

ANNALES DE L'I. H. P., SECTION A

FRIEDRICH W. HEHL

PAUL VON DER HEYDE

Spin and the structure of space-time

Annales de l'I. H. P., section A, tome 19, n° 2 (1973), p. 179-196

http://www.numdam.org/item?id=AIHPA_1973__19_2_179_0

© Gauthier-Villars, 1973, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section A » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

Spin and the Structure of Space-Time (*)

by

Friedrich W. HEHL
and **Paul von der HEYDE**

Institute for Theoretical Physics
of the Technische Universität Clausthal,
D-3392 Clausthal-Zellerfeld, West Germany

“ Newton successfully wrote apple=moon, but
you cannot write apple=neutron. ”

J. L. SYNGE

“ ... la torsion de l'Univers continue à être nulle
dans le vide. ”

E. CARTAN

ABSTRACT. — The gravitational field in General Relativity (GR) is coupled to the energy-momentum tensor of matter, i. e. it is the dynamical manifestation of energy-momentum. In Section 1 we try to collect all arguments which show that it is very plausible to look also for a dynamical manifestation of spin-angular momentum of matter.

In Section 2 it turns out that this program can be fulfilled by generalizing Riemannian geometry of space-time to a Riemann-Cartan geometry. The affine connection is now asymmetric and chosen in such a way that the covariant derivative of the metric still vanishes. The newly introduced contortion tensor or, equivalently, Cartan's torsion tensor describe independent rotational degrees of freedom of the space-time continuum. Hence we couple contortion to spin-angular momentum in a similar way as metric to energy-momentum. In Section 3 we discuss certain situations in 3-dimensional continuum mechanics where the Riemann-Cartan geometry has already been used. We especially get a clear idea of the physical interpretation of torsion.

(*) This article is based on a Seminar given by the first named author (F.W.H.) at the Institut Henri Poincaré, Paris on November 28, 1972. This author would like to thank for the invitation.

In Section 4, with the help of an action principle, GR is appropriately generalized. The new field equation (4.9) generalizing the Einstein equation is derived. In Section 5 we compare the emerging U_4 -theory with GR. The U_4 -theory describes in a unified way usual gravitational interaction and a very weak universal spin-spin contact interaction. For very high matter densities, spin becomes the dominant source of the gravitational field. Hence in the neighborhood of singularities, the metric of space-time is expected to be determined to a large extent by the spin distribution of matter.

1. A DYNAMICAL THEORY OF SPIN ?

The only property of matter which enters General Relativity (GR) is its energy-momentum distribution. This is sufficient and works quite well in macroscopic physics. If one wants to penetrate into more microscopic domains, however, not much is experimentally known and one has to extrapolate in one or another direction.

The theory we shall speak of is an attempt to find a gravitational field theory for microphysics. Nevertheless, we will still work in the framework of classical field theory with a matter field $\psi(x^k)$ ($k = 0, 1, 2, 3$). The process of field quantization, if it is necessary at all in the context of GR, is not yet applied to this theory.

In macrophysics we can describe matter by quantities such as matter density, velocity, pressure, etc. In microphysics, in the framework of classical field theory, we start from Special Relativity (SR). Using the representations of the Poincaré group it turns out that fields, or let us say elementary particles, are labelled by m and s , m being the mass, connected with the translational part, s being the spin, connected with the rotational part of the Poincaré group. Hence m as well as s are *kinematically* related to the Minkowski space-time continuum of SR.

In the context of a field theoretical formalism m corresponds to the (canonical) energy-momentum tensor Σ_i^j and s to the (canonical) spin-angular momentum tensor τ_{ij}^k ($= -\tau_{ji}^k$). This can be recognized, as is well known, in a field theoretical Lagrangian formalism with the help of Noether's theorem :

(1.1) mass m \leftarrow translation \rightarrow (canonical) energy-momentum Σ_i^j ,

(1.2) spin s \leftarrow rotation \rightarrow (canonical) spin-angular momentum τ_{ij}^k .

Macroscopically we live in an unpolarized world, if we disregard such fairly untypical things as ferromagnets. In macrophysics spin usually averages out and the dynamical properties of matter are correctly described by energy-momentum alone. It is due to this fact, as it

fact, as it appears to us, that the description of space-time by the Riemannian geometry of GR remains valid.

In conventional GR the gravitational field is the *dynamical* manifestation of energy-momentum [see Sakurai (1960), e. g.]. Consequently the gravitational field is coupled to the energy-momentum tensor of matter. Let be given the potential of the gravitational field, i. e. the metric $g_{ij}(x^k)$ of the Riemannian space V_4 of GR. If $\mathcal{L}(\psi(x^k), \overset{\{ \}}{\nabla} \psi(x^k))$ is the Lagrangian density of matter, then the (metric) energy-momentum tensor σ^{ij} is defined according to Hilbert (1915) by

$$(1.3) \quad e \sigma^{ij} \stackrel{\text{def}}{=} \frac{2 \delta \mathcal{L}}{\delta g_{ij}}.$$

Here $e \stackrel{\text{def}}{=} \sqrt{-\det g_{ij}}$ and $\overset{\{ \}}{\nabla}$ means the covariant derivative with respect to the V_4 . The kinematical definition of energy-momentum (1.1) is now superseded by the dynamical definition (1.3). Weyl (1961) has put this in the following sentence: "The general theory of relativity alone, which allows the process of variation to be applied to the metrical structure of the world, leads to a true definition of energy". A variation of the metric leads to a variation of the mutual distances of the events of space-time. Hence such a "deformation" is of a translational type.

But if spin does not average out, i. e. if we look for the gravitational properties is microphysics? Would it not be tempting to consider the motions "translation" and "rotation" on an equal footing? Why should translations and energy-momentum be of more fundamental importance than rotations and spin-angular momentum? "There does not exist any known property of particles showing that spin is less important than mass", as Lurçat (1964) has spelled it out. If one studies the notion of Regge trajectories, e. g., one recognizes that spin does not seem to have a lower position than mass.

Accordingly let us look for a dynamical manifestation of spin in an analogous manner, as it was done for energy-momentum in Einstein's GR. [Technically speaking we are looking for a local gauge theory (Yang-Mills theory) for the Poincaré group. See Utiyama (1956), Sciama (1962, 1964) and Kibble (1961)]. This dynamical theory of spin should be a general relativistic field theory, which in the limit of macrophysics (and normal matter densities) goes over into GR, i. e. GR should be valid in the macrophysical region, whereas in microphysics a more detailed theory should be appropriate.

2. THE AFFINE CONNECTION $\{ \overset{k}{i} \}$ — K_{ij}^k

Experience seems to teach us that the length of measuring rods and the angle between two of them do not change under parallel transfer.

In an affine and metric space(-time) this is fulfilled, as soon as the metric g_{ij} is covariantly constant with respect to the affine connection Γ_{ij}^k . Hence we postulate

$$(2.1) \quad \overset{\Gamma}{\nabla}_k g_{ij} = 0.$$

If we furthermore assume a symmetric connection,

$$\text{i. e. } \frac{1}{2} (\Gamma_{ij}^k - \Gamma_{ji}^k) \equiv \Gamma_{[ij]}^k = 0,$$

(2.1) leads to the connection of a Riemannian space V_4

$$(2.2) \quad \{ \overset{k}{ij} \} \stackrel{\text{def}}{=} \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),$$

i. e. to the Christoffel symbol of the 2nd kind, which represents the gravitational field strength in GR.

If we drop the assumption $\Gamma_{[ij]}^k = 0$, (2.1) yields the connection of a Riemann-Cartan space U_4 ⁽¹⁾:

$$(2.3) \quad \Gamma_{ij}^k = \{ \overset{k}{ij} \} - K_{ij}^{\bullet\bullet k}.$$

The *contortion* tensor $K_{ij}^{\bullet\bullet k}$ can be expressed through Cartan's *torsion* tensor

$$(2.4) \quad S_{ij}^{\bullet\bullet k} \stackrel{\text{def}}{=} \Gamma_{[ij]}^k$$

according to

$$(2.5) \quad K_{ij}^{\bullet\bullet k} \stackrel{\text{def}}{=} -S_{ij}^{\bullet\bullet k} + g^{kl} g_{im} S_{jl}^{\bullet\bullet m} - g^{kl} g_{jm} S_{il}^{\bullet\bullet m}.$$

The Riemann-Christoffel curvature tensor of a U_4 is defined in the usual way as "vector vortex". We finally get for it [see Schouten (1954), e. g.] :

$$(2.6) \quad R_{ijk}^{\bullet\bullet l} = 2 \partial_{[i} \Gamma_{j]k}^l + 2 \Gamma_{[i|m}^l \Gamma_{j]k}^m.$$

There exist the following arguments (not all of them being independent of each other, however) to believe that (2.3) is the correct space-time connection :

a. If we look for a dynamical manifestation of spin, we expect the spin $\tau_{ij}^{\bullet\bullet k}$ to correspond to a geometrical quantity of the same rank, i. e. a 3rd

⁽¹⁾ A connection of this type has already been used by Infeld in an attempt to formulate a unified field theory, cf. Tonnelat (1965), e. g. Nevertheless, the formalism and the interpretation of Infeld's theory and the U_4 -theory presented here are completely different.

rank tensor antisymmetric in two indices (24 independent components), as is the case with the contortion $K_{ij}^{\cdot\cdot k} = -K_{i\cdot}^{\cdot k}$ (or the torsion $S_{ij}^{\cdot k} = -S_{i\cdot}^{\cdot k}$).

- b. The contortion tensor $K_{ij}^{\cdot\cdot k}$ describes *rotational* degrees of freedom of space-time, as will be recognized if one parallelly transfers tetrads from one point to another. This fact will also be illustrated with the help of a model in Section 3 [compare (3.9)].
- c. Thus in analogy with (1.3) we postulate the dynamical definition of spin-angular momentum according to

$$(2.7) \quad e \tau_k^{\cdot ij} = \frac{\delta \mathcal{L}}{\delta K_{ij}^{\cdot\cdot k}},$$

where $\mathcal{L} = \mathcal{L}(\psi, \nabla\psi)$ [compare (3.12)]. This definition which couples contortion to spin, works in the sense that the quantity $\tau_k^{\cdot ij}$ defined by (2.7) later on turns out to be identical with the canonical spin tensor mentioned already in (1.2).

- d. The kinematical relation between spin-angular momentum and rotations mentioned in Section 1 allows for an interpretation of spin as a real internal rotation, as used, for instance, by de Broglie and his coworkers in the rotating liquid droplet model of elementary particles [see de Broglie (1963) and Halbwachs (1960), e. g.]. This semiclassical understanding of spin is also inherent in our considerations.

According to an axiom due to Minkowski (1958), it should always be possible to transform substance, i. e. massive particles to rest. If we interpret spin as an internal motion, this is no longer possible for spinning massive particles, and the axiom breaks down. Then the proportionality of 4-momentum and 4-velocity is no longer valid, a fact which immediately leads to an asymmetric energy-momentum tensor as was repeatedly stressed by de Beauregard (1942, 1943, 1959, 1963) and others. Hence one should expect for spinning massive particles a slight modification of conventional SR. Of course this automatically would lead to a corresponding modification of GR, too.

Let us consider as an example a ferromagnet in a freely falling and non-rotating laboratory. Across the ferromagnet there is a non-vanishing macroscopic spin density.

If we take the affine connection of space-time (2.3), in the case of vanishing contortion $K_{ij}^{\cdot\cdot k}$ we are able to choose geodesic coordinates for the laboratory and thereby to transform away Christoffel's symbol $\{\cdot\}_{ij}^k$. Then SR should be valid in accordance with the equivalence principle. But for a non-vanishing contortion tensor, Γ_{ij}^k cannot be transformed to zero any longer, and for that reason SR is modified.

This is the case of the ferromagnet with its intrinsic spin motion, because here spin and therefore, according to (2.7), also contortion cannot vanish. Then a slight violation of the equivalence principle is to be expected. Morgan and Peres (1962) have shown, inter alia, that such a possibility is not ruled out as yet experimentally for test bodies with aligned spins. Since for vanishing spin conventional GR should result, spin should depend algebraically (and not via a differential equation) on the contortion of space-time.

- e. By the theorems on holonomy [cf. Lichnerowicz (1955), e. g.] curvature and torsion of a U_4 are related, respectively, to the groups of rotations and translations in the tangent spaces of the U_4 , as was nicely worked out by Trautman (1972 *a, b*) :

$$(2.8) \quad \text{rotation} \rightarrow \text{curvature} \rightarrow \text{energy-momentum},$$

$$(2.9) \quad \text{translation} \rightarrow \text{torsion} \rightarrow \text{spin-angular momentum}.$$

Curvature according to GR corresponds to energy-momentum, and comparing (2.8) and (2.9) with (1.1) and (1.2), respectively, we are led to the correspondence torsion \sim spin-angular momentum, as was already to be expected according to (d).

Such being assured that there is good evidence for the affine connection (2.3), let us refer to some of the former work on U_4 -theory.

The differential geometric notion of torsion was introduced by E. Cartan (1922, 1923, 1924, 1925). He was aware of the fact that torsion should have something to do with what we today call spin. But he did not present a genuine theory. After certain investigations mainly by de Beauregard (1942, 1943, 1959), Papapetrou (1949), and Weyl (1950), Sciama (1962, 1964) and Kibble (1961) gave a dynamical theory of spin. Kröner and one of the authors [Hehl and Kröner (1965), Hehl (1966, 1970)] gave the explicit form of the affine connection and rederived the theory from a different starting point using another formalism. For more recent work see for instance Clerc (1971, 1972 *a, b*), Datta (1971), Hayashi and Bregman (1973), Hehl and Datta (1971), Lenoir (1971), Kopczyński (1972), and the highly interesting investigations of Trautman (1972 *a, b, c*). For more detailed references see Hehl (1973).

3. DISLOCATIONS : A MODEL OF TORSION

Strain and Force Stress

Dislocations are certain defects in crystals. According to Kondo (1952) and Bilby, Bullough, and Smith (1955) the continuum theory of dislocations can be formulated in a 3-dimensional space with torsion. In

this context a physical interpretation of torsion was discovered for the first time. Compare the review article of Kröner (1964), e. g..

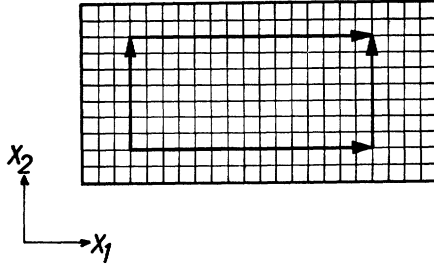


Fig. 1. — Model of an elastic body : An ideal cubic crystal in an undeformed state. The part shown is supposed to be a volume element dV in the continuum limit.

Let us first consider classical elasticity theory. In order to describe the deformation of the crystal in figure 1, we use a cartesian coordinate system x_α fixed in space. The lattice constant of the crystal is assumed to be small in comparison with dx_α . All deformations are performed isothermally.

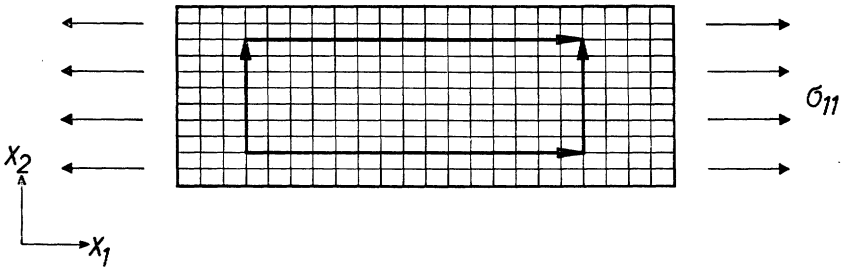


Fig. 2. — The homogeneous dilatation of the crystal in x_1 -direction is maintained by the force stress σ_{11} . The mean distances of the lattice points have changed; strain and force stress occur macroscopically.

If we displace the arbitrary material point with the coordinates x_α to $x'_\alpha = x_\alpha + s_\alpha(x_\beta)$, the crystal in general will be deformed (fig. 2). The mutual distance ds of the points x_α and $x_\alpha + dx_\alpha$ after the deformation process turns out to be

$$(3.1) \quad ds^2 = (\delta_{\alpha\beta} + 2\varepsilon_{\alpha\beta}) dx^\alpha dx^\beta \quad (\alpha, \beta, \dots = 1, 2, 3).$$

In classical elasticity, by (3.1), the strain tensor $\varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha}$ can be expressed for small relative displacements as

$$(3.2) \quad \varepsilon_{\alpha\beta} = \partial_{(\alpha} s_{\beta)}.$$

In order to preserve the deformation, there acts a force dF^α on each arbitrarily oriented area element in the crystal df_β according to

$$(3.3) \quad dF^\alpha = \sigma^{\alpha\beta} df_\beta.$$

The force stress $\sigma^{\alpha\beta}$ is the static response of the crystal to the strain $\varepsilon^{\alpha\beta}$:

$$(3.4) \quad \sigma^{\alpha\beta} = \frac{\delta L}{\delta \varepsilon_{\alpha\beta}}.$$

Here L is the free energy of the crystal. Shortly : Strain produces force stress and *vice versa*.

Now we make the following interesting observation : (3.1) defines the metric of a 3-dimensional Riemannian space V_3 . This space can be constructed according to the following prescription :

- a. Each material point of the deformed continuum is characterized by the cartesian coordinate x_α which it has before deformation.
- b. Lengths and angles are measured by comparison with the undeformed continuum.

Thereby each deformation is mapped into a certain metric space. Two parallel (lattice) vectors C_β of equal length, attached in the undeformed state to the points x_α and $x_\alpha + dx_\alpha$, after deformation differ by

$$(3.5) \quad dC_\beta = \Gamma_{\alpha\beta\gamma} C^\gamma dx^\alpha.$$

The affine connection $\Gamma_{\alpha\beta\gamma}$ can be expressed in $\varepsilon_{\alpha\beta}$, i. e. $\Gamma_{\alpha\beta\gamma}$ reduces in this case to the connection of a V_3 :

$$(3.6) \quad \{ \alpha\beta, \gamma \} \stackrel{\text{def}}{=} \partial_\alpha \varepsilon_{\beta\gamma} + \partial_\beta \varepsilon_{\alpha\gamma} - \partial_\gamma \varepsilon_{\alpha\beta}.$$

The strain tensor (3.2) is of a very special kind, and the correlated space is correspondingly simple. It is a Euclidean space, since de Saint-Venant's compatibility relations are fulfilled :

$$(3.7) \quad \varepsilon_{\alpha\beta} = \partial_{(\alpha} s_{\beta)} \Leftrightarrow R_{\alpha\beta\gamma\delta} (\{ \}) = 0$$

($R_{\alpha\beta\gamma\delta}$ = curvature tensor).

But of which kind is the strain belonging to a non-Euclidean space V_3 ? Imagine the crystal is "blown up" irregularly by a macroscopic distribution of interstitials. This causes a macroscopic strain $\varepsilon_{\alpha\beta}$ and a corresponding self stress $\sigma^{\alpha\beta}$. In general, the strain produced by these interstitials cannot be derived from a displacement field s_α and thus $R_{\alpha\beta\gamma\delta} \neq 0$, i. e. $\varepsilon_{\alpha\beta}$ now has six independent translational functional degrees of freedom. If we cut the continuum into small pieces dV , it will relax and the self stress will vanish, but the elements dV do not fit together any longer. Hence we conclude : self stress is caused by the fact that the non-Euclidean continuum is forced into a Euclidean space.

Contortion and Moment Stress

Let us look at figure 3 in order to understand that a Riemannian space V_3 is too special to describe all types of deformations occurring

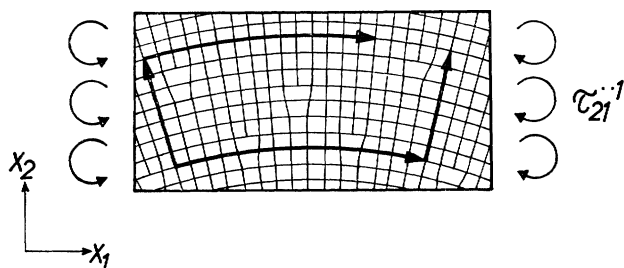


Fig. 3. — Deformation of the crystal by edge dislocations of the α_{21}^{-1} type alters the relative orientation of the crystal structure. Thereby the vector in x_2 -direction, parallelly displaced along the x_1 -axis, will rotate : there occurs a closure failure of the infinitesimal parallelogram. The crystal's deformation will be maintained by the moment stress τ_{21}^{-1} . The mean distances of the lattice points have *not* changed, hence no macroscopic strain and stress are produced.

in a crystal. With respect to the edge dislocations in figure 3 we observe the following :

- a. The infinitesimal parallelogram shown was forced open by the dislocations, which we can imagine to have immigrated from the outside of the volume element. A closure failure db_γ was produced, by which we are able to define the dislocation density $\alpha_{\alpha\beta\gamma}$ according to

$$(3.8) \quad db_\gamma = \alpha_{\alpha\beta\gamma} df^{\alpha\beta}$$

($df^{\alpha\beta} = -df^{\beta\alpha} = \text{area element}$).

- b. The mean distances of the lattice points have *not* changed. Hence there occurs no macroscopic stress and strain : $\{\alpha\beta, \gamma\} = 0$.
- c. The orientation of the lattice structure has changed, however. Consequently the parallel displacement is related to a rotation

$$(3.9) \quad dC_\beta = -K_{\alpha\beta\gamma} C^\gamma dx^\alpha.$$

The deformation measure $K_{\alpha\beta\gamma} = -K_{\alpha\gamma\beta}$ is called contortion and contains nine independent rotational functional degrees of freedom. According to Nye (1953), it can be expressed in terms of the dislocation density

$$(3.10) \quad K_{\alpha\beta\gamma} = -\alpha_{\alpha\beta\gamma} + \alpha_{\beta\gamma\alpha} - \alpha_{\gamma\alpha\beta}.$$

- d. In order to preserve the deformation, there has to act a moment $dM^{\alpha\beta}$

on the area elements :

$$(3.11) \quad dM^{\alpha\beta} = \tau^{\alpha\beta\gamma} df_{\gamma}.$$

The moment stress $\tau^{\alpha\beta\gamma}$ is the static response of the crystal to the contortion $K_{\gamma\beta\alpha}$:

$$(3.12) \quad \tau^{\alpha\beta\gamma} = \frac{\partial L}{\partial K_{\gamma\beta\alpha}}.$$

In short : Contortion produces moment stress and *vice versa* ⁽²⁾.

On the one hand infinitesimal parallelograms are broken up by immigrating dislocations, on the other hand in a space with non-vanishing torsion closed infinitesimal parallelograms are impossible in general. By this and (3.9) it is evident that macroscopically the deformation of a crystal containing dislocations should be mapped into a Riemann-Cartan space U_4 . Its connection reads

$$(3.13) \quad \Gamma_{\alpha\beta\gamma} = \{ \alpha\beta, \gamma \} - K_{\alpha\beta\gamma}.$$

Using (3.10) we have

$$(3.14) \quad \Gamma_{[\alpha\beta]\gamma} = \alpha_{\alpha\beta\gamma},$$

hence Cartan's torsion and dislocation density are identical notions. Accordingly torsion is directly measurable by the closure failure (3.8).

An Elastic Space-Time Continuum Containing Dislocations

We will work out now the close relation between the 3-dimensional statics of a dislocated crystal with point defects and the 4-dimensional "statics" of the space-time continuum. The following facts are well-known :

- (m_1) The 4-dimensional generalization of the force stress tensor $\sigma^{\alpha\beta}$ is the (metric) energy-momentum tensor σ^{ij} .
- (m_2) The generalization of the definition (3.4) of the force stress tensor immediately leads to the dynamical definition (1.3) of energy-momentum. The connection seems clear : variation of the metric g_{ij} means a variation of the distances of the space-time continuum. As response to this deformation, through each 3-dimensional hypersurface element, there acts an infinitesimal 4-momentum, identifying space-time as a sort of an elastic continuum.

⁽²⁾ A theory of a continuum with independent rotational degrees of freedom and moment stress was developed already at the beginning of our century by the Cosserats (1909). Cartan was influenced by their work in introducing torsion in GR.

- (m_3) The affine connection (3.6) corresponds to the Christoffel symbol of GR (2.2).
- (s_1) The 4-dimensional generalization of the moment stress tensor $\tau^{\alpha\beta\gamma}$ is the spin-angular momentum tensor τ^{ijk} . At regions where spin-angular momentum is present, there acts an infinitesimal spin-angular momentum through an arbitrary oriented hypersurface element.

An analogous continuation seems now to suggest itself :

- (s_2) The generalization of the definition (3.12) of the moment stress tensor is the dynamical definition (2.7) of spin-angular momentum. Thus spin is the response of space-time to a variation of the contortion K_j^k , identifying space-time as a sort of a dislocated continuum.
- (s_3) Consistently (3.13) leads to the asymmetric connection (2.3). This is an additional argument for the correctness of the affine connection (2.3).

4. U_4 -THEORY

Let us start with the special relativistic material action function in cartesian coordinates ($c =$ velocity of light, $d\Omega =$ 4-volume element) :

$$(4.1) \quad \frac{1}{c} \int d\Omega \mathcal{L}(\psi, \partial\psi).$$

According to our hypothesis all events should take place in a U_4 . We go over to curvilinear coordinates and apply the minimal substitution

$$(4.2) \quad \partial \rightarrow \nabla.$$

We end up with the material action function

$$(4.3) \quad W_m = \frac{1}{c} \int d\Omega \mathcal{L}(\psi, \nabla\psi, g) = \frac{1}{c} \int d\Omega \mathcal{L}(\psi, \partial\psi, g, \partial g, S).$$

Let be $k \approx 2 \times 10^{-48}$ dyn⁻¹ the relativistic gravitational constant and \mathcal{R} the density of the curvature scalar of the U_4 . Then we get for the total action function, using the conventional simplicity arguments, in analogy with GR :

$$(4.4) \quad W = \frac{1}{c} \int d\Omega \left[\mathcal{L}(\psi, \partial\psi, g, \partial g, S) + \frac{1}{2k} \mathcal{R}(g, \partial g, \partial\partial g, S, \partial S) \right].$$

The independent variables are ψ , g , and S .

Hamilton's principle

$$(4.5) \quad \delta W = 0$$

leads finally to

$$(4.6) \quad \frac{\delta \mathcal{L}}{\delta \psi} = 0 \quad (\text{matter equation})$$

and, after some computation ⁽³⁾, to

$$(4.7) \quad R_{kij}^{\dots k} - \frac{1}{2} g_{ij} R_{kl}^{\dots l k \text{ def}} \equiv \boxed{G_{ij} = k \Sigma_{ij}}$$

(Einstein tensor = $k \times$ energy-momentum),

$$(4.8) \quad S_{ij}^{\dots k} + \partial_i^k S_{j\cdot}^{\dots l} - \partial_j^k S_{i\cdot}^{\dots l \text{ def}} \equiv \boxed{T_{ij}^{\dots k} = k \tau_{ij}^{\dots k}}$$

(modified torsion tensor = $k \times$ spin).

(4.7) is of the general form of Einstein's field equation, but it has additionally an antisymmetric part. (4.8) states that *space-time possesses torsion at those points where spin is present*. As expected, we get an algebraic relation between torsion and spin. Apart from a constant, modified torsion and spin are synonyms. Substitution of (4.8) into (4.7) leads to one single field equation [Hehl (1970)]:

$$(4.9) \quad G^{ij} (\{ \}) = k \sigma^{ij} + k^2 \left[-4 \tau_{\dots l}^{ik} \tau_{\dots k l}^{j\cdot} - 2 \tau^{ikl} \tau_{\cdot kl}^j + \tau^{kli} \tau_{kl}^{\dots j} + \frac{1}{2} g^{ij} (4 \tau_{m\cdot l}^{\cdot k} \tau_{\cdot k l}^{m\cdot} + \tau^{mkl} \tau_{mkl}) \right].$$

Here $G^{ij} (\{ \})$ is the conventional Einstein tensor of the V_k and σ^{ij} the metric (and symmetric) energy-momentum tensor (1.3) taken with respect to the Lagrangian density entering the action function (4.3). (4.9) is suggestive in the sense that for vanishing spin one immediately recognizes that (4.9) goes over to Einstein's field equation of GR. The explicit U_k terms in the bracket are clearly exhibited as corrections to the original V_k -theory.

Using Noether's theorem one is able to derive identities for the Lagrangian density in the usual manner. This leads to the observation

⁽³⁾ In deriving (4.7) Σ_{ij} is defined according to $\Sigma_{ij} \stackrel{\text{def}}{=} \sigma_{ij} + \overset{*}{\nabla}_k (\tau_{ij}^{\dots k} - \tau_{j\cdot i}^k + \tau_{\cdot ij}^k)$; here $\overset{*}{\nabla}_k \stackrel{\text{def}}{=} \nabla_k + 2 S_{kl}^{\dots l}$. Later on Σ_i turns out to be identical with the canonical energy-momentum tensor. The antisymmetric part of (4.7), expressing angular momentum conservation, supplies no independent components to the field equation. This is clear from the derivation, since we varied with respect to the symmetric field g_{ij} . Sciama (1962) and Kibble (1961), combining the tetrad formalism with a Palatini technique, were the first who derived the field equations (4.7) and (4.8).

that Σ_{ij} and τ_{ij}^{*k} are identical with the canonical energy-momentum and spin-angular momentum tensors, respectively. Furthermore one gets the conservation theorems in the U_4 as follows ⁽⁴⁾ :

$$(4.10) \quad \overset{+}{\nabla}_j \Sigma_i^{*j} \equiv \tau_{jk}^{*l} R_{il}^{*jk} \quad (\text{energy-momentum conservation});$$

$$(4.11) \quad \overset{*}{\nabla}_k \tau_{ij}^{*k} - \Sigma_{[ij]} \equiv 0 \quad (\text{angular momentum conservation}).$$

Incidentally (4.10) can be put in another form. If we denote the right-hand side of (4.9) by $k \bar{\sigma}^{ij}$, we have $\overset{\{ \}}{\nabla}_j \bar{\sigma}^{ij} \equiv 0$.

Integrating (4.10) for spinning dust on the background of external curvature and contortion fields, one is led to the equation of motion [Hehl (1970, 1971), see also Trautman (1972 c)]. Since therein contortion is coupled to the antisymmetric part of the energy-momentum distribution, basically according to $K_{jk}^l \Sigma^{[kj]}$, it is possible in principle to measure contortion and torsion of space-time, respectively.

Two remarks should be added :

- a. For a photon the spin is a quantity not gauge invariant. A closer look on the field equations shows that for the photon the substitution (4.2) is not valid, hence photons do not produce torsion ⁽⁵⁾.
- b. The field equations (4.7) and (4.8) are also valid for matter described by spinor fields. One only has to refer the spinors to tetrads and to modify the dynamical definition of the energy-momentum tensor. No new features do appear.

5. UNIVERSAL SPIN-SPIN CONTACT INTERACTION

Let us compare our results with conventional GR. Already in (4.9) we have noticed that U_4 -theory, among other things, supplies correction

⁽⁴⁾ $\overset{+}{\nabla} \Sigma_i^{*j} \stackrel{\text{def}}{=} \overset{*}{\nabla}_j \Sigma_i^{*j} + 2 S_j^i \Sigma_k^{*j}$. We get for

$$g^{il} \overset{+}{\nabla}_j \Sigma_i^{*j} = \frac{1}{e} \partial_j (e \Sigma^{ij}) + \{ \overset{i}{j} \overset{k}{k} \} \Sigma^{(kj)} + K_{jk}^l \Sigma^{[kj]},$$

what is important for deriving the equation of motion.

⁽⁵⁾ 1969 Imbert discovered a transvers shift in the total reflection of a circularly polarized light beam as proposed by de Beauregard [see Imbert (1972) and references given therein]. In the interpretation of this Imbert-shift due to de Beauregard, Imbert, and Ricard (1971) it is assumed, that the energy-momentum distribution of the photons within Fresnel's evanescent wave, in spite of being in the vacuum, has to be described by the *asymmetric* energy-momentum tensor of de Broglie. If this interpretation is correct, photons in the vacuum would produce torsion in contrast to our conclusion above.

It is not clear to us, however, whether this interpretation is compulsory. Interestingly the Imbert-shift may be calculated by only using Maxwell's theory and the appropriate boundary conditions, see Imbert, *loc. cit.* Thus a non-Maxwellian behaviour of photons seems not to be present.

terms quadratic in the spin. If we trace this back to the total action function, it turns out that the U_i -Lagrangian differs from the V_i -Lagrangian of GR to 1st order in the term

$$(5.1) \quad \frac{1}{2} \tau_k^{ijl} K_{ij}^{..k} = k \left(-\frac{1}{2} \tau_{ijk} \tau^{ijk} + \tau_{ijk} \tau^{jki} + \tau_{ik}^{..k} \tau_{..l}^{ll} \right).$$

Accordingly (5.1) represents a very weak *universal* spin-spin contact interaction characteristic for a U_i and proportional to the gravitational constant. In this theory there is no "spin field" which is emitted and thereby the carrier of a new interaction; there is rather a very weak interaction as soon as spinning matter is in contact with each other. In particular there results a universal self-interaction of spinning matter, which leads to non-linearities in an analogous manner as energy-momentum does via gravitational interaction. Hence a U_i -theory supplies a unified description of the universal long-range gravitational interaction and a universal weak spin-spin interaction of vanishing range.

The deviation of the U_i -theory from V_i -theory is very small indeed, as can be seen from (4.9) because the 2nd term carries a factor k^2 . In order to estimate the relative contribution of the spin interaction to the right hand side of (4.9), let us use for spinning dust the semi-classical approximation $\sigma^{ij} = \rho c^2 u^i u^j$ and $\tau_{ij}^{..k} = s_{ij} c u^k$ (ρ = matter density, s_{ij} = spin density, u^i = velocity of the dust). We get

$$(5.2) \quad \frac{k^2 \tau^2}{k \sigma} \approx \frac{k^2 (c s)^2}{k \rho c^2} = \frac{k s^2}{\rho}.$$

For particles with mass m carrying a spin of the order of \hbar , we expect

$$(5.3) \quad s = \frac{\hbar}{m} \rho$$

if the spins are parallelly oriented. (5.3) substituted in (5.2) leads to

$$(5.4) \quad \frac{k^2 \tau^2}{k \sigma} \approx \frac{k \hbar^2}{m^2} \rho.$$

The spin terms are of the same order of magnitude as the matter density term as soon as

$$(5.5) \quad \rho \approx \frac{m^2}{k \hbar^2}.$$

For electrons this occurs at the huge density of about 10^{48} g/cm³.

This estimate shows that the contribution from spin to gravitational interaction can be neglected in the case of normal matter densities. In the region of densities of the order of (5.5) or higher, however, *spin* becomes the *dominant* source of the gravitational field. Hence it is to be expected that the metric of a U_4 , in the neighborhood of *singularities*, is determined to a large extent by the spin distribution of matter.

There are indications ⁽⁶⁾ that the gravitational interaction of parallel spins is of a repulsive type. If so, one could speculate whether thereby the occurrence of singularities might be prevented in gravitational collapse and cosmology. A first attempt in this direction has been undertaken by Kopczyński (1972) [see also Trautman (1972 b)]. His very simple cosmological model of the Robertson-Walker type with a spatially constant spin density is somewhat unrealistic in its assumptions, but it shows how spin could prevent singularities in principle. In the meantime we have proved that in Kopczyński's model it is at densities of the order of (5.5) that spin becomes effective.

For a Dirac particle, as is well-known, the canonical spin-angular momentum tensor is totally antisymmetric and hence equivalent to an axial vector. In this case (5.1) is the exact difference between the U_4 - and V_4 -theory and we get an *axial vector interaction*

$$(5.6) \quad \frac{1}{2} \tau_k^{jl} K_{ij}^{*k} = \frac{k}{2} \tau^{(ljk)} \tau_{ijk} = -\frac{3l^k}{16k} (\Psi^+ \gamma_3 \gamma_\alpha \Psi) (\Psi^+ \gamma_3 \gamma^\alpha \Psi)$$

($l = \sqrt{\hbar ck} \approx 10^{-32}$ cm, $\Psi^+ \stackrel{\text{def}}{=} \Psi^* \gamma^0$, $\Psi =$ Dirac spinor, γ^α , $\gamma_3 =$ Dirac matrices).

The axial vector interaction (5.6) can also be discovered in another way. One starts with the Dirac Lagrangian and applies the formalism of Section 4. Then the matter equation, after substitution of (4.8), turns out to be a non-linear spinor equation of the Heisenberg-Pauli type ⁽⁷⁾, ($i^2 = -1$, $\overset{\{ \}}{\nabla}$ = covariant derivative with respect to the V_4) :

$$(5.7) \quad \left[\gamma^\alpha \overset{\{ \}}{\nabla}_\alpha - \frac{3}{8} i l^2 (\Psi^+ \gamma_3 \gamma^\alpha \Psi) \gamma_3 \gamma_\alpha \right] \Psi = im \Psi.$$

⁽⁶⁾ Spin at rest can be shown to produce time-like "screw dislocations" in space-time [Hehl (1970)]. Parallel screw dislocations, according to 3-dimensional dislocation theory, repel each other.

⁽⁷⁾ In the original Heisenberg-Pauli equation there enters a fundamental length $L \approx 10^{-13}$ cm. We can arrive at a spinor equation with such a constant by a suitable choice of the Lagrangian density of the field. If we choose in (4.4) instead of $\alpha(\Gamma) \rightarrow \alpha(\cdot; \cdot) + e T_k^{jl} K_{ij}^{*k} / \alpha^2$ with a dimensionless constant α , we get (5.7) with $l \rightarrow \alpha l$. In (4.9), for the 2nd term on the right-hand-side, we would have $k^2 \rightarrow (\alpha k)^2$. Such a choice of the Lagrangian would seem artificial to us, however. All this has already been remarked on by Peres (1962). For still more general Lagrangians, we refer to Hayashi and Bregman (1973), see also Hehl (1973).

This equation is equivalent to $\gamma^\alpha \nabla_\alpha \Psi = im \Psi$. In (5.7) it can be seen directly : torsion leads to a self-interaction introducing additional non-linear terms in the Dirac equation. Thus in this special case the affine connection of space-time is influenced by an axial vector.

Let us just note that it is appealing to speculate whether this axial vector interaction is a classical analogue of the weak interaction of elementary particle physics.

There is another deviation from GR. The right-hand-side of (4.10)

$$(5.8) \quad f_i \stackrel{\text{def}}{=} \tau_j^i R_{ik}^{jk}$$

is expected to be a volume force acting on each spinning particle in the presence of a gravitational field. This presumptive force is of the Mathisson type [Mathisson (1937)]. Nevertheless, it is different from it, because (5.8) vanishes for $\hbar \rightarrow 0$. (5.8) would correspond to a slight violation of the equivalence principle, a fact being mentioned in Section 2. With today's experimental techniques (5.8) does not seem to be measurable.

6. SUMMARY

In one respect, U_i -theory seems to give a final answer : if space-time possesses torsion, then torsion has something to do with spin-angular momentum of matter. Hence all theories which try to connect torsion, say with the electromagnetic field, are obsolete. Consequently we understand the potential physical meaning of torsion. The utility of an asymmetric connection has been shown, the U_i as a physically reasonable generalization of the V_i seems to be near at hand.

On the other side we could make plausible the introduction of torsion in connection with material spin, i. e. a dynamical manifestation of spin seems to be a natural thing. We know as yet no argument, however, which could enforce a torsion upon space-time.

Hence we just have to wait for experiments or astronomical observations referring to extremely high matter densities, which will show the existence of the *very weak universal spin contact interaction* characteristic for a space-time continuum obeying the Riemann-Cartan geometry.

ACKNOWLEDGEMENTS

One author (F.W.H.) would like to thank Prof. O. Costa de Beauregard, Dr. R.-L. Clerc, and Dr. J. Madore for interesting discussions about questions on the possible space-time geometry and Dr. C. Imbert for

showing to him his experiment on the "Imbert-shift" of photons. He also highly acknowledges a discussion with Dr. P. Rosenthal on the organization of this article. We would additionally like to thank him for critically reading the manuscript.

REFERENCES

- O. COSTA DE BEAUREGARD :
- (1942) *C. R. Acad. Sc. (Paris)*, t. 214, p. 904.
 (1943) *J. Math. pures et appl.*, t. 22, p. 85.
 (1959) *Cah. Phys.*, t. 13, p. 209.
 (1963) *Phys. Rev.*, t. 129, p. 466.
- O. COSTA DE BEAUREGARD, Ch. IMBERT and J. RICARD (1971) *Int. J. Theor. Phys.*, t. 4, p. 125.
- B. A. BILBY, R. BULLOUGH and E. SMITH (1955) *Proc. Roy. Soc. (London)*, A, t. 231, p. 263.
- L. DE BROGLIE (1963) *Introduction to the Vigier Theory of Elementary Particles* (transl. from French), Elsevier Publ. Comp., Amsterdam, etc.
- E. CARTAN :
- (1922) *C. R. Acad. Sc. (Paris)*, t. 174, p. 593.
 (1923) *Ann. scient. Éc. Norm. Sup.*, t. 40, p. 325.
 (1924) *Ibid.*, t. 41, p. 1.
 (1925) *Ibid.*, t. 42, p. 17.
- R.-L. CLERC :
- (1971) *C. R. Acad. Sc. (Paris)*, t. 272, série A, p. 1145 et 1760.
 (1972 a) *Ibid.*, t. 274, série A, p. 525.
 ✕ (1972 b) *Ann. Inst. H. Poincaré*, série A, t. 17, p. 227 et 259.
- E. et F. COSSERAT (1909) *Théorie des corps déformables*, Hermann, Paris.
- B. K. DATTA (1971) *Nuovo Cimento*, t. 6 B, p. 1 et 16.
- F. HALBWACHS (1960) *Théorie relativiste des fluides à spin*, Gauthier-Villars, Paris.
- K. HAYASHI and A. BREGMAN (1973) *Ann. Phys. (N. Y.)*, t. 75, p. 562.
- F. W. HEHL :
- (1966) *Abh. Braunsch. Wiss. Ges.*, t. 18, p. 98.
 (1970) *Spin und Torsion in der allgemeinen Relativitätstheorie oder die Riemann-Cartansche Geometrie der Welt [Habilitation Thesis (mimeographed), Techn. Univ. Clausthal, August 1970]*.
 (1971) *Phys. Lett.*, t. 36 A, p. 225.
 ✕ (1973) *Spin and Torsion in General Relativity. I. Foundations. II. Geometry and Field Equations. General Relativity and Gravitation J.*, t. 4, p. 333 and t. 5 (to be published).
- F. W. HEHL and B. K. DATTA (1971) *J. Math. Phys.*, t. 12, p. 1334.
- F. W. HEHL and E. KRÖNER (1965) *Z. Physik*, t. 187, p. 478.
- D. HILBERT (1915) *Nachr. Ges. Wiss. Göttingen, Math.-phys. Kl.*, p. 395.
- Ch. IMBERT (1972) *Phys. Rev.*, D, t. 5, p. 787.
- T. W. B. KIBBLE (1961) *J. Math. Phys.*, t. 2, p. 212.
- K. KONDO (1952) *Proc. 2nd Japan Nat. Congr. Appl. Mechanics*, p. 41.
- W. KOPCZYŃSKI (1972) *Phys. Lett.*, t. 39 A, p. 219.
- E. KRÖNER (1964) *Plastizität und Versetzung*. In A. SOMMERFELD, *Vorlesungen über theoretische Physik*, vol. 2, chap. 9, 5th ed., Akad. Verlagsges., Leipzig.
- M. LENOIR (1971) *C. R. Acad. Sc. (Paris)*, t. 273, série A, p. 943.

- A. LICHNEROWICZ (1955) *Théorie globale des connexions et des groupes d'holonomie*, Edizioni Cremonese, Roma.
- F. LURÇAT (1964) *Physics*, t. 1, p. 95.
- M. MATHISSON (1937) *Acta Phys. Polon.*, t. 6, p. 163.
- H. MINKOWSKI (1958) *Raum und Zeit*. In H. A. LORENTZ, etc., *das Relativitätsprinzip*, p. 54. 6th ed., Wiss. Buchges., Darmstadt.
- T. A. MORGAN and A. PERES (1962) *Phys. Rev. Lett.*, t. 9, p. 79.
- J. F. NYE (1953) *Acta Met.*, t. 1, p. 153.
- A. PAPANETROU (1949) *Phil. Mag.*, (7), t. 40, p. 937.
- A. PERES (1962) *Suppl. Nuovo Cimento*, t. 24, p. 389.
- J. J. SAKURAI (1960) *Ann. Phys. (N. Y.)*, t. 11, p. 1.
- J. A. SCHOUTEN (1954) *Ricci-Calculus*, 2nd ed., Springer, Berlin, etc.
- J. A. SCIAMA :
- (1962) *Recent Developments in General Relativity*, p. 415, Pergamon-PWN, London, etc.
- (1964) *Rev. Mod. Phys.*, t. 36, p. 463 et 1103.
- M.-A. TONNELAT (1965) *Les théories unitaires de l'électromagnétisme et de la gravitation*, Gauthier-Villars, Paris.
- A. TRAUTMAN :
- (1972 a) *General Relativity and Gravitation J.*, t. 3, p. 167.
- (1972 b) *On the structure of the Einstein-Cartan Equations* (with an appendix by W. KOPCZYŃSKI) (*Based on a Report on "Convegno di Relativita"* held in Rome in February 1972, Preprint, Warsaw University, April 1972).
- (1972 c) *Bull. Acad. Polon. Sci., Série Sci. math. astr. phys.*, t. 20, p. 185, 503 et 895.
- R. UTIYAMA (1965) *Phys. Rev.*, t. 101, p. 1597.
- H. WEYL :
- (1950) *Phys. Rev.*, t. 77, p. 699.
- (1961) *Raum, Zeit, Materie*, Nachdruck der 5. Aufl. (1923), Wiss. Buchges, Darmstadt.

(Manuscrit reçu le 22 janvier 1973,
version révisée le 24 mai 1973.)