal{U}_1 y] - \mathcal{U}_1[x, y] \}$$

One obtains a Lie algebra $\mathscr{L}_1$ the law of which is:

$$[x, y]_1 = q[\mathscr{U}_1 x, \mathscr{U}_1 y] + p \{ [\mathscr{U}_1 x, y] + [x, \mathscr{U}_1 y] - \mathscr{U}_1[x, y] \}$$

(App. 11°, column V; $a = 2$).

R_1 and R_2 are two subalgebras of $\mathscr{L}_1$; S_1 and S_2 are two abelian ideals of $\mathscr{L}_1$.

If $a = 1$ one finds again the Inonu-Wigner contraction with respect to R_2.

Here, for $\mathscr{U}_1$ one has:

$$m(X) = X(X - 1)(X - 2); \quad q(X) = X(X - 2); \quad \gamma(X) = X$$

and:

$$\hat{\Sigma} = \{ X(X - 2), X - 2, X, 1 \} \qquad \text{(boolean lattice).}$$

Likewise one may define $\mathscr{L}_2$ with $\mathscr{U}_2 = q + ap_1$ $(a \neq 1)$ which is isomorphic to $\mathscr{L}_1$ in the present case.

The alone properties of $\mathscr{L}$ which interfere here are:

$$R_1 \text{ and } R_2 \text{ subalgebras of } \mathscr{L}, \qquad R_1 + R_2 = E.$$

Otherwise, $\mathscr{L}$ is an unification of R_1 and R_2 with intersection H [*8*].

One will see further a generalization of this category of operations, bound to diagonalizable endomorphisms.

Remarks. — In a more general way, the existence of a boolean lattice of subalgebras of $\mathscr{L}$ is necessary, but also sufficient to determine (u, P) equivalents. Besides one may construct such lattices from subalgebras given in $\mathscr{L}$.

The examples I and II proceed from a primitive filtration existence of sl $(3, \mathbb{C})$. We shall explicit the method in the frame of the simple Lie algebras on the complex filed in an other paper. The structure theorem given in § VII will be used in order to compute the set S of the endomorphisms of E which satisfie (10) (P given), up to an equivalence.

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APPENDIX

RECALL ABOUT LATTICES [4]

1° DEFINITIONS.

A set L ordered by $\leqslant$ is called a lattice if for any

$$x, y \in \mathrm{L},\ x \vee y = l.u.b(x, y) \quad \text{and} \quad x \wedge y = g.l.b(x, y)$$

exist.

A subset S of L is called a sublattice of L, if it is a lattice for the restriction of $\leqslant$ to S.

One will use the sublattices $[a, b] = \{ x ; a \leqslant x \leqslant b \}$ for $a, b, x \in \mathrm{L}$ in the same way as

$$] \rightarrow a] \quad \text{and} \quad [a \rightarrow [.$$

If in a set L, one has two laws $\wedge$ and $\vee$, each of therm idempotent, associative, commutative, and such that: $x \wedge (x \vee y) = x \vee (x \wedge y) = x$, then L is a lattice for $x \leqslant y$ defined

$$x \wedge y = x \Leftrightarrow x \vee y = y.$$

In a lattice a maximum element I, a minimum element $\mathcal{O}$ may exist.

2° DUALITY.

If in a lattice L, a property $\mathrm{P}(\vee, \wedge, \leqslant)$ is true, then dual property $\mathrm{P}^* = \mathrm{P}(\wedge, \vee, \leqslant)$ is true.

3° MODULAR LATTICES.

In a lattice L one has:

$$x \leqslant y \Rightarrow x \vee (z \wedge y) \leqslant (x \vee z) \wedge y.$$

If in the second member one has the equality, L is called *modular*.

4° EXAMPLE.

The set $\mathcal{S}(\mathcal{M})$ of subspaces of a vector space $\mathcal{M}$ is a modular lattice for $(+, \cap, \subseteq)$ and more generally the set of invariant subgroups of a group G is a modular lattice for $(.\,, \cap, \subseteq)$.

5° DISTRIBUTIVE (AND BOOLEAN) LATTICES.

In a lattice L one has:

$$x \vee (y \wedge z) \leqslant (x \vee y) \wedge (x \vee z) \qquad \text{and the dual relation.}$$

If one has the equality, L is called *distributive*.

If for any $x \in \mathrm{L}$, an element $\bar{x} \in \mathrm{L}$ such that $x \vee \bar{x} = \mathrm{I}$ and $x \wedge \bar{x} = \mathcal{O}$ exist (complement) the distributive lattice L is called *boolean*.

6° EXAMPLE.

The set $\mathbb{K}^0[\mathrm{X}]$ of polynomials in X, with leading coefficient 1, on a commutative field $\mathbb{K}$ with:

$$a \leqslant b \qquad \text{iff a multiple of } b.$$

$$a \vee b = g.c.d(a, b) \quad \text{and} \quad a \wedge b = l.c.m(a, b)$$

is a distributive lattice with I but without $\mathcal{O}$.

If L is modular (resp. distributive) a sublattice S is modular (resp. distributive).

7° LATTICE MORPHISMS.

An application θ of a lattice L in a lattice M is called a morphism iff:

$$\theta(x \vee y) = \theta(x) \vee \mathrm{A}(y) \quad x, y \in \mathrm{L}$$

$$\theta(x \wedge y) = \theta(x) \wedge \theta(y)$$

One defines also the: epi, mono, iso, endo, auto-morphisms.
A morphism is an isotone application; an isotone bijection the inverse of which is isotone is an isomorphism.

8° LATTICE IDEALS.

On the contrary of the algebras, there are two sorts of « ideals ».
a) J is an *ideal* of L iff:

i) $J \neq \emptyset$

ii) $a \in J,\ x \in L,\ x \leqslant a \Rightarrow x \in J$

iii) $a \in J,\ b \in J \Rightarrow a \vee b \in J.$

a)* J* is a *dual-ideal* iff: (dual statement)

j) $J^* \neq L$

jj) $a \in J^*,\ x \in L,\ x \geqslant a \Rightarrow x \in J^*.$

jjj) $a \in J^*,\ b \in J^* \Rightarrow a \wedge b \in J^*.$

One may state various properties relatives to *a*) (and also dual properties relatives to *a*)*)
b) $J \neq \emptyset$ ideal of $L \Leftrightarrow (a \vee b \in J \Leftrightarrow a \in J$ and $b \in J)$.
c) If θ is a morphism of L in M having a minimum element $\mathcal{O}$, then

$$\operatorname{Ker} \theta = \{ x;\ x \in L;\ \theta(x) = \mathcal{O} \}$$

is an ideal of L.

9° CONGRUENCE RELATIONS.

An equivalence relation C in a lattice L, is called a congruence relation iff:

$$a \equiv b \,(\text{mod } C) \Rightarrow \begin{array}{l} a \vee x \equiv b \vee x \,(\text{mod } C) \\ a \wedge x \equiv b \wedge x \,(\text{mod } C) \end{array} \quad \text{for} \quad a, b, x \in L.$$

One has the following proposition:

Let J be an ideal of a distributive lattice L.

i) The equivalence relation $a \equiv b$ (mod J) defined by: $h \in J$ exist such that $a \vee h = b \vee h$ is a congruence relation.
ii) The quotient set $M = L/(J)$ is a distributive lattice having a minimum element J.
iii) The canonical projection π of L on M is an epimorphism of kernel J.

However we have to note, at the difference of groups and algebras, that if θ is a lattice morphism, the congruence relation mod (Ker θ) does not generally determine the congruence relation associated to the morphism θ.
In the example used here, this does not happen.

10° EXAMPLE.

Let $u \in \operatorname{End} E$, finite dimensional vector space on characteristic zero field $\mathbb{K}$.
The ring epimorphism $\mathbb{K}[X] \to \mathbb{K}[u]$ defined by $\bar{a} = a(u)$ for $a(X) \in \mathbb{K}[X]$ admits a kernel which is generated by a polynomial $m(X)$ (of leading coefficient fixed to 1) called minimum polynomial of u.
Then let the lattice morphisms, defined on $\mathbb{K}[X]$

$$\Theta : a \to \operatorname{Im} \bar{a} \quad \text{in } \mathcal{S}(E); \qquad \text{one will set} \quad \mathcal{I} = \operatorname{Im} \Theta$$
$$\Xi : a \to \operatorname{Ker} \bar{a} \quad \text{in } \mathcal{S}^*(E); \qquad \text{one will set} \quad \mathcal{N} = \operatorname{Im} \Xi$$

$\mathcal{S}^*(E)$ is the lattice where $\cap$ is the *l.u.b* and + is the *g.l.b* and the order is inverted.
Then one has, for $a, b \in \mathbb{K}^0[X]$:

(1) $a \leqslant b \Rightarrow \operatorname{Im} \bar{a} \subseteq \operatorname{Im} \bar{b}$ and $\operatorname{Ker} \bar{a} \supseteq \operatorname{Ker} \bar{b}$

(2) $\operatorname{Im} \overline{a \wedge b} = \operatorname{Im} \bar{a} \cap \operatorname{Im} \bar{b}$; $\operatorname{Ker} \overline{a \vee b} = \operatorname{Ker} \bar{a} \cap \operatorname{Ker} \bar{b}$

(3) $\operatorname{Im} \overline{a \vee b} = \operatorname{Im} \bar{a} + \operatorname{Im} \bar{b}$; $\operatorname{Ker} \overline{a \wedge b} = \operatorname{Ker} \bar{a} + \operatorname{Ker} \bar{b}$
(direct sum iff: $a \wedge b \leqslant m$); (direct sum iff: $a \vee b \vee m = 1$)

(4) $\operatorname{Im} \bar{a} = \operatorname{Im} \bar{b} \Leftrightarrow a \vee m = b \vee m \Leftrightarrow \operatorname{Ker} \bar{a} = \operatorname{Ker} \bar{b}$

(5) $\operatorname{Im} \bar{a} \cap \operatorname{Ker} \bar{b} = \bar{a} \operatorname{Ker} (\bar{a}\bar{b})$

$g.l.b(\mathcal{N}) = g.l.b\mathcal{S}^*(\mathrm{E}) = \mathrm{E}$; Θ and Ξ have same kernel: the set of multiples of m, of leading coefficient 1.

It is obvious that $\dfrac{\mathbb{K}^0[\mathrm{X}]}{(\operatorname{Ker} \Theta)}$ is isomorphic to $[m, 1]$, the set of divisors of m of leading coefficient 1.

From the last proposition and from (4), the distributive lattices $\mathcal{I}$, $\mathcal{N}$ and $[m, 1]$ are isomorphic.

At last let us note that $\operatorname{Ker} \Theta = \operatorname{Ker} \Xi$ is the set of $a(\mathrm{X}) \in \mathbb{K}^0[\mathrm{X}]$ such that $a(u) = 0$; $\operatorname{Ker}^* \Theta = \operatorname{Ker}^* \Xi = \{ a;\ a \in \mathbb{K}^0[\mathrm{X}];\ a \vee m = 1 \}$ is a dual-ideal of $\mathbb{K}^0[\mathrm{X}]$. If is the set of $a(\mathrm{X})$ such that $a(u)$ is regular.

11° EXAMPLES ON sl (3, $\mathbb{C}$).

		I $\mathcal{L} = \mathrm{sl}(3, \mathbb{C})$	II $\mathcal{L}'$	III $\mathcal{L}''$	IV	V($a = 2$) $\mathcal{L}_1$
h_1	e_1	$-2e_1$	0	0	0	$-2e_1$
h_1	e_{12}	$-e_{12}$	0	0	0	$-e_{12}$
h_2	e_1	e_1	0	0	0	e_1
h_2	e_{12}	$-e_{12}$	0	0	0	$-e_{12}$
e_2	e_1	$-e_{12}$	0	0	0	0
$\bar{e}_2$	e_{12}	e_1	0	0	0	$2e_1$
e_1	$\bar{e}_1$	h_1	$2e_1 - 3e_{12} + h_1$	$e_1 - 2e_{12} + h_1$	h_1	0
e_1	$\bar{e}_{12}$	$-\bar{e}_2$	$-e_1 + 2e_{12} - \bar{e}_2$	$-e_1 + 2e_{12} - \bar{e}_2$	$-\bar{e}_2$	0
e_{12}	$\bar{e}_1$	e_2	$e_1 - 2e_{12} + e_2$	$-2_{12} + e_2$	e_2	$2e_2$
e_{12}	$\bar{e}_{12}$	$h_1 + h_2$	$e_{12} + h_1 + h_2$	$e_{12} + h_1 + h_2$	$h_1 + h_2$	0
h_1	e_2	e_2	0	0	0	e_2
h_1	$\bar{e}_2$	$-\bar{e}_2$	0	0	0	$-\bar{e}_2$
h_2	e_2	$-2e_2$	0	0	0	$-2e_2$
h_2	$\bar{e}_2$	$2\bar{e}_2$	0	0	0	$2\bar{e}_2$
e_2	$\bar{e}_2$	h_2	0	0	0	0
h_1	$\bar{e}_1$	$2\bar{e}_1$	$4h_2 + 3e_2 + 3\bar{e}_2$	$4h_2 + 3e_2 + 3\bar{e}_2$	0	$2e_1$
h_1	$\bar{e}_{12}$	$\bar{e}_{12}$	$-h_2 + 2e_2$	$-h_2 + 2e_2$	0	$\bar{e}_{12}$
h_2	$\bar{e}_1$	$-\bar{e}_1$	$-2h_2 + 3e_2 - 3\bar{e}_2$	$-2h_2 + 3e_2 - 3\bar{e}_2$	0	$-\bar{e}_1$
h_2	$\bar{e}_{12}$	$\bar{e}_{12}$	$-h_2 - 6e_2$	$-h_2 - 6e_2$	0	$\bar{e}_{12}$
e_2	$\bar{e}_1$	0	$-h_2 - 4e_2$	$-h_2 - 4e_2$	0	0
e_2	$\bar{e}_{12}$	$\bar{e}_1$	$2h_2 + 5e_2 + \bar{e}_2$	$2h_2 + 5e_2 + \bar{e}_2$	0	0
$\bar{e}_2$	$\bar{e}_1$	$-\bar{e}_{12}$	$4h_2 + 2e_2 + 4\bar{e}_2$	$4h_2 + 2e_2 + 4\bar{e}_2$	0	$-2\bar{e}_{12}$
$\bar{e}_2$	$\bar{e}_{12}$	0	$-2h_2 - 2\bar{e}_2$	$-2h_2 - 2\bar{e}_2$	0	0
$\bar{e}_1$	$\bar{e}_{12}$	0	$-2\bar{e}_1 - 2\bar{e}_{12}$	$-2\bar{e}_1 - 2\bar{e}_{12}$	0	0

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