ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 19, nº 4 (1973), p. 357-373

<http://www.numdam.org/item?id=AIHPA_1973__19_4_357_0>

Ann. Inst. Henri Poincaré,
Vol. XIX, n° 4, 1973, p. 357-373.

Section A :
Physique théorique.

A generalization of entropy using traces on von Neumann algebras (*)

by

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ABSTRACT. — We show that a normal faithful semi-finite trace on a von Neumann algebra can be used to define the entropy of a positive operator with trace one. The usual definitions of the entropy in both classical and quantum statistical mechanics can be obtained as special cases of our definition for an appropriate choice of algebra and trace. We discuss the properties of this generalized entropy. In particular, convexity and subadditivity inequalities are proved. Counterexamples to those properties which are not true in general are also given.

I. INTRODUCTION

A number of useful properties are known for the entropy of both classical and quantum systems ([*1*]-[*6*]). Thus far each case has been considered separately with different definitions, proofs, etc. We will give a more general definition of the entropy which includes all the usual statistical mechanical systems as special cases. We then consider the problem of proving various properties which depend only on the definition of the entropy and not on the dynamics of the system. In particular, we prove some

(*) Work supported in part by U. S. National Science Foundation Grants GP-31674 X, GP-36144, GP-31239 X; Air Force Office of Scientific Research Contract AF 44620-70-C 0030; and National Research Council of Canada Grant No. NRC-A 6595.

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convexity and subadditivity inequalities. However, only a few « essential » properties remain true in general and we construct counterexamples to many others.

The physical significance of many of these properties was discussed in a recent article [4], and will not be repeated here.

Let τ be a normal, faithful, semi-finite trace on a von Neumann algebra $\mathfrak{A}$. Define a density operator ρ as a positive operator satisfying $\tau(\rho) = 1$. Then the entropy associated with ρ is given by

$$\text{(1.1)} \qquad S(\rho) = -\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\|\rho\|} \lambda \log \lambda \tau(dE_\lambda)$$

where $\{ E_\lambda \}$ are the spectral projections of ρ. Formally $S(\rho) = -\tau(\rho \log \rho)$ and we often write the formal expression for simplicity.

Classical systems are described by commutative algebras [7]. Although the trace is not unique in this case, we can always choose to find a trace so that (1.1) agrees with the usual definition of the entropy ([1], [4], [7]). Quantum lattice systems are described by the algebra of bounded operators in a finite-dimensional Hilbert space, and quantum continuous systems by the algebra of bounded operators on a separable Hilbert space [7]. Again, if the trace is normalized appropriately, (1.1) agrees with the usual definition ([2]-[5]).

In order to consider the algebras and entropies associated with different regions, we introduce the concept of a partial trace [8]. Suppose $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are commuting subalgebras of $\mathfrak{A}$ and that τ_1, τ_2 and τ are traces on the respective algebras. Then the partial traces $\hat{\tau}_1$ and $\hat{\tau}_2$ are maps (not necessarily everywhere defined) from $\mathfrak{A}$ into, respectively, $\mathfrak{A}_2$ and $\mathfrak{A}_1$ such that:

(1) If $\tau(A)$ exists, $\hat{\tau}_1(A)$ and $\hat{\tau}_2(A)$ are defined.

(2)

$$\text{(1.2)} \qquad \tau_1[\hat{\tau}_2(A)] = \tau_2[\hat{\tau}_1(A)] = \tau(A) \equiv \tau_{12}(A).$$

(3) If B is in $\mathfrak{A}_2$, then

$$\text{(1.3)} \qquad \hat{\tau}_1(AB) = [\hat{\tau}_1(A)]B.$$

For simplicity, we often drop the caret and write τ_1 for $\hat{\tau}_1$. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are algebras of bounded operators on separable Hilbert spaces and

$$\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2,$$

it is easy to define such partial traces (by identifying $\mathfrak{A}_1$ and $\mathfrak{A}_1 \otimes \mathbb{1}_2$ by isomorphism) and their properties have been discussed in detail [9]. Similarly we consider partial traces on commuting subalgebras $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ of $\mathfrak{A}$.

If $\rho \geq 0$ in $\mathfrak{A}$ and partial traces into $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are defined as above, we write $\rho \equiv \rho_{12}$ and define

$$\rho_1 \equiv \tau_2(\rho_{12}), \tag{1.4}$$

etc. Then we write $S(\rho_{12}) \equiv S_{12}$ and $S(\rho_1) \equiv S_1$. Similarly, we can define ρ_{123}, ρ_{12}, S_{123}, S_{12}, etc. It follows from (1.2) that $\tau_{12}(\rho_{12}) = 1$ implies $\tau_1(\rho_1) = 1$, so that ρ_1 is a density operator if ρ_{12} is.

A summary of the relevant properties of the trace is given in Appendix A. The convexity and subadditivity inequalities are discussed in part II. Some miscellaneous properties are considered in III. Counterexamples to theorems which are true in special cases, but not in general, are given in IV. The proofs of certain technical lemmas and the theorems in II are given in Appendices B and C respectively.

II. INEQUALITIES

The proofs of the theorems in this section are similar to the proofs given for the usual trace on a Hilbert space; however, a number of technical difficulties make them rather messy. Therefore, all proofs are postponed to Appendix C. Convexity, concavity, and weak subadditivity all remain true. However, Araki-Lieb subadditivity [3] is true only in a weaker form. In part IV, we will show that this weaker form is in fact the best one can hope to do in general.

THEOREM 1 (Concavity). — *Let ρ, ρ', ρ'' be density matrices with*

$$\rho = \alpha\rho' + (1 - \alpha)\rho'', \qquad 0 \leq \alpha \leq 1 \tag{2.1}$$

and let S, S′, S″ *be the corresponding entropies. Then*

$$\alpha S' + (1 - \alpha)S''$$

$$\leq S, \tag{2.2}$$

$$\leq \alpha S' + (1 - \alpha)S'' - \alpha \log \alpha - (1 - \alpha) \log (1 - \alpha), \tag{2.3}$$

$$\leq \alpha S' + (1 - \alpha)S'' + \log 2. \tag{2.4}$$

In the next two theorems, we use the partial trace and reduced density matrix formalism introduced above.

THEOREM 2 (Weak Subadditivity):

$$S_{12} \leq S_1 + S_2. \tag{2.5}$$

THEOREM 3 (Araki-Lieb subadditivity):

$$S_{123} \leq S_{12} + S_{23} + \log \| \rho_2 \| \tag{2.6}$$

if $S_1 < \infty$ *or* $S_3 < \infty$.

Two additional theorems, which were known for commutative algebras, have recently been proven ([4], [5]) when $\mathfrak{A}$ is the algebra of bounded operators in a separable Hilbert space. Unfortunately, the proofs are indirect in infinite dimensions and these results cannot be proven with the techniques used here. However, we believe they are true and state them as conjectures. Recent results of Epstein [10], which give new proofs of the convexity theorems of Lieb [11] which were used to prove these conjectures [5], do generalize to finite traces von Neumann algebras. Therefore, one can prove these conjectures for finite traces. However, the semi-finite case is unclear.

CONJECTURE 1 (Strong Subadditivity):

$$S_{123} + S_2 \leq S_{12} + S_{23}\,. \tag{2.7}$$

CONJECTURE 2. — *The function from the set of density operators into* $\mathbb{R}$ *given by* $\rho_{12} \to (S_1 - S_{12})(\rho_{12}$ *is convex.*

The closely related Wigner-Yanase-Dyson ([4], [11]) conjecture is true even for semi-finite trace. Since the proof is identical to Lieb's [11] we will not repeat it here.

THEOREM 4. — *If* $0 \leq p,\ r \leq 1$, *and* $0 \leq p + r \leq 1$ *then the function*

$$C \to \tau(C^{r/2}A^*C^pAC^{r/2}),$$

from the set of positive operators in $\mathfrak{A}$ *into* $\mathbb{R}$ *is concave in* $C \geq 0$.

III. NORMALIZATION AND POSITIVITY

Many of the entropy inequalities proven previously depend on the fact that the norm of a density matrix is ≤ 1. Unfortunately, this is not true in general. $\tau(\rho) = 1$ and $\rho \geq 0$ do not imply $\|\rho\| \leq 1$. On the contrary, if $\mathfrak{A}$ is a factor of type II, $\{\, \|\rho\| : \rho \geq 0,\ \tau(\rho) = 1 \,\}$ is unbounded (1). Furthermore, we will provide examples to show that virtually all inequalities which use the fact that $\|\rho\| \leq 1$ are not true in general.

We begin by considering the conditions under which $\|\rho\| \leq 1$ and the conditions under which $S(\rho)$ is positive or negative. Note that although the condition (*a*) does not hold in general (in fact it implies the existence of minimal projections), it is satisfied in certain relevant cases, namely by the usual trace on a Hilbert space and by an appropriate choice of trace for the commutative algebra of diagonal operators on a Hilbert space.

(1) To prove this, note that in a factor of type II, there exist projections, E, with arbitrarily small trace, ε, and let $\rho = (1/\varepsilon)E$.

THEOREM 5:

(A)	$\tau(E) \geq 1$ *if* E *is a non-zero projection*	(*a*)
	$\Rightarrow \|\| \rho \|\| \leq 1$	(*b*)
	$\Rightarrow \rho^2 \leq \rho$	(*c*)
	$\Rightarrow \tau(\rho^2) \leq \tau(\rho)$	(*d*)
	$\Rightarrow S(\rho) \geq 0$	(*e*)
(B)	$\tau(\mathbb{1}) \leq 1$	(*f*)
	$\Rightarrow S(\rho) \leq 0$	(*g*)
	$\Rightarrow \tau(\rho) \leq \tau(\rho^2)$	(*h*)
	$\Rightarrow \|\| \rho \|\| \geq 1$	(*i*)

Proof :

(1) Note that the following proofs are trivial:

$$b \Leftrightarrow c; \qquad c \Rightarrow d; \qquad h \Rightarrow i.$$

(2) If $x \geq 0$ then

$$x - x^2 \leq x \log x \leq 1 - x.$$

It then follows from the spectral theorem that

$$\tau(\rho - \rho^2) \leq S(\rho) \leq \tau(\mathbb{1}) - 1.$$

This proves $d \Rightarrow e$, $f \Rightarrow g$, and $g \Rightarrow h$.

(3) Finally, assume that (*a*) holds. Let (α_k) be a sequence of numbers increasing to $|| \rho ||$, let $\{ E(\lambda) \}$ be the spectral projections of ρ, and

$$F_k = \mathbb{1} - E(\alpha_k).$$

Then $\alpha_k F_k \leq \rho F_k$ and $\tau(F_k) \geq 1$ implies

$$\alpha_k \leq \alpha_k \tau(F_k) \leq \tau(\rho F_k) < || F_k || \tau(\rho) = 1 \qquad \text{for all } k.$$

Thus $\lim\limits_{k \to \infty} \alpha_k = || \rho || \leq 1$.

One might wonder if there is any connection between $|| \rho_{12} ||$ and $|| \rho_1 ||$. In general there is not. If $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$, with $\mathfrak{A}_1$, $\mathfrak{A}_2$ both factors of type II_∞, then

$$\{ || \rho_1 || : || \rho_{12} || = 1, \qquad \tau(\rho_{12}) = 1, \quad \rho_1 = \tau_2(\rho_{12}) \}$$

is unbounded (2). The best one can do is the following theorem. Unfortunately, the hypotheses can only be satisfied if $\tau_1(\mathbb{1}_1) \geq 1$ and $\tau_2(\mathbb{1}_2) \leq 1$, so the result is not of much interest.

THEOREM 6. — *If* $\tau_2(\mathbb{1}_2) \leq 1$, $\tau_{12}(\rho_{12}) = 1$ *and* $|| \rho_{12} || \leq 1$, *then* $|| \rho_1 || \leq 1$.

(2) Let E_1 be a projection with $\tau(E_1)$ arbitrarily small and E_2 a projection with $\tau(E_2) = 1/\varepsilon$. Let $\rho_{12} = E_1 \otimes E_2$.

Proof. — Let (α_k) be a sequence increasing to $\|\rho_1\|$, $E(\lambda)$ be the spectral projections of ρ_1, and $F_k = \mathbb{1} - E(\alpha_k)$:

$$\begin{aligned}\alpha_k \tau_1(F_k) &\leq \tau_1(\rho_1 F_k) = \tau_{12}(\rho_{12} F_k) \\ &\leq [\tau_{12}(F_k \mathbb{1}_2)]^{1-1/2^n}[\tau_{12}(\rho_{12}^{2^n} F_k)]^{1/2^n} \leq [\tau_1(F_k)]^{1-1/2^n}.\end{aligned}$$

Therefore $\alpha_k \leq [\tau_1(F_k)]^{-1/2^n}$ for all k, n.

Since $[\tau_1(F_k)]^{-1/2^n}$ can be made arbitrarily close to 1,

$$\lim_{k\to\infty} \alpha_k = \|\rho_1\| \leq 1.$$

IV. COUNTEREXAMPLES

We have already remarked that many properties of the entropy in special cases are not true in general. We now give a list of counterexamples. Most of the inequalities we consider are drawn from [*3*] and [*4*].

It will become apparent that only those properties which are independent of the normalization of the trace will remain true in general. Thus, if there do not exist projections with arbitrarily small trace, it will be possible to renormalize the trace so that the inequalities remain true. For factors of type II, however, this is not possible and one can always find a density matrix for which the inequalities are false.

It is worth noting that the normalization of the trace affects both the definition of S and ρ. If $\tilde{\tau} = \lambda\tau$ then $\tau(\rho) = 1$ implies $\tilde{\rho} = (1/\lambda)\,\rho$ satisfies $\tau(\tilde{\rho}) = 1$. Thus $S(\rho) \to \tilde{S}(\tilde{\rho}) = \tau(\rho \log \rho) - \log \lambda$.

We now give our example. By appropriate choice of algebra one can make a, b, c anywhere in $(0, \infty)$.

EXAMPLE. — Let

$$\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \mathfrak{A}_3$$

and

$$\tau_{123}(A \otimes B \otimes C) = \tau_1(A)\tau_2(B)\tau_3(C).$$

Let E_1, E_2, E_3 be projections in, respectively, $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, such that:

$$\tau_1(E_1) = \frac{1}{a},$$

$$\tau_2(E_2) = \frac{1}{b},$$

$$\tau_3(E_3) = \frac{1}{c},$$

Let $\rho_{123} = abc(E_1 \otimes E_2 \otimes E_3)$.

Then, for example

$$\rho_{12} = abE_1 \otimes E_2,$$
$$\rho_1 = aE_1$$

and

$$\begin{aligned} \|\rho_{123}\| &= abc, \\ \|\rho_{12}\| &= ab, \\ \|\rho_1\| &= a. \end{aligned}$$

The relative entropies are, for example

$$S_{123} = -\log abc,$$
$$S_{12} = -\log ab$$

and

$$S_1 = -\log a.$$

Thus they have the following properties:

(1) $S_{123} - S_{12} - S_{23} = \log b$
< 0 if $b < 1$, and
> 0 if $b > 1$.

(2) $S_{123} - S_{12} - S_{13} - \log \|\rho_2\| = 0$.

(3) $S_2 + S_{123} = S_{12} + S_{23}$.

(4) $S_2 - S_{23} - S_{12} = \log abc$
< 0 if $abc < 1$, and
> 0 if $abc > 1$.

(5) $S_2 - S_{23} - S_{12} - \log \|\rho_{123}\| = 0$.

(6) $S_2 - S_{23} - S_{13} = \log ac^2$
< 0 if $ac^2 < 1$, and
> 0 if $ac^2 > 1$.

(7) $S_2 - S_{23} - S_{13} - \log \|\rho_1\| = 2 \log c$
< 0 if $c < 1$, and
< 0 if $c > 1$.

(8) $S_2 - S_1 - S_{12} = 2 \log a$
< 0 if $a < 1$, and
> 0 if $a > 1$.

(9) $|S_1 - S_2| = |\log (a/b)|$
$\geq S_{12} = -\log ab$ if $a > 1$ or $b > 1$, and
$\leq S_{12} = -\log ab$ if $a < 1$ and $b < 1$.

(10) Suppose $abc = 1$. Then

$$\rho_{123}^2 = \rho_{123}$$

but

$$-\log a = S_1 \neq S_{23} = -\log bc = +\log a$$

unless $a = bc = 1$.

(11) $S_1 - S_{12} = \log b$
< 0 if $b < 1$, and
> 0 if $b > 1$.

We note that equality in (3) implies that conjecture 1, strong subadditivity, is satisfied in this example for any choice of *a*, *b*, *c*. At first glance (11) might appear to contradict conjecture 2 since the logarithm is concave; however a convex combination of such density matrices does not give a density matrix of the same form. On the contrary $(S_1 - S_{12})(\rho_1 \otimes \rho_2) = -S_2$. Thus if $\rho'_{12} = \rho_1 \otimes \rho'_2$ and $\rho''_{12} = \rho_1 \otimes \rho''_2$, conjecture 2 follows from theorem 1, i. e. the concavity of S_2.

V. CONCLUSION

One of the reasons for studying entropy inequalities is that they can then be used to prove the existence of the infinite-volume limit of the entropy per unit volume ([*1*]-[*4*]). Unfortunately, we can not do this here. The proof in [*3*] depends on the inequality $S_2 \leq S_{23} + S_{12}$ which we have shown to be false in general. A proof of the existence of the infinite-volume limit would therefore seem to require a general proof of the strong subadditivity conjecture ([*1*], [*2*]).

In our counterexamples, those inequalities which fail do so because the algebra contains projections with arbitrarily small trace. This problem does not affect conjectures 1 and 2; on the contrary, we have already remarked that they are true if the trace is finite. Furthermore, they are true for traces on a separable, infinite-dimensional Hilbert space, but a « direct » proof has not been given. The proof in [*5*] uses the finite-dimensional result and a special limiting process. Thus, we are convinced that these conjectures are true for semi-finite traces, but the techniques used here are inadequate for proving them.

ACKNOWLEDGEMENTS

The author is grateful to Professor D. Ruelle for suggesting this problem and many helpful discussions. This work was begun while the author was Battelle Fellow at the Institut de Physique Theorique, University of Geneva; much of it was done during the 1971 Battelle Rencontres in Mathematics and Statistical Mechanics in Seattle, where the author profitted from discussions with many of the participants. The final version was written while the author was guest at the Institut des Hautes Etudes Scientifiques in France and at the Lyman Laboratory of Physics, Harvard University.

APPENDIX A

A trace τ on a von Neumann algebra, $\mathfrak{A}$, of operators on a Hilbert space H is a function, defined on $\mathfrak{A}^+ = \{ A : A \geq 0 \}$ and extended to the 2-sided ideal, whose positive part is $M^+ = \{ A : A \geq 0 \text{ and } \tau(A) < \infty \}$, with the following properties:

(A.1) $$\tau(A) \geq 0 \quad \text{if} \quad A \geq 0.$$

(A.2) $$\tau(A + \lambda B) = \tau(A) + \lambda\tau(B)$$

if

i) λ in $\mathbb{C}$; A, B in M

or

ii) $\lambda \geq 0$; A, B ≥ 0.

(A.3) $$\tau(A) = \tau(UAU^*) \quad \text{if} \quad A \geq 0; \ U \text{ is unitary} \in \mathfrak{A}.$$

We will be primarily interested in traces with the following additional properties.

(A.4) (normal): If $\{ A_i \}$ is a bounded increasing net of positive operators, then

$$\sup_i . \tau(A_i) = \tau(\sup_i . A_i).$$

(A.5) (semi-finite) (3): If $A \geq 0$ and $\tau(A) = \infty$, then there exists a B such that $0 < B \leq A$ and $\tau(B) < \infty$.

(A.6) (faithful): $\tau(A) = 0$ and $A \geq 0 \Rightarrow A = 0$.

One can then show that τ has the following useful well-known properties (4):

(A.7) $$\tau(AB) = \tau(BA)$$

if

i) A in M, B in $\mathfrak{A}$

or

ii) $B = A^*$ in $\mathfrak{A}$.

(A.8) $A \to (AB)$ is ultraweakly continuous for A in $\mathfrak{A}$, B in M. In particular if A_k is a bounded net converging to A strongly, then $\lim_{k \to \infty} \tau(A_k B) = \tau(AB)$ if B is in M.

(A.9) $|\tau(A^*B)|^2 \leq \tau(A^*A)\tau(B^*B)$ if A^*B is in M.

(A.10) There exists a family, (x_k), of vectors in H such that

$$\tau(A) = \sum_k \langle x_k A x_k \rangle \qquad \text{if} \qquad A \geq 0.$$

(A.11) $\| \tau(AB) | \leq \tau(| AB |) \leq \| A \| \tau(| B |) \qquad \text{if} \qquad$ A in $\mathfrak{A}$, B in M.

Recently [*13*], the following useful theorems were proved:

(A.12) (Golden-Thompson inequality) $\tau(e^{A+B}) \leq \tau(e^{A/2}e^{B}e^{A/2})$

if (*a*) A, B are self-adjoint operators, bounded above, and

(*b*) A + B is essentially self-adjoint.

Further, if $\tau(a^A) < \infty$ or $\tau(e^B) < \infty$ then $\tau(e^{A+B}) \leq \tau(e^A e^B)$.

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(3) This definition of semi-finiteness is valid only for normal traces.

(4) *See* [*12*]: Proposition 1, p. 82; Theorem 2, p. 88; Corollary, p. 85; Theorem 8, p. 106.

(A.13) (Holder Inequality): If $0 < \alpha < 1$,

$$\begin{aligned} |\tau(AB)| &\leq \tau(|AB|) \\ &\leq [\tau(|A|^{1/\alpha})]^{\alpha}[\tau(|B|^{1/1-\alpha})]^{1-\alpha}. \end{aligned}$$

(A.14) (Peierls-Bogolyubov Inequality): If $A \in \mathfrak{A}$, $\tau(e^{A}) < \infty$, and B is a self-adjoint operator, bounded above and associated with $\mathfrak{A}$, then:

$$\log\left[\frac{\tau(e^{A+B})}{\tau(e^{A})}\right] \geq \frac{\tau(e^{A}B)}{\tau(e^{A})}.$$

APPENDIX B

Some technical lemmas.

LEMMA 1. — *Let* A, B *be fixed elements of* M^+ *and* $\mathscr{D}(0, \infty)$ *the space of* C^∞ *functions on* $\mathscr{D}(0, \infty)$ *with compact support. Then there exists a unique positive measure* $\mu(da\,db)$ *such that*

$$\tau[\varphi(A)\psi(B)] = \int_0^\infty \int_0^\infty \mu(da\,db)\varphi(a)\psi(b) \tag{B.1}$$

for all φ, ψ *in* $\mathscr{D}(0, \infty)$.

Proof. — Let $\omega(\varphi, \psi) = \tau[\varphi(A)\psi(B)]$. Then

(*i*) $\omega(\varphi, \psi)$ is a bilinear functional on $\mathscr{D}(0, \infty)$;

(*ii*) $\omega(\varphi, \psi)$ is positive since if $\varphi \geq 0$;
$\omega(\varphi, \varphi) = \tau([\varphi(A)]^{1/2}\varphi(B)[\varphi(A)]^{1/2} \geq 0$;

(*iii*) $\omega(\varphi, \psi)$ is separately continuous in each variable. Since $\tau(\varphi(A)]$ and $\tau(\psi(B)] < \infty$ and convergence of $\varphi_\alpha \to \varphi$ in $\mathscr{D}(0, \infty)$ implies $\| \varphi_k(A) - \varphi(A) \| \to 0$, property (A.8) implies $\omega(\varphi_k, \psi) \to \omega(\varphi, \psi)$.

Thus, it follows from the Schwartz nuclear theorem that there exists an unique distribution T on $\mathscr{D}(0, \infty) \otimes \mathscr{D}(0, \infty)$ such that $T(\varphi \otimes \psi) = \omega(\varphi, \psi)$. Now suppose δ_ε is a net in $\mathscr{D}(-\infty, \infty)$ such that $\delta_\varepsilon \geq 0$, sup. $\delta_\varepsilon \subset [-\varepsilon, \varepsilon]$, and $\int \delta_\varepsilon = 1$. Then (*see* e. g. [*14*], p. 166). T is the limit of the regularized distributions,

$$T_\varepsilon = T^*(\delta_\varepsilon \otimes \delta_\varepsilon),$$

which, since ω is positive, are positive functions.

Thus, T is a positive distribution and [*14*] (p. 29, Theorem V) can be identified with a positive measure μ on $\mathscr{D}(0, \infty) \otimes \mathscr{D}(0, \infty)$ such that

$$\omega(\varphi, \psi) = \int_0^\infty \int_0^\infty \mu(da\,db)\varphi(a)\psi(b).$$

Lemma B.1 can be used to provide a generalization of Klein's inequality ([7], Theorem 2.5.2) as follows: If f, g are positive functions $\mathscr{D}(0, \infty)$ it then follows from Lemma 1 that

$$\begin{aligned} 0 \leq \int\int \mu(da\,db)f(a)g(b)[a \log a - a \log b - (a - b)] \\ = \tau(f(A)[A \log A - A \log B - (A - B)]g(B)). \end{aligned} \tag{B.2}$$

Now replace f in (B.2) by an increasing sequence $f^{(k)}(a)$ tending to $f_n(a)$ where

$$f_n(a) = \begin{cases} 0 & \text{if} \quad a \in \left(0, \frac{1}{n}\right] \cup \left[\, \| A \| + 1, \infty \right), \\ 1 & \text{if} \quad a \in \left(\frac{1}{n}, \| A \| + 1\right). \end{cases}$$

Then $f^k(A) \to f_n(A)$ strongly and it follows from (A.8) that

$$\begin{aligned} \lim_{k\to\infty} \tau(f^k(A)[A \log A - A \log B - (A - B)]g(B)) \\ = \tau(f_n(A)[A \log A - A \log B - (A - B)]g(B)) \geq 0. \end{aligned} \tag{B.3}$$

Similarly, replace g in (B.3) by an increasing sequence

$$g^l(b) \quad \text{tending to } g_m(b) \text{ where } g_m(b) = \begin{cases} 0 & \text{if } \ b \in \left(0, \dfrac{1}{m}\right] \cup \left[\| \mathrm{B} \| + 1, \infty \right), \\ 1 & \text{if } \ b \in \left(\dfrac{1}{m}, \| \mathrm{B} \| + 1 \right). \end{cases}$$

Then $g^l(\mathrm{B})$ and $(\log \mathrm{B})g^l(b)$ are both bounded nets converging strongly to $g_m(\mathrm{B})$ and $(\log \mathrm{B})g_m(\mathrm{B})$ respectively. Thus

$$\text{(B.4)} \quad \lim_{l \to \infty} \lim_{k \to \infty} \tau(f^k(\mathrm{A})(\mathrm{A} \log \mathrm{A} - \mathrm{A} \log \mathrm{B} - (\mathrm{A} - \mathrm{B}))g^l(\mathrm{B})] = \tau(f_n(\mathrm{A})(\mathrm{A} \log \mathrm{A} - \mathrm{A} \log \mathrm{B} - (\mathrm{A} - \mathrm{B}))g_m(\mathrm{B})] \geq 0.$$

LEMMA B.2. — *If* A, B *are bounded self-adjoint operators with* $0 \leq \mathrm{A} \leq \mathrm{B}$, *then*:

(*a*) $\mathrm{A}^{1/2} (\log \mathrm{A})\mathrm{A}^{1/2} \leq \mathrm{A}^{1/2} (\log \mathrm{B})\mathrm{A}^{1/2}$.

(*b*) $\mathrm{A}^{1/2} (\log \mathrm{B})\mathrm{A}^{1/2}$ *is a bounded self-adjoint operator.*

(*c*) $(- \log \mathrm{B} + \log \| \mathrm{B} \|)^{1/2}\mathrm{A}^{1/2}$ *is a bounded operator where* $(\log \mathrm{B})\mathrm{A}^{1/2}$ *is defined to be* 0 *on the null space of* A.

Proof. — Let $\mathscr{R}(x)$ = range of x;
$\mathscr{D}(x)$ = domain of x

(*i*) We show $\mathscr{R}(\mathrm{A}^{1/2}) \subset \mathscr{R}(\mathrm{B}^{1/2})$. If $\mathrm{C} \geq 0$ and $x \in \mathscr{R}(\mathrm{C})^\perp$, $\langle x, \mathrm{C}x \rangle = 0$. Therefore $\overline{\mathscr{R}(\mathrm{A})} \subset \overline{\mathscr{R}(\mathrm{B})}$ and we can assume without loss of generality that

$$\overline{\mathscr{R}(\mathrm{B})} = \overline{\mathscr{R}(\mathrm{B}^{1/2})} = \overline{\mathscr{D}(\mathrm{B}^{-1/2})} = \mathrm{H}.$$

Now suppose x is in H and y is in $\mathscr{D}(\mathrm{B}^{-1/2})$.
Then

$$\begin{aligned} | \langle \mathrm{A}^{1/2}x, \mathrm{B}^{-1/2}y \rangle | &= | \langle x, \mathrm{A}^{1/2}\mathrm{B}^{-1/2}y \rangle | \\ &\leq \| x \| [\langle \mathrm{A}^{1/2}\mathrm{B}^{-1/2}y, \mathrm{A}^{1/2}\mathrm{B}^{-1/2}y \rangle]^{1/2} \\ &\leq \| x \| [\langle \mathrm{B}^{-1/2}y, \mathrm{B}\,\mathrm{B}^{-1/2}y \rangle]^{1/2} \\ &= \| x \| \cdot \| y \| \ \forall\, y \text{ in } \mathscr{D}(\mathrm{B}^{-1/2}). \end{aligned}$$

Therefore $y \to \langle \mathrm{A}^{1/2}x, \mathrm{B}^{-1/2}y \rangle$ is continuous for all x in H. Therefore $(\mathrm{B}^{-1/2})^*$ is defined on $\mathrm{A}^{1/2} x$. Since $(\mathrm{B}^{-1/2})^* = \mathrm{B}^{-1/2}$.

$$\mathscr{R}(\mathrm{A}^{1/2}) \subset \mathscr{D}(\mathrm{B}^{-1/2}) = \mathscr{R}(\mathrm{B}^{1/2}).$$

(*ii*) It follows from the spectral theorem that if C is a bounded positive operator, then $\mathrm{C}^{1/2} (\log \mathrm{C})\mathrm{C}^{1/2}$ is a bounded self-adjoint operator with

$$\| \mathrm{C}^{1/2} (\log \mathrm{C})\mathrm{C}^{1/2} \| \leq \max \{ e^{-1}, \| \mathrm{C} \| \log \| \mathrm{C} \| \}.$$

Furthermore $\mathscr{R}(\mathrm{C}^{1/2}) \subset \mathscr{D}(\log \mathrm{C})$. (*See* e. g. [*15*], p. 165, problem 5.10.)

(*iii*) $\langle x, \log (\mathrm{A} + \varepsilon\mathrm{I})x \rangle \leq \langle x, \log (\mathrm{B} + \varepsilon\mathrm{I})x \rangle$, $\forall x$ in H. (*See* e. g. [7], Theorem 2.5.8).

Now suppose $x \in \mathscr{D}(\mathrm{A}^{-1/2})$. Then it follows from the spectral theorem and Jensen's inequality that:

$$\text{(B.5)} \quad \begin{aligned} &| \langle x, \log (\mathrm{A} + \varepsilon\mathrm{I})x \rangle - \langle x, \log \mathrm{A}x \rangle | \\ &= \langle x, \log (\mathrm{I} + \varepsilon\mathrm{A}^{-1})x \rangle \\ &\leq \log \langle x, (\mathrm{I} + \varepsilon\mathrm{A}^{-1})x \rangle \\ &= \log (\| x \|^2 + \varepsilon \| \mathrm{A}^{-1/2}x \|^2). \end{aligned}$$

Now for x in $\mathscr{D}(\mathrm{A}^{-1/2})$, $\| \mathrm{A}^{-1/2}x \| < \infty$ so (B.5) converges to 0 as $\varepsilon \to 0$. Now let $x = \mathrm{A}^{1/2}w$. Then $x \in \mathscr{D}(\mathrm{A}^{-1/2})$ and the above argument implies

$$\lim_{\varepsilon \to 0} \langle \mathrm{A}^{1/2}w, \log (\mathrm{A} + \varepsilon\mathrm{I})\mathrm{A}^{1/2}w \rangle = \langle w, \mathrm{A}^{1/2} (\log \mathrm{A})\mathrm{A}^{1/2}w \rangle.$$

Since $\mathscr{R}(A^{1/2}) \subset \mathscr{D}(B^{-1/2})$, $A^{1/2}w \in \mathscr{D}(B^{-1/2})$ and a similar argument gives

$$\lim_{\varepsilon \to 0} \langle A^{1/2}w, \log (B + \varepsilon I)A^{1/2}w \rangle = \langle w, A^{1/2} \log BA^{1/2}w \rangle.$$

Thus

$$\begin{aligned} &\langle w, A^{1/2} (\log A)A^{1/2}w \rangle - \langle w, A^{1/2} (\log B)A^{1/2}w \rangle \\ &\qquad = \lim_{\varepsilon \to 0} [\langle A^{1/2}w \log (A + \varepsilon I)A^{1/2}w \rangle - \langle A^{1/2}w, \log (B + \varepsilon I)A^{1/2}w \rangle] < 0, \end{aligned}$$

which proves (*a*).

(*iv*) To prove (*b*) and (*c*) note that if $\| x \| = 1$,

$$\begin{aligned} &\langle x \mid A^{1/2}(-\log B)A^{1/2} \mid x \rangle \\ &\quad \leq \langle x, A^{1/2}(-\log A)A^{1/2}x \rangle \\ &\quad \leq \| A^{1/2}(-\log A)A^{1/2} \| < \infty. \end{aligned}$$

LEMMA B.3. — *Let* $A \in H_1 \otimes H_2$, $A_1 = \tau_2(A)$, *and* $\eta(x) =$ *the null space of* x. *Then* $\eta(A_1) \subset \eta(A)$.

Proof. — Let E be the orthogonal projection on $\eta(A_1)$. Then $0 = A_1E = [\tau_2(A)]E$. Therefore

$$0 = \tau_1([\tau_2(A)]E) = \tau_{12}(AE) = \tau_{12}(EAE),$$

and $EAE = 0$ since τ_{12} is faithful.

APPENDIX C

Proof of Theorem 1. — For simplicity, define $\hat{\rho} = \rho/\|\rho\|$ and similarly for $\hat{\rho}'$, $\hat{\rho}''$.

(*a*) Let $A = \rho'$, $B = \rho$ in Klein's inequality (B.4).

Then

$$(C.1)\qquad -\tau[F_n(\rho' \log \rho')G_m] \leq -\tau[F_n(\rho' \log \rho)G_m] + \tau[F_n(\rho - \rho')G_m] \qquad \text{for all} \quad n, m.$$

Note that F_n increases ultrastrongly to F, the projection on $\overline{\mathscr{R}(\rho')}$, and G_m increases ultrastrongly to G, the projection on $\overline{\mathscr{R}(\rho)}$.

When $\alpha \neq 0$,

$$\langle x\rho x \rangle = 0 \Rightarrow \langle x\rho' x \rangle = 0.$$

Therefore $F \leq G$.

Now

$$\lim_{n\to\infty} \lim_{m\to\infty} \tau[F_n(\rho - \rho')G_m] = \tau[F(\rho - \rho')G] = \tau[F(\rho - \rho')] = \tau(F\rho) - \tau(\rho') \leq \tau(\rho) - 1 = 0.$$

and

$$\begin{aligned}(C.2)\qquad \lim_{n\to\infty} \lim_{m\to\infty} -\tau(F_n\rho' \log \rho' G_m) &= -\lim_{n\to\infty} \tau[F_n(\rho' \log \rho')G] \\ &= \lim_{n\to\infty} \tau[(-\rho' \log \hat{\rho}')^{1/2}F_n(-\rho' \log \hat{\rho}')^{1/2}] - \log \|\rho'\| \\ &= \tau(-\rho' \log \hat{\rho}') - \log \|\rho'\| = -\tau(\rho' \log \rho') = S'.\end{aligned}$$

Since $\alpha\rho' < \rho$, it follows from Lemma 2 that $(\rho')^{1/2} \log \rho(\rho')^{1/2}$ and $(-\log \hat{\rho})^{1/2}(\rho')^{1/2}$ are bounded operators. Thus

$$\begin{aligned}\lim_{n\to\infty} \lim_{m\to\infty} -[F_n\rho' \log \rho G_m] &= \lim_{n\to\infty} -\tau[F_n(\rho')^{1/2} \log \rho(\rho')^{1/2}] \\ &= \lim_{n\to\infty} \tau[(-\log \hat{\rho})^{1/2}(\rho')^{1/2}F_n(\rho')^{1/2}(-\log \hat{\rho})^{1/2}] - \log \|\rho\| \\ &= \tau[(-\log \hat{\rho})^{1/2}\rho'(-\log \hat{\rho})^{1/2}] - \log \|\rho\|.\end{aligned}$$

Therefore, it follows from (C.1) and (C.2) that

$$(C.3)\qquad S' \leq \tau[(-\log \hat{\rho})^{1/2}\rho'(-\log \hat{\rho})^{1/2}] - \log \|\rho\|.$$

Multiplying by α and combining this with the corresponding expression for ρ'' one gets:

$$\begin{aligned}(C.4)\qquad & \alpha S' + (1 - \alpha)S'' \\ & \leq \tau[(-\log \hat{\rho})^{1/2}[\alpha\rho' + (1 - \alpha)\rho''](-\log \hat{\rho})^{1/2}] - \log \|\rho\| \\ & = -\tau[\rho \log \hat{\rho}] - \log \|\rho\| = -\tau(\rho \log \rho) = S.\end{aligned}$$

(*b*) Let $A = \alpha\rho'$, $B = \rho$ in Lemma B.2,

$$\begin{aligned}(C.5)\qquad & \tau[(-\log \hat{\rho})^{1/2}(\alpha\hat{\rho}')(-\log \hat{\rho})^{1/2}] - \alpha \log \|\rho\| \\ & = \tau[\alpha\rho')^{1/2}(-\log \rho)(\alpha\rho')^{1/2}] \\ & \leq \tau[\alpha\rho')^{1/2}(-\log (\alpha\rho'))(\alpha\rho')^{1/2}] \\ & = -\alpha\tau(\rho' \log \rho') - \alpha \log \alpha.\end{aligned}$$

Combining this with the corresponding result for $(1 - \alpha)\rho''$ and

$$S = \tau[(-\log \hat{\rho})^{1/2}(\alpha\rho' + (1 - \alpha)\rho'')(-\log \hat{\rho})^{1/2}] - \log \|\rho\|$$

one gets

$$(C.6)\qquad S \leq \alpha S' + (1 - \alpha)S'' - \alpha \log \alpha - (1 - \alpha) \log (1 - \alpha).$$

(*c*) The last result follows from the fact that

$$-\alpha \log \alpha - (1 - \alpha) \log (1 - \alpha) \leq \log 2 \qquad \text{on } (0, 1).$$

Proof of Theorem 2:

(*a*) We first note that it suffices to prove the theorem under the assumption $\eta(\rho_{12}) = \{0\}$. Although this assumption can not be satisfied in general, it suffices to prove the theorem for ρ_{12} in the algebra $Q_1Q_2\mathfrak{A}Q_1Q_2$ where Q_i projects on the range of $\rho_i (i = 1, 2)$. Let

$$\rho_{12} = (1 - \lambda)\rho_{12} + \lambda\rho_1\rho_2 .$$

Then

$$\eta(\rho_{12}^\lambda) = \{0\} \qquad \text{on } Q_1Q_2H \quad \text{for} \quad 0 < \infty < 1.$$

Furthermore, Lemma B.3 implies $\eta(\rho_i^\lambda) = \{0\}$ also. Since Theorem 1 implies

$$\lim_{\lambda\to 0} \rho_{12}^\lambda = \rho_{12} \qquad \text{and} \qquad \lim_{\lambda\to 0} \rho_i^\lambda = \rho_i ,$$

it suffices to assume

$$\eta(\rho_{12}) = \eta(\rho_1) = \eta(\rho_{12}) = \{0\}.$$

(*b*) We note that inequality (2.5) is unchanged if the partial traces are renormalized

$$\text{(i. e.} \quad \hat{\tau}_i = a_i\tau_i ; \ \hat{\rho}_i = a_i^{-1}\rho_i; \qquad \hat{\tau}_{12} = a_1a_2\tau_{12}, \ \hat{S}_i = S_i - \log a_i).$$

Thus, it suffices to prove the theorem under the assumption that $\|\rho_{12}\| \leq 1$ and $\|\rho_i\| \leq 1$. Thus one can assume without loss of generality that $-\log \rho_{12}$, $-\log \rho_1$ and $-\log \rho_2$ are all non-negative.

(*c*) Use (B.1) with $A = \rho_{12}$ and $B = \rho_1\rho_2$. Since ρ_1 and ρ_2 commute and we have assumed $\eta(\rho_i) = \{0\}$, $\log B = \log \rho_1 + \log \rho_2$ is a densely defined self-adjoint operator. Thus

$$\begin{aligned} -\tau[F_l(\rho_{12} \log \rho_{12})G_k] \leq & -\tau[F_l\rho_{12} (\log \rho_1 + \log \rho_2)G_k] \\ & - \tau[F_l\rho_{12}G_k] + \tau[F_l\rho_1\rho_2G_k] \end{aligned}$$

where

$$F_l = \text{spectral projection of } \rho_{12} \text{ at } \frac{1}{l},$$

and

$$G_k = \text{spectral projection of } \rho_1\rho_2 \text{ at } \frac{1}{k}.$$

Since the inequality is true for all k, l it is true in $\lim\limits_{k\to\infty}\lim\limits_{l\to\infty}$. We consider each term separately using properties (A.4) and (A.8) repeatedly as in the proof of Theorem 1.

(*d*)
$$\begin{aligned} \lim_{k\to\infty}\lim_{l\to\infty} \quad & -\tau[F_l(\rho_{12}\log\rho_{12})G_k] \\ & = \lim_{k\to\infty} \tau[(-\rho_{12}\log\rho_{12})^{1/2}G_k(-\rho_{12}\log\rho_{12})^{1/2}] \\ & = -\tau(\rho_{12}\log\rho_{12}) = S_{12} . \end{aligned}$$

(*e*)
$$\begin{aligned} \lim_{k\to\infty}\lim_{l\to\infty} \quad & -\tau[F_l\rho_{12} (\log\rho_1 + \log\rho_2)G_k] \\ & = +\tau[\rho_{12}^{1/2}G_k(-\log\rho_1 - \log\rho_2)G_k\rho_{12}^{1/2}] \\ & = +\tau[\rho_{12}^{1/2}(-\log\rho_1 - \log\rho_2)\rho_{12}^{1/2}] = S_1 + S_2 , \end{aligned}$$

where we have used the fact that

$$[G_k , B] = 0 \Rightarrow [G_k , \log B] = 0 \Rightarrow G_k(-\log B)G_k$$

is a monotone increasing net if G_k is an increasing net of projections and $-\log B \geq 0$.

(*f*)
$$\lim_{k\to\infty}\lim_{l\to\infty} -\tau[F_l\rho_{12}G_k] = -\tau(\rho_{12}) = -1.$$

(*g*)
$$\lim_{k\to\infty}\lim_{l\to\infty} \tau[F_l\rho_1\rho_2G_k] = \tau(\rho_1\rho_2) = 1.$$

(*h*) Combining limits gives

$$S_{12} \leq S_1 + S_2 - 1 + 1 = S_1 + S_2 .$$

Proof of Theorem 3:

(*i*) Let α denote any subset of $\{1, 2, 3\}$. We can again assume without loss of generality that $\eta(\rho_\alpha) = \{ 0 \}$, $\| \rho_\alpha \| \leq 1$, and $-\log \rho_\alpha$ is non-negative. In particular, it suffices to prove

$$S_{123} \leq S_{12} + S_{23}$$

when $\| \rho_2 \| \leq 1$.

(*ii*) We again use (B.1) but with $A = \rho_{123}$ and

$$B = B_{\varepsilon\omega} = \exp [W^\varepsilon_{12} + \log \rho_{23} + X^\omega_1] \tag{C.7}$$

where

$$W^\varepsilon_{ij} = \int_\varepsilon^{\|\rho_{ij}\|} \log \lambda dE_{ij}(\lambda), \tag{C.8}$$

$$X^\omega_i = \int_0^\omega \log \lambda dE_j(\lambda) \tag{C.9}$$

and $E_\alpha(\lambda)$ are the spectral projections of ρ_α. Note that $\eta(\rho_\alpha) = \{ 0 \}$ implies that $\log \rho_{23}$ and X^ω_i are densely defined. Since $[\rho_1, \rho_{23}] = 0$, $\log \rho_{23} + X^\omega_1$ is a densely-defined self-adjoint operator. Since W^ε_{12} is bounded, $\log B_{\varepsilon\omega}$ is a densely-defined self-adjoint operator and $B_{\varepsilon\omega}$ can be extended to a bounded operator on all of $Q_1Q_2Q_3H$.

(*iii*) We proceed as in Theorem 2, taking limits as $k, l \to \infty$ and then take limits as $\varepsilon, \omega \to 0$. Only the terms involving $B_{\varepsilon,\omega}$ [i. e. parts (*e*) and (*g*)] will be different.

(*iv*) Changes in part (*e*): Define positive measures μ, μ_γ on $\mathscr{D}(0, \infty)$ such that

$$\int \mu(dx)\varphi(x) = \tau_{123}[\rho^{1/2}_{123}(-\log B)\varphi(-\log B)\rho^{1/2}_{123}] \tag{C.10}$$

and

$$\int \mu_\gamma(dx)\varphi(x) = \tau_{123}[\rho^{1/2}_{123}K^\gamma_1K^\gamma_{23}(-\log B)\varphi(-\log B)K^\gamma_1K^\gamma_{23}\rho^{1/2}_{123}] \tag{C.11}$$

where $K^\gamma_\alpha = \mathbb{1}_\alpha - E_\alpha(\gamma)$. Recall that we have assumed that $-\log B \geq 0$ and note that sup. φ compact implies that $-\log B\varphi(-\log B)$ is a bounded operator which implies

$$\int \mu(dx)\varphi(x) < \infty \qquad \text{since} \qquad \tau_{123}(\rho_{123}) = 1.$$

Now

$$\begin{aligned} \| \mu_\gamma \| &= \int \mu_\gamma(dx) \\ &= \tau_{123}[\rho_{123}K^\gamma_1K^\gamma_{23}(-\log B_{\varepsilon\omega})K^\gamma_1K^\gamma_{23}\rho^{1/2}_{123}] \\ &= \tau_{123}(\rho_{123}K^\gamma_1K^\gamma_{23}(-W^\varepsilon_{12})) \\ &\quad + \tau_{123}[\rho^{1/2}_{123}K^\gamma_1K^\gamma_{23}(-\log \rho_{23})K^\gamma_1K^\gamma_{23}\rho^{1/2}_{123}] \\ &\quad + \tau_{123}[\rho_{123}K^\gamma_1K^\gamma_{23}(-X^\omega_1)] \\ &\leq \tau_{123}[\rho_{123}K^\gamma_1K^\gamma_{23}(-W^\varepsilon_{12})] + \tau_{23}[\rho_{23}K^\gamma_{23}(-\log \rho_{23})] \\ &\quad + \tau_1[\rho_1K^\gamma_1(-X^\omega_1)] \leq \| W^\varepsilon_{12} \| + S_{23} + S_1 . \end{aligned} \tag{C.12}$$

Thus, $\{ \| \mu_\gamma \| \}$ is uniformly bounded. Furthermore

$$\lim_{k\to\infty} \tau(\rho^{1/2}_{123}G_k(-\log B_{\varepsilon\omega})G_k\rho^{1/2}_{123}) = \| \mu \|.$$

Since sup. φ is compact,

$$\lim_{\gamma\to 0} \int \mu_\gamma(dx)\varphi(x) = \int \mu(dx)\varphi(x)$$

and $\mu_\gamma \to \mu$ in the vague topology ([*16*], [*17*]). Since the vague topology is compact in the unit ball

$$\| \mu \| \leq \lim_{\gamma\to 0} \| \rho_\gamma \|, \quad \text{i. e.}$$

(C.14) $$\lim_{k\to\infty} \tau[\rho_{123}G_k(-\log B_{\varepsilon\omega})]$$
$$\leq \lim_{\gamma\to 0} [\rho_{123}^{1/2}K_1^\gamma K_{23}^\gamma(-\log B_{\varepsilon\omega})K_1^\gamma K_{23}^\gamma\rho_{123}^{1/2}],$$
(C.15) $$= -\tau_{12}(\rho_{12}W_{12}^\varepsilon) + S_{23} - \tau_1(\rho_1 X_1^\varepsilon).$$

Thus

(C.16) $$\lim_{\omega\to 0}\lim_{\varepsilon\to 0}\lim_{k\to\infty}\lim_{l\to\infty}\tau_{123}[F_l\rho_{123}(-\log B_{\varepsilon\omega})G_k]$$
$$\leq \lim_{\omega\to 0}\lim_{\varepsilon\to 0}[-\tau_{12}(\rho_{12}W_{12}^\varepsilon) + S_{23} - \tau_1(\rho_1 X_1^\omega) = S_{12} + S_{23},$$

since
$$\lim_{\omega\to 0} -\tau_1(\rho_1 X_1^\omega) = \lim_{\omega\to 0} -\int_0^\omega \lambda \log \lambda dE_1(\lambda) = 0 \quad \text{if} \quad S_1 < \infty.$$

(v) [Changes in part (g)]: First note that it follows from the « Golden inequality » (A.12), that

(C.17) $$\tau(B_{\varepsilon\omega}) \leq \tau[(\exp W_{12}^\varepsilon)\exp(\log\rho_{23} + X_1^\omega)]$$
$$= \tau[(\exp W_{12}^\varepsilon)\rho_{23}(\exp X_1^\omega)] < \infty,$$

since $\rho_{23}\exp X_1^\omega$ is in M^+. Thus it follows from (A.8) that
$$\lim_{k\to\infty}\lim_{l\to\infty}\tau[F_l B_{\varepsilon\omega}G_k] = \tau(B_{\varepsilon\omega}).$$

From (C.17) above we then find
$$\lim_{\omega\to 0}\lim_{\varepsilon\to 0}\tau(B_{\varepsilon\omega}) \leq \lim_{\omega\to 0}\lim_{\varepsilon\to 0}\tau_{12}[(\exp W_{12}^\varepsilon)\rho_2 \exp X_1^\omega]$$
$$= \lim_{\omega\to 0}\tau_{12}(\rho_{12}\rho_2 \exp X_1^\omega)$$
$$= \tau_{12}(\rho_{12}\rho_2) = \tau_2(\rho_2^2) \leq \|\rho_2\| \leq 1.$$

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(Manuscrit reçu le 4 juin 1973).