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Section A :
Physique théorique.

On the infrared problem in a model of scalar electrons and massless, scalar bosons

by

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ABSTRACT. — In this paper the infrared problem is investigated in the framework of simple models (closely related to Nelson's model, [28]). These models describe the interaction of conserved, charged, scalar particles (here called electrons) and relativistic, neutral, scalar bosons of restmass 0. They exhibit an " infra-particle situation " in the sense of [35] and proper infrared problems in the construction of dressed one electron states and scattering amplitudes.

A renormalized Hilbert space is constructed such that the spectrum of the energy-momentum operator on this Hilbert space contains a unique one electron shell corresponding to dressed one electron states. However, the spectrum of the energy-momentum operator on the physical Hilbert space does *not* contain a one electron shell.

Several concepts for a collision theory on the charge one sector are developped. These concepts are compared with the proposals of Faddeev and Kulish, [12], and a list of interesting, yet unsolved problems is presented.

RÉSUMÉ. — Nous étudions le problème infrarouge dans le cadre de deux modèles simples (correspondant essentiellement au modèle de Nelson avec des bosons de masse nulle, [28]). Ces modèles décrivent les interactions de particules scalaires, chargées (que nous appelons « électrons ») avec des bosons scalaires, neutres de masse nulle. Ces interactions conservent la charge.

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La dynamique de ces modèles nous amène à une situation « d'infra-particules » dans le sens de [35] et à des problèmes infrarouges dans la construction d'électrons habillés et d'amplitudes de diffusion.

Un espace hilbertien renormalisé est construit tel que le spectre de l'opérateur d'énergie-impulsion sur cet espace contienne une couche de masse d'électron non relativiste. Cependant, le spectre de l'opérateur d'énergie-impulsion sur l'espace hilbertien physique *ne contient pas* une couche de masse d'électron.

Plusieurs concepts pour une théorie de diffusion de ce modèle sont élaborés. En particulier, nous discutons une théorie de Haag-Ruelle généralisée et un cadre algébrique pour une théorie de diffusion.

Nous comparons nos concepts avec les idées de Faddeev et Kulish, [12], et nous discutons quelques problèmes importants qui ne sont pas encore résolus.

ORGANISATION OF THE PAPER

CHAPTER 0

INTRODUCTION

In this paper we discuss two models which are modified versions of Nelson's model ([28], [4], [15]). They describe a system of conserved, scalar electrons interacting with neutral, massless, scalar bosons (1).

(1) *See* also [1], [11].

The electrons are described by a quantized, complex, scalar field $\psi(x, t)$ and the bosons by a quantized, real, scalar field $\varphi(x, t)$. The model of main interest in this paper treats the (free) electrons as non-relativistic particles. Formally the dynamics of the fields $\psi(x, t)$ and $\varphi(x, t)$ is then given by the following field equations :

$$(0.1)\quad \begin{cases} \square\, \varphi(x, t) = \lambda \psi^*(x, t)\, \psi(x, t), \qquad x \in \mathbf{R}^3, \quad t \in \mathbf{R}, \quad \lambda \in \mathbf{R}; \\ \left(i\dfrac{\partial}{\partial t} + \dfrac{1}{2\,\mathrm{M}} \Delta_x\right) \psi(x, t) = (\lambda \varphi(x, t) + \mathrm{E}_1(\lambda))\, \psi(x, t), \end{cases}$$

where $\mathrm{E}_1(\lambda)$ is an infinite counter term corresponding to a self energy renormalization of the electron. It is evident from the equations (0.1) that the creation and annihilation of pairs of electrons and positrons are neglected. However, we hope that these processes are of minor relevance for the qualitative understanding of the behavior of our system at small boson energies; [12], [25].

At $t = 0$ we shall impose as initial conditions for the field equations (0.1) the ones of the corresponding free field equations;

$$(0.2)\qquad \square\, \varphi(x, t) = 0, \qquad \left(i\frac{\partial}{\partial t} + \frac{1}{2\,\mathrm{M}} \Delta_x\right) \psi(x, t) = 0$$

namely :

$$(0.3)\quad \begin{cases} \varphi(x) = \varphi(x, 0) = (2\pi)^{-3/2} \displaystyle\int \frac{d^3 k}{\sqrt{2\,|k|}}\, e^{ikx} \{ b^*(k) + b(-k) \}, \\ [b(k), b^*(l)] = \delta(k - l), \qquad [b^\#(k), b^\#(l)] = 0, \qquad b^\# = b \text{ or } b^* \end{cases}$$

and

$$(0.4)\quad \begin{cases} \psi(x) = \psi(x, 0) = (2\pi)^{-3/2} \displaystyle\int d^3 p\, e^{-ipx}\, n(p), \\ \psi^*(x) = (\psi(x, 0))^* = (2\pi)^{-3/2} \displaystyle\int d^3 p\, e^{ipx}\, n^*(p). \end{cases}$$

Although the statistics of the electron does not play any role, it is considered to be a fermion; whence :

$$(0.5)\quad \begin{cases} \{ n(p), n^*(q) \} = n(p)\, n^*(q) + n^*(q)\, n(p) = \delta(p - q), \\ \qquad \{ n^\#(p), \; n^\#(q) \} = 0, \qquad n^\# = n \text{ or } n^*. \end{cases}$$

In Chapter 1, we shall start our analysis of the equations (0.1) within the Hamiltonian formalism on the Fock-space. It should however be noted that the choice of the Fock-space as an underlying Hilbert-space of the theory is somewhat arbitrary due to the fact that the restmass of the bosons is 0; *see* [14], *a*.

In order to show that, like e. g. Quantum Electrodynamics (QED), our model actually leads to non trivial IR problems, we want to present here the results obtained from a simple approximation of the field equations (0.1) which exhibit clearly the presence of IR divergencies and moreover have some predictive power on the kind of phenomena we shall meet in the discussion of the equations (0.1): We replace the " charge density operator " $\lambda\psi^* (x, t)\, \psi (x, t)$ by a c-number charge density $\rho (x, t)$ corresponding to a classical electron of finite size and solve the equation

$$\square\, \varphi (x, t) = \rho (x, t), \tag{0.6}$$

where $\rho (x, t)$ is a real-valued distribution.

DEFINITIONS :

$$\left\{ \begin{aligned} \hat{\rho} (k, k^0) &= (2\pi)^{-3/2} \int d^3 x\, dt\, e^{i(k^0 t - k.x)} \rho (x, t), \\ \tilde{\rho} (k, t) &= (2\pi)^{-3/2} \int d^3 x\, e^{-ik.x} \rho (x, t). \end{aligned} \right. \tag{0.7}$$

We immediately get for the time evolution operator in the interaction picture :

$$\mathrm{U} (t, 0) = \exp - i \int \frac{d^3 k}{\sqrt{2 |k|}} \int_0^t ds\, [\tilde{\rho} (k, s)\, e^{i|k|s}\, b^* (k) + \tilde{\rho} (-k, s)\, e^{-i|k|s}\, b (k)] \tag{0.8}$$

whence for the S-matrix :

$$\begin{aligned} \mathrm{S} &= \mathrm{U}^* (\infty, 0)\, \mathrm{U} (0, -\infty) \\ &= \exp i \int \frac{d^3 k}{\sqrt{2 |k|}} [\hat{\rho} (k, |k|)\, b^* (k) + \hat{\rho} (-k, -|k|)\, b (k)]. \end{aligned} \tag{0.9}$$

Since $\rho (x, t)$ is real-valued, $\hat{\rho} (-k, -|k|) = \overline{\hat{\rho} (k, |k|)}$, etc. Therefore S and U $(t, 0)$ are unitary operators on the Fock-space $\mathcal{F}_b$ of the bosons iff $\int d^3 k\, (2 |k|)^{-1}\, |\hat{\rho} (k, |k|)|^2 < \infty$, etc. However, for realistic charge-densities, the integral

$$\int_{|k| \geq \lambda} d^3 k\, (2 |k|)^{-1}\, |\hat{\rho} (k, |k|)|^2 \quad \text{diverges like } \ln \lambda^{-1}, \text{ as } \lambda \to 0.$$

For finite times U $(t, 0)$ is in general still unitary on $\mathcal{F}_b$, but asymptotically, as $t \to \pm \infty$, the operators U $(\pm \infty, 0)$ map $\mathcal{F}_b$ on new Hilbert spaces $\mathcal{H}_\pm$ of scattering states which carry representations $\pi_\pm$ of the canonical commutation relations (CCR) (0.3) (or more precisely of the corresponding Weyl relations) that are disjoint from the Fock representation and in general disjoint from each other. The S-matrix intertwines the two representations π_+, π_-. In order to get some more explicit results we assume that the electron has a finite size and is scattered at small times from initial velocity v_i to final velocity v_f. The Fourier transform of $\rho(x, 0)$ is chosen to be a function $w(k)$ in $\mathcal{S}(\mathbf{R}^3)$ such that $w(k=0) = \int d^3 x \, \rho(x, 0) = 1$. This yields :

$$\tilde{\rho}(k, t) = w(k) \, e^{-i(k, v_i)t} \qquad \text{for} \quad t < 0$$

and

$$\tilde{\rho}(k, t) = w(k) \, e^{-i(k, v_f)t} \qquad \text{for} \quad t > 0.$$

Hence :

$$(0.10) \quad \left\{ \begin{aligned} \mathrm{U}(\pm\infty, 0) &= \exp \int d^3 k \frac{w(k)}{\sqrt{2|k|}} \\ &\quad \times (|k| - (k, v_{f/i}))^{-1} [b(k) - b^*(k)], \\ \mathrm{S} &= \exp \int d^3 k \frac{w(k)}{\sqrt{2|k|}} \\ &\quad \times [(|k| - (k, v_f))^{-1} \\ &\qquad - (|k| - (k, v_i))^{-1}] [b^*(k) - b(k)]. \end{aligned} \right.$$

Obviously $\mathrm{U}(\pm\infty, 0)$ map $\mathcal{F}_b$ onto new Hilbert spaces $\mathcal{H}_\pm$, orthogonal to $\mathcal{F}_b$.

For the transition probabilities $\mathrm{P}(n, \mathrm{K}_{\lambda,\Lambda})$ of emission of n bosons with momenta in the region $\mathrm{K}_{\lambda,\Lambda} = \{ k \in \mathbf{R}^3 \mid \lambda \leq |k| \leq \Lambda \}$ we get

$$(0.11) \qquad \mathrm{P}(n, \mathrm{K}_{\lambda,\Lambda}) = \frac{c(\mathrm{K}_{\lambda,\Lambda})^n}{n!} e^{-c(\mathrm{K}_{\lambda,\Lambda})},$$

where

$$c(\mathrm{K}_{\lambda,\Lambda}) = \int_{\lambda \leq |k| \leq \Lambda} d^3 k \, (2|k|)^{-1} |\hat{\rho}(k, |k|)|^2,$$

$\mathrm{P}(n, \mathrm{K}_{\lambda,\Lambda})$ is a *Poisson distribution*.

We shall compare these results with the results of a rigorous discussion of the translationally invariant model. Formulas (0.10) and (0.11) suggest that in our models the scattering states (in- and out states) are in Hilbert spaces which are determined by (generalized) coherent states. (*See* [39], [6], [24], [25], [2], [12], [31], and others.)

Since this paper is ambitious in as much as we hope to learn something about the IR problem which should hold under general circumstances, [16], it is necessary and fair to refer to some of the classical ideas which determined the history of the IR problem.

The fact that accelerated charged particles radiate and that the radiation field which is emitted is somewhat singular in its nature could already been observed on semiclassical grounds, namely by combining classical electrodynamics with the Planck-Einstein equation for the energy of the light quantum :

$$\mathrm{E}(\nu) = h\nu \qquad (\nu : \text{frequency of the photon}).$$

This combination yields that the total number of photons $\mathrm{N}(\lambda, \Lambda)$ emitted in the frequency interval $[\lambda, \Lambda]$ by an accelerated charged particle diverges like $\ln \lambda^{-1}$, as λ tends to 0 (for fixed $\Lambda < \infty$).

The next step (a test for this prediction) was to calculate the transition probabilities for a quantized radiation field in the framework of a simplified version of QED. In essence one considered the interaction of the quantized radiation field with a classical current. This has been done for the first time by Bloch and Nordsieck in two famous papers, [3]. They got a Poisson distribution for the transition probabilities [in accordance with (0.11)]. The mean number of photons emitted in the spectral interval $[\lambda, \Lambda]$ obtained from these calculations agrees with the result for $\mathrm{N}(\lambda, \Lambda)$ of the semiclassical argument.

Investigations of this type on a higher level of mathematical rigor were done among others by T. W. B. Kibble, [24], who also analyzed the structure of the space of scattering states obtained from these calculations.

Conservation of total energy in the scattering process was taken into account by Pauli and Fierz, [29]. They also observed that within this framework the assumption that the electron is a point particle leads to the conclusion : $\mathrm{S} = \mathrm{I}$.

Similar calculations and a discussion of the ultraviolet cutoff, i. e. the formfactor of the electron, were given in [22].

The dipole approximation of non relativistic QED was analyzed by Blanchard, [2]; *see* also Shale, [36]. The physical meaning of their results remained somewhat unclear.

In all these calculations one could not avoid the introduction of a formfactor for the electron in order to obtain reasonable results. However in a more complete, translationally invariant — therefore less singular — theory such a formfactor should not be necessary. A step in the direction of such a theory was the discussion of the singularity structure of the Green's functions of QED and its relations to the properties of the scattering states by Kibble, [25], and others. These calculations seem to show furthermore that there are no stable one particle states of charged particles in the physical Hilbert space; [35]. This is equivalent to the fact that the scattering states do not form a Fock space; *see* chapter 4 and [16].

Chung, [6], analyzed perturbation expansions in QED and showed how to get rid of infrared divergencies in the S-matrix elements by assuming that the scattering states expressed in terms of asymptotic fields are essentially generalized coherent states (with respect to the asymptotic electromagnetic field). The assumption that the scattering states are essentially generalized coherent states is equivalent to the postulate that the classical theory should be correct in the infrared limit (correspondence principle), [16].

A nice heuristic recipe for the calculation of the scattering states and time dependent (logarithmically divergent) Coulomb phases that cancel divergent phases in the S-matrix in the limit $t = \pm \infty$ was invented by Faddeev and Kulish, [12], by comparison with the non relativistic Coulomb scattering.

All these investigation made obvious that a naive application of perturbation theory for the calculation of radiative corrections in cross sections ought to lead to wrong results : Logarithmically divergent transition probabilities for the emission of some undetected soft photons instead of vanishing probabilities as predicted by the results mentioned so far. These divergencies, the famous divergence of $\int_0^a \frac{d\omega}{\omega}$, were found by Mott, [27], and Sommerfeld, [38], in 1931.

It is well known that in QED one found later a " simple " recipe to get rid of all IR divergencies in a perturbation theoretic calculation of *cross sections* in finite orders of the Feinstructure constant; *see* [40]. It was natural to try to relate this recipe to the structure of scattering states, [6]. The cross section approach in its *perturbation theoretic version* is however not important for what we are going to present here, since we do not use perturbation theory and since we try to construct scattering amplitudes.

After this short historical excursion we add some more precise remarks concerning the content of this paper :

In *chapter* 1 we define our models. We show that one can solve the field equations (0.1) within the Hamiltonian formalism on the Fock space $\mathcal{H} = \bigoplus_{Z=0}^{\infty} \mathcal{H}^{(Z)}$, where $\mathcal{H}^{(Z)}$ is the Z electron sector (also called charge Z sector).

The field equations determine a Hamiltonian H. We also define an IR cutoff Hamiltonian $H(\sigma)$ with the property that bosons with momenta $k \in K_\sigma = \{ q \in \mathbf{R}^3 \mid |q| \leq \sigma \}$ do not interact at all. We show

THEOREM A. — *For all* $\sigma \geqq 0$ *the following holds :* $e^{it\mathrm{H}(\sigma)}$ *is a strongly continuous unitary group on* $\mathcal{H}$ *which conserves the number of electrons (i. e. the charge) and commutes with the total momentum operator* P. $\mathrm{H}(\sigma) \upharpoonright \mathcal{H}^{(\mathrm{Z})}$ *is bounded below, for all* Z *in* $\mathbf{Z}^{+}$.

THEOREM B. — *For all* $\sigma > 0$ $(\mathrm{H}(\sigma), \mathrm{P}) \upharpoonright \mathcal{H}^{(1)}$ *has a unique one particle shell corresponding to dressed one electron states (DES).*

A DES of momentum p is denoted by $\psi_1(\sigma, p)$.

In *chapter* 2 we define a natural operator algebra $\overline{\Delta_{\mathrm{B}}(\mathrm{V})}$ contained in $\{ \mathrm{P} \}' \cap \mathrm{B}(\mathcal{H}^{(1)})$, where $\{ \mathrm{P} \}'$ is the commutant of the total momentum operator and $\mathrm{B}(\mathcal{H}^{(1)})$ is the algebra of all bounded operators on $\mathcal{H}^{(1)}$. Hence $\overline{\Delta_{\mathrm{B}}(\mathrm{V})}$ consists of operators leaving the total momentum unchanged and will be shown to be isometrically isomorphic to the CCR algebra generated by the boson Weyl operators over a test function space V, [26], [37].

We define $\omega_{\sigma, p}(\mathrm{A}) = (\psi_1(\sigma, p), \mathrm{A}\,\psi_1(\sigma, p))$, $\mathrm{A} \in \overline{\Delta_{\mathrm{B}}(\mathrm{V})}$ and $\omega_{\sigma, p}$ determines a state on $\overline{\Delta_{\mathrm{B}}(\mathrm{V})}$ (provided $|p|$ is sufficiently small).

We shall analyze the properties of the sequence $\{ \omega_{\sigma, p} \}_{\sigma > 0}$ of states on $\overline{\Delta_{\mathrm{B}}(\mathrm{V})}$ for different values of p.

In *chapter* 3 we prove :

THEOREM C :

$$\omega_p(\mathrm{A}) = \lim_{\sigma \downarrow 0} \omega_{\sigma, p}(\mathrm{A})$$

exists for all A *in* $\overline{\Delta_{\mathrm{B}}(\mathrm{V})}$ *and all p in a certain set* $\mathcal{E}$ *of positive Lebesgue measure;*

$\omega_p(.)$ *defines a unique DES without IR cutoff. This DES is essentially a generalized coherent state with respect to the algebra* $\overline{\Delta_{\mathrm{B}}(\mathrm{V})}$.

THEOREM D. — *It is impossible to construct a Hilbert space containing DES for all momenta p in* $\mathcal{E}$ *in the limit* $\sigma = 0$ *and such that the dynamics on this Hilbert space is non trivial and is compatible with a scattering theory* (*see* section 3.3).

In *chapter* 4 we prove an LSZ asymptotic condition for the boson field and the boson Weyl operators for the dynamics given by $e^{it\mathrm{H}(\sigma)}$, $\sigma \geqq 0$. We prove a strong convergence asymptotic condition in time in the sense of Haag-Ruelle for the dynamics $e^{it\mathrm{H}(\sigma)}$, $\sigma > 0$.

We shall explain the reasons why in the limit $\sigma = 0$ there seem to exist states in $\mathcal{H}^{(1)}$ which converge strongly in time to scattering states (" generalized " Haag-Ruelle theory). Furthermore we try to approximate these scattering states by a certain sequence of scattering states for $\sigma > 0$. We discuss the nature of the scattering states (generalized coherent states) and define transition amplitudes.

The most interesting part of chapter 4 contains the construction of an algebraic framework for the scattering theory in the charge one sector. We give new definitions of " particle interpretations of a theory ", of " asymptotic completeness ", etc.

We show that the LSZ asymptotic condition for the boson Weyl operators together with a detailed knowledge of the dynamics determined by e^{itH} on $\mathcal{H}^{(1)}$ provide a *complete information about the scattering on* $\mathcal{H}^{(1)}$, the scattering of the charge (or the charged particle) included. We calculate cross sections for the scattering of a charge and the bosons and we construct a *scattering isomorphism* (which however does *not* seem to be spatial, i. e. implementable by a unitary S-matrix).

In *chapter* 5 we compare our results with the proposals of Faddeev and Kulish, [12], and the approximate calculations of chapter 0, (0.10), (0.11).

We give an outlook to the future.

CHAPTER 1

DEFINITION OF THE MODELS; THE DYNAMICS; DES FOR THE MODELS WITH AN INFRARED CUTOFF DYNAMICS

1.1. Definitions; the time evolution

The formal field equations (0.1) show that the system we want to describe consists of an arbitrary but conserved number of non relativistic electrons interacting with neutral, massless, scalar bosons. In this section we shall construct the time evolution for this system. Since the interaction is such that there is no vacuum polarization, it is justified (though not necessary; [14], *a*, hereafter referred to as II, *a*) to start the investigation of the models on the Fock space $\mathcal{H}$ of electrons and bosons :

$$(1.1) \qquad \mathcal{H} = \mathcal{F}_n \otimes \mathcal{F}_b \begin{cases} \mathcal{F}_n = \bigoplus_{Z=0}^{\infty} L^2(\mathbf{R}^3)^{(a)Z}, \\ \mathcal{F}_b = \bigoplus_{m=0}^{\infty} L^2(\mathbf{R}^3)^{(s)m}, \end{cases}$$

where (*a*) denotes antisymmetric and (*s*) symmetric tensor product.

Clearly

$$(1.2) \qquad \mathcal{H} = \bigoplus_{Z=0}^{\infty} \mathcal{H}^{(Z)}, \qquad \mathcal{H}^{(Z)} = L^2(\mathbf{R}^3)^{(a)Z} \otimes \mathcal{F}_b,$$

Z counts the number of bare electrons and is called the " charge ". Since the charge will turn out to be a constant of the motion and the electron field [defined in (0.4)] is not observable, the spaces $\mathcal{H}^{(Z)}$ are super selection sectors. The operators

$$(1.3) \qquad n(f) = \int d^3 p\, \overline{f(p)}\, n(p), \qquad n^*(g) = \int d^3 p\, g(p)\, n^*(p)$$

are bounded in norm by $\|f\|_2$, $\|g\|_2$ respectively, on $\mathcal{F}_n$.

The operators $\{ I, n(f), n^*(g) \mid f, g \text{ in } L^2(\mathbf{R}^3) \}$ generate a norm separable C* algebra which acts irreducibly on $\mathcal{F}_n$.

The free Hamiltonian for the electron is

$$(1.4) \qquad 0 \leq H_{0n} = \int d^3 p\, n^*(p)\, \Omega(p)\, n(p),$$

where $\Omega(p) = \frac{p^2}{2M}$ [we shall mention at some places results for the case $\Omega(p) = \sqrt{p^2 + M^2}$, which has some advantages].

Electron momentum and position operators are given by

$$(1.5) \qquad P_n = \int d^3 p\, n^*(p)\, p\, n(p), \qquad Q_n = \int d^3 x\, \psi^*(x)\, x\, \psi(x)$$

and the number — or charge operator is

$$0 \leq N_n = \int d^3 p\, n^*(p)\, n(p).$$

All these operators are selfadjoint (s. a.) on $\mathcal{F}_n$.

Boson operators. — The operators

$$b(f) = \int d^3 k\, \overline{f(k)}\, b(k), \qquad b^*(g) = \int d^3 k\, g(k)\, b^*(k)$$

are densely defined on $\mathcal{F}_b$, if f and g are in $L^2(\mathbf{R}^3)$ and

$$\{ I, b(f), b^*(g) \mid f, g \text{ in } L^2(\mathbf{R}^3) \}$$

generates a *algebra which acts irreducibly on $\mathcal{F}_b$. As usual one defines the following s. a. operators :

$$(1.6) \quad \begin{cases} 0 \leq H_{0b} = \int d^3 k\, b^*(k)\, |k|\, b(k), \\ P_b = \int d^3 k\, b^*(k)\, k\, b(k), \qquad 0 \leq N_b = \int d^3 k\, b^*(k)\, b(k). \end{cases}$$

All the operators defined on $\mathcal{F}_n$ or $\mathcal{F}_b$ have a natural extension to $\mathcal{H}$: If A is a densely defined, (s. a., positive, ...) operator on $\mathcal{F}_n$, $(\mathcal{F}_b)$, then $A \otimes I$, $(I \otimes A)$, is densely defined, (s. a., positive, ...) on $\mathcal{H}$. We write again A instead of $A \otimes I$, $I \otimes A$.

The *algebra generated by

$$\{ I, n(f), n^*(g), b(f'), b^*(g') \mid f^{(\prime)}, g^{(\prime)} \text{ in } L^2(\mathbf{R}^3) \}$$

acts irreducibly on $\mathcal{H}$. The usual vacuum state in $\mathcal{H}$ is denoted by φ_0.

The field equations (0.1) lead to the following *interaction Hamiltonian* :

$$H_I = \lambda \int d^3 x\, \psi^*(x)\, \varphi(x)\, \psi(x).$$

In order to start our analysis with a well defined total Hamiltonian we introduce an ultraviolet and an IR cutoff in H_I.

DEFINITIONS :

a. $v(k)$ is a real valued, rotation invariant C^∞ function such that $0 \leq v(k) \leq 1$, $v(k) = v(-k)$, $v(k = 0) = 1$.

b. $g_\sigma(k)$ is a real valued, rotation invariant C^∞ function such that $0 \leq g_\sigma(k) \leq 1$, $g_\sigma(k) = g_\sigma(-k)$, $g_\sigma(k) = 1$, $|k| \geq 2\sigma$ and $g_\sigma(k) = 0$, $|k| \leq \sigma$. σ is restricted to the interval [0, 1/2], and we put $v_\sigma = v\, g_\sigma$.

c. In the following $\hat{f}$ denotes the Fourier transformed of f (from configuration space to momentum space) and ˇ is the inverse of ˆ.

d. The interaction *cutoff Hamiltonian :*

$$(1.7)\quad H_I(v_\sigma) = \lambda \int d^3 x\, \psi^*(x)\, (\check{v}_\sigma \star \varphi)(x)\, \psi(x)$$
$$= \lambda \int d^3 p\, d^3 k\, \{ n^*(p-k)\, b^*(k) \times v_\sigma(k)\, (2|k|)^{-1/2}\, n(p) + \text{h. c.} \},$$

where $\star$ denotes convolution.

If $\| v_\sigma |k|^{-1/2} \|_2 < \infty$, then $H_I(v_\sigma)$ is a s. a. operator on $\mathcal{H}$.

e. Formal definition of the Hamiltonian :

$$(1.8)\qquad H(v_\sigma) = H_{0n} + H_{0b} + H_I(v_\sigma) + E_I(v)\, N_n,$$

where

$$(1.9)\qquad E_I(v) = \lambda^2 \int_{|k| \geq 1} d^3 k\, v(k)^2\, (2|k|)^{-1}\, (\Omega(k) + |k|)^{-1}$$

is the *selfenergy renormalization* of the electron.

Remark. — We could also study more general interaction Hamiltonians, e. g.

$$\int d^3 x\, \psi^* (x) : \mathrm{P}\,(\dot{v}_\sigma \star \varphi) : (x)\, \psi\, (x),$$

where P is some positive polynomial and $\| v \mid k \mid^{-1/2} \|_2 < \infty$, or

$$\int d^3 p\, d^3 k\, \{ n^* (p - k)\, b^* (k)\, w\, (p, k)\, n\, (p) + \text{h. c.} \}$$

where w is a positive C^∞ function, $[w\,(p, 0) > 0]$ and

$$\left\| \sup_p w\,(p, \,.\,) \mid k \mid^{-2/3} \right\|_2 < \infty.$$

The reason why we take $\mathrm{H_I}\,(v_\sigma)$ to be linear in φ and v_σ to be independent of the electron momentum is that generalizations of the type mentioned above do *not* seem to change the *IR behaviour* of our system in an essential way. They can be included in the analysis of chapters 1, 2, and parts of chapters 3, 4. We shall indicate wich of our results extend to the Pauli-Fierz model of non relativistic QED with ultraviolet cutoff.

Analysis of the Hamiltonian $\mathrm{H}\,(v_\sigma)$. — Our first task is to define $\mathrm{H}\,(v_\sigma)$ in a rigorous manner as a selfadjoint operator on $\mathcal{H}$. If $\| v_\sigma \mid k \mid^{-1/2} \|_2 < \infty$ then $\mathrm{E_I}\,(v) < \infty$.

The operator $\mathrm{H}\,(v_\sigma)$ is symmetric and conserves the number of bare electrons. For all $\mathrm{Z} < \infty$, $\mathrm{H^Z}\,(v_\sigma) = \mathrm{H}\,(v_\sigma) \restriction \mathcal{H}^{(\mathrm{Z})}$ is a selfadjoint operator which is bounded below by some finite constant $\delta\,(\lambda, \mathrm{Z})$ on $\mathcal{H}^{(\mathrm{Z})}$. This is a simple application of the Kato-Rellich theorem.

We try to choose $v\,(k)$ as in definition (a) and moreover :

$$(1.10)\quad \begin{cases} \text{if } \Omega\,(p) = \dfrac{p^2}{2\,\mathrm{M}} : & v\,(k) \equiv 1; \\ \text{if } \Omega\,(p) = \sqrt{p^2 + \mathrm{M}^2} : & v\,(k) \propto \mid k \mid^{-1/2}, \qquad \text{as} \quad \mid k \mid \to \infty. \end{cases}$$

DEFINITION :

$$(1.11)\quad \tau\,(k) = \begin{cases} \mid k \mid, & \mid k \mid \leq 1 \\ 1, & \mid k \mid \geq 1 \end{cases}, \qquad \mathrm{N}_\tau = \int d^3 k\, b^* (k)\, \tau\,(k)\, b\,(k).$$

THEOREM 1.1 :

(i) *Assume that* $\mathrm{Z} < \infty$, $\Omega\,(p) = \dfrac{p^2}{2\,\mathrm{M}}$ and $v\,(k) \equiv 1$. *Let* $\chi_{0,\,\mathrm{R}}\,(k)$ *be the characteristic function of* $\{ k / \mid k \mid \leq \mathrm{R} \}$. *Then for all real* ε, $\mid \varepsilon \mid < 1$, for all $\sigma \geqq 0$ and for all ζ *such that* $\operatorname{Im} \zeta \neq 0$, *or* $\operatorname{Re} \zeta$ *is small enough,* $(\zeta - \mathrm{H}^{\mathrm{Z}}(\chi_{0,\mathrm{R}}\, g_\sigma) - \varepsilon \mathrm{N}_\tau)^{-1}$ *converges strongly to an operator* $\mathrm{R}(\zeta, \sigma, \varepsilon)$, *as* $\mathrm{R} \to \infty$. $\mathrm{R}(\zeta, \sigma, \varepsilon)$ *is a pseudo-resolvent. Actually it is invertible and the operator*

$$\mathrm{H}^{\mathrm{Z}}(g_\sigma) + \varepsilon \mathrm{N}_\tau = \zeta - \mathrm{R}(\zeta, \sigma, \varepsilon)^{-1}$$

is independent of ζ *and defines a s. a. operator which is bounded from below. On* $\mathrm{D}(\mathrm{R}(\zeta, \sigma, \varepsilon)^{-1})^{\times 2}$ *the equation*

$$\mathrm{H}^{\mathrm{Z}}(g_\sigma) + \varepsilon \mathrm{N}_\tau = \zeta - \mathrm{R}(\zeta, \sigma, 0)^{-1} + \varepsilon \mathrm{N}_\tau$$

holds in the sense of sesquilinear forms (which justifies notation).

(ii) *If* $\Omega(p) = \sqrt{p^2 + \mathrm{M}^2}$ *and* $v(k)$ *is as in definition* (*a*) *and* $v(k) \propto |k|^{-1/2}$, *as* $|k| \to \infty$, *if moreover* $\varepsilon \in (-1, 1)$ *and if* $\operatorname{Re} \zeta$ *is small enough then* $(\zeta - \mathrm{H}^{\mathrm{Z}}(v_\sigma \chi_{0,\mathrm{R}}) - \varepsilon \mathrm{N}_\tau)^{-1}$ *converges in norm to the resolvent of a s. a. operator, also denoted by* $\mathrm{H}^{\mathrm{Z}}(v_\sigma) + \varepsilon \mathrm{N}_\tau$, *as* $\mathrm{R} \to \infty$, *for all* $\sigma \geqq 0$. $\mathrm{H}^{\mathrm{Z}}(v_\sigma) + \varepsilon \mathrm{N}_\tau$ *is bounded from below, for all* $\mathrm{Z} \in \mathbf{Z}^+$, $\sigma \geqq 0$.

Proof. — *see* II, *a* chap. 1, theorems 1.2 and 1.3.

We have used for the proof of (i) techniques of Nelson, [28], in particular his canonical transformation $e^{\mathrm{T}_{\mathrm{K},\varepsilon}}$,

$$(1.12)\quad \mathrm{T}_{\mathrm{K},\varepsilon} = \int d^3 p\, d^3 k \,\{ n^*(p-k)\, b^*(k)\, \beta_\varepsilon(k)\, \chi_{\mathrm{K},\infty}(k)\, n(p) - \text{h. c.} \},$$

where the functions β_ε and $\chi_{\mathrm{K},\infty}$ can be found in II, *a*, chap. 1. We have studied

$$\mathrm{H}'(\mathrm{K}, \infty, \varepsilon) = e^{\mathrm{T}_{\mathrm{K},\varepsilon}} (\mathrm{H}^{\mathrm{Z}}(g_\sigma) + \varepsilon \mathrm{N}_\tau)\, e^{-\mathrm{T}_{\mathrm{K},\varepsilon}}$$

in the spirit of [28].

We have proven (ii) along the lines of [11].

Let now $\mathrm{H}^{\mathrm{Z}}(\sigma)$ denote either of the Hamiltonians obtained in theorem 1.1. It is obvious that $\mathrm{H}^{\mathrm{Z}}(\sigma)$ commutes with the total momentum operator $\mathrm{P} = \mathrm{P}_n + \mathrm{P}_b$.

We have :

COROLLARY 1.2 :

(i) $\mathrm{H}(\sigma) = \bigoplus_{\mathrm{Z}=0}^{\infty} (\mathrm{H}^{\mathrm{Z}}(\sigma) + \varepsilon \mathrm{N}_\tau)$ *is a selfadjoint operator on* $\mathcal{H}$, *for all* $\varepsilon \in (-1, 1)$.

(ii) *If* $\Omega(p) = \dfrac{p^2}{2\mathrm{M}}$ *and if* $\psi(x, t)$ *and* $\varphi(x, t)$ *are the Heisenberg picture fields corresponding to the Hamiltonian* $\mathrm{H}(\sigma = 0)$ *the field equations* (0.1) *hold in the sense of equations between operator valued tempered distributions.*

Proof :

(i) is a trivial consequence of theorem 1.1.

(ii) has been proven by Cannon, [4], for the case of Nelson's model with massive bosons. His techniques apply to the present case.

Q. E. D.

Remark. — Theorem 1.1 and corollary 1.2 establish *Theorem A* of chapter 0.

Analysis of the Hamiltonian $H^1(\sigma)$. — The operators Q_n and P are selfadjoint on $\mathcal{H}^{(1)}$ and their spectra are equal to $\mathbf{R}^3$.

spec P (= spectrum of P) and spec Q_n are absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^3$ and of infinite multiplicity. We can therefore decompose $\mathcal{H}^{(1)}$ on spec P or spec Q_n in the form of a direct integral :

(A) $\mathcal{H}^{(1)} = \int^{\oplus} d^3 x\, \mathcal{H}_x^{(1)}$, where $x \in \operatorname{spec} Q_n$ is the position of the bare electron. There is a natural isomorphism :

$$\mathcal{H}_x^{(1)} \cong \mathcal{F}_b.$$

If $\theta \in \mathcal{H}^{(1)}$, it has the decomposition :

$$\theta = \{ \theta(x) \mid x \in \mathbf{R}^3,\ \theta(x) \in \mathcal{F}_b \text{ for almost all } x \},$$

(B) $\mathcal{H}^{(1)} = \int^{\oplus} d^3 p\, \mathcal{H}_p^{(1)}$, where $p \in \operatorname{spec} P$ is the total momentum. There is a natural isomorphism : $\mathcal{H}_p^{(1)} \cong \mathcal{F}_b$. If $\theta \in \mathcal{H}^{(1)}$, it has the following decomposition

$$\theta = \{ \theta(p) \mid p \in \mathbf{R}^3,\ \theta(p) \in \mathcal{F}_b \text{ for almost all } p \}, \qquad \ldots.$$

Connection between (A) (configuration space representation, [28]) and (B) (total momentum representation) :

$$(1.13) \quad \begin{cases} \theta(x) = (2\pi)^{-3/2} \int d^3 p\, (e^{i P_n \cdot x}\, \theta)(p) \\ \qquad\;\; = (2\pi)^{-3/2} \int d^3 p\, e^{ipx}\, e^{-i P_b \cdot x}\, \theta(p), \\ \theta(p) = (2\pi)^{-3/2} \int d^3 x\, e^{-ipx}\, e^{i P_b \cdot x}\, \theta(x). \end{cases}$$

Since P commutes with $H^1(\sigma)$, $H^1(\sigma)$ can be decomposed on spec P :

$$H^1(\sigma) = \int^{\oplus} d^3 p\, H_p^1(\sigma),$$

where $H^1_p(\sigma)$ is s. a. and bounded from below [by inf (spec $H^1(\sigma)$)] on $\mathcal{H}^{(1)}_p$, for almost all $p \in \mathbf{R}^3$.

Our goal is now to write down an explicit expression for $H^1_p(\sigma)$, to analyze the properties of its spectrum and to prove *Theorem B* of chapter 0.

In order to avoid confusions we define a concrete isomorphism

$$I_p : \mathcal{F}_b \rightarrow \mathcal{H}^{(1)}_p. \tag{1.14}$$

The following operators are densely defined on $\mathcal{H}^{(1)}_p$ and commute with P, provided f and g are in $L^2(\mathbf{R}^3)$:

$$\begin{cases} B(f) = \int d^3x\, \psi^*(x) \left(\check{f} \star \check{b}\right)(x)\, \psi(x), \\ B^*(g) = (B(g))^*; \qquad B^\# = B \text{ or } B^*. \end{cases} \tag{1.15}$$

Obviously :

$$[B(f), B^*(g)] = (f, g), \qquad [B^\#(f), B^\#(g)] = 0.$$

It is easy to see that there is a state φ_{0p} in $\mathcal{H}^{(1)}_p$ [formally corresponding to the plane wave $n^*(p)\,\varphi_0$] which is cyclic for the *algebra $\mathcal{Q}_{L^2}$ generated by $\{ I, B(f), B^*(g) \mid f, g \text{ in } L^2(\mathbf{R}^3)\}$ and has the properties :

$$\| \varphi_{0p} \|_{\mathcal{H}^{(1)}_p} = 1, \qquad B(f)\,\varphi_{0p} = \vec{0}, \quad \text{for all } f \in L^2(\mathbf{R}^3), \qquad (P\,\varphi_{0p} = p\,\varphi_{0p}).$$

$\mathcal{H}^{(1)}_p$ is the Fock space corresponding to $\mathcal{Q}_{L^2}$, φ_{0p}.

The operators

$$\begin{cases} \Phi(f) = \dfrac{1}{\sqrt{2}} \{ B^*(f) + B(f) \} \\ \text{and} \\ \Pi(f) = \dfrac{i}{\sqrt{2}} \{ B^*(f) - B(f) \} \end{cases} \tag{1.16}$$

are selfadjoint on $\mathcal{H}^{(1)}_p$ if $f \in L^2(\mathbf{R}^3)$ is real valued.

Therefore the corresponding *Weyl operators*

$$\begin{cases} U(f) = \exp i\,\Phi(f), \\ V(f) = \exp i\,\Pi(f). \end{cases} \tag{1.17}$$

are unitary.

The isomorphism I_p is now uniquely defined by the following equations :

$$I_p\,\varphi_0 = \varphi_{0p}, \qquad I_p\, b^\#(f)\, I^*_p = B^\#(f) \quad \text{for all } f \in L^2(\mathbf{R}^3),$$

$$I_p \exp \frac{i}{\sqrt{2}} \{ b^*(f) + b(f) \}\, I^*_p = U(f)$$

and

$$I_p \exp \frac{1}{\sqrt{2}} \{ b(f) - b^*(f) \} I_p^* = V(f)$$

for all real square integrable functions f.

If the index p is irrelevant we write :

$$(1.18) \quad \begin{cases} \psi_0 & \text{instead of } \varphi_{0p}, \\ \mathcal{F}_B & \text{instead of } \mathcal{H}_p^{(1)}, \\ I_{b,B} & \text{instead of } I_p. \end{cases}$$

If A_b is an operator on $\mathcal{F}_b$ then $A_B = I_{b,B} A_b I_{b,B}^*$ denotes the corresponding operator on $\mathcal{F}_B$.

An explicit expression for $H_p^1(\sigma)$. — We want to assume first that the interaction kernel v is such that $\| v \mid k \mid^{-1/2} \|_2 < \infty$. $H_p^1(v_\sigma)$ denotes the corresponding cutoff Hamiltonian on $\mathcal{H}_p^{(1)}$. Using the fact that [according to definition (*a*)] v is real valued and $v(k) = v(-k)$ one can easily check that

$$(1.19) \quad H_p^1(v_\sigma) = \Omega(p - P_B) + H_{0B} + \Phi(v_\sigma \mid k \mid^{-1/2}) + E_1(v),$$

$H_p^1(v_\sigma)$ is of course s. a. and bounded below on $\mathcal{F}_B$.

THEOREM 1.1'. — *If* $\operatorname{Re}\zeta$ *is small enough,* $\varepsilon \in (-1, 1)$ *and* $\sigma \geqq 0$ *then :*

(i) *if* $\Omega(p) = \frac{p^2}{2M}$, $(\zeta - H_p^1(\chi_{0,R}\, g_\sigma) - \varepsilon N_{\tau,B})^{-1}$ *converges strongly to the resolvent of a s. a. operator denoted by* $H_{\sigma,p} + \varepsilon N_{\tau,B}$, *which is bounded from below (as* $R \to \infty$).

(ii) *if* $\Omega(p) = \sqrt{p^2 + M^2}$ *and* $v(k) \propto |k|^{-1/2}$, *as* $|k| \to \infty$ *(*v *fixed), then* $(\zeta - H_p^1(v_\sigma \chi_{0,R}) - \varepsilon N_{\tau,B})^{-1}$ *converges in norm to the resolvent of a s. a. operator again denoted by* $H_{\sigma,p} + \varepsilon N_{\tau,B}$, *which is bounded from below.* (*See* II, *a*, chap. 2.)

DEFINITION :

$$(1.20) \quad \begin{cases} H_{0B,\rho,R} = \int_{\rho \leq |k| \leq R} d^3k\, B^*(k) \mid k \mid B(k), \\ P_{B,\rho,R} = \int_{\rho \leq |k| \leq R} d^3k\, B^*(k)\, k\, B(k), \\ H_{\sigma,p,\rho,R} = \Omega(p - P_{B,\rho,R}) \\ \qquad + H_{0B,\rho,R} + \Phi(v_\sigma \mid k \mid^{-1/2} \chi_{\rho,R}) + E_1(v\,\chi_{\rho,R}); \\ H_{\sigma,p,0,\infty} \equiv H_{\sigma,p} \end{cases}$$

$\chi_{\rho, R}$ is the characteristic function of $\{ k/\rho \leq |k| \leq R \}$ and **e** is an arbitrary unit vector in $\mathbf{R}^3$.

LEMMA 1.3. — *If* $\Omega(p) = \frac{p^2}{2M}$, $v(.) = 1$ *then*

(i) $H_{0B, \rho, R} \leq a (H_{\sigma, p, \rho', R'} + b)$;

(ii) $\pm (\mathbf{e}, P_{B, \rho, R}) \leq \tilde{a} (H_{\sigma, p, \rho', R'} + \tilde{b})$

uniformly in $0 \leq \sigma \leq \frac{1}{2}$, $0 \leq \rho' \leq \rho < R \leq R' \leq \infty$; a, $\tilde{a}$, b *and* $\tilde{b}$ *depend only on* p.

Proofs. — *See* II, *a*, chap. 2, theorem 2.4, corollary 2.5, theorem 2.6.

Remarks. — Since the canonical transformation $e^{T_{K,\varepsilon}}$ [defined in (1.12)] commutes with P, it determines a unitary mapping V : $\mathcal{H}_p^{(1)} \to \mathcal{H}_p^{(1)}$ (independent of p).

Using the transformation V we prove theorem 1.1′ and lemma 1.3 in the spirit of [28].

The proof of theorem 1.1′, (ii) is essentially the same as the one of theorem 1.1, (ii).

We have stated lemma 1.3, since it allows us to decouple IR divergences from ultraviolet divergences in the construction of dressed one electron states without cutoffs. (*See* chap. 2; the decoupling mechanism is clearly explained in [14], *b*, hereafter referred to as II, *b*.)

DEFINITION :

$$E(\sigma, p) = \inf \operatorname{spec} H_{\sigma, p}.$$

In the next section we shall show that $E(\sigma, p)$ is a simple eigenvalue of $H_{\sigma, p}$, provided $\sigma > 0$ and $|p| < \rho_0(\lambda)$, where $\rho_0(\lambda)$ is a constant which can be estimated by $\rho_0(\lambda) \geq (\sqrt{3} - 1) M$ if $\Omega(p) = \frac{p^2}{2M}$, and by $\rho_0(\lambda) = \infty$ if $\Omega(p) = \sqrt{p^2 + M^2}$.

This will establish *Theorem B*, i. e. the existence of one particle shells. It is intuitively clear that the fact that $E(\sigma, p)$ is an eigenvalue of $H_{\sigma, p}$, for $|p| < \rho_0(\lambda)$, must be related to certain basic properties of the energy function $E(\sigma, p)$ of p. These properties and their connection with the existence of one particle shells are investigated in the next section.

1.2. Properties of $E(\sigma, p)$; existence and uniqueness of groundstates for the Hamiltonians $H_{\sigma, p}$; $\sigma > 0$.

Let $E(\sigma, p) = \inf \operatorname{spec} H_{\sigma, p}$ and $H_{\sigma, p}$ either of the Hamiltonians obtained in theorem 1.1′. The proofs of all the following properties of $E(\sigma, p)$ are contained in II, *a*, chap. 3 and hold for all $\sigma \geq 0$.

(i) $E(\sigma, 0) \leqq E(\sigma, p)$, for all p in $\mathbf{R}^3$.

$E(\sigma, p)$ is rotation invariant. Therefore one can fix a unit vector $\mathbf{e}$ in $\mathbf{R}^3$ and put $p = x.\mathbf{e}$. We define

$$E(\sigma, x) = E(\sigma, x.\mathbf{e})$$

$E(\sigma, x)$ is absolutely continous in x. $E(\sigma, x) \downarrow E(\sigma = 0, x) \equiv E(x)$ as $\sigma \downarrow 0$, for all x in $\mathbf{R}$.

(ii) $E(\sigma, x)$ is differentiable in x, except on a set of measure 0 (which is a simple consequence of the fact that it is absolutely continuous in x).

(iii) If $\Omega(p) = \frac{p^2}{2M}$, we define $T_{\sigma,p} = H_{\sigma,p} - \frac{p^2}{2M}$,

$$t(\sigma, p) = \inf \operatorname{spec} T_{\sigma,p} = E(\sigma, p) - \frac{p^2}{2M}. \tag{1.21}$$

Then $t(\sigma, 0) = E(\sigma, 0) \geqq t(\sigma, p)$, for all p in $\mathbf{R}^3$. We put

$$t(\sigma, x) = t(\sigma, x.\mathbf{e});$$

$t(\sigma, x)$ is *concave* in x. Hence $\frac{\partial}{\partial x} t(\sigma, x)$ is monotonically decreasing and $E(\sigma, x)$ is the difference of two functions $\frac{x^2}{2M}$, $-t(\sigma, x)$ which are *convex* in x.

Therefore $\frac{\partial}{\partial x} E(\sigma, x)$ is of *bounded variation.*

(iv) There is a constant $\rho_1(\lambda) > 0$ such that :

$$\left| \frac{\partial}{\partial x} E(\sigma, x) \right| < 1, \quad \text{whereever } \frac{\partial}{\partial x} E(\sigma, x) \text{ exists,}$$

provided $|x| < \rho_1(\lambda)$. Furthermore the following estimates hold :

$$\begin{cases} \rho_1(\lambda) = \infty & \text{if} \quad \Omega(p) = \sqrt{p^2 + M^2}, \\ \text{and} & \\ \rho_1(\lambda) \geqq M & \text{if} \quad \Omega(p) = \frac{p^2}{2M}, \end{cases} \tag{1.22}$$

for all real coupling constants λ.

Remark. — The first part of (1.22) is a consequence of the fact that $|\nabla_p \sqrt{p^2 + M^2}| < 1$, if $|p| < \infty$. The second part of (1.22) is a rather simple consequence of the facts that $E(\sigma, 0) \leqq E(\sigma, x)$, for all x in $\mathbf{R}$ and that $t(\sigma, x)$ is *concave* and

$$E(\sigma, 0) = t(\sigma, 0) = \max_x t(\sigma, x).$$

For details *see* II, *a*, chap. 3, theorems 3.4 and 3.5.

Finally, for all ε in the interval (0, 1], there is a constant $c_\varepsilon(\lambda) > -\infty$ such that

$$(1-\varepsilon)\,|x| + c_\varepsilon(\lambda) \leqq E(\sigma, x) \leqq E(\sigma, 0) + |x|.$$

(v)

$$\inf_{|k|\geqq\rho} (E(\sigma, p-k) + |k| - E(\sigma, p)) \equiv \Delta(\sigma, p, \rho) > 0, \tag{1.23}$$

where $\Delta(\sigma, p, \rho) \geqq \Delta(p, \rho)$, uniformly in $\sigma \geqq 0$, and $\Delta(p, \rho)$ is *positive*, for all $\rho > 0$, if $|p| < \rho_0(\lambda)$.

Here $\rho_0(\lambda)$ is some constant estimated by $\rho_0(\lambda) \geqq (\sqrt{3}-1)\,\rho_1(\lambda)$ [which follows easily from (i) and (iv)].

Hence :

$$\rho_0(\lambda) = \infty \qquad \text{if} \quad \Omega(p) = \sqrt{p^2 + M^2},$$

and

$$\rho_0(\lambda) \geqq (\sqrt{3}-1)\,M \qquad \text{if} \quad \Omega(p) = \frac{p^2}{2M}.$$

In the following we shall keep λ fixed.

DEFINITION :

$$\rho_0 = \sup \left\{ |p| \,\middle|\, \Delta(\sigma = 0, p, \rho) > 0, \text{ for all } \rho > 0 \right\}.$$

It is easy to show that (if $|p| < \rho_0$) there is a $\sigma_p > 0$ such that $\Delta(\sigma, p, \rho) > 0$, for all $\rho > 0$ and all $\sigma \leqq \sigma_p$.

Remark. — We are able to show that (besides theorems 1.1 and 1.1′) properties (i)-(v) hold for the Pauli-Fierz model of non relativistic QED with an ultraviolet cutoff, as well.

DEFINITION. — Let

$$K_\sigma = \{ k/|k| \geqq \sigma \} \qquad \text{and} \qquad K_\sigma^{(c)} = \{ k/|k| \leqq \sigma \}.$$

We define :

$$\mathcal{H}_p^{(1)}(K_\sigma^{(c)}) \equiv \mathcal{F}_B(K_\sigma^{(c)}) = \bigoplus_{m=0}^{\infty} L^2(K_\sigma^{(c)}, d^3k)^{sm}.$$

It is obvious that $\mathcal{F}_B(K_\sigma^{(c)})$ reduces $H_{\sigma,p}$ and

$$\mathcal{F}_B = \mathcal{F}_B(K_\sigma)\,(s)\,\mathcal{F}_B(K_\sigma^c).$$

We are now prepared to state the main theorem of this section

THEOREM 1.4. — *Let* $H_{\sigma,p}$ *be either of the Hamiltonians obtained in theorem* 1.1′, *let* σ *be positive and* $|p| < \rho_0(\lambda)$ (*or* $0 < \sigma \leq \sigma_p$, *and* $|p| < \rho_0$). *Then*

(i)

$$E(\sigma, p) = \inf \operatorname{spec}(H_{\sigma,p} \upharpoonright \mathcal{F}_B(K_\sigma)),$$

$\operatorname{spec}(H_{\sigma,p} \upharpoonright \mathcal{F}_B(K_\sigma)) \cap [E(\sigma, p), E(\sigma, p) + \Delta(\sigma, p, \sigma))$ *consists of isolated eigenvalues of finite multiplicity.*

(ii) $E(\sigma, p)$ *is a* simple *eigenvalue of* $H_{\sigma,p}$; *the corresponding ground-state* $\psi_1(\sigma, p) \in \mathcal{F}_B(K_\sigma)$ *is therefore unique. There is a choice of the phase of* $\psi_1(\sigma, p)$ *such that*

$$(\psi_1(\sigma, p), \varphi_{0p}) \equiv (\psi_1(\sigma, p), \psi_0) > 0.$$

Remark. — It is an immediate corollary of this inequality that

$$(1.24) \quad \psi_1(\sigma, p) = \int_\Gamma d\zeta\,(\zeta - H_{\sigma,p})^{-1}\psi_0 \left\| \int_\Gamma d\zeta\,(\zeta - H_{\sigma,p})^{-1}\psi_0 \right\|^{-1},$$

where Γ is a circle in the complex plane around $E(\sigma, p)$ not containing any other point of $\operatorname{spec}(H_{\sigma,p} \upharpoonright \mathcal{F}_B(K_\sigma))$. We shall choose the diameter of Γ, $d(\Gamma)$, to be smaller than $\Delta(\sigma, p, \sigma)$.

The vector $\psi_1(\sigma, p)$ is called a dressed one electron state (DES) of momentum p (with IR cutoff σ).

Proof of theorem 1.4. — *See* II, *a* theorem 2.7, theorem 3.5. (*See* also Glimm and Jaffe, [17], where some of the essential techniques are developped.)

Obviously theorem 1.4 establishes *Theorem B* of chapter 0.

Remark. — Theorem 1.4, (i) [for all $|p| \leq (\sqrt{3} - 1)M$] and theorem 1.4, (ii) (for $p = 0$) hold for the Pauli-Fierz model of non relativistic QED with ultraviolet cutoff.

We now want to study the properties of the wave functions of DES. A convenient tool for this analysis is formula (1.24) together with the following :

Pull-through formula (Schweber, [34], p. 359, Glimm and Jaffe, [19]; II, *a*, chap. 1 and 3).

Let

$$R_{\sigma,q}(\zeta) = (\zeta - H_{\sigma,q})^{-1}.$$

Suppose that $\operatorname{Re}\zeta < E(\sigma, 0)$ or $\operatorname{Im}\zeta \neq 0$, that ψ is in the domain of $B(k)$ and that $B(k)\psi$ is strongly continuous in k.

Then $B(k) R_{\sigma,p}(\zeta) \psi$ exists for all $k \neq 0$ and

$$(1.25) \quad B(k) R_{\sigma,p}(\zeta) \psi = R_{\sigma,p-k}(\zeta - |k|) B(k) \psi + R_{\sigma,p-k}(\zeta - |k|) \xi_\sigma(k) R_{\sigma,p}(\zeta) \psi$$

for all $\sigma \geq 0$. Here

$$\xi_\sigma(k) = \lambda v_\sigma(k) (2|k|)^{-1/2}.$$

Proof. — Formally this follows directly from

$$B(k) \Omega(p - P_B) = \Omega(p - k - P_B) B(k),$$
$$B(k) H_{0B} = (H_{0B} + |k|) B(k),$$
$$[B(k), H_I(v_\sigma)] = \xi_\sigma(k),$$

whence

$$B(k) H_{\sigma,p} = (H_{\sigma,p-k} + |k|) B(k) + \xi_\sigma(k).$$

The details are left to the reader [but *see* also II, *a* : chap. 1, sect. 1.4; chap. 3, (3.34)].

We now apply the pull-through formula (1.25) to formula (1.24) :

$$B(k) \psi_1(\sigma, p) = \int_\Gamma d\zeta\, R_{\sigma,p-k}(\zeta - |k|) \xi_\sigma(k) R_{\sigma,p}(\zeta) \psi_0 \times \left\| \int_\Gamma d\zeta\, R_{\sigma,p}(\zeta)\, \psi_0 \right\|^{-1}.$$

Because of property (v) of $E(\sigma, p)$ (which holds, since $|p| < \rho_0$, $\sigma \leq \sigma_p$) and since $\operatorname{Re} \zeta < E(\sigma, p) + \Delta(\sigma, p, \sigma)$, for all $\zeta \in \Gamma$, $R_{\sigma,p-k}(\zeta - |k|)$ is holomorphic (in norm) in ζ on $\{\zeta / |\zeta - E(\sigma, p)| \leq d(\Gamma)\}$, provided $|k| \geq \sigma$, i. e. for all $k \in \operatorname{supp} \xi_\sigma$.

Thus :

$$(1.26) \quad B(k) \psi_1(\sigma, p) = R_{\sigma,p-k}(E(\sigma, p) - |k|) \xi_\sigma(k) \psi_1(\sigma, p).$$

We want to generalize this equation and calculate

$$\prod_{i=1}^{m} B(k_i) \psi_1(\sigma, p).$$

DEFINITION :

$$k^m = (k_1, \ldots, k_m), \qquad k_\Sigma = \sum_{i=1}^{m} k_i,$$

$\pi \in \gamma_m$ is an arbitrary permutation of $(1, \ldots, m)$:

$$k_{\pi,j} = \sum_{i=j}^{m} k_{\pi(i)}, \qquad d_\sigma(p, \pi, j) = \mathrm{E}(\sigma, p) - \sum_{i=j}^{m} | k_{\pi(i)} |.$$

With these definitions one can easily verify by complete induction that

$$(1.27) \quad \prod_{i=1}^{m} \mathrm{B}(k_i)\, \psi_1(\sigma, p) = \sum_{\pi \in \gamma_m} \left\{ \prod_{j=1}^{m} \mathrm{R}_{\sigma, p-k_{\pi,j}}(d_\sigma(p, \pi, j))\, \xi_\sigma(k_{\pi(j)}) \right\} \psi_1(\sigma, p).$$

Now inf spec $\mathrm{H}_{\sigma, p-k_{\pi,j}} = \mathrm{E}(\sigma, p - k_{\pi,j})$,

$$\| \mathrm{R}_{\sigma, p-k_{\pi,j}}(d_\sigma(p, \pi, j)) \| = (\mathrm{E}(\sigma, p - k_{\pi,j}) - d_\sigma(p, \pi, j))^{-1}.$$

There is a $\rho > 0$ and an $\mathrm{A}(p) < 1$ such that

$$| \mathrm{E}(\sigma, p - k_{\pi,j}) - \mathrm{E}(\sigma, p) | \leq \mathrm{A}(p) | k_{\pi,j} | \leq \mathrm{A}(p) \left\{ \sum_{i=j}^{m} | k_{\pi(i)} | \right\}$$

if $| p | < \rho_0$ and $| k_{\pi,j} | < \rho$ [*see* property (iv)].

Thus :

$$(\mathrm{I}) \qquad \mathrm{E}(\sigma, p - k_{\pi,j}) - d_\sigma(p, \pi, j) \geq (1 - \mathrm{A}(p)) \left\{ \sum_{i=j}^{m} | k_{\pi(i)} | \right\}.$$

It follows from property (i) of $\mathrm{E}(\sigma, p)$ that there is an $\mathrm{R} < \infty$ such that

$$(\mathrm{II}) \qquad \mathrm{E}(\sigma, p - k_{\pi,j}) - d_\sigma(p, \pi, j) \geq (1 - \mathrm{A}(p)) \left\{ \sum_{i=j}^{m} | k_{\pi(i)} | \right\}$$

if $| k_{\pi,j} | \geq \mathrm{R}$.

If $\rho \leq | k_{\pi,j} | \leq \mathrm{R}$, then :

$$(\mathrm{III}) \qquad \mathrm{E}(\sigma, p - k_{\pi,j}) - d_\sigma(p, \pi, j) \geq \frac{\Delta(p, \rho)}{\mathrm{E}(\sigma, p) - \mathrm{E}(\sigma, 0) + \Delta(p, \rho)} \left\{ \sum_{i=1}^{m} | k_{\pi(i)} | \right\}$$

which follows from property (v) of $\mathrm{E}(\sigma, p)$, (1.23), (and the spectral theorem).

Clearly (I), (II) and (III) imply that, given $|p| < \rho_0$, there is a $D(p) < \infty$ such that

$$\text{(IV)} \qquad (E(\sigma, p - k_{\pi,j}) - d_\sigma(p, \pi, j))^{-1} \leq D(p) \left(\sum_{i=j}^{m} |k_{\pi(i)}| \right)^{-1}$$

uniformly in $0 \leq \sigma \leq \frac{\sigma_p}{2}$.

But

$$\sum_{\pi \in \gamma_m} \left\{ \prod_{j=1}^{m} \left(\sum_{i=j}^{m} |k_{\pi(i)}| \right)^{-1} \right\} = \prod_{i=1}^{m} |k_i|^{-1}.$$

[Observe that

$$|k_1|^{-1} (|k_1| + |k_2|)^{-1} + |k_2|^{-1} (|k_2| + |k_1|)^{-1} = |k_1|^{-1} |k_2|^{-1};$$

complete the proof by complete induction.]

We have thus proven the following :

LEMMA 1.5. — *For all* $0 < \sigma \leq \frac{\sigma_p}{2}$, $|p| < \rho_0$,

$$\left\| \prod_{i=1}^{m} B(k_i) \psi_1(\sigma, p) \right\|_{\mathcal{F}_B} \leq \left(\sum_{\pi \in \gamma_m} \prod_{j=1}^{m} \| R_{\sigma, p - k_{\pi,j}} (d_\sigma(p, \pi, j)) \| \right) \times \prod_{i=1}^{m} \xi_\sigma(k_i) \| \psi_1(\sigma, p) \|_{\mathcal{F}_B}$$
$$\leq D(p)^m \prod_{i=1}^{m} \xi_\sigma(k_i) |k_i|^{-1}.$$

Since the m particle wave function of $\psi_1(\sigma, p)$ is given by $\frac{1}{\sqrt{m!}} \left(\psi_0, \prod_{i=1}^{m} B(k_i) \psi_1(\sigma, p) \right)_{\mathcal{F}_B}$, lemma 1.5 yields obvious estimates on the m particle wave function of $\psi_1(\sigma, p)$.

Finally we want to analyze $\psi_1(\sigma, p)$ as an $\mathcal{F}_B$-valued function of p, for all $\sigma > 0$. Our results can be summarized in the following

LEMMA 1.6. — *For all* $\sigma > 0$, $E(\sigma, p)$ *is holomorphic in* p *in some* σ*-dependent complex neighbourhood of*

$$M_{\rho_0(\lambda)} = \{ p \in \mathbf{R}^3 \mid |p| < \rho_0(\lambda) \},$$

$$(1.28) \qquad \nabla_p E(\sigma, p) = (\psi_1(\sigma, p), (\nabla_p H_{\sigma,p}) \psi_1(\sigma, p))$$
$$= (\psi_1(\sigma, p), (\nabla_p \Omega(p - P_B)) \psi_1(\sigma, p)).$$

The $\mathcal{F}_B$*-valued function* $\psi_1(\sigma, p)$ *is strongly holomorphic in* p *on some* σ*-dependent neighbourhood of* $M_{\rho_0(\lambda)}$.

Proof. — We apply standard analytic perturbation theory : $E(\sigma, p)$ is an isolated point in spec $(H_{\sigma,p} \upharpoonright \mathcal{F}_B(K_\sigma))$ and it is a simple eigenvalue of $H_{\sigma,p}$, for all $|p| < \rho_0(\lambda)$, $\sigma > 0$.

The resolvent $R_{\sigma,q}(c)$ (where c is a real number not contained in spec $H_{\sigma,p}$) has a norm convergent power series expansion in $(q - p)$ with some finite c-dependent convergence radius. [Since $\{R_{\sigma,q}(c)\}_{q \in \mathbf{R}^3}$ is a family of s. a. operators, it is of type A in the sense of Kato.] The lemma now follows from results of analytic perturbation theory.

A remark concerning (1.28) :

$$E(\sigma, q) \approx E(\sigma, p) + (\psi_1(\sigma, p), (H_{\sigma,q} - H_{\sigma,p})\psi_1(\sigma, p)),$$

for small $|p - q|$. Actually :

$$E(\sigma, q) = E(\sigma, p) + (\psi_1(\sigma, p), (H_{\sigma,q} - H_{\sigma,p})\psi_1(\sigma, p)) + O(|p - q|^2)$$

whence (1.28).

Q. E. D.

More details concerning the proof of lemma 1.6 are given in II, *a*, chap. 3, theorem 3.6, (3.45).

Remark. — Lemma 1.5 has an obvious generalization to the case where $\psi_1(\sigma, p)$ is replaced by

$$\frac{\partial^{|m|}}{(\partial p^1)^{m_1}(\partial p^2)^{m_2}(\partial p^3)^{m_3}}\psi_1(\sigma, p)$$

where p^i is the i-th component of $p \in \mathbf{R}^3$ and $|m| = \sum_{i=1}^{3} m_i$. (*See* II, *a*, chap. 3.)

This result, lemma 1.5 and lemma 1.6 have interesting applications in chapter 4 (Haag-Ruelle theory for $\sigma > 0$).

Chapter 2

ALGEBRAIC PRELIMINARIES; SOLUTION OF A SIMPLIFIED MODEL; ALGEBRAIC REMOVAL OF THE IR CUTOFF IN THE DES

In this chapter we want to define a suitable, norm closed, selfadjoint subalgebra $\mathfrak{A} \subset B(\mathcal{F}_B)$. We shall study the family of states

$$\omega_{\sigma,p}(A) = (\psi_1(\sigma, p), A\psi_1(\sigma, p))_{\mathcal{F}_B}, \qquad A \in \mathfrak{A}, \qquad \sigma > 0.$$

We shall choose $\mathfrak{A}$ such that :

The family of states $\{\omega_{\sigma,\rho}\}_{\sigma>0}$ on $\mathfrak{A}$ has reasonable topological properties with respect to the w^* topology on the dual $\mathfrak{A}^*$ of $\mathfrak{A}$; $\mathfrak{A}$ contains sufficiently many operators of physical significance such as $\{e^{itH_{0B,\rho,R}}\}_{t\in\mathbf{R}}$ $(0<\rho<R<\infty)$.

In section 2.1 we summarize some rather well known algebraic preliminaries. In section 2.2 we solve a simplified model which has some predictive power and motivates our procedure in section 2.3, where the family of states $\{\omega_{\sigma,\rho}\}_{\sigma>0}$ is studied and some properties of the accumulation points of $\{\omega_{\sigma,\rho}\}_{\sigma>0}$ are derived.

2.1. Algebraic preliminaries

In (1.17) we have introduced the Weyl operators $U(f)$ and $V(g)$.

DEFINITION :

$$W(f, g) = \exp i\,(\Phi(f) + \Pi(g)),$$

where $\Phi(f)$ and $\Pi(g)$ are defined in (1.16) and f and g are real valued, square integrable functions.

The following well known relations hold :

$$(2.1)\quad \begin{cases} U(f)\,V(g) = V(g)\,U(f)\,e^{-i(f,g)}, \\ W(f, g)\,W(f_1, g_1) = W(f+f_1, g+g_1)\,e^{-\frac{i}{2}\{(f,g_1)-(f_1,g)\}}, \\ W(f, g)^* = W(-f, -g), \qquad W(0, 0) = I. \end{cases}$$

If $\mathcal{D}$ is any space of test functions in $L^2(\mathbf{R}^3)$ then $\mathcal{D}_r$ denotes the real part of $\mathcal{D}$ and $D = \mathcal{D}_r \times \mathcal{D}_r$.

DEFINITION. — $\Delta(D)$ is the *algebra generated algebraically by the operators $\{W(f, g) \mid [f, g] \in D\}$.

We shall need the following test function spaces :

$$(a)\quad \mathcal{S}_0(\mathbf{R}^3) = \{f \in \mathcal{S}(\mathbf{R}^3) \mid f(k) = 0, \text{ if } |k| \leq \sigma_0(f),\ \sigma_0(f) > 0\},$$
$$S_0 = \mathcal{S}_{0r}(\mathbf{R}^3)^{\times 2}.$$

(b) We define

$$(2.2)\quad \begin{cases} K_{\rho,R} = \{k \mid \rho \leq |k| \leq R;\ 0<\rho<R\leq\infty\}, \\ K^c_{\rho,R} = \overline{\mathbf{R}^3 \setminus K_{\rho,R}}, \\ \mathcal{H}_{\rho,R} = L^2(K_{\rho,R}, d^3k) \quad \text{and} \quad \mathcal{H}^{\perp}_{\rho,R} = L^2(K^c_{\rho,R}, d^3k). \end{cases}$$

(*c*) Furthermore :

$$K_\nu = K_{2^{-\nu}, 2^{-\nu+1}} \quad \text{if } \nu = 1, 2, 3, \ldots, \qquad \text{and} \qquad K_0 = K_{1,\infty};$$

$$\mathfrak{V}_\nu = L^2 (K_\nu, d^3 k), \qquad \mathfrak{V} = \bigcup_{\nu=0}^{\infty} \mathfrak{V}_\nu, \qquad \text{and} \qquad V = \mathfrak{V}_r^{\times 2}.$$

All the operator algebras Δ (D) defined so far are subalgebras of B ($\mathcal{F}_B$). However one can abstract from their representations as subalgebras of B ($\mathcal{F}_B$) and just preserve their algebraic structure determined by (2.1). One can define a C* norm on Δ (D) :

For all $A \in \Delta$ (D), put $\| A \| = \sup_\omega \sqrt{\omega (A^* A)}$, where ω varies over the states on Δ (D).

The closure of Δ (D) in this norm is a C* algebra denoted by $\overline{\Delta (D)}$.

π_F denotes the Fock representation of Δ (D) as a subalgebra of B ($\mathcal{F}_B$). The closure of $\pi_F (\Delta (D))$ in the operator norm of B ($\mathcal{F}_B$) is isometrically isomorphic to $\overline{\Delta (D)}$. Therefore we may write for notational simplicity $\overline{\Delta (D)}$ for both, the abstract C* algebra and its representation on $\mathcal{F}_B$, W (*f*, *g*) for both, the element of $\overline{\Delta (D)}$ and its representative in B ($\mathcal{F}_B$).

For these and the following facts we recommend ([26], [37]).

We have

$$\overline{\Delta (V)} = \bigotimes_{\nu=0}^{\infty} \overline{\Delta (V_\nu)}$$

is non separable and *simple.*

$\overline{\Delta (V_{\rho, R})}$ acts irreducibly on

$$\mathcal{F}_B (K_{\rho, R}) = \bigoplus_{m=0}^{\infty} L^2 (K_{\rho, R}, d^3 k)^{(s) m}.$$

$\overline{\Delta (V_\nu)}$ acts irreducibly on $\mathcal{F}_B (K_\nu)$.

If $\mathfrak{B}$ is some algebra of operators on a Hilbert space $\mathfrak{B}'$ denotes its commutant and $\mathfrak{B}''$ its double commutant. (If $\mathfrak{B}$ is selfadjoint) $\mathfrak{B}'$ and $\mathfrak{B}''$ are von Neumann algebras.

DEFINITION :

$$\mathfrak{A}_{\rho, R} = \pi_F \left(\overline{\Delta (V_{\rho, R})}\right)'' \cong B (\mathcal{F}_B (K_{\rho, R})).$$

Clearly $\mathfrak{A}'_{\rho, R} = \pi_F \left(\overline{\Delta (V^{\perp}_{\rho, R})}\right)'' \cong B (\mathcal{F}_B (K^c_{\rho, R}))$, ... $\mathfrak{A}_{\rho, R}$ and $\mathfrak{A}'_{\rho, R}$ are factors of type I_∞, hence of type $I_{\infty, \infty}$.

PROPOSITION. — *Isomorphisms between factors of type* $I_{\infty, \infty}$ *are spatial (i. e. " unitarily implementable ").*

DEFINITION :

$$\mathring{\mathfrak{A}} = \bigcup_{0<\rho<R<\infty} \mathfrak{A}_{\rho,R}, \qquad \mathfrak{A} = \overline{\mathring{\mathfrak{A}}}.$$

Furthermore $\mathfrak{A}_\infty = \overline{\bigcup_{0<\rho<R\leq\infty} \mathfrak{A}_{\rho,R}}$, where the closures can be taken in the norm of $B(\mathcal{F}_B)$. $\mathfrak{A}_\infty$ contains the simple C* algebra $\overline{\Delta(V)}$. If $W(f, g) \in \mathfrak{A}$, then $f(k = 0) = g(k = 0) = 0$; etc.

Infinite tensor products [39] :

DEFINITION :

$$\hat{\mathcal{H}} = \bigotimes_{\nu=0}^{\infty} \mathcal{F}_B(K_\nu),$$

$\hat{\mathcal{H}}$ is called complete tensor product space (or complete direct product space; for short CDPS). Clearly $\hat{\mathcal{H}}$ is non separable. $\hat{\mathcal{H}}$ contains uncountably many orthogonal subspaces which are representation spaces for *inequivalent, irreducible* representations π of $\overline{\Delta(V)}$ such that

$$\pi\left(\overline{\Delta(V_\nu)}\right)'' \cong \pi_F\left(\overline{\Delta(V_\nu)}\right)'' \equiv \mathfrak{A}_\nu \qquad \text{(is type } I_{\infty,\infty}),$$

for all ν.

$$\pi(\Delta(V))'' = \pi\left(\bigcup_{N=0}^{\infty} \bigotimes_{\nu=0}^{N} \mathfrak{A}_\nu\right)'' = \bar{\bigotimes}_{\nu=0}^{\infty\, \zeta_\pi} \mathfrak{A}_\nu$$

where $\bar{\otimes}$ denotes tensor product of von Neumann algebras; ([33], [9], p. 24-26) and ζ_π is the so called product reference vector; [39.3]. Representations π of this type are called " infinite direct product representations " (IDPR) and the corresponding representation spaces in $\hat{\mathcal{H}}$ " incomplete direct product spaces " (IDPS).

DEFINITION. — Let α be an automorphism of $\overline{\Delta(V)}$ and ω a state on $\overline{\Delta(V)}$, then $\alpha^* \circ \omega(A) = \omega(\alpha(A))$, for all A in $\overline{\Delta(V)}$ defines a mapping of the dual $\overline{\Delta(V)}^*$ of $\overline{\Delta(V)}$ into itself. We call α (α^*) (locally) normal with respect to $\overline{\Delta(V_\nu)}$ if α can be extended to an ultra-weakly continuous automorphism of $\mathfrak{A}_\nu$.

2.2. **A simplified model**

We want to study the simplified Hamiltonians

$$H^{s}_{\sigma,p} = \frac{1}{2M} p^2 + H_{0B} - \frac{1}{M}(p, P_B) + \lambda\, \Phi(v_\sigma \mid k\mid^{-1/2})$$

where the interaction kernel v is such as in definition a, section 1.1, and $\| v \,|k|^{-1/2} \|_2 < \infty$; $|p| < M$. Then :

$$H^s_{\sigma,p} = H^1_p(v_\sigma) - \frac{1}{2M} P^2_B - E_1(v).$$

The following facts about $H^s_{\sigma,p}$ are easily proven. (*See* e. g. Schweber, [34], p. 339).

(I) $H^s_{\sigma,p}$ is s. a. and bounded below and has a unique ground-state $\psi^s_1(\sigma, p) = V^s_{\sigma,p}\,\psi_0$ corresponding to the simple eigenvalue

$$E^s(\sigma, p) = \frac{1}{2M} p^2 - \lambda^2 \int d^3 k \,|v_\sigma(k)|^2 (2|k|)^{-1} \left(|k| - \frac{1}{M}(k, p)\right)^{-1}.$$

Here

$$V^s_{\sigma,\nu} = \exp i\,\Pi\left(\lambda\, v_\sigma\, |k|^{-1/2} \left(|k| - \frac{1}{M}(k, p)\right)^{-1}\right)$$

and we define :

$$\alpha^s_{\sigma,p}(A) = (V^s_{\sigma,p})^* A V^s_{\sigma,p}$$

for all A in $\overline{\Delta(V)}$.

(II) As $\sigma \downarrow 0$, $\alpha^s_{\sigma,p}(A)$ converges on $\overline{\Delta(V)}$ to an automorphism α^s_p which is normal with respect to $\overline{\Delta(V_\nu)}$, for all $\nu < \infty$ (in the sense of the definition in section 2.1) $(\alpha^s_{\sigma,p})^* \circ (\psi_0 . \psi_0)$ converges w^* to a state ω^s_p on $\overline{\Delta(V)}$ which corresponds to the vector

$$\Omega^s_p = \left(\bigotimes_{\nu=0}^{\infty} V^s_{p,\nu}\right) \psi_0 \in \hat{\mathcal{H}},$$

where

$$V^s_{p,\nu} = \exp i\,\Pi\left(\lambda\, v\, |k|^{-1/2} \left(|k| - \frac{1}{M}(k, p)\right)^{-1} \chi_\nu\right)$$

and χ_ν is the characteristic function of K_ν. The vector Ω^s_p is in an IDPS denoted by $\mathcal{H}^{(1)}_{p,\text{ren},s}$ (which is separable). It is called a *generalized coherent state*. The IDPR of $\overline{\Delta(V)}$ on $\mathcal{H}^{(1)}_{p,\text{ren},s}$ is called π^s_p.

(III) If $p \neq q$ ($|p|$ and $|q| < M$) then the IDPR π^s_p and π^s_q are disjoint.

(IV) The groups $e^{itH_{0B}}$, e^{ixP_B}, $e^{itH^s_q}$ ($|q| < M$) can be defined and are strongly continuous and unitary on uncountably many IDPS in $\hat{\mathcal{H}}$, in particular on $\mathcal{H}^{(1)}_{p,\text{ren},s}$; (*see* II, a, chap. 1).

Now

$$H^1_p(v_\sigma) = H^s_{\sigma,p} + \frac{1}{2M} P^2_B + E_1(v).$$

The perturbation $\frac{1}{2M} P_B^2$ is "small" on states with soft (but no hard) bosons. It is however *not* possible to calculate the groundstate of $H_p^1(v_{\sigma=0})$ from the one of H_p^s by means of perturbation theory expansions. These expansions are IR-divergent, except for $p = 0$. Nevertheless the IR properties of the models defined by the Hamiltonians H_p^s, $H_p^1(v)$ and H_p are expected to be qualitatively the same.

Therefore (I)-(IV) suggest the following predictions :

(II') $\{ \omega_{\sigma, p} \restriction \mathfrak{A}_\nu \}_{\sigma>0}$ is an essentially norm compact set in the dual of $\mathfrak{A}_\nu$ [which corresponds to the fact that α_p^s is normal with respect to $\overline{\Delta(V_\nu)}$, for all $\nu = 0, 1, 2, \dots$].

The states $\{ \omega_{\sigma, p} \restriction \overline{\Delta(V)} \}_{\sigma>0}$ converge to a state ω_p on $\overline{\Delta(V)}$, as $\sigma \downarrow 0$. The corresponding cyclic vector Ω_p is a unique groundstate for H_p. It defines an IDPR π_p of $\overline{\Delta(V)}$ which is unitarily equivalent to the one defined by a coherent state in $\hat{\mathcal{H}}$, (if $|p| < \rho_0$).

(III') If $p \neq q$ (and $|p|$ and $|q| < \rho_0$) π_p and π_q are *disjoint, irreducible* representations of $\overline{\Delta(V)}$.

(IV') The Hilbert space $\mathcal{H}_{p,\text{ren}}^{(1)}$ corresponding to $(\Omega_p, \overline{\Delta(V)})$ is a separable space and $e^{itH_{0B}}$, e^{ixP_B}, ... extend to unitary groups $e^{it\hat{H}_{0B}}$, $e^{ix\hat{P}_B}$, ... on $\mathcal{H}_{p,\text{ren}}^{(1)}$, etc.

2.3. Algebraic removal of the infrared cutoff σ in DES

In this section we summarize results of a rather technical character concerning the existence of DES in the limit $\sigma = 0$. Most of the proofs are given in II, *b*, chap. 1.

All results are independent of whether

$$\Omega(p) = \frac{p^2}{2M} \qquad \text{or} \qquad \Omega(p) = \sqrt{p^2 + M^2}.$$

1° (*See* II, *b*, chap., 1 section 1.1; [18]).

In the following p is a fixed momentum, $|p| < \rho_0$, $(\sigma \leq \sigma_p)$. We want to consider the nets $\{ \omega_{\sigma, p} \restriction \mathfrak{A}_{\rho, R} \mid 0 < \rho < R < \infty \}_{\sigma>0}$ contained in $\mathfrak{A}_{\rho, R}^*$, where $\omega_{\sigma, p} \restriction \mathfrak{A}_{\rho, R}$ is the restriction of $\omega_{\sigma, p}$ to $\mathfrak{A}_{\rho, R}$ and is a normal state on $\mathfrak{A}_{\rho, R}$, for all $\sigma > 0$.

We show in II, *b*, chap. 1, that the operator

$$C_{\rho, R} = \int d^3 y \, d^3 z \, \check{B}^*(y) \, w(y, z) \, \check{B}(z) \geq I,$$

where

$$w(y, z) = \int d^3 x \, \check{\chi}_{\rho, R}(y - x) \, (|x| + 1)^{1/2} \, \check{\chi}_{\rho, R}(x - z),$$

has the properties :

$C_{\rho, R}$ is affiliated with $\mathfrak{A}_{\rho, R}$ and $C_{\rho, R}^{-1} \upharpoonright \mathcal{F}_B(K_{\rho, R})$ is *compact*. One then shows that

$$\omega_{\sigma, p}(C_{\rho, R}) \leq M_{\rho, R} < \infty, \quad \text{uniformly in } \sigma > 0.$$

Therefore $\{ \omega_{\sigma, p} \upharpoonright \mathfrak{A}_{\rho, R} \}_{\sigma > 0} \subset \mathfrak{A}^*_{\rho, R}$ is essentially *norm compact* (*see* [18]; since $\mathfrak{A}_{\rho, R}$ is a von Neumann algebra, $\mathfrak{A}^*_{\rho, R}$ is a Banach space !).

Thus if $\{ \sigma_i \}_{i=0}^{\infty}$ is an arbitrary sequence converging to 0 there is a subsequence $\{ \sigma_{ik} \}_{k=0}^{\infty}$ converging to 0 such that $\{ \omega_{\sigma_{ik}, p} \upharpoonright \mathfrak{A}_{\rho, R} \}_{k=0}^{\infty}$ converges to a normal state $\omega_{p, \rho, R}$ on $\mathfrak{A}_{\rho, R}$.

Therefore there is a density matrix $\Sigma^2_{p, \rho, R}$ in $\mathfrak{A}_{\rho, R}$ such that :

$$\omega_{p, \rho, R}(A) = \operatorname{Tr}(\Sigma^2_{p, \rho, R} A), \quad \text{for all } A \in \mathfrak{A}_{\rho, R}. \tag{2.3}$$

We now choose $\rho_n = n^{-1}$, $R_n = n + 1$, $n \in \mathbf{N}$. By Cantor's diagonal procedure there is a subsequence $\{ \sigma_{ij} \}_{j=0}^{\infty}$ converging to 0 such that $\{ \omega_{\sigma_{ij}, p} \}_{j=0}^{\infty}$ converges on $\mathfrak{A}_{n^{-1}, n+1}$, for all $n < \infty$. Hence $\{ \omega_{\sigma_{ij}, p} \}_{j=0}^{\infty}$ converges on $\mathring{\mathfrak{A}} = \bigcup_n \mathfrak{A}_{n^{-1}, n+1}$, thus on $\mathfrak{A}$, to a state $\omega_p \in \mathfrak{A}^*$ and $\omega_{p, \rho, R} = \omega_p \upharpoonright \mathfrak{A}_{\rho, R}$ is a normal state on $\mathfrak{A}_{\rho, R}$. $\mathcal{H}^{(1)}_{p, \text{ren}}$ is defined to be the G. N. S. space corresponding to $(\omega_p, \mathfrak{A})$, π_p the G. N. S. representation of $\mathfrak{A}$ on $\mathcal{H}^{(1)}_{p, \text{ren}}$.

The cyclic vector corresponding to $(\omega_p, \mathfrak{A})$ in $\mathcal{H}^{(1)}_{p, \text{ren}}$ is denoted by Ω_p :

$$(\Omega_p, \pi_p(A) \Omega_p)_{\mathcal{H}^{(1)}_{p, \text{ren}}} = \omega_p(A), \quad \text{for all } A \text{ in } \mathfrak{A}.$$

DEFINITION :

$$N_{\rho, R} = \int_{\rho \leq |k| \leq R} d^3 k \, B^*(k) B(k), \tag{2.4}$$

$e^{-tN_{\rho, R}}$ is in $\mathfrak{A}_{\rho, R}$ $(0 < \rho < R < \infty$; ρ is kept fixed). Therefore $\pi_p(e^{-tN_{\rho, R}})$ exists $\pi_p(N_{\rho, R})$ is a positive, s. a. number operator for $\pi_p(\overline{\Delta(V_{\rho, R})})$; [5].

Suppose that $\Omega(p) = \frac{p^2}{2M}$. It is then easy to deduce from lemma 1.3 (i) and (2.3) that $\underset{R \to \infty}{s\text{-lim}} \, \pi_p(e^{-tN_{\rho, R}}) \equiv e^{-t\hat{N}_\rho}$ exists and is a s. a. contraction semi-group on $\mathcal{H}^{(1)}_{p, \text{ren}}$; [if $\Omega(p) = \sqrt{p^2 + M^2}$, the proof is even simpler; *see* II, *b*, chap. 1. One makes use of lemma 1.5].

The infinitesimal generator $\hat{N}_\rho$ is a positive, s. a. number operator for $\pi_p\left(\overline{\Delta(V_{\rho,\infty})}\right)$.

Applying theorems of Dell'Antonio *et al.*, [7], [8], we conclude that ω_p can be extended to a *normal* state on $\mathfrak{A}_{\rho,\infty}$, for all $\rho > 0$, hence ω_p extends to $\mathfrak{A}_\infty$ and therefore to $\overline{\Delta(V)}$.

Since Ω_p is cyclic for $\mathring{\mathfrak{A}}$ and $\omega_p \upharpoonright \mathfrak{A}_{\rho,R}$ is normal, $\mathcal{H}^{(1)}_{p,\mathrm{ren}}$ is a separable Hilbert space. Since $\overline{\Delta(V)}$ is simple $\pi_p\left(\overline{\Delta(V)}\right)$ is faithful. π_p is also a faithful representation of $\mathfrak{A}$.

2° (*See* II, *b*, chap. 1, section 1.2).

We want to verify prediction (IV′) of section 2.2.

DEFINITION :

$$(2.5)\quad H_{q,\rho,R} = \Omega(q - P_{B,\rho,R}) + H_{0B,\rho,R} + \lambda, \Phi(\chi_{\rho,R} \mid k\mid^{-1/2}) + E_1(\chi_{\rho,R}),$$

where $\chi_{\rho,R}$ is the characteristic function of $K_{\rho,R}$ and $H_{0B,\rho,R}$ and $P_{B,\rho,R}$ have been defined in (1.20). We only consider the model with $\Omega(p) = \frac{p^2}{2M}$. The other case is treated in the same way.

Obviously $e^{i(tH_{0B,\rho,R} - xP_{B,\rho,R})}$ and $e^{itH_{q,\rho,R}}$ are strongly continuous unitary groups in $\mathfrak{A}_{\rho,R}$.

We use lemma 1.3 for the limit $R \to \infty$ and lemma 1.5 for the limit $\rho \to 0$ and we use the facts that $\omega_p \upharpoonright \mathfrak{A}_{\rho,R}$ is normal and Ω_p is cyclic with respect to $\mathring{\mathfrak{A}}$ and arrive at

$$\underset{\rho \downarrow 0,\, R \uparrow \infty}{s\text{-lim}}\ \pi_p\left(e^{i(tH_{0B,\rho,R} - xP_{B,\rho,R})}\right) \equiv e^{i\left(t\hat{H}_{0B} - x\hat{P}_B\right)}$$

and

$$\underset{\rho \downarrow 0,\, R \uparrow \infty}{s\text{-lim}}\ \pi_p\left(e^{itH_{q,\rho,R}}\right) \equiv e^{it\hat{H}_q}$$

exist and are strongly continuous unitary groups on $\mathcal{H}^{(1)}_{p,\mathrm{ren}}$ which are in $\pi_p(\mathfrak{A})''$, i. e. $(\hat{H}_{0B}, \hat{P}_B)$ and $\hat{H}_q$ are affiliated with $\pi_p(\mathfrak{A})''$.

Properties of spec $(\hat{H}_{0B}, \hat{P}_B)$ *and* spec $\hat{H}_q$:

(2.6)
- (i) spec $(\hat{H}_{0B}, \hat{P}_B)$ is in the forward cone $\overline{V}_+$.
- (ii) $\hat{H}_q$ is bounded from below by $E(q)$.
- (iii) $E(p)$ is an eigenvalue of $\hat{H}_p$ and Ω_p is an eigenstate of $\hat{H}_p$ corresponding to $E(p)$.

For details of the proofs *see* II, *b*, chap. 1, section 1.2.

3° (*See* II, *b*, chap. 1, section 1.3.)

We establish some properties of the state ω_p on $\mathfrak{A}$ with respect to the unbounded annihilation and creation operators. We prove the convergence of $\{ \omega_{\sigma_{ij}, p} (\mathrm{A}) \}$, as $j \to \infty$, for certain unbounded operators A.

Suppose that s is in **R** and $[f, g]$ is in V. Without danger of confusion the infinitesimal generators of the strongly continuous unitary groups $\pi_p (\mathrm{U} (sf))$ and $\pi_p (\mathrm{V} (sg))$ are denoted by $\Phi (f)$, $\Pi (g)$, respectively, and we define in the same way as on $\mathcal{F}_{\mathrm{B}}$:

$$\mathrm{B}^* (f) = 2^{-1/2} (\Phi (f) - i \, \Pi (f)), \qquad \mathrm{B} (f) = 2^{-1/2} (\Phi (f) + i \, \Pi (f)).$$

For f in $\mathcal{V}$, B (f) and B* (f) are densely defined operators on $\mathcal{H}^{(1)}_{p, \mathrm{ren}}$. If $\{ f_j^{(m)} \} \subset \mathcal{V}$ and $\{ g_i^{(n)} \} \subset \mathcal{V}$ then

$$\mathrm{P} = c_0 + \sum_{1 \leq m+n \leq \mathrm{M}} \prod_{j=1}^{m} \mathrm{B}^* (f_j^{(m)}) \prod_{i=1}^{n} \mathrm{B} (g_i^{(n)})$$

is densily defined on $\mathcal{H}^{(1)}_{p, \mathrm{ren}}$.

These results follow of course from the fact that $\omega_p \restriction \mathfrak{A}_{\rho, \infty}$ is normal, for all $\rho > 0$, and from the definition of $\mathcal{V}$. $\mathfrak{P}_{\mathcal{V}}$ denotes the *algebra generated by all Wick polynomials of the form of P.

THEOREM 2.1 :

(i)

$$\omega_p \left(\prod_{j=1}^{m} \mathrm{B}^* (k_j) \prod_{i=1}^{n} \mathrm{B} (k_i) \right)$$

$$= \prod_{j=1}^{m} \xi (k_j) \prod_{i=1}^{n} \xi (k_i) \left\{ \sum_{\substack{\pi \in \gamma_m \\ \pi' \in \gamma_n}} \omega_p \left(\prod_{j=1}^{m} \hat{\mathrm{R}}_{p - k_{\pi, j}} (d (p, \pi, j)) \right. \right.$$

$$\left. \left. \times \prod_{i=1}^{n} \hat{\mathrm{R}}_{p - k'_{\pi', i}} (d (p, \pi', i)) \right) \right\},$$

where

$$\hat{\mathrm{R}}_q (\zeta) = (\zeta - \hat{\mathrm{H}}_q)^{-1}, \qquad \xi (k) = \lambda \, v (k) (2 \, | \, k \, |)^{-1/2}$$

$$\left[\textit{and } \xi (k) = \lambda (2 \, | \, k \, |)^{-1/2}, \textit{ if } \Omega (p) = \frac{p^2}{2 \mathrm{M}} \right],$$

$$d (p, \pi, j) = \mathrm{E} (p) - \sum_{l=j}^{m} | \, k_{\pi (l)} \, | ;$$

$k_{\pi, j}$ *is as in lemma* 1.5.

(ii) Ω_p *is cyclic for* $\mathfrak{L}_{\mathfrak{V}}$.

Proof :

(i) If $\sigma > 0$, $A \in \mathring{\mathfrak{A}}$, then (1.26) yields

$$\omega_{\sigma, p} (AB (k)) = \xi_\sigma (k) \, \omega_{\sigma, p} (AR_{\sigma, p-k} (E (\sigma, p) - | k |)).$$

If $| k | > 0$, then $\xi_\sigma (k)$ tends to $\xi (k)$ and $(E (\sigma, p) - | k |)$ tends to $E (p) - | k |$, as $\sigma \downarrow 0$.

Since A is in $\mathring{\mathfrak{A}}$, there are numbers ρ' and R′ such that A is in $\mathfrak{A}_{\rho', R'}$.

DEFINITION :

$$H_{\sigma, q, \rho, R} = \Omega (q - P_{B, \rho, R}) + H_{0 B, \rho, R} + \lambda \Phi (g_\sigma \chi_{\rho, R} | k |^{-1/2}) + E_1 (\chi_{\rho, R}),$$
$$R_{\sigma, q, \rho, R} (\zeta) = (\zeta - H_{\sigma, q, \rho, R})^{-1}.$$

We choose $0 < \rho \leq \rho' < R' \leq R < \infty$.

Then the results of section 2.3, 1° imply :

$$(2.7) \qquad \lim_{j \to \infty} \omega_{\sigma_{ij}, q} (AR_{\sigma_{ij}, p-k, \rho, R} (\zeta))$$
$$= \lim_{j \to \infty} \omega_{\sigma_{ij}, p} (AR_{p-k, \rho, R} (\zeta)) = \omega_p (AR_{p-k, \rho, R} (\zeta))$$

[for $A R_{p-k, \rho, R} (\zeta)$ is in $\mathfrak{A}_{\rho, R}$ if $\operatorname{Im} \zeta \neq 0$, or $\operatorname{Re} \zeta < E (0)$].

We have shown in II, *b*, chap. 1, section 1.2 that, given $\varepsilon > 0$, there are numbers $\hat{\rho} (\varepsilon)$, $\hat{R} (\varepsilon)$ such that

$$(2.8) \qquad | \omega_{\sigma, p} (AR_{p-k, \rho, R} (\zeta)) - \omega_{\sigma, p} (AR_{p-k, \rho_1, R_1} (\zeta)) | < \varepsilon$$

if $\rho, \rho_1 < \hat{\rho} (\varepsilon)$, $R, R_1 > \hat{R} (\varepsilon)$, $A \in \mathfrak{A}_{\rho', R'}$ is fixed, and uniformly in $\sigma > 0$ and in k in an arbitrary fixed compact set.

From section 2.3, 2° it follows that

$$(2.9) \qquad \lim_{\rho \downarrow 0, R \uparrow \infty} \omega_p (AR_{p-k, \rho, R} (\zeta)) = \omega_p \big(A \hat{R}_{p-k} (\zeta)\big).$$

From (2.7), (2.8) and (2.9) we conclude :

$$\lim_{j \to \infty} \omega_{\sigma_{ij}, p} (AR_{\sigma_{ij}, p-k} (\zeta)) = \omega_p \big(A \hat{R}_{p-k} (\zeta)\big).$$

From section 2.3, 1° and the boundedness of $\omega_\sigma (N_{\rho, \infty})$ in σ it follows that

$$\lim_{j \to \infty} \omega_{\sigma_{ij}, p} (AB (k)) = \omega_p (AB (k))$$

in the sense of bounded linear functionals on $\mathfrak{V}$.

Hence :

$$\lim_{i \to \infty} \omega_{\sigma_{ij}, p} (\mathrm{AB}(k)) = \xi(k)\, \omega_p \big(\mathrm{A}\, \hat{\mathrm{R}}_{p-k} (\mathrm{E}(p) - |k|)\big) = \omega_p (\mathrm{AB}(k))$$

if $|k| > 0$.

Since ω_p is cyclic for $\mathring{\mathfrak{A}}$, $\{ \mathrm{A}\, \Omega_p \mid \mathrm{A} \in \mathring{\mathfrak{A}} \}$ is dense in $\mathcal{H}^{(1)}_{p, \mathrm{ren}}$ and therefore :

$$\mathrm{B}(k)\, \Omega_p = \xi(k)\, \hat{\mathrm{R}}_{p-k} (\mathrm{E}(p) - |k|)\, \Omega_p.$$

It is easy to show (II, *b*, chap. 1, section 1.3) that the pull-through formula

$$\mathrm{B}(k)\, \hat{\mathrm{R}}_q(\zeta) = \hat{\mathrm{R}}_{q-k}(\zeta - |k|)\, \mathrm{B}(k) + \hat{\mathrm{R}}_{q-k}(\zeta - |k|)\, \xi(k)\, \hat{\mathrm{R}}_q(\zeta) \tag{2.10}$$

holds on $\mathcal{H}^{(1)}_{p, \mathrm{ren}}$.

But now (i) follows by induction (as in lemma 1.5).

Remarks. — An immediate corollary is

$$\left\| \prod_{i=1}^{m} \mathrm{B}(k_i)\, \Omega_p \right\|_{\mathcal{H}^{(1)}_{p, \mathrm{ren}}} \leq \mathrm{D}(p)^m \prod_{i=1}^{m} \xi(k_i)\, |k_i|^{-1}. \tag{2.11}$$

It is shown in II, *b* that $\hat{\mathrm{R}}_{p-k}(\mathrm{E}(p) - |k|)$ is holomorphic in k in norm in some complex neighbourhood of $\{ k \in \mathbf{R}^3 \setminus \{0\} \}$. Hence the distributions $\omega_p \left(\prod_{j=1}^{m} \mathrm{B}^*(k_j) \prod_{i=1}^{n} \mathrm{B}(k'_i) \right)$ are C^∞ in $k_1 \ldots k_m$, $k'_1 \ldots k'_n$, except on $\{ k'^m, k'^n \mid \text{some } k'_j \text{ or some } k_i \text{ are } 0 \}$.

Proof of (ii). — Since $\pi_p \big(\overline{\Delta(\mathrm{S}_0)}\big)'' = \pi_p(\mathfrak{A})''$ (which is easily shown), Ω_p is cyclic for $\overline{\Delta(\mathrm{S}_0)}$. The estimate (2.11) implies that Ω_p is an entire vector of $\Phi(f)$ and $\Pi(g)$; $[f, g]$ in S_0. From this and the definition of $\mathrm{W}(f, g)$ it can be easily deduced that $\mathrm{W}(f, g)\, \Omega_p$ has a convergent power series expansion in terms of B's and B*'s applied to Ω_p. But this completes the proof of (ii).

Q. E. D.

Remark. — From the equation

$$\omega_p (\mathrm{B}^*(k)\, \mathrm{B}(k)) = \xi(k)^2\, \omega_p \big(\hat{\mathrm{R}}_{p-k} (\mathrm{E}(p) - |k|)^2\big)$$

we conclude that $\omega_p(\mathrm{N}_{\rho, \mathrm{R}})$ does not diverge more than logarithmically as $\rho \downarrow 0$ (for fixed $\mathrm{R} \leq \infty$). We shall prove in the next chapter that it actually does diverge logarithmically and that moreover the probability of finding finitely many (virtual) bosons in the state Ω_p vanisches. Therefore π_p cannot be equivalent to the Fock representation.

LEMMA 2.2 :

(i)

$$\lim_{j\to\infty} \omega_{\sigma_{ij},p}\left(e^{i\left(t\mathrm{H}_{0\mathrm{B}}-x\mathrm{P}_{\mathrm{B}}\right)}\right) = \omega_p\left(e^{i\left(t\hat{\mathrm{H}}_{0\mathrm{B}}-x\hat{\mathrm{P}}_{\mathrm{B}}\right)}\right)$$

uniformly for (t, x) *on bounded sets of* $\mathbf{R}^4$.

Furthermore

$$\lim_{j\to\infty} \omega_{\sigma_{ij},p}((\mathbf{e}, \mathrm{P}_{\mathrm{B}})) = \omega_p((\mathbf{e}, \hat{\mathrm{P}}_{\mathrm{B}})),$$

where $\mathbf{e}$ *is a unit vector in* $\mathbf{R}^3$.

(ii)

$$\lim_{j\to\infty} \omega_{\sigma_{ij},p}(\nabla_p \Omega(p - \mathrm{P}_{\mathrm{B}})) = \omega_p(\nabla_p \Omega(p - \hat{\mathrm{P}}_{\mathrm{B}})). \tag{2.12}$$

Remark. — If $\Omega(p) = \frac{p^2}{2\mathrm{M}}$ (2.12) is an easy consequence of the second part of (i). If $\Omega(p) = \sqrt{p^2 + \mathrm{M}^2}$, $\nabla_p \Omega(p - \mathrm{P}_{\mathrm{B}})$ is a bounded function of P_{B} and then (2.12) is proven in the same way as the first part of (i). This part of (i) is a consequence of lemma 1.5, (2.11), lemma 1.3, (i) and the fact that $\omega_{\sigma_{ij},p}$ converges on $\mathfrak{A}_{\rho,\mathrm{R}}$. The techniques are similar to the ones used in the proof of theorem 2.1, (i). A complete proof of lemma 2.2 is given in II, *b*, chap. 1, section 1.3 and is of no interest for what we are going to do in the following chapters.

COROLLARY 2.3 :

$$\begin{aligned}\lim_{j\to\infty} \nabla_p \mathrm{E}(\sigma_{ij}, p) &= \lim_{j\to\infty} \omega_{\sigma_{ij},p}(\nabla_p \Omega(p - \mathrm{P}_{\mathrm{B}})) \\ &= \omega_p(\nabla_p \Omega(p - \hat{\mathrm{P}}_{\mathrm{B}})) = \omega_p(\nabla_p \hat{\mathrm{H}}_p).\end{aligned} \tag{2.13}$$

Proof. — The first equation follows from lemma 1.6. The second equation is a consequence of lemma 2.2, (ii).

Proof of the third equation :

From lemma 1.3, (ii) we infer that

$$|\Omega(q - \mathrm{P}_{\mathrm{B},\rho,\mathrm{R}}) - \Omega(p - \mathrm{P}_{\mathrm{B},\rho,\mathrm{R}})| \leq \frac{|q - p|}{\mathrm{M}} \tilde{a}(\mathrm{H}_{p,\rho,\mathrm{R}} + \tilde{b}) \tag{2.14}$$

uniformly in $0 < \rho < \mathrm{R} < \infty$.

From the results of 2° it follows that $\Omega(q - \mathrm{P}_{\mathrm{B},\rho,\mathrm{R}}) - \Omega(p - \mathrm{P}_{\mathrm{B},\rho,\mathrm{R}})$ converges strongly to $\Omega(q - \hat{\mathrm{P}}_{\mathrm{B}}) - \Omega(p - \hat{\mathrm{P}}_{\mathrm{B}})$ on $\mathcal{H}^{(1)}_{p,\mathrm{ren}}$, and $(\zeta - \mathrm{H}_{p,\rho,\mathrm{R}})^{-1}$ converges strongly to $\hat{\mathrm{R}}_p(\zeta)$ on $\mathcal{H}^{(1)}_{p,\mathrm{ren}}$ (if $\zeta \notin \operatorname{spec} \hat{\mathrm{H}}_p$). The principle of cutoff independence, [32], and (2.14) now imply that

$$|\Omega(q - \hat{\mathrm{P}}_{\mathrm{B}}) - \Omega(p - \hat{\mathrm{P}}_{\mathrm{B}})| \leq \frac{|q - p|}{\mathrm{M}} \tilde{a}(\hat{\mathrm{H}}_p + \tilde{b}).$$

Hence, if $\frac{|q-p|}{M}\tilde{a} < 1$, the unique Friedrichs extension of $\hat{H}_p + \Omega(q - \hat{P}_B) - \Omega(p - \hat{P}_B)$ is $\hat{H}_q$; moreover $\hat{H}_p$ and the Friedrichs extension of $\hat{H}_p + \Omega(q - \hat{P}_B) - \Omega(p - \hat{P}_B)$ have the same form domain which agrees with the form domain of $\hat{H}_q$.

Therefore

$$\hat{H}_q - \hat{H}_p = \Omega(q - \hat{P}_B) - \Omega(p - \hat{P}_B)$$

holds on the form domain of $\hat{H}_p \left(\text{provided} \frac{|q-p|}{M}\tilde{a} < 1\right)$. Obviously the pair $[\Omega_p, \Omega_p]$ is in the form domain of $\hat{H}_p$. Thus

$$\omega_p(\hat{H}_q - \hat{H}_p) = \omega_p(\Omega(q - \hat{P}_B) - \Omega(p - \hat{P}_B))$$

and

$$|\omega_p(\hat{H}_q - \hat{H}_p)| \leq O(|q-p|).$$

This completes the proof of (2.13).

Q. E. D.

We have now collected the technical results which we need in order to start our hard work and to prove the most interseting results on DES.

CHAPTER 3

PROPERTIES OF THE DES IN THE LIMIT $\sigma = 0$; UNIQUENESS OF THE DES FOR $\sigma = 0$; ABSENCE OF DES IN THE PHYSICAL HILBERT SPACE

In chapter 2 we have constructed a state ω_p on the C* algebra $\mathfrak{A}$. Applying the G. N. S. construction we have obtained a Hilbert space $\mathcal{H}^{(1)}_{p,\text{ren}}$, a representation π_p of $\mathfrak{A}$ and a cyclic vector Ω_p in $\mathcal{H}^{(1)}_{p,\text{ren}}$ which is a groundstate for $\hat{H}_p$ corresponding to the eigenvalue $E(p)$. Ω_p is called a DES of momentum p ($|p| < \rho_0$) without cutoffs.

In this chapter we want to analyze the representations π_p of $\overline{\Delta(V)}$ defined by ω_p and we want to prove ω_p is essentially unique. We recall predictions (II′) and (III′) of section 2.2 as a motivation of our procedure.

We shall restrict our analysis in chapters 3 and 4 to the model with $\Omega(p) = \frac{p^2}{2M}$, since the proof of the basic lemma 3.1 (section 3.1) is only given for this model. However this result is likely to hold for the model with $\Omega(p) = \sqrt{p^2 + M^2}$, as well. In this case all the subsequent results hold for both models.

3.1. Determination of the representations π_p of $\overline{\Delta(V)}$

The key to an analysis of the representations π_p ($|p| < \rho_0$) is the following :

LEMMA 3.1. — *Suppose that* $\Omega(p) = \frac{p^2}{2M}$.

Then there is a rotation invariant set $\mathcal{E} \subset \overline{M}_{\rho_0} = \{ q / |q| \leq \rho_0 \}$ *with the property that* $\overline{M}_{\rho_0} \setminus \mathcal{E}$ *is of Lebesgue measure* 0 *and a sequence* $\{\sigma_l\}_{l=0}^{\infty}$ *converging to* 0 *such that for all p in* $\mathcal{E}$ $\nabla E(p) = \nabla_p E(\sigma = 0, p)$ *exists and*

$$(3.1) \qquad \nabla E(p) = \lim_{l \to \infty} \nabla_p E(\sigma_l, p) = \lim_{l \to \infty} \omega_{\sigma l, p}(\nabla_p \Omega(p - P_B)).$$

The measurable function $\nabla E(q)$ *is Lipshitz at* $q = p$ *in* $\mathcal{E}$, *i. e. there is a constant* $C(p) < \infty$ *such that*

$$(3.2) \qquad |\nabla E(q) - \nabla E(p)| \leq C(p) |q - p| \qquad [\textit{for all } q, |q - p| \leq 1].$$

The *proof* of this lemma is contained in appendix 1.

In the following we always assume that p is in $\mathcal{E}$ and that ω_p is an accumulation point of $\{\omega_{\sigma l, p}\}_{l=0}^{\infty}$ which is known to exist according to section 2.3, 1°.

Combining now (3.1) with lemma 2.2, (ii) and corollary 2.3 we arrive at

$$(3.3) \qquad \nabla E(p) = \omega_p(\nabla_p \Omega(p - \hat{P}_B)) = \omega_p(\nabla \hat{H}_p).$$

One of our final goals in this chapter is

$$\lim_{l \to \infty} \omega_{\sigma l, p} = \omega_p \quad \text{on } \mathfrak{A}.$$

But let us start now with the analysis of the representations

$$\{ \pi_p \mid p \in \mathcal{E} \} \quad \text{of } \overline{\Delta(V)}.$$

THEOREM 3.2. — *We assume that lemma* 3.1 *holds. Then :*

(i) $|\omega_p(B^*(k) B(k)) - \omega_p(B(k))^2|$ *is integrable at* $k = 0$, *i. e.*

$$\int_{|k| \leq 1} d^3 k \, |\omega_p(B^*(k) B(k)) - \omega_p(B(k))^2| < \infty.$$

(ii) *Suppose that*

$$\int_{|k| \leq 1} d^3 k \, | \omega_p (B^* (k) B (k)) - \omega_p (B (k))^2 | < \infty$$

and set

$$\hat{w}_p (k) = - \sqrt{2} \, \omega_p (B (k)) \, \Im (k),$$

where $\Im$ *is a non negative* C^∞ *function and*

$$\Im (k) = \begin{cases} 1, & | k | \leq 1, \\ 0, & | k | \geq 2. \end{cases}$$

Then the representation π_p *of* $\overline{\Delta (V)}$ *defined by the state* ω_p *is quasi equivalent to the one defined by the generalized coherent state*

$$\exp i \, \Pi (\hat{w}_p) \, \psi_0 = \bigotimes_{\nu=0}^{\infty} \exp i \, \Pi (\hat{w}_p \, \chi_\nu) \, \psi_0 \in \hat{\mathcal{H}}.$$

Proof :

(i) Theorem 2.1, (i) tells us that

$$\omega_p (B (k)) = \lambda \, v (k) \, (2 \, | \, k \, |)^{-1/2} \, \omega_p \left(\hat{R}_{p-k} (E (p) - | \, k \, |)\right)$$

and

$$\omega_p (B^* (k) B (k)) = \lambda^2 \, v (k)^2 \, (2 \, | \, k \, |)^{-1} \, \omega_p \left(\hat{R}_{p-k} (E (p) - | \, k \, |)^2\right).$$

Now $| \, \omega_p (B^* (k) B (k)) - \omega_p (B (k))^2 \, | \, \Im (k)^2 \in L^1 (\mathbf{R}^3)$ if

$$| \, \omega_p (B^* (k) B (k)) - \omega_p (B (k))^2 \, | \leq K_p \, | \, k \, |^{-3} \, | \, k \, |^{\varepsilon}$$

for some $\varepsilon > 0$ and $K_p < \infty$. But this holds if

$$(3.4) \quad \left| \omega_p \left(\hat{R}_{p-k} (E (p) - | \, k \, |)^2\right) - \omega_p \left(\hat{R}_{p-k} (E (p) - | \, k \, |)\right)^2 \right| \leq K_p \, | \, k \, |^{-2+\varepsilon}.$$

Actually we shall show that lemma 3.1 implies $\varepsilon = 1$. Since $\hat{H}_{p-k}$ is selfadjoint and bounded from below by $E (p - k)$ on $\mathcal{H}^{(1)}_{p, \mathrm{ren}}$, it has a spectral decomposition :

$$\hat{H}_{p-k} = \int_{E (p-k)}^{\infty} \lambda \, dF_{p-k} (\lambda),$$

where $\{ F_{p-k} (\lambda) \}$ are the spectral projections of $\hat{H}_{p-k}$.

Therefore :

$$(3.5)\quad \left|\omega_p\left(\hat{R}_{p-k}(E(p)-|k|)^2\right)-\omega_p\left(\hat{R}_{p-k}(E(p)-|k|)\right)^2\right|$$

$$\leq \int_{E(p-k)}^{\infty}(E(p)-|k|-\lambda)^{-1}\, d\omega_p(F_{p-k}(\lambda))$$

$$\times\left[(E(p)-|k|-\lambda)^{-1}\right.$$

$$\left.-\int_{E(p-k)}^{\infty}(E(p)-|k|-\lambda')^{-1}\, d\omega_p(F_{p-k}(\lambda'))\right]$$

$$\leq (E(p-k)+|k|-E(p))^{-1}\int_{E(p-k)}^{\infty} d\omega_p(F_{p-k}(\lambda))$$

$$\times\left(E(p)-|k|-\lambda)^{-1}\right.$$

$$\left.-\int_{E(p-k)}^{\infty} d\omega_p(F_{p-k}(\lambda'))(E(p)-|k|-\lambda')^{-1}\right)$$

$$\leq 2(E(p-k)+|k|-E(p))^{-1}\int_{E(p-k)}^{\infty} d\omega_p(F_{p-k}(\lambda))$$

$$\times[(E(p)-|k|-\lambda)^{-1}-(E(p)-|k|-E(p-k))^{-1}]$$

$$= 2(E(p-k)+|k|-E(p))^{-1}\int_{E(p-k)}^{\infty} d\omega_p(F_{p-k}(\lambda))$$

$$\times(\lambda-E(p-k))(E(p)-|k|-\lambda)^{-1}$$

$$\times(E(p)-|k|-E(p-k))^{-1}$$

$$\leq 2(E(p-k)+|k|-E(p))^{-3}\,\omega_p\left(\hat{H}_{p-k}-E(p-k)\right).$$

If $\omega(p)=\dfrac{p^2}{2M}$:

$$\omega_p\left(\hat{H}_{p-k}-E(p-k)\right)$$

$$=\omega_p\left(\hat{H}_{p-k}-\hat{H}_p\right)+E(p)-E(p-k)$$

$$=-\left(k,\ \omega_p\left(\frac{1}{M}(p-\hat{P}_B)\right)\right)+E(p)-E(p-k)+\frac{k^2}{2M}$$

$$=-\left(k,\ \omega_p(\nabla\hat{H}_p)\right)+E(p)-E(p-k)+\frac{k^2}{2M}$$

$$=-(k,\nabla E(p))+E(p)-E(p-k)+\frac{k^2}{2M}.$$

The last two equations follow from (3.3). The second follows from

$$\hat{H}_{p-k}-\hat{H}_p=\Omega(p-k-\hat{P}_B)-\Omega(p-\hat{P}_B).$$

Now

$$E(p) - E(p-k) = \int_0^1 d\mu \, (k, \nabla E(p - \mu k)).$$

Therefore :

$$| E(p) - E(p-k) - (k, \nabla E(p)) |$$

$$\leq \int_0^1 d\mu \, |k| . | \nabla E(p - \mu k) - \nabla E(p) | \leq C(p) \, |k|^2 \qquad (\text{for } |k| \leq 1),$$

if p is in $\mathcal{E}$, which follows from lemma 3.1.

In step (IV) of the proof of lemma 1.5 we have shown that

$$(E(p-k) + |k| - E(p))^{-1} \leq D(p) \, |k|^{-1} \qquad [\text{for some } D(p) < \infty].$$

Put now

$$K_p = 4 . \max\left(C(p), \frac{1}{2M} \right) D(p)^3 < \infty.$$

Then for all k ($|k| \leq 1$) we have

$$\left| \omega_p \left(\hat{R}_{p-k} (E(p) - |k|)^2 \right) - \omega_p \left(R_{p-k} (E(p) - |k|)^2 \right) \right| \leq K_p \, |k|^{-1}.$$

If $\Omega(p) = \sqrt{p^2 + M^2}$, p in $\mathcal{E}$, we proceed as follows :

$$F(k) = \omega_p \left(\hat{H}_{p-k} - E(p-k) \right).$$

Now if lemma 3.1 holds for this choice of $\Omega(p)$ the function F has the properties :

$$F(k = 0) = 0, \qquad \nabla F(k = 0) = 0,$$

$\nabla F(k)$ is continuous at $k = 0$ and

$$| \nabla F(k) | \leq M_p \, |k|.$$

Hence :

$$\left| \omega_p \left(\hat{H}_{p-k} - E(p-k) \right) \right| = | F(k) | \leq K_p \, |k|^2$$

which implies (3.1).

Proof of (ii). — $\chi_{\rho, \infty}$ is the characteristic function of $K_{\rho, \infty} \left(0 \leq \rho \leq \frac{1}{2} \right)$ and

$$\hat{V}_{p, \rho} = \exp i \, \Pi \, (\hat{w}_p \, \chi_{\rho, \infty}) \in \pi_p \left(\overline{\Delta(V)} \right).$$

We define

$$\sigma_{\rho,p}(A) = \omega_p(\hat{V}_{p,\rho} A \hat{V}^*_{p,\rho}) = (\hat{V}^*_{p,\rho}\Omega_p, \pi_p(A)\hat{V}^*_{p,\rho}\Omega_p)$$

where A is in $\overline{\Delta(V)}$ (or in $\mathfrak{A}$).

We show :

(*a*) If $0 \leq \rho \leq \rho'$ $\{\sigma_{\rho,p} \restriction \mathfrak{A}_{\rho',R}, R \leq \infty\}$ is independent of ρ and is given by a density matrix on $\mathfrak{A}_{\rho',R}$. Hence $\{\sigma_{\rho,p}\}_{\rho>0}$ converges w^* on $\mathring{\mathfrak{A}}$, thus on $\mathfrak{A}$ and on $\overline{\Delta(V)}$ to a state

$$\sigma_p = (\hat{\alpha}_p)^* \circ \omega_p,$$

where

$$\hat{\alpha}_p(A) = n\text{-}\lim_{\rho\downarrow 0} \hat{V}_{p,\rho} A \hat{V}^*_{p,\rho} \qquad (A \in \mathring{\mathfrak{A}}).$$

Proof of (*a*) : $\chi_{\rho,\rho'}$ is the characteristic function of $K_{\rho,\rho'}$ and

$$\hat{V}_{p,\rho} = \exp i\,\Pi(\hat{\omega}_p \chi_{\rho,\rho'}) \exp i\,\Pi(\hat{\omega}_p \chi_{\rho',\infty}).$$

But $\exp i\,\Pi(\tilde{\omega}_p \chi_{\rho,\rho'})$ is in $\pi_p(\mathfrak{A}_{\rho,\rho'}) \subset \pi_p(\mathfrak{A}_{\rho',R})'$. Since $\exp i\,\Pi(\hat{\omega}_p \chi_{\rho',\infty})$ is unitary and since $\omega_p \restriction \mathfrak{A}_{\rho',\infty}$ is normal, the representation of $\mathfrak{A}_{\rho',\infty}$ defined by $\sigma_p = (\hat{\alpha}_p)^* \circ \omega_p$ is quasi equivalent to the Fock representation.

The rest of (*a*) is obvious.

(*b*) We show that the representation $\pi_{\sigma_p} = \pi_p \circ \hat{\alpha}_p$ has a positive, s. a., total number operator $\hat{N}$. It is therefore quasi equivalent to the Fock representation and hence given by some density matrix Σ_p^2 on $B(\mathcal{F}_B)$.

But then the equations

$$\omega_p(.) = \sigma_p((\hat{\alpha}_p)^{-1}(.)) = w^*\text{-}\lim_{\rho\downarrow 0} \sigma_p(\hat{V}^*_{p,\rho}.\hat{V}_{p,\rho})$$

prove (ii).

Proof of (*b*) : Let Σ_p denote the cyclic vector and $\mathcal{H}^{(1)}_{\sigma_p}$ the Hilbert space corresponding to $(\sigma_p, \mathfrak{A})$ by G. N. S. construction. $\{\pi_{\sigma_p}(A)\Sigma_p \mid A \in \mathring{\mathfrak{A}}\}$ is dense in $\mathcal{H}^{(1)}_{\sigma_p}$. We define

$$N_\rho = \int_{\rho \leq |k| \leq \infty} d^3k\, B^*(k) B(k), \qquad \exp itN_\rho \in \mathfrak{A}_{\rho,\infty}$$

hence $\pi_{\sigma_p}(\exp itN_\rho)$ is a unitary group on $\mathcal{H}^{(1)}_{\sigma_p}$. We show : $\pi_{\sigma_p}(\exp itN_\rho)\pi_{\sigma_p}(A)\Sigma_p$ converges strongly as $\rho \downarrow 0$, for all A in $\mathring{\mathfrak{A}}$, uniformly in t in some compact set in $\mathbf{R}^1$.

Then the limit, denoted by exp $it\,\hat{N}$, is a strongly continuous unitary group on $\mathcal{H}^{(1)}_{\sigma_p}$.

Since

$$\inf \operatorname{spec} \pi_{\sigma_p}(N_\rho) = \inf \operatorname{spec} (N_\rho \upharpoonright \mathcal{F}_B) = 0,$$
$$\hat{N} \geq 0 \qquad \text{and} \qquad \inf \operatorname{spec} \hat{N} = 0.$$

Since $\pi_{\sigma_p}(N_\rho)$ is a number operator for $\mathfrak{A}_{\rho,\varkappa}$, $\hat{N}$ is a number operator for $\mathring{\mathfrak{A}}$, hence for $\mathfrak{A}$ and therefore for $\overline{\Delta(V)}$. Thus $\hat{N} = N_B$.

Assume now that A is in $\mathring{\mathfrak{A}}$. Then there is a $\rho_1 > 0$ such that A is in $\mathfrak{A}_{\rho_1,\varkappa}$. Pick $\rho' \leq \rho \leq \rho_1$ an define

$$N_{\rho',\rho} = \int_{\rho' \leq |k| \leq \rho} d^3 k\, B^*(k)\, B(k).$$

Then :

$$\| \pi_{\sigma_p}((e^{itN_\rho} - e^{itN_{\rho'}})\,A)\,\Sigma_p \|$$
$$= 2\,\sigma_p(A^*A) - \sigma_p(A^*\,e^{itN_{\rho',\rho}}\,A) - \sigma_p(A^*\,e^{-itN_{\rho',\rho}}\,A).$$

But

$$\pi_{\sigma_p}(I - e^{itN_{\rho',\rho}}) = -\,i \int_0^t ds\, \pi_{\sigma_p}(e^{isN_{\rho',\rho}}\,N_{\rho',\rho}).$$

Therefore :

$$| \sigma_p(A^*A) - \sigma_p(A^*\,e^{itN_{\rho',\rho}}\,A) | \leq \int_0^t ds\, | \sigma_p(A^*\,e^{isN_{\rho',\rho}}\,N_{\rho',\rho}\,A) |$$

$$\leq \int_0^t ds \int_{\rho' \leq |k| \leq \rho} d^3 k\, | \sigma_p(B^*(k)\,A^*\,e^{is(N_{\rho',\rho}+1)}\,AB(k)) |$$

$$\leq |t| . \|A\|^2 \int_{\rho' \leq |k| \leq \rho} d^3 k\, \sigma_p(B^*(k)\,B(k))$$

$$= |t| . \|A\|^2 \int_{\rho' \leq |k| \leq \rho} d^3 k \Big\{ \omega_p(B^*(k)\,B(k)) + \frac{2}{\sqrt{2}}\,\hat{w}_p(k)\,\omega_p(B(k)) + \frac{1}{2}\,\hat{w}_p(k)^2 \Big\}$$

$$= |t| . \|A\|^2 \int_{\rho' \leq |k| \leq \rho} d^3 k\, \{ \omega_p(B^*(k)\,B(k)) - \omega_p(B(k))^2 \}$$

which tends to 0 by hypothesis, as $\rho \downarrow 0$. Q.E.D.

Remark. — It follows from the proof of (i) that

$\hat{w}_p(k) - \lambda\, v(k)\, \mathfrak{Z}(k)\, |k|^{-1/2}\,(E(p-k) + |k| - E(p))$ is in $L^2(\mathbf{R}^3)$.

Because of continuity of $\nabla E(q)$ at $q = p$ (for all p in $\mathcal{E}$) this implies that

$$\hat{w}_p(k) - w_p(k) \quad \text{is in } L^2(\mathbf{R}^3),$$

where

$$(3.6) \qquad w_p(k) = \lambda\, v(k)\, \mathfrak{J}(k)\, |k|^{-1/2} (|k| - (k, \nabla E(p)))^{-1}.$$

Hence the representation π_p of $\mathfrak{A}$ is quasi equivalent to the representation determined by the generalized coherent state :

$$(3.7) \qquad \exp i\, \Pi(w_p)\, \psi_0 = \bigotimes_{\nu=0}^{\infty} \exp i\, \Pi(w_p\, \chi_\nu)\, \psi_0 \in \hat{\mathcal{H}}$$

corresponding to the automorphism $\alpha_p : B(f) \mapsto B(f) - \dfrac{(f, w_p)}{\sqrt{2}}$.

Definition :

$$\mathfrak{I}_p = (\alpha_p^{-1})^* \circ \omega_p,$$

$\mathfrak{I}_p$ is a density matrix on $B(\mathcal{F}_B)$ (by theorem 3.2).

$V_p = \exp i\, \Pi(w_p)$ is a unitary operator on $\hat{\mathcal{H}}$ mapping $\mathcal{F}_B$ to $\mathcal{H}^{(1)}_{p,\mathrm{ren}}$.

$V_{p,\rho} = \exp i\, \Pi(w_p\, \chi_{\rho,\infty})$ is unitary on $\mathcal{F}_B$.

Corollary 3.3. — *Suppose that* $t \geq 0$. *Then* $\exp - t(V^*_{p,\rho} H_{p,\rho,\infty} V_{p,\rho})$ *converges strongly on* $\mathcal{F}_B$ *to a selfadjoint semigroup* $\exp - t\, H'_p$, *as* $\rho \downarrow 0$.

H'_p *is s. a. and bounded below by* $E(p)$ *on* $\mathcal{F}_B$ *and the cyclic vector* θ_p *associated to the pair* $(\mathfrak{I}_p, \mathfrak{A})$ *by G. N. S. construction is a groundstate for* H'_p.

$\mathfrak{I}_p$ *is a density matrix on* $B(\mathcal{F}_B)$, *i. e.*

$$\mathfrak{I}_p(.) = \sum_{n=0}^{N} s_{n,p}\, (\theta_{n,p}, \theta_{n,p}); \qquad s_{n,p} > 0, \qquad \sum_{n=0}^{N} s_{n,p} = 1;$$

$\theta_{n,p} \in \mathcal{F}_B$ *and* $(\theta_{n,p}, \theta_{m,p}) = \delta_{mn}$; $N \leq \infty$.

$\theta_{n,p}$ *is a groundstate for* H'_p *corresponding to the eigenvalue* $E(p)$, *for all* $n \leq N$.

Proof. — If $\rho' \leq \rho$ then $\exp - t(V^*_{p,\rho'} H_{p,\rho,\infty} V_{p,\rho'})$ does *not* depend on ρ'. Therefore, for $0 \leq \rho'' < \rho' \leq \rho$,

$$\begin{aligned} &\exp - t(V^*_{p,\rho'} H_{p,\rho',\infty} V_{p,\rho'}) - \exp - t(V^*_{p,\rho} H_{p,\rho,\infty} V_{p,\rho}) \\ &\quad = \exp - t(V^*_{p,\rho''} H_{p,\rho',\infty} V_{p,\rho''}) - \exp - t(V^*_{p,\rho''} H_{p,\rho,\infty} V_{p,\rho''}). \end{aligned}$$

Hence the strong convergence of $\exp - t(V^*_{p,\rho} H_{p,\rho,\infty} V_{p,\rho})$ on $\mathcal{F}_B$ follows from the strong convergence of $\exp - t\, H_{p,\rho,\infty}$ on $\mathcal{H}^{(1)}_{p,\mathrm{ren}}$ (as $\rho \downarrow 0$)

which was established in section 2.3, 2º. In 2º we have also stated that

$$e^{-t H_{p,\varrho,\infty}} \Omega_p \xrightarrow{s} e^{-t \hat{H}_p} \Omega_p = e^{-t E(p)} \Omega_p.$$

This implies

$$\exp - t (V^*_{p,\varrho} H_{p,\varrho,\infty} V_{p,\varrho}) \theta_p = V^*_{p,\varrho} e^{-t H_{p,\varrho,\infty}} V_{p,\varrho} \theta_p = V^*_p e^{-t H_{p,\varrho,\infty}} \Omega_p$$

which tends to

$$V^*_p e^{-t \hat{H}_p} \Omega_p = V^*_p e^{-t E(p)} \Omega_p = e^{-t E(p)} \theta_p.$$

Thus θ_p is a groundstate for H'_p.

If $Q_{n,p}$ denotes the selfadjoint projection onto $\theta_{n,p}$ and A is in $\mathfrak{A}$, then

$$s_{n,p} e^{-t E(p)} (A \theta_{n,p}, \theta_{n,p}) = e^{-t E(p)} \mathfrak{I}_p (Q_{n,p} A^*)$$
$$= \mathfrak{I}_p (Q_{n,p} A^* e^{-t H'_p}) = s_{n,p} (A \theta_{n,p}, e^{-t H'_p} \theta_{n,p}).$$

Since $s_{n,p} > 0$, we have

$$e^{-t E(p)} (A \theta_{n,p}, \theta_{n,p}) = (A \theta_{n,p}, e^{-t H'_p} \theta_{n,p}).$$

But $\mathfrak{A}$ acts irreducibly on $\mathcal{F}_B$ and therefore $\{ A \theta_{n,p} \mid A \in \mathfrak{A} \}$ is dense in $\mathcal{F}_B$.

Thus

$$e^{-t H'_p} \theta_{n,p} = e^{-t E(p)} \theta_{n,p}$$

Q. E. D.

Explicit construction of H'_p. — Assume for simplicity that the interaction kernel v is such that $\| v \mid k \mid^{-1} \|_2 < \infty$.

We define :

$$V^*_p C V_p = \underset{\varrho \downarrow 0}{s\text{-lim}} V^*_{p,\varrho} C V_{p,\varrho},$$

for all densely defined operators C on $\mathcal{F}_B$ for which the limit exists. Then :

$$V^*_p P_B V_p = P_B - \Phi(\mathbf{w}_p) + \frac{1}{2}(w_p, \mathbf{w}_p), \tag{3.8}$$

where

$$\mathbf{w}_p(k) = k w_p(k);$$

$$V^*_p H_{0B} V_p = H_{0B} - \Phi(|k| w_p) + \frac{1}{2} \| \, |k|^{1/2} w_p \|_2^2, \tag{3.9}$$

$$V^*_p H_I(v) V_p = H_I(v) - \lambda (v \mid k \mid^{-1/2}, w_p). \tag{3.10}$$

Clearly $P_B - \Phi(\mathbf{w}_p) + \frac{1}{2}(w_p, \mathbf{w}_p)$ is densely defined and symmetric. Since supp $\mathbf{w}_p \subseteq K_{0,2}$ standard estimates show that

$$P_B - \Phi(\mathbf{w}_p) + \frac{1}{2}(w_p, \mathbf{w}_p)$$

has a dense set of analytic vectors. Therefore it is selfadjoint.

$H_{0B} - \Phi(|k| w_p) + \frac{1}{2}\| |k|^{1/2} w_p \|_2^2$ is selfadjoint by Kato's theorem. The operators $\Omega(V_p^* P_B V_p - p)$ and $V_p H_{0B} V_p$ commute and therefore their sum is selfadjoint.

Finally $H_I(v) - \lambda(v | k|^{-1/2}, w_p)$ is a densely defined, real quadratic form which is dominated by and infinitely small with respect to $V_p^* H_{0B} V_p$; therefore it is infinitely small with respect to

$$\Omega(p - V_p^* P_B V_p) + V_p^* H_{0B} V_p.$$

Thus the unique Friedrichs extension of

$$\Omega(p - V_p^* P_B V_p) + V_p^* H_{0B} V_p + V_p^* H_I(v) V_p + E_1(v) \tag{3.11}$$

is selfadjoint and bounded from below by $E(p)$.

It is equal to H'_p. If $\Omega(p) = \frac{p^2}{2M}$ one can extend this explicit construction to the case $v \equiv 1$ by use for the transformation (1.12). (*See* [28]; appendix 2.)

3.2. Uniqueness of the dressed one electron states

In this section we want to establish *Theorem C* of chapter 0.

DEFINITIONS :

A. $\mathcal{H}^{(1)}(w_p)$ is the IDPS in $\hat{\mathcal{H}}$ corresponding to the product reference vector

$$V_p \psi_0 = \exp i \Pi(w_p) \psi_0 \equiv \bigotimes_{\nu=0}^{\infty} \exp i \Pi(w_p \chi_\nu) \psi_0. \tag{3.12}$$

B. We define a dense set $\mathfrak{M} \subset \mathcal{H}^{(1)}(w_p)$ by

$$\mathfrak{M} = \{ \psi \in \mathcal{H}^{(1)}(w_p) \mid \exists \rho > 0, \text{ such that } \psi = \exp i \Pi(w_p \chi_{0,\rho}) \psi_0(s) \theta \},$$

where θ is an arbitrary state in $\mathcal{F}_B(K_{\rho,\infty})$.

Since $V_p \psi_0$ is cyclic for $\mathring{\mathfrak{A}}$ [or $\overline{\Delta(V)}$], $\mathfrak{M}$ is dense in $\mathcal{H}^{(1)}(w_p)$.

Let $\psi = \exp i \Pi(w_p \chi_{0,\rho}) \psi_0(s) \theta$ be in $\mathfrak{M}$.

We define

$$J_0 \psi = \exp i \Pi (w_p \chi_{0,\rho}) \psi_0 (s) \bar{\theta}$$

and $\bar{\theta}$ is obtained from θ by complex conjugation of wave functions of θ.

J is the unique closure of J_0 and is a conjugation on $\mathcal{H}^{(1)}(w_p)$, since J is antilinear and $J^2 = I$.

Re $\mathcal{H}^{(1)}(w_p)$ is the eigenspace of J corresponding to the eigenvalue $+1$.

C. Re $\mathfrak{M} \subset$ Re $\mathcal{H}^{(1)}(w_p)$ contains the cone

$$(3.13) \quad \mathfrak{K}_0 = \{ \psi \in \mathfrak{M} \mid \psi = \exp i \Pi (w_p \chi_{0,\rho}) \psi_0 (s) \theta \text{ for some } \rho > 0, \\ \theta \in \mathcal{F}_B (K_{\rho,\infty}) \text{ has non negative wave functions} \}.$$

$\mathfrak{K}$ denotes the closure of $\mathfrak{K}_0$ in Re $\mathcal{H}^{(1)}(w_p)$, and it is almost trivial to show that $\mathfrak{K}$ is a Hilbert cone in the sense of W. Faris [13] (*see* also II, *b*, i. e. $\varphi \in$ Re $\mathcal{H}^{(1)}(w_p)$ has the unique decomposition :

$$\varphi = \varphi_+ - \varphi_-, \qquad (\varphi_+, \varphi_-) = 0, \qquad \varphi_\pm \in \mathfrak{K}$$

(which is obvious for all φ in Re $\mathfrak{M}$)).

D. Without loss of generality we may assume that the interaction kernel v is positive and $\lambda < 0$. (There is a symmetric between the cases $\lambda < 0$ and $\lambda > 0$.)

An operator A is called *positivity preserving* iff for all ψ, θ in $\mathfrak{K}$:

$$(\psi, A \theta) \geqq 0$$

and *positive ergodic* iff there is a $j < \infty$ such that

$$(\psi, A^j \theta) > 0.$$

We first verify that $e^{-t\hat{H}_q}$ is positivity preserving for all q in $\mathbf{R}^3$. Let

$$\hat{H}_{0q} = \Omega (q - \hat{P}_B) + \hat{H}_{0B} \qquad \text{and} \qquad \hat{H}_q (l) = \hat{H}_{0q} + H_I (v_l) + E_I (v_l)$$

where $\{ v_l \}_{l=0}^{\infty}$ is a sequence of *positive* C^∞ functions and $v_l (k) > 0$, $v_l (k) = v_l (-k)$, $v_l (k) \to 1$, for all k in $\mathbf{R}^3$, and $\| v_l \|_2 < \infty$, for all $l < \infty$, $[v_l (0) = 1]$.

It follows from theorem 1.1′ and section 2.3, 2° that

$$e^{-t\hat{H}_q} = s\text{-}\lim_{l \to \infty} e^{-t\hat{H}_q (l)}.$$

It is now sufficient to show that $e^{-t\hat{H}_q(l)}$ is positivity preserving. But

$$(3.14)\qquad e^{-t\hat{H}_q(l)} = \sum_{n=0}^{\infty} \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n$$
$$\times\, e^{-(t-s)\hat{H}_{0q}} (-\lambda\, \Phi(v_l \mid k\mid^{-1/2})) \ldots$$
$$\times\, e^{-(s_{n-1}-s_n)\hat{H}_{0q}} (-\lambda\, \Phi(v_l \mid k\mid^{-1/2}))\, e^{-s_n \hat{H}_{0q}}.$$

For any $l < \infty$ this expansion converges in norm. We see by inspection that each term in the expansion is *positivity preserving.*

For all ψ and θ in $\mathfrak{K}$ we define

$$F^{(n)}_{\psi,\theta}(t - s_1, \ldots, s_n) = (-\lambda)^n \big(\psi, e^{-(t-s_1)\hat{H}_{0q}} \Phi(v_l \mid k\mid^{-1/2}) \ldots$$
$$\times\, e^{-(s_{n-1}-s_n)\hat{H}_{0q}} \Phi(v_l \mid k\mid^{-1/2})\, e^{-s_n\hat{H}_{0q}}\, \theta\big) \geq 0,$$

$F^{(n)}_{\psi,\theta}(t - s_1, \ldots, s_n)$ is continuous on the set $\{ t, s_1, \ldots, s_n / t - s_1 > 0, s_n > 0 \}$.

We keep $t - s_1 \equiv \tau > 0$ and $s_n \equiv \sigma > 0$ fixed.

Let ψ and θ be some vectors in $\mathfrak{K}$. Since $\Delta(V)$ acts irreducibly on $\mathcal{H}^{(1)}(w_p)$, there is an operator C in $\Delta(V)$ such that

$$\big(e^{-\tau\hat{H}_{0q}}\psi, C\, e^{-\sigma\hat{H}_{0q}}\theta\big) > 0.$$

Using the definition of $\Delta(V)$ and an expansion of C in terms of the operators B(.) and B*(.) (which converges on $e^{-\sigma\hat{H}_{0q}}\theta$, for $\sigma > 0$) it is rather easy to conclude that there is an $n < \infty$ such that

$$F^{(n)}_{\psi,\theta}(\tau, 0, \ldots, 0, \sigma) > 0, \qquad \tau = t - s_1 \leq t, \qquad \sigma = s_n > 0.$$

Because of continuity properties of $F^{(n)}_{\psi,\theta}$ it follows that

$$\int_0^t ds_1 \ldots \int_0^{s_{n-1}} ds_n\, F^{(n)}_{\psi,\theta}(t - s_1, \ldots, s_n) > 0.$$

Hence $\big(\psi, e^{-t\hat{H}_q(l)}\theta\big) > 0$, i. e. $e^{-t\hat{H}_q(l)}$ is *positive ergodic* (for all $l < \infty$).

It is technically quite difficult to remove the cutoff l and to show that $e^{-t\hat{H}_q}$ is *positive ergodic.* But it can be done. The details of this and all the arguments given in this section are explained and proven in II, *b*, chap. 2, " main theorem ".

Putting $p = q$ we get $e^{-t\hat{H}_p}$ is *positive ergodic* on $\mathfrak{K}$.

$\hat{H}_p$ has at least one groundstate $V_p\, \theta_{n,p}$ [where $\theta_{n,p}$ was obtained in corollary 3.2 and $V_p\, \theta_{n,p} \in \mathcal{H}^{(1)}(w_p)$].

For a proper choice of the phase of $\theta_{n,p}$ we get therefore :

The vector $V_p\, \theta_{n,p}$ belongs to $\mathfrak{K}$.

A generalization of the Perron-Frobenius theorem (Glimm and Jaffe, [17]; *see* also Faris, [13], and refs. given there, II, *b*) tells us that the groundstate of $\hat{H}_p$ is *unique.*

We have now proven (II, *b*, chap. 2, " main theorem ") :

THEOREM 3.4. — *Let p be in* $\mathcal{E}$ *and let* $\{ \sigma_l \}_{l=0}^{\infty}$ *be the sequence constructed in lemma* 3.1. *Then :*

$$\omega_{\sigma_l, p} \xrightarrow{w^*} \omega_p \quad \text{on } \mathfrak{A}, \qquad \text{as} \quad l \to \infty.$$

The representation π_p *of* $\mathfrak{A}$ *is irreducible and determined by the generalized coherent state* $\exp i\, \Pi(w_p)\, \psi_0 \equiv V_p\, \psi_0 \in \hat{\mathcal{H}}$.

The groundstate $\Omega_p \in \mathcal{H}^{(1)}(w_p)$ *of* $\hat{H}_p$ *is* unique.

$$(3.15) \quad \begin{cases} \Omega_p = V_p\, \theta_p, \\ \text{where } \theta_p \text{ is in } \mathcal{F}_B \text{ and is the unique groundstate of } H'_p. \end{cases}$$

Remarks. — The convergence of $\{ \omega_{\sigma_l, p} \}_{l=0}^{\infty}$ to ω_p on $\mathfrak{A}$ follows from the following facts : Any accumulation point of $\{ \omega_{\sigma_l, p} \}_{l=0}^{\infty}$ determines a representation which is quasi equivalent to the one determined by $V_p\, \psi_0$. Hence all groundstates of $\hat{H}_p$ are in $\mathcal{H}^{(1)}(w_p)$. But in this space there is at most one groundstate of $\hat{H}_p$. Therefore $\{ \omega_{\sigma_l, p} \}_{l=0}^{\infty}$ has a unique accumulation point. It follows from theorem 3.2 that the probability of finding finitely many virtual bosons in Ω_p vanishes. Compare this with the remark following theorem 2.1 and with prediction (II′), section 2.2. Theorem 3.4 establishes a precise version of *Theorem C.*

3.3. Absence of DES in the physical Hilbert space

In this section we want to establish Theorem D of chapter 0. Our analysis of DES (in the limit $\sigma = 0$) is of course motivated by the requirements of scattering theory : In order to apply the methods of the conventional Haag-Ruelle theory one has to construct a Hilbert space $\mathcal{H}_{\text{ren}}$ such that the spectrum of the energy momentum operator $(H, P) \restriction \mathcal{H}_{\text{ren}}$ [the extension of (H, P) to $\mathcal{H}_{\text{ren}}$] contains a one particle shell, i. e. such that $\mathcal{H}_{\text{ren}}$ contains DES.

We shall show : There are two ways of constructing a Hilbert space $\mathcal{H}_{\text{ren}}$ containing DES. The first way consists in constructing Wightman distributions which, by the reconstruction theorem, determine a Hilbert space $\mathcal{H}_{\text{ren}}$ and a unitary space-time translation group $e^{i(t\text{H}-x\text{P})}$ such that spec ((H, P) $\restriction \mathcal{H}_{\text{ren}}$) contains a one particle shell. However, it turns out that the dynamics determined by $e^{it\text{H}} \restriction \mathcal{H}_{\text{ren}}$ is the one of a system of one free electron and free bosons without any interaction.

The second way consists in glueing together the spaces $\mathcal{H}^{(1)}(w_p)$ [with the dynamics $e^{it\hat{\text{H}}_p}$ on $\mathcal{H}^{(1)}(w_p)$] for all possible values of p. However, this leads to a theory, where the total momentum is a super selection rule. Furthermore the theory is incompatible with a scattering theory. This is of course not acceptable from the physical point of view.

Our conlusion is :

The " physical Hilbert space " $\mathcal{H}$ has to be defined in such a way that the energy momentum operator (H, P) $\restriction \mathcal{H}^{(1)}$ has a unique extension to $\mathcal{H}$ and the dynamics determined by $e^{it\text{H}} \restriction \mathcal{H}$ is not the one of a non interacting system and is compatible with a scattering theory. Then $\mathcal{H}$ does *not* contian DES and therefore (*a fortiori*) the scattering theory cannot consist in an application of the conventional Haag-Ruelle theory. In chapter 4 we try to show that $\mathcal{H} = \mathcal{H}^{(1)}$ is compatible with a scattering theory (other canditates for $\mathcal{H}$ are given in II, *a*).

We start this section by proving a precise version of prediction (III') of section 2.3 which we use in the proof of Theorem D.

1° Suppose that p is in $\mathcal{E}$ and $\nabla \text{E}(p) \neq 0$. If $p \neq p'$, but $|p| = |p'|$, then p' is in $\mathcal{E}$ (since $\mathcal{E}$ is rotation invariant). However π_p and $\pi_{p'}$ are *disjoint*, since $w_p(k) - w_{p'}(k)$ is *not* square integrable at $k = 0$.

Therefore in a proper sense of the word, [39 : 1, 2], the IDPS $\mathcal{H}^{(1)}(w_p)$ and $\mathcal{H}^{(1)}(w_{p'})$ are orthogonal; in particular $(\Omega_p, \Omega_{p'}) = 0$. We now want to show that there is an open set $\mathcal{E}_2 \subseteq \{ p / |p| < \rho_0 \}$ such that the set

$$\{ p \in \mathcal{E}_2 \,|\, \pi_p \text{ is equivalent to } \pi_{p_0}, \text{ for fixed } p_0 \in \mathcal{E}_2 \}$$

is of *Lebesgue measure* 0.

DEFINITION :

$$\mathcal{E}_1 = \mathcal{E} \setminus \{ p / |p| \leq \rho_0, |\nabla \text{E}(p)| = 0 \}.$$

Since $|\nabla \text{E}(p)|$ is a measurable function of p, the set

$$\{ p / |p| \leq \rho_0, |\nabla \text{E}(p)| = 0 \}$$

is measurable. $\mathcal{E}$ is measurable, as well. Therefore $\mathcal{E}_1$ is measurable. Clearly $\mathcal{E}_1$ is invariant under rotations in momentum space.

We show that $\mathcal{E}_1$ has *positive Lebesgue measure.*

Proof. — It follows from property (iv), section 1.2, of E (p) that E $(p) \to \infty$, as $|p| \to \infty$. Since E (p) is absolutely continuous in p, the proof is complete if $\rho_0 = \infty$ [in particular the proof is complete if $\Omega(p) = \sqrt{p^2 + M^2}$].

If $\Omega(p) = \dfrac{p^2}{2M}$ we have shown that $\mathcal{E}_1$ has positive Lebesgue measure if either $\rho_0 = \infty$, or there is a p, $|p| < \rho_0$ such that E $(p) >$ E (0).

Assume now that $\rho_0 < \infty$ and E $(p) \equiv$ E (0), for all $|p| \leq \rho_0$. From property (i), section 1.2; of E (p) we know that E $(p) \geq$ E (0), for all p. It then follows from the definition of ρ_0 that there is an $\varepsilon > 0$ such that E $(p) >$ E (0), for all p such that $\rho_0 < |p| < \rho_0 + r$.

Let $\mathbf{e}$ be a unit vector in $\mathbf{R}^3$ and put

$$E(x) = E(x.\mathbf{e}), \qquad t(x) = t(x.\mathbf{e}) = E(x) - \frac{x^2}{2M}.$$

Property (iii), section 1.2, tells us that $t'(x)$ is monotonically decreasing. Hence $t'(x) \leq -\dfrac{\rho_0}{M}$, for all $x > \rho_0$.

Thus, if $\rho_0 < x < \rho_0 + \varepsilon$,

$$E(0) < E(x) \leq \frac{1}{M}\int_{\rho_0}^{x} (y - \rho_0)\, dy + E(0) = \frac{1}{2M}(x - \rho_0)^2 + E(0).$$

From this it obviously follows that there is a δ, $0 < \delta < \varepsilon$, such that $\inf\limits_{|k| \geq \rho} (E(p - k) + k - E(p)) > 0$, for all $\rho > 0$ and for all $|p| < \rho_0 + \delta$.

But this contradicts the definition of ρ_0.

Q. E. D.

Because of lemma 3.1 there is a set $\mathfrak{N}$ of Lebesgue measure 0 such that $\mathcal{E}_1 \setminus \mathfrak{N}$ contains a (maximal) set $\mathcal{E}_2$ which is left invariant under rotations and such that the measures of $\mathcal{E}_2$ and $\overline{\mathcal{E}}_2$ are equal. Clearly $\overline{\mathcal{E}}_2$ has the property.

(3.16) $\{ p \in \overline{\mathcal{E}}_2 \mid \pi_p$ is equivalent to π_{p_0}, for some fixed $p_0 \in \overline{\mathcal{E}}_2 \}$

is of measure 0, which is what we wanted to prove.

This establishes a weaker form of prediction (III′), section 2.2.

CONJECTURE :

(3.17) $$\overline{\mathcal{E}}_2 = \{ p / |p| \leq \rho_0 \}.$$

2° " Wightman distributions " in the super selection sector of DES.

DEFINITION :

$$B(k, +1) = B^*(k), \qquad B(k, -1) = B(k), \qquad \delta = \pm 1.$$

The only possible way to define " Wightman distributions " in the super selection sector of DES which determine a theory with a reasonable physical interpretation is

$$(3.18) \qquad W^{(n)}_{p,p'}(t_0, k_1, \delta_1, t_1, k_2, \delta_2, \ldots, k_n, \delta_n, t_n)$$
$$= \left(\Omega_p, e^{it_0 \hat{H}_p} \prod_{j=1}^{n} \left(B(k_j, \delta_j) e^{it_j \hat{H}_{p - \sum_{i=1}^{j} \delta_i k_i}} \right) \Omega_{p'} \right)$$
$$\times \delta \left(p' - p - \sum_{i=1}^{n} \delta_i k_i \right).$$

Since the DES Ω_p and $\Omega_{p'}$ exist only if $|p| < \rho_0$ and $|p'| < \rho_0$, one can define the distributions $W^{(n)}_{p,p'}(\ldots)$ in a rigorous manner only if $|p| \leq \rho_0$ and $|p'| \leq \rho_0$. We restrict the values of p and p' to the set $\overline{\mathcal{E}}_2$. This is no restriction of generality if conjecture (3.17) holds.

We use the following results in order to calculate $W^{(n)}_{p,p'}(\ldots)$:

(*a*) $\{ q \in \overline{\mathcal{E}}^2 \mid \pi_q$ is equivalent to π_p, for fixed p in $\overline{\mathcal{E}}_2 \}$ is of measure 0. If π_q and π_p are disjoint and $\theta_p \in \mathcal{H}^{(1)}(w_p)$, we define

$$(\Omega_q, \theta_p) = 0.$$

$$(b) \qquad [B(k, \delta), B(l, \delta')] = \frac{1}{2}(\delta' - \delta)\,\delta(k - l)$$

$\hat{H}_q$ is selfadjoint on $\mathcal{H}^{(1)}(w_p)$, for all $q \in \mathbf{R}^3$.

Hence $e^{it\hat{H}_q}$ and $\hat{R}_q(\zeta)$ $(\zeta \notin \operatorname{spec} \hat{H}_q)$ leave $\mathcal{H}^{(1)}(w_p)$ invariant and

$$e^{it\hat{H}_p}\,\Omega_p = e^{it E(p)}\,\Omega_p.$$

(*c*) We have shown in Theorem 2.1, (i) that

$$B(k, -1)\,\Omega_p = \lambda\, v(k)\,(2|k|)^{-1/2}\,\hat{R}_{p-k}(E(p) - |k|)\,\Omega_p.$$

It follows from the pull-through formula for $\hat{R}$ that

$$B(k, -1)\, e^{i\hat{H}_q t} = e^{i(\hat{H}_{q-k} + |k|)t}\, B(k, -1)$$
$$+ i\lambda\, v(k)\,(2|k|)^{-1/2} \int_0^t ds\, e^{i(\hat{H}_{q-k} + |k|)s}\, e^{i\hat{H}_q(t-s)}.$$

We are now ready for the following

THEOREM 3.5. — *Suppose that p and p' are in* $\bar{\mathcal{E}}_2$.
Then (a), (b) and (c) imply that

$$(3.19)\quad W^{(n)}_{p,p'}(t_0, k_1, \delta_1, t_1, k_2, \ldots, k_n, \delta_n, t_n) = \delta(p-p')\left(\psi_0, e^{it_0(H_{0B}+E(p))}\prod_{j=1}^{n}\left(B(k_j,\delta_j)\,e^{it_j(H_{0B}+E(p))}\right)\psi_0\right)$$

as a bounded multilinear functional on $L^2(\bar{\mathcal{E}}_2, d^3 q)^{\times 2}\, \mathcal{S}_0(\mathbf{R}^{3n})$, *where*

$$\mathcal{S}_0(\mathbf{R}^{3n}) = \{ f \in \mathcal{S}(\mathbf{R}^{3n}) \mid f(k^n) = 0 \text{ in some neighbourhood of } k_1 = \ldots = k_n = 0 \}.$$

Proof. — For the " one point function " $W^{(1)}_{p,p'}(t, k, \delta, s) = W^{(1)T}_{p,p'}(t, k, \delta, s)$ the theorem is trivial.

For the " n point function " we construct a linked cluster expansion in terms of " truncated m point functions " $(m \leq n)$.

We prove the theorem for $n = 2$:

$$\begin{aligned} W^{(2)}_{p,p'}&(s, k, -1, t, l, +1, u) \\ &= (\Omega_p, B(k)\,e^{it\hat{\Pi}_{p+k}}\,B^*(l)\,\Omega_{p'})\,e^{i(uE(p')-sE(p))}\,\delta(p+k-l-p') \\ &= \delta(p-p')\,\delta(k-l)\,e^{it(|k|+E(p))}\,e^{+i(u-s)E(p)} \\ &\quad + W^{(2)T}_{p,p'}(s, k, -1, l, +1, u). \end{aligned}$$

But

$$\begin{aligned} W^{(2)T}_{p,p'}&(s, k, -1, l, +1, u) \\ = {}& e^{i(uE(p')-sE(p))}\,\delta(p+k-p'-l)\,\frac{1}{2}\lambda^2\,\frac{v(k)}{|k|^{1/2}}\,\frac{v(l)}{|l|^{1/2}} \\ &\times\Bigg\{\Bigg(\Omega_p, \int_0^t ds' \int_0^{t-s'} ds''\, e^{is'(\hat{\Pi}_p+|k|)} \\ &\qquad \times e^{+i(t-s'-s'')\hat{\Pi}_{p+k}}\,e^{is''(\hat{\Pi}_{p+k-l}+|l|)}\,\Omega_{p+k-l}\Bigg) + \text{similar terms} \\ &\quad + \Bigg(\hat{R}_{p-l}(E(p)-|l|)\,\Omega_p, \\ &\qquad \int_0^t ds'\, e^{is'(\hat{\Pi}_{p-l}+|k|+|l|)}\,e^{i(t-s')(\hat{\Pi}_{p+k-l}+|l|)}\,\Omega_{p+k-l}\Bigg) \\ &\quad + \text{similar terms} + \Big(\hat{R}_{p-l}(E(p)-|l|)\,\Omega_p, \\ &\qquad\qquad e^{it(\hat{\Pi}_{p-l}+|k|+|l|)}\,\hat{R}_{p-l}(E(p)-|k|)\,\Omega_{p+k+l}\Big\}. \end{aligned}$$

All terms in $\{ \dots \}$ vanish unless $k = l$ [because of (a) and (b)]. On $k = l$ they take finite values except at $k = l = 0$.

Therefore $W^{(2)\,T}_{p,p'}(s, k, -1, t, l, +1, u)$ vanishes as a multilinear functional on $L^2(\overline{\mathcal{E}}_2, d^3 p) \times L^2(\overline{\mathcal{E}}_2, d^3 p') \times \mathcal{S}_0(\mathbf{R}^6)$.

It is now rather straightforward to complete the proof by complete induction.

Q. E. D.

Excluding infinitely many possible bosons of momentum 0 (Doplicher, [10]), we get :

COROLLARY. — *The distribution*

$$W^{(n)}_{p,p'}(t_0, k_1, \delta_1, t_1, k_2, \dots, k_n, \delta_n, t_n)$$

has a unique extension to a multilinear functional on

$$L^2(\overline{\mathcal{E}}_2, d^3 p) \times L^2(\overline{\mathcal{E}}_2, d^3 p') \times L^2(\mathbf{R}^3)^{\times n}$$

which is given by the expression (3.19).

These " Wightman distributions " determine a Hilbert space, (reconstruction theorem), which contains DES, and a dynamics which describes a system of one free electron [with momentum p in $\overline{\mathcal{E}}_2$ and energy $E(p)$] and essentially finitely many free, scalar bosons without any interaction.

3° One can glue together DES by constructing a direct integral of the spaces $\mathcal{H}^{(1)}(w_p)$, p in $\overline{\mathcal{E}}_2$, with the dynamics $e^{it\hat{H}_p}$ on $\mathcal{H}^{(1)}(w_p)$. This determines a Hilbert space $\mathcal{H}_{\text{ren}}$ (containing DES) and a dynamics e^{itH} on $\mathcal{H}_{\text{ren}}$; $\mathcal{H}_{\text{ren}}$ determines a representation $\pi_{\mathcal{H}_{\text{ren}}}$ of $\overline{\Delta_B(V)}$. Because of (3.16) $\pi_{\mathcal{H}_{\text{ren}}}(\overline{\Delta_B(V)})'$ is abelian and is generated by $e^{i\lambda P}$, where P is the total momentum operator. Hence P is a super selection rule. The dynamics e^{itH} on $\mathcal{H}_{\text{ren}}$ is incompatible with an asymptotic condition in time.

Theorem 3.5 and 3° establish a precise version of *Theorem D*, chap. 0.

Remark. — The absence of DES in the physical Hilbert space $\mathcal{H}$ $(= \mathcal{H}^{(1)})$ has the following heuristic motivation : On the Hilbert space $\mathcal{H}$ the interaction between the electron and the bosons is non trivial. Trying to observe an electron means constructing a localized counter interacting with it. The electron is perturbed and gets localized in space and time. Since it is coupled to the quantized boson field and since the bosons have the restmass 0, this perturbation of the electron leads to the emission of infinitely many soft bosons of finite total energy, as the time t tends to $+\infty$.

A possible point of view could then be : Only the cloud of soft bosons emitted by the electron and moving with it is really observable and indicates (because of the absence of coincidence events in a proper system of counters) the presence of a charge (or a " charged particle ", namely the electron).

Compare this picture with section 4.4, (5), subsection C, (4.49), (4.51), proposition 4.10 and with [12].

CHAPTER 4

SOME ASPECTS OF A COLLISION THEORY OF ONE ELECTRON AND INFINITELY MANY BOSONS

In this chapter we want to study several different aspects of a collision theory on $\mathcal{H}^{(1)}$. In particular we shall construct some steps towards a generalized Haag-Ruelle theory describing the scattering of one charged particle and a cloud of infinitely many bosons. Furthermore we shall try to approximate the scattering states in $\mathcal{H}^{(1)}$ of the model without infrared cutoff by a sequence of rigorously constructed scattering states of the corresponding infrared cutoff models. Neither the first nor the second approach for a scattering theory on $\mathcal{H}^{(1)}$ leads to a final success. Assuming however that one of two possible conjectures hold, we are able to reconfirm completely the infrared folklore, [6], [12], [25] : The scattering states in $\mathcal{H}^{(1)}$ are superpositions of generalized coherent states with respect to the algebra generated by the asymptotic free boson field (up to asymptotic hard bosons), i. e. asymptotically the charged particle is always surrounded by a cloud of infinitely many real soft bosons forming a generalized coherent state.

Because of the lack of a final success in the " conventional " approach to a collision theory on $\mathcal{H}^{(1)}$ described so far we shall construct a different, *algebraic* framework for a scattering theory on $\mathcal{H}^{(1)}$ which is based on the LSZ asymptotic condition for the boson field. This framework has furthermore the advantage that it can be generalized to a framework which might be suitable for the description of collision processes in field theories without a massgap and without mass shells for the charged particles. *See* [16].

4.1. **LSZ asymptotic condition for the boson field**

We define the Weyl operators associated with the annihilation and creation operators of the boson field :

DEFINITION :

$$(4.1)\quad \begin{cases} U_b(f) = \exp \frac{i}{\sqrt{2}} \{ b^*(f) + b(f) \}, \\ V_b(g) = \exp \frac{1}{\sqrt{2}} \{ b(g) - b^*(g) \}. \end{cases}$$

The operators $U_b(f)$ and $V_b(g)$ are unitary on $\mathcal{H}^{(1)}$ if f and g are square integrable real functions on $\mathbf{R}^3$, and they satisfy the usual Weyl relations.

Let $\mathcal{D}$ be any test function space such as defined in section 2.1 and $D = \mathcal{D}_r \times \mathcal{D}_r$.

$\overline{\Delta_b(D)}$ denotes the unique (minimal) CCR C* algebra generated by $\{ U_b(f) V_b(g) \mid [f, g] \in D \}$.

We shall choose for D both S_0 and V which were defined in section 2.1 (*a*) and (*c*).

Let A be in $\overline{\Delta_b(D)}$. On $\mathcal{H}^{(1)}$ we define

$$(4.2)\qquad A_t = e^{-iH_{0b}t} A e^{iH_{0b}t}.$$

Let $H(\sigma)$ $(\sigma \geqq 0)$ denote the Hamiltonian obtained in theorem 1.1 and corollary 1.2 and (for the sake of concreteness) we put throughout this chapter $\Omega(p) = \frac{p^2}{2M}$, $v(.) \equiv 1$.

LEMMA 4.1. — *Let* A *be in* $\overline{\Delta_b(S_0)}$ *and let the Hilbert space be* $\mathcal{H}^{(1)}$. *Then for all* $\sigma \geqq 0$:

(i) $n\text{-}\lim_{t \to \pm\infty} e^{iH(\sigma)t} A_t e^{-iH(\sigma)t} \equiv \mu_{\sigma,\pm}(A) \equiv A_{\sigma,\pm} \in B(\mathcal{H}^{(1)})$ *exists.*

(ii) $s\text{-}\lim_{\sigma \downarrow 0} \mu_{\sigma,\pm}(A) = A_{\sigma=0,\pm} \equiv \mu_{\pm}(A)$.

Proof. — Our proof consists in a slight extension of Hoegh-Krohn's methods, [21], [23].

We assume that f is in $\mathcal{S}_{0r}(\mathbf{R}^3)$. We then show that

$$e^{iH(\sigma)t} U_b(f)_t e^{-iH(\sigma)t} \quad \text{and} \quad e^{iH(\sigma)t} V_b(f)_t e^{-iH(\sigma)t}$$

converge in norm [the operator norm of $B(\mathcal{H}^{(1)})$] as $t \to \pm\infty$. Since both cases are similar, it suffices to show norm convergence of $e^{iH(\sigma)t} U_b(f)_t e^{-iH(\sigma)t}$ as $t \to \pm\infty$, for all $\sigma \geqq 0$.

Adapting the arguments of [21] to our situation we easily verify that in some dense set in $\mathcal{H}^{(1)} \times \mathcal{H}^{(1)}$.

$e^{iH(\sigma)t} U_b(f)_t e^{-iH(\sigma)t}$ is weakly differentiable in t with a weakly continuous derivative.

We define in the sense of densely defined sesquilinear forms

$$H^{a}_{I,\sigma} = \lambda \int d^3 p\, d^3 k\, n^* (p)\, g_\sigma (k)\, (2\,|\,k\,|)^{-1/2}\, b(k)\, n(p-k)$$

and

$$H^{c}_{I,\sigma} = H_I (g_\sigma) - H^{a}_{I,\sigma}.$$

It is straightforward to calculate

$$(4.3)\quad \frac{d}{dt}\left(e^{i H(\sigma) t}\, U_b (f)_t\, e^{-i H(\sigma) t}\right)$$
$$= -\frac{1}{\sqrt{2}} e^{i H(\sigma) t} \{ [H^{a}_{I(\sigma)}, b^* (f_t)]\, U_b (f)_t + U_b(f)_t\, [H^{c}_{I(\sigma)}, b(f_t)] \}\, e^{-i H(\sigma) t}$$

weakly on some dense domain in $\mathcal{H}^{(1)} \times \mathcal{H}^{(1)}$, where

$$f_t (k) = f(k)\, e^{-i|k|t}.$$

(All domain problems are treated as in [21].)

It is well known, [21], that

$$(4.4)\quad \|\, [H^{a}_{I,\sigma}, b^* (f_t)]\, \| = \|\, [H^{c}_{I,\sigma}, b(f_t)]\, \|$$
$$= \sup_{x \in \mathbf{R}^3} \left| \int d^3 k\, g_\sigma (k)\, (2\,|\,k\,|)^{-1/2} f(k)\, e^{i(kx - |k| t)} \right|$$
$$\leqq C_{3/2} (f)\, (1 + |\,t\,|)^{-3/2},$$

where $C_{3/2} (f)$ is some norm which is bounded on $\mathcal{S}_0 (\mathbf{R}^3)$ and is independent of $\sigma \geqq 0$.

The right hand side of (4.3) therefore defines a family of operators which are uniformly bounded in σ and t and (4.4) implies that they are strongly integrable in t on $\mathbf{R}$ with uniform bounds on the integral (that are independent of σ).

Hence

$$e^{i H(\sigma) t}\, U_b (f)_t\, e^{-i H(\sigma) t} - U_b (f)$$

converges in norm as $t \to \pm \infty$, for all $\sigma \geqq 0$ and

$$U_{\sigma,\pm} (f) = n\text{-}\lim_{t \to \pm \infty} e^{i H(\sigma) t}\, U_b (f)_t\, e^{-i H(\sigma) t}$$

exists and defines a unitary operator on $\mathcal{H}^{(1)}$ for all $\sigma \geqq 0$.

[$U_{\sigma,\pm} (f)$ is unitary, since $U_{\sigma,\pm} (-f)$ exists and

$$U_{\sigma,\pm} (-f)\, U_{\sigma,\pm} (f) = U_{\sigma,\pm} (f)\, U_{\sigma,\pm} (-f) = I \quad \text{for all } \partial \geqq 0].$$

One can now replace $U_b(f)$ by $V_b(g)$ and repeat essentially the same arguments. Since $\{U_b(f)V_b(g) \mid [f, g] \in S_0\}$ is norm dense in $\overline{\Delta_b(S_0)}$, the proof of (i) is complete.

Proof of (ii). — It suffices to show that

$$s\text{-}\lim_{\sigma \downarrow 0} U_{\sigma,\pm}(f) = U_{\sigma=0,\pm}(f) \equiv U_\pm(f)$$

since the other case is similar.

Since f is in $\mathcal{S}_{0r}(\mathbf{R}^3)$, $[H^a_{I(\sigma)}, b^*(f_t)]$ and $[H^c_{I(\sigma)}, b(f_t)]$ are independent of σ if $\sigma \leq \sigma_0(f)$, for some

(4.5) $$\sigma_0(f) > 0.$$

We now show : $(\zeta - H(\sigma))^{-1}$ converges in norm to $(\zeta - H)^{-1}$ if

(4.6) $$\zeta \notin \operatorname{spec} H.$$

Obviously

$$(\zeta - H(\sigma))^{-1} - (\zeta - H)^{-1} = -(\zeta - H(\sigma))^{-1} H_I (1 - g_\sigma)(\zeta - H)^{-1}.$$

Hence

$$\begin{aligned} &\| (\zeta - H(\sigma))^{-1} - (\zeta - H)^{-1} \| \\ &\leq 2 \| (1 - g_\sigma)\tau^{-1/2} \|_2 \| (\zeta - H(\sigma))^{-1} (N_\tau + I)^{1/2} \| \\ &\quad \times \| (N_\tau + I)^{1/2} (\zeta - H)^{-1} \|, \end{aligned}$$

where

$$\tau(k) = \begin{cases} |k|, & |k| \leq 1, \\ 1, & \text{otherwise.} \end{cases}$$

If ζ is not in $\operatorname{spec} H(\sigma) \cup \operatorname{spec} H$ the second and the last term of this inequality are bounded because of theorem 1.1, (i). But the first term $2\|(g_\sigma - 1)\tau^{-1/2}\|_2$ tends to 0 as $\sigma \downarrow 0$.

This proves (4.6) and therefore

(4.7) $$e^{iH(\sigma)t}\theta \xrightarrow{s} e^{iHt}\theta, \qquad \text{as } \sigma \downarrow 0$$

uniformly in t in bounded subsets of $\mathbf{R}$ and for all θ in $\mathcal{H}^{(1)}$.

Combining now (4.3)-(4.7) and applying the triangle inequality we conclude

(4.8) $$w\text{-}\lim_{\sigma \downarrow 0} U_{\sigma,\pm}(f) = U_\pm(f) \quad \text{for all } f \text{ in } \mathcal{S}_{0r}(\mathbf{R}^3).$$

But since $U_{\sigma,\pm}(f)$ and $U_{\pm}(f)$ are unitary operators, (4.3) yields strong convergence

$$s\text{-}\lim_{\sigma \downarrow 0} U_{\sigma,\pm}(f) = U_{\pm}(f) \quad \text{for all } f \text{ in } \mathcal{S}_{0r}(\mathbf{R}^3).$$

We complete the proof of (ii) with the same arguments that we have used to complete the proof of (i).

Q. E. D.

Remark. — Lemma 4.1, (ii) represents a first step in our programm of approximating the scattering theory in the limit $\sigma = 0$ by the ones for $\sigma > 0$.

Let $\mathcal{V} = \bigcup_{\nu=0}^{\infty} L^2(K_\nu, d^3 k)$, where K_ν has been defined in section 2.1, (c). This space is much bigger than $\mathcal{S}_0(\mathbf{R}^3)$. Therefore the following lemma is not a trivial consequence of lemma 4.1.

LEMMA 4.2. — *Let* θ *be in* $D(H^{1/2})$. *Then for all* $\sigma \geq 0$:

(i) $$s\text{-}\lim_{t \to \pm\infty} e^{iH(\sigma)t} b^{\#}(f_t) e^{-iH(\sigma)t} \theta = b^{\#}_{\sigma,\pm}(f)\theta$$

exists for all f *in* $\mathcal{V}$.

(ii) *If* f *is in* $\mathcal{V}_r$ *then* $s\text{-}\lim_{t \to \pm\infty} e^{iH(\sigma)t} U_b(f)_t e^{-iH(\sigma)t}, \ldots$ exist and

$$\sqrt{2}\, b^*_{\sigma,\pm}(f)\theta = \frac{1}{i}\frac{d}{ds}(U_{\sigma,\pm}(sf) - i V_{\sigma,\pm}(sf))\theta\,|_{s=0}, \qquad \ldots$$

Proof. — Let f be in $\mathcal{S}_0(\mathbf{R}^3)$. Then the results of [21] imply directly that $e^{iH(\sigma)t} b^{\#}(f_t) e^{-iH(\sigma)t}\theta$ converges strongly as $t \to \pm\infty$, for all $\sigma \geq 0$ and all θ in $D(H^{1/2})$ [and all θ in $D(N_\tau^{1/2})$].

If f is in $\mathcal{V}$ then there is a sequence $\{g_n\}_{n=0}^{\infty} \subset \mathcal{S}_0(\mathbf{R}^3)$ such that

$$|f - g_n|_{1/2} = \|f - g_n\|_2 + \|(f - g_n)\tau^{-1/2}\|_2 \to 0 \qquad \text{as} \quad n \to \infty.$$

Obviously $|f_t|_{1/2} = |f|_{1/2}$ such that

$$|f_t - g_{n,t}|_{1/2} = |(f - g_n)_t|_{1/2} = |(f - g_n)|_{1/2} \to 0 \qquad \text{as} \quad n \to \infty.$$

If θ is in $D(H^{1/2})$ then

$$\begin{aligned}
&\{e^{iH(\sigma)t} b^{\#}(f_t) e^{-iH(\sigma)t} - e^{iH(\sigma)t} b^{\#}(g_{n,t}) e^{-iH(\sigma)t}\}\theta \\
&\quad = e^{iH(\sigma)t} b^{\#}((f - g_n)_t) e^{-iH(\sigma)t}\theta \\
&\quad = e^{iH(\sigma)t}\left(b^{\#}((f - g_n)_t)\right)(N_\tau + I)^{-1/2}((N_\tau + I)^{1/2}(H(\sigma) + c)^{-1/2}) \\
&\qquad \times e^{-iH(\sigma)t}(H(\sigma) + c)^{1/2}\theta.
\end{aligned}$$

Thus :

$$(4.9)\quad \| e^{iH(\sigma)t}\big(b^{\#}(f_t) - b^{\#}(g_{n,t})\big) e^{-iH(\sigma)t}\theta \|$$
$$\leq \| b^{\#}((f-g_n)_t)(N_\tau + I)^{-1/2}\| . \| (N_\tau + I)^{1/2}$$
$$\times (H(\sigma)+c)^{-1/2}\| . \| (H(\sigma)+c)^{1/2}\theta\|$$
$$\leq | f - g_n |_{1/2} . \| (N_\tau + I)^{+1/2}(H(\sigma)+c)^{-1/2}\| . \| (H(\sigma)+c)^{1/2}\theta \|.$$

From theorem 1.1 we know that $(N_\tau + I)^{1/2}(H(\sigma)+c)^{-1/2}$ is a bounded operator if $c > -\inf \operatorname{spec} H(\sigma)$.

Since $H(\sigma) \leq \text{const.} \times (H + I)$ and since θ is in $D(H^{1/2})$, $\|(H(\sigma)+c)^{1/2}\theta\|$ is bounded for all $\sigma \geq 0$.

Finally $|f - g_n|_{1/2}$ tends to 0 as $n \to \infty$.

Since $\operatorname*{s-lim}_{t\to\pm\infty} e^{iH(\sigma)t} b^{\#}(g_{n,t}) e^{-iH(\sigma)t}\theta$ exists for all $n < \infty$ we conclude that

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{iH(\sigma)t} b^{\#}(f_t) e^{-iH(\sigma)t}\theta \equiv b^{\#}_{\sigma,\pm}(f)\theta \quad \text{exists for all } \sigma \geq 0.$$

This completes the proof of (i).

Proof of (ii). — If f is in $\mathcal{V}_r$ there is a sequence $\{ g_n \}_{n=0}^{\infty}$ in $\mathcal{S}_{0r}(\mathbf{R}^3)$ such that $|f - g_n|_{1/2}$ tends to 0 as $n \to \infty$.

Now

$$(4.10)\quad U_b(f) - U_b(g_n) = \frac{i}{\sqrt{2}}\int_0^1 ds\, U_b(sf)\, U_b((1-s)g_n)$$
$$\times \{ b(f-g_n) + b^*(f-g_n) \}.$$

Let θ be a vector in $D(H^{1/2})$. Then we get from (4.10) and (4.9) :

$$\| e^{iH(\sigma)t}(U_b(f)_t - U_b(g_n)_t) e^{-iH(\sigma)t} \|$$
$$\leq \text{const.} \times | f - g_n |_{1/2} . \| (H(\sigma)+c)^{1/2}\theta \|,$$

which tends to 0 as $n \to \infty$, uniformly in t.

Since $e^{iH(\sigma)t} U_b(g_n)_t e^{-iH(\sigma)t}$ converges in norm to $U_{\sigma,\pm}(g_n)$, as $t \to \pm\infty$ for all $n < \infty$, we conclude that $e^{iH(\sigma)t} U_b(f)_t e^{-iH(\sigma)t}\theta$ converges strongly to $U_{\sigma,\pm}(f)\theta$, as $t \to \pm\infty$ for all θ in $D(H^{1/2})$.

But $D(H^{1/2})$ is dense in $\mathcal{H}^{(1)}$ and therefore

$$(4.11)\quad \operatorname*{s-lim}_{t\to\pm\infty} e^{iH(\sigma)t} U_b(f)_t e^{-iH(\sigma)t} \equiv U_{\sigma,\pm}(f) \quad \text{exists.}$$

Let now θ be in $D(H^{1/2})$ and f in $\mathcal{V}_r$:

$$\frac{\partial}{\partial s} e^{iH(\sigma)t} U_b(sf)_t e^{-iH(\sigma)t}\theta$$
$$= \frac{i}{\sqrt{2}}(e^{iH(\sigma)t} U_b(sf)_t e^{-iH(\sigma)t})(e^{iH(\sigma)t}\{ b^*(f_t) + b(f_t)\} e^{-iH(\sigma)t})\theta.$$

Applying lemma 4.2, (i) and (4.11) we get :

$$s\text{-}\lim_{t \to \pm\infty} \frac{\partial}{\partial s} e^{i\mathrm{H}(\sigma)t}\, \mathrm{U}_b(sf)_t\, e^{-i\mathrm{H}(\sigma)t}\, \theta = \frac{i}{\sqrt{2}} \mathrm{U}_{\sigma,\pm}(sf) \{ b^*_{\sigma,\pm}(f) + b_{\sigma,\pm}(f) \} \theta$$

and we may easily convince ourselves that the convergence is uniform in s in bounded subsets of **R**. Therefore we may interchange limits and get :

$$\frac{d}{ds} \mathrm{U}_{\sigma,\pm}(sf)\, \theta = \frac{i}{\sqrt{2}} \mathrm{U}_{\sigma,\pm}(sf) \{ b^*_{\sigma,\pm}(f) + b_{\sigma,\pm}(f) \} \theta.$$

Since $\mathrm{U}_{\sigma,\pm}(sf)$ is a strongly continuous unitary group in s and $s\text{-}\lim_{s \downarrow 0} \mathrm{U}_{\sigma,\pm}(sf) = \mathrm{I}$ we conclude :

$$\frac{d}{ds} \mathrm{U}_{\sigma,\pm}(sf)\, \theta \Big|_{s=0} = \frac{i}{\sqrt{2}} \{ b^*_{\sigma,\pm}(f) + b_{\sigma,\pm}(f) \} \theta.$$

We now repeat the same arguments with $\mathrm{U}_b(sf)$ replaced by $\mathrm{V}_b(sf)$ This complete the proof of (ii).

Q. E. D.

Remarks :

(1) Lemma 4.2, (ii) implies that $\mathrm{U}_{\sigma,\pm}(f)$ and $\mathrm{V}_{\sigma,\pm}(g)$ satisfy the Weyl relations and that the mappings

$$\mu_{\sigma,\pm} \begin{cases} \overline{\Delta_b(\mathrm{V})} \to \overline{\Delta_{\sigma,\pm}(\mathrm{V})} \\ \left.\begin{array}{l} \mathrm{U}_b(f) \mapsto \mathrm{U}_{\sigma,\pm}(f) \\ \mathrm{V}_b(g) \mapsto \mathrm{V}_{\sigma,\pm}(g) \end{array}\right\} [f, g] \in \mathrm{V} \end{cases}$$

are C* isomorphisms. The algebras $\overline{\Delta_{\sigma,\pm}(\mathrm{V})}$ are called algebras of asymptotic boson observables.

(2) Lemmas 4.1 and 4.2 still hold on the spaces $\mathcal{H}^{(Z)}$, $0 \leq Z < \infty$ (by essentially the same proofs).

4.2. Haag-Ruelle theory on the space $\mathcal{H}^{(1)}$ for the models with IR cutoff $\sigma > 0$

In this section we want to define a family of time dependent states converging strongly to a scattering state, as $t \to \pm\infty$. This result combined with the results of section 4.1 establishes a strong convergence asymptotic condition in $\mathcal{H}^{(1)}$ in the sense of Haag and Ruelle. Since we are not primarily interested in the scattering theory on $\mathcal{H}^{(1)}$ for $\sigma > 0$ but rather in the one for $\sigma = 0$ our procedure in this section will be such that it suggests the correct procedure in the limit $\sigma = 0$.

DEFINITIONS :

$$(4.12)\quad \begin{cases} w_{\sigma,p}(k) = \lambda\, g_\sigma(k)\,|k|^{-1/2}\,(|k| - (k, \nabla_p \mathrm{E}(\sigma, p)))^{-1}\,\mathfrak{I}(k), \\ \mathrm{C}_{\sigma,p}(t) = \exp - i\,\{\Pi(w_{\sigma,p}\cos(|k|\,t)) + \Phi(w_{\sigma,p}\sin(|k|\,t))\}, \\ \mathrm{V}_{\sigma,p}(x) = \exp i\,\{\Pi(w_{\sigma,p}\cos(kx)) + \Phi(w_{\sigma,p}\sin(kx))\}, \\ \Phi_{\sigma,p}(t) = (g_\sigma\,|k|^{-3/2},\ w_{\sigma,p}\sin(|k|\,t)). \end{cases}$$

The dressed one electron state for $\sigma > 0$ in the configuration space representation is given by

$$e^{ixp}\,\mathrm{I}_p^*\,e^{-ix\mathrm{P_B}}\,\psi_1(\sigma, p) = e^{ixp}\,\mathrm{I}_p^*\,\mathrm{V}_{\sigma,p}(x)\,e^{-ix\mathrm{P_B}}\,\theta_{\sigma,p},$$

where $\theta_{\sigma,p} = \mathrm{V}^*_{\sigma,p}\,\psi_1(\sigma, p)$ is a state in $\mathcal{F}_\mathrm{B}$.

$\mathrm{Q}_{\sigma,p}$ denotes the selfadjoint projection onto $\theta_{\sigma,p}$ and

$$\psi_{\sigma,p}(x) = e^{-ix\mathrm{P_B}}\,\mathrm{Q}_{\sigma,p}\,\psi_0.$$

We have shown in II, *a* that $(\psi_1(\sigma, p), \mathrm{V}_{\sigma,p}\,\psi_0) > 0$, such that $\psi_{\sigma,p}(x) \neq \vec{0}$.

We pick a C^∞ function h of compact support such that supp h is in $\mathrm{M}_{\rho_0(\lambda)} = \{\, p/\,|p| < \rho_0(\lambda)\,\}$ and form the state

$$(4.13)\qquad \theta_{\sigma,t}(h\,|\,x) = (2\pi)^{-3/2}\int d^3p\,h(p)\,e^{i(xp - t\mathrm{E}(\sigma,p))} \times \mathrm{I}_p^*\,\mathrm{C}_{\sigma,p}(t)\,\mathrm{V}_{\sigma,p}(x)\,\psi_{\sigma,p}(x).$$

Obviously $\{\,\theta_{\sigma,t}(h\,|\,x)\,\}$ is a family of states contained in $\mathcal{F}_b$ (for all $|x| < \infty$ and $|t| < \infty$) which is strongly continuous in x. We are now ready for :

THEOREM 4.3. — *Let σ be positive and h be a C^∞ function of compact support such that* supp h *is in* $\mathrm{M}_{\rho_0(\lambda)}$.

Then :

(i) $\theta_{\sigma,t}(h) = \theta_{\sigma,t}(h\,|\,.)$ *is a vector in* $\mathcal{H}^{(1)}$, *in particular*

$$\|\,\theta_{\sigma,t}(h)\,\|^2_{\mathcal{H}^{(1)}} = \int d^3x\,\|\,\theta_{\sigma,t}(h\,|\,x)\,\|^2_{\mathcal{F}_b} < \infty \quad \textit{for all } |t| < \infty.$$

(ii) $\text{s-}\lim_{t\to\pm\infty} e^{i\mathrm{H}(\sigma)t}\,\theta_{\sigma,t}(h) \equiv \theta_{\sigma,\pm}(h)$ *exists and defines a scattering state in* $\mathcal{H}^{(1)}$.

(iii) *Let* A *be in* $\overline{\Delta_b(\mathrm{V})}$ *and* $\mathrm{A_B} = \mathrm{I}_{b,\mathrm{B}}\,\mathrm{A}\,\mathrm{I}^*_{b,\mathrm{B}}$ *its image in* $\overline{\Delta(\mathrm{V})}$.

Then

$$(4.14)\quad \lim_{t\to\pm\infty}(\theta_{\sigma,t}(h),\ \mathrm{A}_t\,\theta_{\sigma,t}(h')) = \int d^3p\,\overline{h(p)}\,h'(p)\,(\psi_0,\ \mathrm{Q}_{\sigma,p}\,\psi_0)\,(\mathrm{C}_{\sigma,p}\,\psi_0,\ \mathrm{A_B}\,\mathrm{C}_{\sigma,p}\,\psi_0)$$

Furthermore the following intertwining relations hold :

$$e^{-i\mathrm{H}(\sigma)t}\,\mu_{\sigma,\pm}(\mathrm{A})\,e^{i\mathrm{H}(\sigma)t} = \mu_{\sigma,\pm}(\mathrm{A}_t) \tag{4.15}$$

and

$$(\theta_{\sigma,\pm}(h),\ \mathrm{A}_{\sigma,\pm}\,e^{i\mathrm{H}(\sigma)t}\,\theta_{\sigma,\pm}(h')) = \int d^3p\,h(p)\,h'(p)\,e^{it\mathrm{E}(\sigma,p)}\,(\psi_0,\ \mathrm{Q}_{\sigma,p}\,\psi_0)\,(\mathrm{C}_{\sigma,p}\,\psi_0,\ \mathrm{A}_{\mathrm{B}}\,\mathrm{C}_{\sigma,p}(t)\,\psi_0). \tag{4.16}$$

The *proof* of theorem 4.3 is given in appendix 2.

Remarks :

(1) Because of theorem 4.3, (i) we can represent the vector $\theta_{\sigma,t}(h)$ in the following way :

$$\theta_{\sigma,t}(h) = \int d^3p\,e^{-it\mathrm{H}_{0b}}\exp\frac{1}{\sqrt{2}}\{\,b^*(w_{\sigma,p}) - b(w_{\sigma,p})\,\}\,e^{it\mathrm{H}_{0b}} \times n^*(p-\mathrm{P}_b)\,\mathrm{I}_p^*\,\psi_1(\sigma,p)\,h(p)\,e^{-it\mathrm{E}(\sigma,p)}.$$

Interpretation of $\theta_{\sigma,t}(h)$: $\theta_{\sigma,t}(h)$ represents a *dressed electron* [described by the $\mathcal{H}^{(1)}$-valued distribution $n^*(p-\mathrm{P}_b)\,\mathrm{I}_p^*\,\psi_1(\sigma,p)$] with wave function $h(p)\,e^{-it\mathrm{E}(\sigma,p)}$ which is surrounded by a *cloud of real bosons* $\left[\text{described by the family of operators } \exp\frac{1}{\sqrt{2}}\{\,b^*(w_{\sigma,p}) - b(w_{\sigma,p})\,\}\right]$.

The vector $\theta_{\sigma,t}(h)$ is the image of $\theta_{\sigma,t=0}(h)$ under the "*free*" time evolution (at ime t).

(2) The relations (4.15) and (4.16) show that the dynamics determined by $e^{it\mathrm{H}(\sigma)}$ is the one of a free electron and free bosons with respect to the *asymptotic* boson observables in the algebra $\overline{\Delta_{\sigma,\pm}(\mathrm{V})}$ and the *scattering states*

$$\mathrm{A}_{\sigma,\pm}\,\theta_{\sigma,\pm}(h) \qquad [\mathrm{A}\in\overline{\Delta_b(\mathrm{V})}].$$

Combination of lemma 4.2, (ii) and theorem 4.3, (ii) yields :

$$s\text{-}\lim_{t\to\pm\infty} e^{i\mathrm{H}(\sigma)t}\,\mathrm{A}_t\,\theta_{\sigma,t}(h) = \mathrm{A}_{\sigma,\pm}\,\theta_{\sigma,\pm}(h). \tag{4.17}$$

for all A in $\overline{\Delta_b(\mathrm{V})}$.

It is rather easy to show that one can replace in (4.17) the operator A_t by a polynomial in the operators

$$b(f_t),\quad b^*(f_t),\quad f \text{ in } \mathcal{S}_0(\mathbf{R}^3).$$

We shall not prove this here. (For similar results *see* [1], [11], [15].)

We can now define spaces of scattering states :

(4.18) $\mathcal{H}^{(1)}_{\sigma,\pm} = \{ A_{\sigma,\pm}\, \theta_{\sigma,\pm}(h) \mid A \in \overline{\Delta_b(V)},\ h \in L^2(M_{\rho_0(\lambda)}, d^3k) \}$

and transition amplitudes :

(4.19) $(\psi_{\sigma,+}, \varphi_{\sigma,-})_{\mathcal{H}^{(1)}}, \quad \varphi_{\sigma,-} \in \mathcal{H}^{(1)}_{\sigma,-}, \quad \psi_{\sigma,+} \in \mathcal{H}^{(1)}_{\sigma,+}.$

We conclude this section with a *remark on the scattering theory on* $\mathcal{H}^{(Z)}$, $1 < Z < \infty$:

Let K be an arbitrary compact set in $M_{\rho_0(\lambda)}$ and

$$\Delta_K = \inf_{p \in K} \{ \inf(\text{spec}(H_{\sigma,p} \restriction \mathcal{F}_B(K_\sigma)) \setminus \{ E(\sigma, p) \}) - E(\sigma, p) \},$$

where $\mathcal{F}_B(K_\sigma)$ has been defined in section 1.2.

From theorem 1.4 we know that $\Delta_K > 0$ (i. e. on the region K in momentum space the one particle shell is isolated).

Let $\mathcal{G}_K$ be the class of C^∞ functions $g(p, p^0)$ on $\mathbf{R}^4$ such that

$$g(p, E(\sigma, p)) > 0 \quad \text{for all } p \text{ in supp } g$$

and

$$\text{supp } g \subseteq \left\{ (p, p^0) \mid p \in K, \middle| \, | p^0 - E(\sigma, p) | \leq \frac{\Delta_K}{2} \right\}.$$

DEFINITION :

$$g_t(p, p^0) = g(p, p^0)\, e^{-itE(\sigma,p)}.$$

The function $\check{g}_t(x, s)$ denotes the four dimensional Fourier transformed of $g_t(p, p^0)$ and

$$\psi^*(\check{g}_t) = \int d^3x\, ds\, e^{iH(\sigma)s}\, \psi^*(x)\, e^{-iH(\sigma)s}\, \check{g}_t(x, s).$$

Then one can show

$$\text{s-lim}_{t \to \pm\infty} e^{iH(\sigma)t} \prod_{i=1}^{Z} \psi^*(\check{g}_{i,t})\, \varphi_0 \equiv \prod_{i=1}^{Z} \psi^*_{\sigma,\pm}(g_i)\, \varphi_0$$

exists if $g_i \in \mathcal{G}_K$, $1 \leq i \leq Z$.

This result and lemma 4.2 lead to a strong convergence asymptotic condition in time on $\mathcal{H}^{(Z)}$ in the sense of Haag and Ruelle and to a reasonable collision theory (with an S-matrix which has spatial cluster decomposition properties); *see* [15].

These methods do however *not* have a straightforward generalization to the limiting case $\sigma = 0$.

4.3. Haag-Ruelle theory in the limit $\sigma = 0$

From chapter 3 we know that in the limiting case where $\sigma = 0$ there is a unique dressed one electron state (DES) Ω_p in $\mathcal{H}^{(1)}(w_p)$ for all p in $\mathcal{E}$ (where $\mathcal{E}$ was defined in lemma 3.1 and $\overline{M}_{\rho_0} \setminus \mathcal{E}$ is of measure 0).

$\mathcal{H}^{(1)}(w_p)$ is the IDPS corresponding to the product reference vector

$$\bigotimes_{\nu=0}^{\infty} \exp i\,\Pi(w_p\,\chi_\nu)\,\psi_0 \equiv V_p\,\psi_0,$$

where

$$w_p(k) = \lambda\,|\,k\,|^{-1/2}\,(|\,k\,| - (k,\ \nabla E(p)))^{-1}\,\mathfrak{z}(k)$$

and χ_ν is the characteristic function of K_ν (section 2.1).

$\Omega_p = V_p\,\theta_p$, where θ_p is a vector in $\mathcal{F}_B$.

We define Q_p to be the s. a. projection onto θ_p if p is in $\mathcal{E}$ and onto ψ_0 if p is in $\overline{M}_{\rho_0} \setminus \mathcal{E}$, and we set

$$\psi_p(x) = e^{-ix\cdot P_B}\,Q_p\,\psi_0. \tag{4.20}$$

Since $e^{ix\cdot P_B} \restriction \mathcal{F}_B$ and $e^{ix\hat{P}_B} \restriction \mathcal{H}^{(1)}(w_p)$ are strongly continuous unitary groups,

$$V_p(x) = e^{-ix\cdot \hat{P}_B}\,V_p\,e^{ix P_B} \tag{4.21}$$

is a strongly continuous family of unitary mappings from $\mathcal{F}_B$ onto $\mathcal{H}^{(1)}(w_p)$ (which intertwine the representations π_F and π_p).

$e^{itH_{0B}} \restriction \mathcal{F}_B$ and $e^{it\hat{H}_{0B}} \restriction \mathcal{H}^{(1)}(w_p)$ are strongly continuous unitary groups and therefore

$$C_p(t) = e^{-itH_{0B}} \exp - i\,\Pi(w_p)\,e^{it\hat{H}_{0B}} \tag{4.22}$$

defines a strongly continuous family of unitary mappings from $\mathcal{H}^{(1)}(w_p)$ onto $\mathcal{F}_B$ [and from $\mathcal{F}_B$ to $\mathcal{H}^{(1)}(-w_p)$].

Hence $C_p(t)\,V_p(x)$ is a family of Weyl operators on $\mathcal{F}_B$ which is strongly continuous in t and x.

DEFINITION :

$$\Phi_p(t) = -\lambda \int d^3k\,|\,k\,|^{-3/2}\,w_p(k)\sin(|\,k\,|\,t). \tag{4.23}$$

Obviously $\Phi_p(t)$ is real such that $e^{i\Phi_p(t)}$ is a phase factor of modulus 1,

$$|\,\Phi_p(t)\,| \propto \log t, \qquad \text{as} \quad |\,t\,| \to \infty.$$

We define

$$(4.24)\qquad \theta_t(p \mid x) = e^{i(xp - t\,E(p))}\, e^{i\,\Phi_p(t)}\, C_p(t)\, V_p(x)\, \psi_p(x)$$

which is a vector in $\mathcal{F}_B$ for all t in $\mathbf{R}$ and x in $\mathbf{R}^3$.

The intuitive meaning of $\theta_t(p \mid x)$ is $\theta_t(p \mid x)$ is the configuration space representation of a DES of momentum p at time t surrounded by a cloud of infinitely many real bosons [described by $e^{i\,\Phi_p(t)}\, C_p(t)$].

DEFINITION :

$$(4.125)\qquad \theta_t(h \mid x) = (2\pi)^{-3/2} \int d^3 p\, h(p)\, I_p^*\, \theta_t(p \mid x).$$

We have to verify that this important definition makes sense.

(1) Since the projection Q_p is s. a. and $e^{ix\,P_B}$ is unitary, $\| \psi_p(x) \| \leq 1$ for all p, $| p | \leq \rho_0$, and all x in $\mathbf{R}^3$. The operator $C_p(t)\, V_p(x)$ is unitary. Hence $\| C_p(t)\, V_p(x) \| = 1$. Therefore $\| \theta_t(p \mid x) \|_{\mathcal{F}_B} \leq 1$ for all p, $| p | \leq \rho_0$, and all (t, x).

(2) We want to show that $\theta_t(p \mid x)$ is a weakly measurable $\mathcal{F}_B$-valued function of p for all t and x, $| t | + | x | < \infty$. Since $\| \theta_t(p \mid x) \|_{\mathcal{F}_B} \leq 1$ for all $| p | \leq \rho_0$, it follows that $\theta_t(p \mid x)$ is strongly integrable on each compact set

$$K \subseteq \overline{M}_{\rho_0} = \{ p / | p | \leq \rho_0 \}.$$

LEMMA 4.4. — *For any $\psi \in \mathcal{F}_B$, $Q_p\, \psi$ is weakly measurable in p and strongly integrable on each compact set $K \subseteq \overline{M}_{\rho_0}$.*

$$(4.26)\qquad (\psi_0, Q_p\, \psi_0) > 0 \quad \textit{for all } p, \qquad | p | \leq \rho_0.$$

Remark. — It follows from (4.26) that

$$(4.27)\qquad \int_K d^3 p\, \psi_p(x) \neq \vec{0}$$

for any compact set $K \subseteq \overline{M}_{\rho_0}$ of positive Lebesgue measure.

LEMMA 4.5. — *For all t in $\mathbf{R}$ and x in $\mathbf{R}^3$, $e^{i\,\Phi_p(t)}\, C_p(t)\, V_p(x)$ is weakly measurable in p in $\overline{M}_{\rho_0}$, hence strongly integrable on each compact set $K \subseteq \overline{M}_{\rho_0}$.*

The *proofs* of lemmas 4.4 and 4.5 are given in the appendix 3.

COROLLARY 4.6. — *Let h be in $L^1(\mathbf{R}^3)$ and* supp $h \subseteq \overline{M}_{\rho_0}$. *Then*

$$(4.28) \quad (2\pi)^{-3/2} \int d^3 p \, I_p^* \theta_t (p \mid x) h(p)$$

$$= (2\pi)^{-3/2} \int d^3 p \, h(p) e^{i(xp - t E(p))} e^{i \Phi_p(t)} I_p^* C_p(t) V_p(x) \psi_p(x)$$

exists in the strong sense and defines a vector $\theta_t(h \mid x)$ in $\mathcal{F}_b$.

Remark. — Formulas (4.26) and (4.27) imply that $\theta_t(h \mid x) \neq \vec{0}$ if $h \geq 0$ $(h \neq 0)$.

Proof. — Let $\{ \varphi_m \}_{m=0}^{\infty} \subset \mathcal{F}_B$ be a complete orthonormal system and ψ an arbitrary state in $\mathcal{F}_B$. We define :

$$F_N(p) = e^{i \Phi_p(t)} \sum_{m=0}^{N} (V_p(x)^* C_p(t)^* \psi, \varphi_m)(\varphi_m, \psi_p(x)).$$

Obviously $F_N(p)$ is measurable in p and $| F_N(p) | \leq \| \psi \|$ for all $N < \infty$. Furthermore

$$F_N(p) \to F(p) = e^{i \Phi_p(t)} (\psi, \; C_p(t) V_p(x) \psi_p(x))$$

as $N \to \infty$, for all p in $\mathcal{E}$.

Hence $F(p)$ is measurable and therefore $h(p) \theta_t (p \mid x)$ is weakly measurable and strongly integrable.

Thus $\theta_t(h \mid x)$ is in $\mathcal{F}_b$, for all t in $\mathbf{R}$ and x in $\mathbf{R}^3$,

$$\theta_{t=0}(h \mid x = 0) = \int d^3 p \, h(p) I_p^* \psi_p(0) = \int d^3 p \, h(p) I_p^* Q_p \psi_0 .$$

Because of (4.26) it follows that $\theta_{t=0}(h \mid x = 0) \neq \vec{0}$ if $h > 0$. But, since $\theta_t(h \mid x)$ is strongly continuous in t and x,

$$\theta_t(h \mid x) \neq \vec{0} \quad \text{if } |t| + |x| \text{ is sufficiently small.}$$

Q. E. D.

We now estimate $\| \theta_t(h \mid x) \|^2_{\mathcal{F}_b}$:

$$(4.29) \quad \| \theta_t(h \mid x) \|^2_{\mathcal{F}_b}$$

$$= (2\pi)^{-3} \int d^3 p \, dp' \, \overline{h(p)} \, h(p') (\theta_t(p \mid x), \theta_t(p' \mid x))_{\mathcal{F}_b}$$

$$= (2\pi)^{-3} \int d^3 p \, d^3 p' \, \overline{h(p)} \, h(p') \, e^{i(x(p'-p) - t(E(p') - (E(p)))}$$

$$\times e^{i(\Phi_{p'}(t) - \Phi_p(t))} G_t(p, p' \mid x),$$

where

$$(4.30)\quad G_t(p, p' \mid x) = (C_p(t)\, V_p(x)\, \psi_p(x),\ C_{p'}(t)\, V_{p'}(x)\, \psi_{p'}(x))_{\mathcal{F}_B}.$$

Since $\operatorname*{ess\,sup}_{|p| \leq \rho_0, |p'| \leq \rho_0} |(\theta_t(p \mid x), \theta_t(p' \mid x))_{\mathcal{F}_B}| \leq 1,$

$$(4.31)\qquad \| \theta_t(h \mid x) \|^2_{\mathcal{F}_b} \leq (2\pi)^{-3} \| h \|_1^2.$$

Let us assume for the moment that

$$(4.32)\qquad \int d^3 x \,| x |^\varepsilon \,\| \theta_t(h \mid x) \|^2_{\mathcal{F}_b} < \infty$$

for some $\varepsilon > 0$ and all t in $\mathbf{R}$.

Then $\theta_t(h) = \theta_t(h \mid .)$ is in $\mathcal{H}^{(1)}$, and we conjecture :

For any A in $\overline{\Delta_b(S_0)}$:

$$(4.33)\quad \lim_{t \to \pm\infty} (\theta_t(h),\ A_t\, \theta_t(h'))$$

$$= \int d^3 p\, \overline{h(p)}\, h'(p)\, (\psi_0,\ Q_p\, \psi_0)\, (C_p\, \psi_0,\ A_B\, C_p\, \psi_0)_{\mathcal{H}^{(1)}(-w_p)},$$

$\operatorname*{s\text{-}lim}_{t \to \pm\infty} e^{iHt}\, \theta_t(h)$ exist and define vectors $\theta_\pm(h)$ in $\mathcal{H}^{(1)}$.

Unfortunately neither (4.32) nor (4.33) have been rigorously proven, until now. We want to give certain *heuristic arguments* why (4.32) and (4.33) could be true in our model and why our choice of the phase $\Phi_p(t)$, (4.23), seems to be the appropriate one :

$$\int d^3 x \,| x |^\varepsilon \,\| \theta_t(h \mid x) \|^2$$

$$= (2\pi)^{-3} \int d^3 x \,| x |^\varepsilon \int d^3 p\, d^3 p'\, h(p)\, h(p')\, e^{i(\Phi_{p'}(t) - \Phi_p(t))}$$

$$\times\ e^{i(x(p-p') - t(E(p') - E(p)))}\, G_t(p, p' \mid x).$$

Second order perturbation theory indicates that $E(p)$ is C^∞ in p and that $G_t(p, p'/x)$ is C^∞ in p and p' with bounds on the derivatives that increase linearly in x. This is more than we need in order to prove (4.32). It would suffice that $\nabla E(p)$ is twice differentiable in p on M_{ρ_0} and that $G_t(p, p'/x)$ is twice continuously differentiable in p and p' with derivatives that are bounded by const. $\times | x |^\alpha$ $(\alpha < 1)$.

Motivation of (4.33). — For the first part of (4.33) we refer to an analogous result proven in theorem 4.3, (iii). This part of (4.33) is formally true.

In order to motivate the second part of (4.33) we should look at $\frac{d}{dt} e^{i\mathrm{H}t} \theta_t (h)$.

This part of (4.33) holds if $\int dt \left\| \frac{d}{dt} e^{i\mathrm{H}t} \theta_t (h) \right\|_{\mathcal{H}^{(1)}}$ exists.

We cannot prove the existence of this integral but we shall show :

Suppose that h is in $\mathrm{L}^1 (\mathbf{R}^3)$ and supp $h \subset \mathrm{M}_{\rho_0}$ and that φ is a vector in $\mathcal{H}^{(1)}$ such that $\int d^3 p \, \| \varphi (p) \|_{\mathcal{F}_\mathrm{B}} < \infty$, where

$$\varphi (p) = (2\pi)^{-3/2} \, \mathrm{I}_p \int d^3 x \, e^{ix(p - \mathrm{P}_b)} \varphi (x).$$

Then

$$(4.34) \quad \int dt \left| \left((e^{-i\mathrm{H}t} \varphi)(x), \left(e^{-i\mathrm{H}t} \left[\frac{d}{dt} e^{-i\mathrm{H}t} \theta_t (h) \right] \right)(x) \right)_{\mathcal{F}_b} \right| \quad \text{exists.}$$

Remark. — Formula (4.34) is a first step towards a proof of weak convergence of $e^{i\mathrm{H}t} \theta_t (h)$ as $t \to \pm \infty$.

A bound on the integral (4.34) for large values of x that is *uniform in* t is missing.

Proof of (4.34). — It is straightforward to calculate

$$(4.35) \quad \left(e^{-i\mathrm{H}t} \left[\frac{d}{dt} e^{i\mathrm{H}t} \theta_t (h) \right] \right)(x)$$
$$= i (2\pi)^{-3/2} \int d^3 p \, h(p) \, e^{i(xp - t\mathrm{E}(p))} \, e^{i\Phi_p(t)} \, \mathrm{F}_p (t, x)$$
$$\times \mathrm{I}_p^* \, \mathrm{C}_p (t) \, \mathrm{V}_p (x) \, \psi_p (x),$$

where

$$\mathrm{F}_p (t, x) = \left\{ \frac{\partial}{\partial t} \Phi_p (t) + \mathrm{C}_p (t)^* [\mathrm{H}_\mathrm{I} (x), \mathrm{C}_p (t)] \right\} \quad \text{for all } p \text{ in } \mathcal{E},$$

and

$$\mathrm{C}_p (t)^* [\mathrm{H}_\mathrm{I} (x), \mathrm{C}_p (t)]$$
$$= \lambda \int d^3 k \, | k |^{-1/2} \{ \; w_p (k) \cos (kx) \cos (| k | t)$$
$$- w_p (k) \sin (kx) \sin (| k | t) \},$$

whence

$$\mathrm{F}_p (t, x) = \lambda \int d^3 k \, | k |^{-1/2} \{ w_p (k) (\cos (kx) - 1) \cos (| k | t)$$
$$- w_p (k) \sin (kx) \sin (| k | t) \}.$$

(One can prove (4.35) easily by using ultraviolet cutoff approximations for H and removing the cutoffs at the end of the calculations; [21].)

It follows that

$$(4.36) \qquad | F_p(t, x) | \leq \text{const.} \times | x |^{\varepsilon} (1 + | t |)^{-1-\varepsilon'}$$

for some $\varepsilon' > 0$, $1 > \varepsilon(\varepsilon') > 0$ and almost all p in supp h.

Therefore

$$\left| \left((e^{-iHt} \varphi)(x), \left(e^{-iHt} \left[\frac{d}{dt} e^{iHt} \theta_t(h) \right] (x) \right)_{\mathcal{F}_b} \right| \leq \text{const.} \times \| \varphi \|_1 \| h \|_1 | x |^{\varepsilon} (1 + | t |)^{-1-\varepsilon'}.$$

This completes the proof of (4.34).

We now want to formulate two alternative conjectures which lead to a satisfactory collision theory on $\mathcal{H}^{(1)}$ after combination with lemma 4.1 and lemma 4.2. This collision theory formally agrees with perturbation theory predictions of [6] and [12].

CONJECTURE α. — *Formulas* (4.33) *hold.*

We then conclude :

Let A *be in* $\overline{\Delta_b(V)}$. *Then*

$$(\alpha.1) \qquad \underset{t \to \pm\infty}{s\text{-lim}}\, e^{iHt} A_t \theta_t(h) = A_{\pm} \theta_{\pm}(h) \in \mathcal{H}^{(1)},$$

$$(\alpha.2) \qquad e^{-iHt} A_{\pm} e^{iHt} = (A_t)_{\pm}$$

and

$$(\theta_{\pm}(h), A_{\pm} e^{iHt} \theta_{\pm}(h')) = \int d^3 p\, h(p)\, h'(p)\, e^{-itH(p)} (\psi_0, Q_p \psi_0) (C_p \psi_0, A_B C_p(t) \psi_0)_{\mathcal{H}^{(1)}(-w_p)}.$$

The proofs of (α.1) and (α.2) are trivial. [Since the free time evolution is a strongly continuous unitary group on $\mathcal{H}^{(1)}(-w_p)$, $C_p(t)\psi_0$ is in $\mathcal{H}^{(1)}(-w_p)$ and $(C_p \psi_0, A_B C_p(t) \psi_0)_{\mathcal{H}^{(1)}(-w_p)}$ is well defined.]

CONJECTURE β. — *Let* $\theta_{\sigma,\pm}(h)$ *be the scattering states for the dynamics determined by* $e^{iH(\sigma)t}$ *which were obtained in theorem* 4.3, (ii). *Let* h *be a* C^∞ *function and* supp $h \subset M_{\rho_0(\lambda)}$ *and define :*

$$h_\sigma(p) = h(p) \| Q_{\sigma,p} \psi_0 \|^{-1}.$$

Then there is a sequence $\{ \sigma_k \}_{k=0}^{\infty}$ *converging to* 0 *and a sequence of phases* $\{ \Phi_k \}_{k=0}^{\infty}$ *such that*

$$(4.37) \qquad \underset{k \to \infty}{s\text{-lim}}\, e^{i\Phi_k} \theta_{\sigma_k,\pm}(h_{\sigma_k}) \equiv \theta_{\pm}(h \| Q_p \psi_0 \|^{-1}) \quad \textit{exists.}$$

We then conclude :

$$(\beta.1)\qquad \underset{k\to\infty}{s\text{-lim}}\, A'_{\sigma_k,\pm}\, e^{i\Phi_k}\, \theta_{\sigma_k,\pm}(h_{\sigma_k}) = A_{\pm}\,\theta_{\pm}(h\,\|\,Q_p\,\psi_0\,\|^{-1}) \quad \textit{for all } A \textit{ in } \overline{\Delta_b(V)},$$

$$\Big(\underset{t\to\pm\infty}{s\text{-lim}}\, e^{iHt}\, A_t\, e^{-iHt}\Big)\Big(\underset{k\to\infty}{s\text{-lim}}\, e^{i\Phi_k}\,\theta_{\sigma_k,\pm}(h_{\sigma_k})\Big) = A_{\pm}\,\theta_{\pm}(h\,\|\,Q_p\,\psi_0\,\|^{-1}) \quad \textit{for all } A \textit{ in } \overline{\Delta_b(V)}.$$

(β.2) *Formula* (α.2) *hold.*

The proofs of (β.1) and (β.2) follow easily from conjecture β, lemma 4.1 and lemma 4.2.

We do not want to explain our heuristic arguments for the correctness of conjecture β, at this place. [They involve a bound on $(\theta_{\sigma_k,\pm}(h_{\sigma_k}), N_b\,\theta_{\sigma_k,\pm}(h_{\sigma_k}))$ which is uniform in k.]

In complete analogy to (4.18) and (4.19) one can now define spaces of scattering states $\mathcal{H}^{(1)}_{\pm}$ and scattering amplitudes

$$(\psi_+, \varphi_-)_{\mathcal{H}^{(1)}}, \quad \psi_+ \in \mathcal{H}^{(1)}_+, \quad \varphi_- \in \mathcal{H}^{(1)}_-.$$

If we could prove conjecture α or conjecture β and if we knew that $\mathcal{H}^{(1)}_+ = \mathcal{H}^{(1)}_-$ (asymptotic completeness) or at least that $\mathcal{H}^{(1)}_+ \cap \mathcal{H}^{(1)}_-$ is non trivial in the sense that the transition amplitudes for physically relevant processes do not vanish, the scattering problem in the charge one sector would be solved. Keeping in mind that perturbation theory for the calculation of scattering states and the S-matrix expresses all the operators in terms of asymptotic fields and observables [for e. g. $t = -\infty$, corresponding to the algebra $\overline{\Delta_-(V)}$] we observe that, once conjectures α or β are proven, our solution is in complete, formal agreement with the perturbation theoretic proposals of Faddeev and Kulish, [12]; *see* chapter 5.

However, we neither are able to prove conjectures, α or β, nor have we therefore attempted to investigate whether $\mathcal{H}^{(1)}_+ = \mathcal{H}^{(1)}_-$.

Fortunately there are some rigorous results available which in principle contain the complete information on the scattering in $\mathcal{H}^{(1)}$ and which yield some insight into a possible general approach to the scattering problem in theories without a massgap and without one particle shells of charged particles. These results are however not explicit at all.

The type of problem we consider in the next section is the following :

Given a vector ψ in $\mathcal{H}^{(1)}$. Then ψ determines representations π^+_ψ, π^-_ψ of the algebras $\overline{\Delta_+(V)}$, $\overline{\Delta_-(V)}$, respectively.

PROBLEM. — What are the conditions on the representations π_ψ^+, π_ψ^- in order that the vector ψ admits a complete particle interpretation in terms of a charged particle and bosons, as $t \to \pm\infty$?

When does the set of expectation values

$$\{ (\psi, A_+ \psi) \mid A_+ \in \overline{\Delta_+ (V)} \}$$

contain a complete informations about the configuration of asymptotic particles at $t = +\infty$ determined by the vector ψ, the momentum distribution of the *charged particle included* ?

4.4. An algebraic framework for a collision theory on $\mathcal{H}^{(1)}$

In this section we want to study the representation $\pi_{\mathcal{H}^{(1)}}$ of the algebras $\overline{\Delta_\pm (V)}$. We shall state our results in terms of the algebras $\overline{\Delta_+ (V)}$ $\overline{\Delta_+ (V)}'$, wherever this is possible. But these results are of course symmetric under exchange of $\overline{\Delta_+ (V)}$ and $\overline{\Delta_- (V)}$.

(1) *The space-time translation automorphisms of* $\overline{\Delta_+ (V)}$:

DEFINITION. — Let C be a bounded operator on $\mathcal{H}^{(1)}$. We define

$$\tau_{t,x} (C) = e^{i(tH - xP)}\, C\, e^{-i(tH - xP)}.$$

If A is in $\overline{\Delta_b (V)}$ we define

$$\tau^0_{t,x} (A) = e^{i(tH_{0b} - xP_b)}\, A\, e^{-i(tH_{0b} - xP_b)}.$$

From the *intertwining relations* (α.2) we get

$$\tau_{t,x} \circ \mu_+ (A) = \mu_+ \circ \tau^0_{t,x} (A), \quad \text{for all A in } \overline{\Delta_b (V)}.$$

(This follows from the strong convergence of $e^{iHt} A_t e^{-iHt}$ to $\mu_+(A)$, as $t \to \infty$.) We conclude that $\{ \tau_{t,x} \}$ defines an automorphism group of $\overline{\Delta_+ (V)}$.

LEMMA 4.7. — *The automorphisms* $\tau_{t,x}$ *of* $\overline{\Delta_+ (V)}$ *can be unitarily implemented on* $\mathcal{H}^{(1)}$:

$$\tau_{t,x} \circ \mu_+ (A) = e^{i(tH_{0+} - xP_+)}\, \mu_+ (A)\, e^{-i(tH_{0+} - xP_+)} \tag{4.38}$$

in such a way that the generators H_{0+} *and* P_+ *are affiliated with* $\pi_{\mathcal{H}^{(1)}} \left(\overline{\Delta_+ (V)}\right)''$ *and* H_{0+} *is bounded below.*

The operators H_{0+} *and* P_+ *are unique up to selfadjoint operators from the center of* $\pi_{\mathcal{H}^{(1)}} \left(\overline{\Delta_+ (V)}\right)''$. *The portions of* H_{0+} *and* P_+ *belonging to* *the center of* $\pi_{\mathcal{H}^{(1)}}\left(\overline{\Delta_+(\mathrm{V})}\right)''$ *can be chosen such that* H_{0+} *and* P_+ *are given in the weak sense by*

$$(4.39)\qquad \begin{cases} \mathrm{H}_{0+} = \int d^3k\, b_+^*(k)\,|k|\,b_+(k) \geq 0, \\ \mathrm{P}_+ \;\; = \int d^3k\, b_+^*(k)\,k\,b_+(k), \end{cases}$$

whence

$$(4.40)\qquad \begin{cases} \Phi_+(f) \leq \mathrm{H}_{0+} + \frac{1}{2}\|\,|k|^{-1/2} f\|_2^2, \\ \Pi_+(f) \leq \mathrm{H}_{0+} + \frac{1}{2}\|\,|k|^{-1/2} f\|_2^2 \end{cases}$$

for all f *in* $\mathcal{V}$, *where* $\Phi_+(f) = \frac{1}{\sqrt{2}}\{\,b_+^*(f) + b_+(f)\,\},\ldots$

Remark. — The asymptotic creation and annihilation operators $b_+^\#(.)$ have been obtained in lemme 4.2.

Proof. — (4.38) is proven in [33], p. 164, in quite a general context. The general theorem mentioned there is due to Borchers.

For the proof of (4.39) and (4.40) we need the easy estimates

$$\Phi(f_t) \leq \text{const.}\,(\mathrm{H} + c),$$
$$\Pi(f_t) \leq \text{const.}\,(\mathrm{H} + c)$$

for some $c > 0$ and f in $\mathcal{V}_r$ and uniformly in t.

Hence

$$\Phi_+(f) \leq \text{const.}\,(\mathrm{H} + c),$$
$$\Pi_+(f) \leq \text{const.}\,(\mathrm{K} + c).$$

From these estimates one can deduce that $e^{i\Phi_+(g)}\,\mathrm{H}_{0+}\,e^{-i\Phi_+(g)}$, are densely defined, positive quadratic forms, $(\|\,g\,|\,k\,|\,\|_2 < \infty)$. This implies (4.40). The details of the proof are given in [16].

Q. E. D.

(2) *Existence of asymptotic boson counters.*

We shall show that the particle number operators

$$\mathrm{N}_{+\rho} = \int_{|k| \geq \rho} d^3k\, b_+^*(k)\, b_+(k)$$

(i. e. the asymptotic boson counters) exist and are positive selfadjoint operators on $\mathcal{H}^{(1)}$, for all $\rho > 0$.

This is implied by the following lemma :

LEMMA 4.8. — *For any* $\rho > 0$, $\pi_{\mathcal{H}^{(1)}}\left(\overline{\Delta_+(V_{\rho,\infty})}\right)$ *is quasi equivalent to the Fock representation of* $\overline{\Delta_+(V_{\rho,\infty})}$, *i. e.* $\pi_{\mathcal{H}^{(1)}}\left(\overline{\Delta_+(V_{\rho,\infty})}\right)''$ *admits a particle number operator* $N_{+\rho}$.

Hence $\left(\bigcup_{\nu=0}^{\infty} \pi_{\mathcal{H}^{(1)}}\left(\overline{\Delta_+(V_\nu)}\right)''\right)^-$ *is a proper sequential type* I_∞ *funnel in the sense of Takesaki* (2); [41].

Proof. — It is easy to derive from (4.39) and (4.40) that

$$D\left(b_+^{\#}(f)\right) \supset D(H_{0+}), \quad \text{for all } f \text{ in } \mathcal{V},$$

and $b_+^{\#}(f) : D(H_{0+}^2) \to D(H_{0+})$, if $|k| f$ is in $L^2(\mathbf{R}^3)$ and f is in $\mathcal{V}$.

From this one can deduce that there is at least one state ψ_+ in each super selection sector for $\overline{\Delta_+(V_{\rho,\infty})}$ contained in $\mathcal{H}^{(1)}$ such that $b_+(f)\psi_+ = \vec{0}$, for all f in $L^2(K_{\rho,\infty}, d^3k)$ and all $\rho > 0$.

The second part of lemma 4.8 follows then from the definitions given in [41].

Q. E. D.

(3) *Space-time translations for the asymptotic charge.*

We would like to show that $H_+^c = H - H_{0+}$ is selfadjoint and bounded below on $\mathcal{H}^{(1)}$ and that $P_+^c = P - P_+$ is a triple of selfadjoint operators on $\mathcal{H}^{(1)}$. If this is true one would like to identify (H_+^c, P_+^c) with the energy momentum operator of asymptotic charged particles.

LEMMA 4.9 :

(i) *The operators* $e^{ixP}e^{-ixP_+}$ *and* $e^{itH}e^{-itH_{0+}}$ *form unitary groups on* $\mathcal{H}^{(1)}$ *which are in the commutant of* $\pi_{\mathcal{H}^{(1)}}\left(\overline{\Delta_+(V)}\right)$. *Their selfadjoint infinitesimal generators are given by*

$$P_+^c = P - P_+, \qquad H_+^c = H - H_{0+}.$$

These operators are affiliated with $\pi_{\mathcal{H}^{(1)}}\left(\overline{\Delta_+(V)}\right)'$ *and* H_+^c *is bounded below on* $\mathcal{H}^{(1)}$.

(ii) *We denote the spectral projections of* (H_+^c, P_+^c) *by* $F_+^c(\Delta)$, *where* Δ *is a Borel set in* $\mathbf{R}^4$.

Obviously $F_+^c(\Delta)$ *is in* $\pi_{\mathcal{H}^{(1)}}\left(\overline{\Delta_+(V)}\right)'$, $F_+^c(\Delta)\,\pi_{\mathcal{H}^{(1)}}\left(\overline{\Delta_+(V)}\right)''$ is *a von Neumann algebra.*

(2) We thank Prof. Takesaki for reference [41].

The restriction of a normal state ρ *on* $B(\mathcal{H}^{(1)})$ *to* $\overline{\Delta_+(V)}$ *can be decomposed on the spectrum* Σ *of* (H_+^c, P_+^c) :

$$(4.41) \quad \rho\,(e^{i(tH_+^c - xP_+^c)} A_+) = \int_\Sigma e^{i(t\omega - xp)}\, d^4 \rho\,(F_+^c(\omega, p) A_+ F_+^c(\omega, p))$$

for all A_+ *in* $\overline{\Delta_+(V)}$.

Proof :

(i) We first show that e^{ixP_+} commutes with e^{iyP} and e^{isH}, and $e^{itH_{0+}}$ commutes with e^{iyP} and e^{isH}.

Proof : e^{ixP_+} is in $\pi_{\mathcal{H}^{(1)}}(\overline{\Delta_+(V)})''$. From the intertwining relations (4.37) it follows that $e^{iyP} e^{iyP_+}$ is in $\pi_{\mathcal{H}^{(1)}}(\overline{\Delta_+(V)})'$. Hence

$$e^{iyP} e^{-iyP_+} e^{ixP_+} e^{iyP_+} = e^{iyP} e^{ixP_+} = e^{ixP_+} e^{iyP} e^{-iyP_+} e^{iyP_+} = e^{ixP_+} e^{iyP}.$$

In the same way we can show that e^{ixP_+} commutes with e^{isH}, etc., and the proof is complete.

Since e^{ixP} and e^{iyP_+} commute $e^{ixP} e^{-ixP_+}$ is a unitary group on $\mathcal{H}^{(1)}$. It has an infinitesimal generator P_+^c which is the closure of $P - P_+$. Since $e^{ixP} e^{-ixP_+}$ is in $\pi_{\mathcal{H}^{(1)}}(\overline{\Delta_+(V)})'$, P_+^c is affiliated with the same von Neumann algebra.

The same arguments apply to $e^{itH} e^{-itH_{0+}}$ and H_+^c.

It is easy to show that inf spec $H_{0+} = 0$ and that

$$\text{inf spec } H = \text{inf spec } H_{0+} + \text{inf spec } H_+^c.$$

This completes the proof of (i).

(ii) Obviously $F_+^c(\Delta)$ is in $\pi_{\mathcal{H}^{(1)}}(\overline{\Delta_+(V)})'$.

Therefore

$$F_+^c(\Delta)\, \pi_{\mathcal{H}^{(1)}}(\overline{\Delta_+(V)})'' = F_+^c(\Delta)\, \pi_{\mathcal{H}^{(1)}}(\overline{\Delta_+(V)})''\, F_+^c(\Delta)$$

is a von Neumann algebra, for any Borel set $\Delta \subset \mathbf{R}^4$.

(4.41) follows now from the spectral theorem for $e^{i(tH_+^c - xP_+^c)}$.

Q. E. D.

Remarks. — Let C be some operator affiliated with $\pi_{\mathcal{H}^{(1)}}(\overline{\Delta_+(V)})''$, and let ρ be in the form domain of C. Then we can replace A by C. For a dense set of states we can replace A by some asymptotic boson counters $N_{+\lambda}$ $(\lambda > 0)$, etc.

Since all the results of section 4.4 proven so far for $t = +\infty$ hold also for $t = -\infty$, formula (4.41) permits us to compute the *cross sections* for the scattering of a charge being at $t = -\infty$. e. g. in the region Δ of energy momentum space [eigenspace of $F_-^c(\Delta)$] and a configuration of bosons at $t = -\infty$ which is compatible with the representation $\pi_{\mathcal{H}^{(1)}}$ of $\overline{\Delta_-(V)}$ [i. e. which is determined by some projection in $\pi_{\mathcal{H}^{(1)}}(\overline{\Delta_-(V)})''$; *see* also chap. 5].

Furthermore we can define a *scattering isomorphism* :

$$(4.42) \qquad s\begin{cases} \overline{\Delta_-(V)} \to \overline{\Delta_+(V)}, \\ \{ F_-^c(\Delta) \} \to \{ F_+^c(\Delta) \}, \end{cases}$$

defined by

$$s(A_-) = \mu_+ \circ \mu_-^{-1}(A_-), \quad \text{for all } A_- \text{ in } \overline{\Delta_-(V)},$$

and

$$s(F_-(\Delta)) = F_+(\Delta), \quad \text{for all Borel sets } \Delta.$$

Until now nothing guarantees us that the dynamics of the charge determined by $(H_\pm^c, P_\pm^c)$ is compatible with a particle interpretation of the charge at $t = +\infty$. We do not know, yet, if the spectrum of $(H_\pm^c, P_\pm^c)$ contains a particle shell. In order to study this problem we need some basic notions and definitions :

(4) *Asymptotic particle structure of the charge*

Intuitively it is obvious that one would like to identify P_+^c with the momentum operator of the asymptotic electron and its possible excited states. However such an identification is certainly not possible unless the spectrum of P_+^c is purely absolutely continuous. This has not been proven, yet. (In a relativistic model the absolute continuity of spec P_+^c would follow from the covariance of (H_+^c, P_+^c) under Lorentz rotations, [16].)

DEFINITION. — $\sigma_{\text{a. c.}}(P_+^c)$ denotes the absolutely continuous part of spec P_+^c and $\mathcal{H}_{\text{a. c.}}^+ \subseteq \mathcal{H}^{(1)}$ the subspace of $\mathcal{H}^{(1)}$ corresponding to $\sigma_{\text{a. c.}}(P_+^c)$.

Since $e^{itH_+^c}$ and $e^{ixP_+^c}$ commute and since $e^{ixP_+^c}$ is in $\pi_{\mathcal{H}^{(1)}}(\overline{\Delta_+(V)})'$, $\mathcal{H}_{\text{a. c.}}^+$ is left invariant by $e^{itH_+^c}$ and $\pi_{\mathcal{H}^{(1)}}(\overline{\Delta_+(V)})''$.

Obviously the dynamics on $\mathcal{H}^{(1)}$ is compatible with a *partial particle interpretation* for $t = +\infty$ ($\text{PPI}_{+\infty}$) if

$$\text{supp}(F_+^c \restriction \mathcal{H}_{\text{a. c.}}^+) \subseteq \{ (p^0, p) \mid p \in \mathbf{R}^3, p^0 = \lambda_i(p), i = 1, 2, 3, \ldots \},$$

where $\lambda_i(p)$ are continuous functions of p, for all i [and the equation $\lambda_i(p) = \lambda_j(p)$ $(i \neq j)$ has at most countably many solutions].

The operator $F_{+,i}^c$ denotes the selfadjoint projection onto the eigenspace $\mathcal{H}_i^+$ of (H_+^c, P_+^c) corresponding to the portion $\{ (p^0, p) \mid p^0 = \lambda_i(p) \}$ of the spectrum of (H_+^c, P_+^c).

As long as we do not know anything about the representations $\pi_{\mathcal{H}_i^+}$ of $\overline{\Delta_+ (V)}$ and the multiplet of asymptotic charged particles corresponding to the eigenvalue function $\lambda_i (p)$, the notion of particle ($PPI_{+\infty}$) defined above does not agree with the usual notion of asymptotic particles.

In order to elaborate a suitable framework for the investigation of the particle problem we want to assume more than what we actually can prove in our model, at the moment.

Let us assume that $\mathcal{H}^+_{a.c.} \neq \{\vec{0}\}$. We can decompose $\mathcal{H}^+_{a.c.}$ on $\sigma_{a.c.} (P^c_+)$ in the form of a direct integral :

$$(4.43) \qquad \mathcal{H}^+_{a.c.} = \int^\oplus d^3 p \, \mathcal{H}^+_p.$$

Since $e^{ixP^c_+}$ and $e^{itH^c_+}$ commute, we get

$$(4.44) \qquad H^c_+ \restriction \mathcal{H}^+_{a.c.} = \int^\oplus d^3 p \, H_{p,+},$$

$H_{p,+}$ is selfadjoint and $\inf \operatorname{spec} H_{p,+} \geq E(p) = \inf \operatorname{spec} H_p$, for almost all p in $\sigma_{a.c.} (P^c_+)$; *see* [16].

DEFINITION. — $\mathcal{H}^+_{p,d}$ denotes the subspace of $\mathcal{H}^+_p$ consisting of eigenvectors of $H_{p,+}$. It is again easy to show that $\mathcal{H}^+_{p,d}$ is left invariant by $\overline{\Delta_+ (V)}$.

In general it is not acceptable to interpret the eigenvalues $\lambda_i (p)$ of $H_{p,+} \restriction \mathcal{H}^+_{p,d}$ as energies of asymptotic charged particles of momentum p, since we do not know, yet, if the multiplet of particles of energy $\lambda_i (p)$ is finite.

A necessary and sufficient condition for the finiteness of these multiplets is :

$\mathcal{H}^+_{p,d}$ is a tensor product :

$$\mathcal{H}^+_{p,d} = \mathcal{H}^+_{p,1} \otimes \mathcal{H}^+_{p,2}$$

such that for all A in $\overline{\Delta_+ (V)}$,

$$A \restriction \mathcal{H}^+_{p,d} = I \otimes A \restriction \mathcal{H}^+_{p,2}$$

and

$$H_{p,+} \restriction \mathcal{H}^+_{p,d} = H_{p,+} \restriction \mathcal{H}^+_{p,1} \otimes I$$

and the spectrum of $H_{p,+} \restriction \mathcal{H}^+_{p,1}$ consists of isolated eigenvalues of finite multiplicity.

Furthermore we should postulate that the asymptotic observables generating the C* algebra $\overline{\Delta_+ (V)}$ yield a complete information about the configuration of asymptotic bosons, or, in mathematical terms

$$\pi_{\mathcal{H}^+_{p,2}} \left(\overline{\Delta_+ (V)}\right) \text{ acts irreducibly on } \mathcal{H}^+_{p,2}.$$

We are now prepared for the following definition

DEFINITION α. — The dynamics on $\mathcal{H}^{(1)}$ is compatible with a complete particle interpretation for $t = +\infty$ (for short $\mathrm{CPI}_{+\infty}$) iff

$$\operatorname{spec} P^c_+ = \sigma_{\text{a.c.}} (P^c_+),$$

$\pi_{\mathcal{H}^+_p} \left(\overline{\Delta_+ (V)}\right)''$ is a factor of type I_∞, for almost all p in spec P^c_+, such that

$$\mathcal{H}^+_p = \mathcal{H}^+_{p,1} \otimes \mathcal{H}^+_{p,2},$$

where $\overline{\Delta_+ (V)}$ acts irreducibly on the space $\mathcal{H}^+_{p,2}$ and

$$\text{(4.45)} \qquad \operatorname{spec} (H_{p,+} \restriction \mathcal{H}^+_{p,1}) = \sigma_d (H_{p,+} \restriction \mathcal{H}^+_{p,1}),$$

for almost all p in spec P^c_+ and all the eigenvalues $\lambda_i (p)$ of $H_{p,+} \restriction \mathcal{H}^+_{p,1}$ and their multiplicities are piecewise continuous functions of p.

If ψ is a vector in $\mathcal{H}^+_{\text{a.c.}}$ we can decompose it

$$\psi = \int^{\oplus} d^3 p \, \psi^+ (p),$$

where $\psi^+ (p)$ is in $\mathcal{H}^+_p$, for almost all p and

$$\| \psi \|^2 = \int d^3 p \, \| \psi^+ (p) \|^2,$$

$V_{\psi^+(p)}$ denotes the minimal closed subspace of $\mathcal{H}^+_p$ which contains $\psi^+ (p)$ and is invariant under $\overline{\Delta_+ (V)}$ and $e^{itH_{p,+}}$.

The subrepresentation of $\overline{\Delta_+ (V)}$ on $V_{\psi^+(p)}$ is called $\pi_{\psi^+(p)}$.

DEFINITION β. — A vector ψ in $\mathcal{H}^+_{\text{a.c.}}$ admits a $\mathrm{CPI}_{+\infty}$, iff $\pi_{\psi^+(p)} \left(\overline{\Delta_+ (V)}\right)''$ is a factor of type I_∞, for almost all p in $\sigma_{\text{a.c.}} (P^c_+)$ such that

$$V_{\psi^+(p)} = \mathcal{H}^+_{p,1} \otimes \mathcal{H}^+_{p,2},$$

where $\overline{\Delta_+ (V)}$ acts irreducibly on $\mathcal{H}^+_{p,2}$ and (4.45) of definition α holds.

Remarks. — Suppose that ψ admits a $\mathrm{CPI}_{+\infty}$. Then it follows from a theorem of Takesaki, [41], that

$$\pi_{\psi^+(p)}\left(\overline{\Delta_+(\mathrm{V})}\right) \approx \pi_{\mathcal{F}_b} \circ \alpha_p^+\left(\overline{\Delta_b(\mathrm{V})}\right)$$

for some *automorphism α_p^+ of $\overline{\Delta_b(\mathrm{V})}$ and almost all p in $\sigma_{\text{a.c.}}(\mathrm{P}_+^c)$.

It is an interesting and important problem to determine α_p^+.

The class of all vectors in $\mathcal{H}_{\text{a.c.}}^+$ which admit a $\mathrm{CPI}_{+\infty}$ forms a closed subspace $\mathcal{H}_+^{(1)} \subseteq \mathcal{H}_{\text{a.c.}}^+ \subseteq \mathcal{H}^{(1)}$.

$\mathcal{H}_+^{(1)}$ is called the *space of scattering states* for $t = +\infty$.

DEFINITION γ. — The dynamics on $\mathcal{H}^{(1)}$ is called asymptotically complete (a. c.) iff

$$\mathcal{H}_+^{(1)} = \mathcal{H}_-^{(1)} \neq \{\vec{0}\}$$

and strongly a. c. iff

$$\mathcal{H}_+^{(1)} = \mathcal{H}_-^{(1)} = \mathcal{H}^{(1)}.$$

(5) *Consequences of* (4) *for the representation* $\pi_{\mathcal{H}_+^{(1)}}$ *of* $\overline{\Delta_+(\mathrm{V})}$

(5.A)

PROPOSITION 4.10. — *Let the dynamics on* $\mathcal{H}^{(1)}$ *be given by the Hamiltonian* H *obtained in theorem* 1.1, (i) *and suppose that there is a vector* ψ *in* $\mathcal{H}_+^{(1)}$ *such that :*

For all p *in a set* $\mathrm{K} \subseteq \{p / |p| \leq \rho_0\}$ *of positive Lebesgue measure*

$$(4.46)\quad \mathrm{V}_{\psi^+(p)} = \mathcal{H}_{p,2}^+ \neq \{\vec{0}\}, \qquad \mathrm{H}_{p,+} \restriction \mathrm{V}_{\psi^+(p)} = \mathrm{E}(p)\, \mathrm{I} \restriction \mathrm{V}_{\psi^+(p)}.$$

Then the representation $\pi_{\psi_+(p)}$ *of* $\overline{\Delta_+(\mathrm{V})}$ *is* disjoint *from the Fock representation for almost all* p *in* K.

Proof. — We assume the contrary. Then there is a set $\mathrm{K}_1 \subseteq \mathrm{K}$ of positive Lebesgue measure such that $\pi_{\psi^+(p)}$ of $\overline{\Delta_+(\mathrm{V})}$ is *equivalent* to the Fock representation for all $p \in \mathrm{K}_1$.

Therefore, for an arbitrary p in K_1, there is a unitary operator $\mathrm{U}_p : \mathcal{H}_{p,2}^+ \to \mathcal{H}_{p,2}^+$ such that $\mathrm{U}_p\, \psi^+(p) = h(p)\, \Omega_p$ where $h(p) = \|\psi^+(p)\| > 0$ and Ω_p is a vacuum for $\overline{\Delta_+(\mathrm{V})}$, $\|\Omega_p\| = 1$. Since Ω_p is an eigenstate of $\mathrm{H}_{p,+}$ [with eigenvalue $\mathrm{E}(p)$] and $\mathrm{H}_{0+}\, \Omega_p = \mathrm{P}_+\, \Omega_p = \vec{0}$, it is an eigenstate of H_p corresponding to the eigenvalue $\mathrm{E}(p)$. Since $h(p)$ is positive and measurable on K_1 :

$$\vec{0} \neq \int_{\mathrm{K}_1}^{\oplus} d^3 p\, h(p)\, \Omega_p = \int_{\mathrm{K}_1}^{\oplus} d^3 p\, \mathrm{U}_p\, \psi^+(p) \in \mathcal{H}_{\text{a.c.}}^+ \subseteq \mathcal{H}^{(1)}.$$

But from chapter 3, (3.6) and [35] we know that

$$\int_{K_1}^{\oplus} d^3 p\, h(p)\, \Omega_p \notin \mathcal{H}^{1)}.$$

Hence our assumption leads to a contradiction.

Q. E. D.

It is straightforward to show the *converse* :

Assume that the hypothesis (4.46) of proposition 4.10 hold and that $\pi_{\psi^+(p)}$ of $\overline{\Delta^+(V)}$ is disjoint from the Fock representation for almost all p in K. Then the space $\mathcal{H}_p^{(1)}$ does *not* contain a groundstate of H_p, for almost all p in K.

For some detailed theorems of this type, *see* [16].

Remark. — This proposition reflects the general belief that, whenever the physical Hilbert space does not contain DES of the charged particles, the scattering states do *not* form a Fock space with respect to the asymptotically free massless bosons, [i. e. $\pi_{\mathcal{H}^{(1)}}$ of $\overline{\Delta_\pm(V)}$ is *not* quasi equivalent to the Fock representation of the algebras $\overline{\Delta_\pm(V)}$ and hence the wave isomorphisms $\mu_\pm : \overline{\Delta_b(V)} \to \overline{\Delta_\pm(V)}$ are not unitarily implementable] and conservely.

(5.B) If either conjecture α or conjecture β of section 4.3 holds then $\mathcal{H}_+^{(1)}$ is an infinite dimensional space containing all vectors of the form $A_+ \theta_+(h)$, $(A_+ \in \overline{\Delta_+(V)},\ h \in L^2(\overline{M}_{\rho_0}, d^3 p))$.

The representations $\pi_{\psi^+(p)}$ of $\overline{\Delta_+(V)}$ (ψ in $\mathcal{H}_+^{(1)}$) are determined explicitly by (α.2), conjecture α, section 4.3. They are *disjoint* from the Fock representation of $\overline{\Delta_+(V)}$, for almost all p in $\overline{M}_{\rho_0}$ (in agreement with proposition 4.10). Let $\overline{\mathcal{E}}_2 \subseteq \overline{M}_{\rho_0}$ be the closed set defined in chapter 3 section 3.3, (3.16). Then

(4.47) $\quad \{ q \in \overline{\mathcal{E}}_2 \mid \pi_{\psi^+(q)} \text{ is equivalent to } \pi_{\psi^+(p_0)}, \text{ for some fixed } p_0 \text{ in } \overline{\mathcal{E}}_2 \}$

is a set of measure 0. [This follows immediately from the definition (3.16) and the assumption that either conjecture α or conjecture β hold and hence (α.2) is valid.]

Remark. — The same result (4.47) is true under the the following conditions : Suppose H is the Hamiltonian obtained in theorem 1.1, (i), and suppose that for all p in $\mathcal{E}$ H_p has a unique DES given by the state Ω_p obtained in theorem 3.4.

If there is a state ψ in $\mathcal{H}_+^{(1)}$ such that for almost all p in $\{ p / |p| \leq \rho_0 \}$:

$$V_{\psi^+(p)} = \mathcal{H}^+_{p,2} \neq \{\vec{0}\}, \qquad H_{p,+} \restriction V_{\psi^+(p)} = E(p)\, I \restriction V_{\psi^+(p)}$$

then

$$\{ q \in \bar{\mathcal{E}}_2 \mid \pi_{\psi+(q)} \text{ is equivalent to } \pi_{\psi+(p_0)}, \text{ for some fixed } p_0 \in \bar{\mathcal{E}}_2 \}$$

is a set of measure 0, i. e. (4.47) *holds.*

[This is a consequence of definition (3.16), the second part of lemma 4.8 and the remark following definition β. Details are left to the reader.]

(5.C) Assume that there is a subspace W of $\mathcal{H}^{(1)}$ which is invariant under $\overline{\Delta_+ (V)}$ and $e^{i(t H_+^c - x P_+^c)}$ and such that

$$\pi_W (\overline{\Delta_+ (V)})' \restriction W \quad \text{is } abelian. \tag{4.48}$$

Then $\pi_W (\overline{\Delta_+ (V)})''$ is a type I algebra and can be decomposed uniquely on the spectrum of $\pi_W (\overline{\Delta_+ (V)})' \restriction W$ in the form of a direct integral.

The spectrum of $\pi_W (\overline{\Delta_+ (V)})' \restriction W$ is the cartesian product of $\Sigma_W = \text{spec}((H_+^c, P_+^c) \restriction W)$ and some compact Hausdorff space X. If ζ_1 and ζ_2 are two different points in $\Sigma_W \times X$, then the corresponding representations π_{ζ_1} and π_{ζ_2} are *disjoint,* irreducible representations of $\overline{\Delta_+ (V)}$ (for almost all $\zeta_1 \neq \zeta_2$ in $\Sigma_W \times X$; *see* Dixmier, [9]).

X can be interpreted as the spectrum of an abelian, asymptotic dynamical symmetry group.

It is interesting that if ψ is a vector in W, then the state of the charge (or of the asymptotic charged particle) as $t \to +\infty$ determined by the vector ψ is completely determined by

$$\{ (\psi, A_+ \psi) \mid A_+ \in \overline{\Delta_+ (V)} \}, \tag{4.49}$$

i. e. by just measuring the configuration of asymptotic bosons determined by ψ.

Under the condition that (4.47) holds the space

$$W = \int_{\bar{\mathcal{E}}_2}^{\oplus} d^3 p \, V_{\psi+(p)}$$

has the property that $\pi_W (\overline{\Delta_+ (V)})' \restriction W$ is abelian.

The conditions under which (4.48) and (4.49) hold are analyzed in detail in [16].

Finally, let us assume that conjecture α or β of section 4.3 holds. $\bar{\mathcal{E}}_2$ is the set defined in (5.B) with the property (4.47). We put

$$W = \{ A_+ \theta_+ (h) \mid A_+ \in \overline{\Delta_+ (V)}, h \in L^2 (\bar{\mathcal{E}}_2, d^3 p) \} \tag{4.50}$$

[i. e. W is the closure of the linear manifold generated by the scattering states $\{ A_+ \theta_+ (h) \}$, where h has support in $\bar{\mathcal{E}}_2$].

Then for the space W defined by (4.50), (4.48) holds. The proof of this fact follows immediately from (4.47); *see* [9].

Moreover W contains a dense set of vectors θ that are in the form domain of the operator valued distribution

$$\Phi_+(k) = \frac{1}{\sqrt{2}} \{ b_+^*(k) + b_+(k) \}$$

and such that the function

$$(4.51) \qquad F_\theta(k) = (\theta, \Phi_+(k)\,\theta) - \int d^3 p \, | h(p) |^2 \, w_p(k)$$

is square integrable in k, for some function $| h(p) |^2$ in $L^1(\mathcal{E}_2, d^3 p)$:

$$w_p(k) = \lambda \, \mathfrak{z}(k) \, | k |^{-1/2} (| k | - (k, \nabla E(p)))^{-1}.$$

It is now easy to show that the function $| h(p) |^2$ is uniquely determined by the property that F_θ is in $L^2(\mathbf{R}^3)$, and that $| h(p) |^2$ is the *momentum distribution* of the asymptotic electron. [For the proof use the definition (4.50) of W, (α.2), conjecture α, section 4.3 and (4.47).]

(5.D) The existence of a vector $\psi \in \mathcal{H}^{\pm}_{\text{a.c.}}$ which admits a $\text{CPI}_{\pm\infty}$ is of course a necessary condition for an asymptotic condition in time and a scattering theory with a particle interpretation on the space $\mathcal{H}^{(Z)}$, $Z > 1$.

(These problems are analyzed in a general framework in [16].)

This concludes chapter 4.

Chapter 5

COMPARISON OF THE RESULTS OF CHAPTER 4 WITH THE PROPOSALS OF FADDEEV AND KULISH; OUTLOOK

5.1. The formalism of Faddeev and Kulish, [12]

In this chapter we shall first briefly describe the approach of Faddeev and Kulish applied to Nelson's model with bosons of restmass 0 and the dynamics determined by the Hamiltonian $H \upharpoonright \mathcal{H}^{1)}$ obtained in theorem 1.1, (i).

Definition :

$$V_{as}(t) = \lambda \int d^3 p \, d^3 k \, \{ n^*(p) \, b^*(k) \mathfrak{z}(k) (2 | k |)^{-1/2} e^{-i(k, \nabla E(p))t} n(p) + \text{h. c.} \},$$

$$V^I_{as}(t) = e^{i H_{ob} t} \, V_{as}(t) \, e^{-i H_{ob} t}$$

and

$$H_{as}(t) = \tilde{H}_0 + V_{as}(t),$$

where

$$\tilde{H}_0 = \int d^3 p \, n^*(p) \, E(p) \, n(p) + H_{0b}.$$

The operator $V^I_{as}(t)$ seems to be that portion of the interaction in our model which persists in the limit $t = +\infty$ and hence excludes the existence of a conventional scattering theory in the sense of Haag and Ruelle for the dynamics of our model.

Faddeev and Kulish, [12], (and also Blanchard, [2], in a slightly different context) have derived what in the present model reads :

$$U_{as}(t) = e^{i\tilde{H}_0 t} \, e^{-iH_{as}(t)t} = \exp \int d^3 p \, n^*(p) \{ A_p(t) + \tilde{\Phi}_p(t) \} \, n(p),$$

where

$$A_p(t) = \lambda \int d^3 p \, d^3 k \frac{\varsigma(k)}{\sqrt{2|k|}} (|k| - (k, \nabla E(p)))^{-1} \times \{ b^*(k) (e^{i(|k|-(k,\nabla E(p)))t} - 1) - \text{h. c.} \},$$

and $\tilde{\Phi}_p(t)$ is a phase which is important only on $\mathcal{H}^{(Z)}$, $Z > 1$.

The physical significance of the time evolution $U_{as}(t)$ can be illustrated by the calculation of the distribution

$$\begin{aligned}
&(U_{as}(t) \, n^*(p) \, \varphi_0, \varphi(x, t) \, U_{as}(t) \, n^*(p') \, \varphi_0) \\
&= (e^{-iH_{as}(t)t} n^*(p) \, \varphi_0, \varphi(x, 0) \, e^{-iH_{as}(t)t} n^*(p') \, \varphi_0) \\
&= i \int_0^t dt' \, (n^*(p) \, \varphi_0, [V^I_{as}(t'), \varphi(x, t)] \, n^*(p') \, \varphi_0) \\
&= i \lambda \int_0^t dt' \int d^3 y \\
&\quad \times \left(n^*(p) \, \varphi_0, \left[(\check{\varsigma} \star \varphi)(y - \nabla E(p) \, t', t'), \varphi(x, t)\right] n^*(p') \, \varphi_0\right) \\
&= \delta(p - p') \, \lambda \int_0^t dt' \int d^3 y \, d^3 z \, \check{\varsigma}(y - \nabla E(p) \, t' - z) \, D_m(z - x, t' - t),
\end{aligned}$$

where

$$D_m(z - x, t' - t) = i \, [\varphi(z, t'), \varphi(x, t)].$$

We observe that the bare electron is moving freely and that it emits a cloud of bosons moving with it.

According to [12] (analogy to the quantum mechanical Coulomb scattering) one has now to look at

$$\Omega(t) = e^{iHt} e^{-i\tilde{H}_0 t} U_{as}(t) \tag{5.1}$$

and

$$S'(t, -t) = U_{as}^*(t) e^{i\tilde{H}_0 t} e^{-2iHt} e^{i\tilde{H}_0 t} U_{as}(-t). \tag{5.2}$$

If it exists " $\lim_{t \to \pm\infty} \Omega(t)$ " is what corresponds to the wave operators in ordinary quantum scattering theory. However the existence of these limits is an open problem. Actually we cannot see more than formal reasons for the existence of these limits.

Let us therefore look at the matrix elements : $(\theta, S'(t, -t)\psi)$ where θ and ψ are in $\mathcal{H}^{(Z)}$ $(Z \geq 1)$, and then try to pass to the limit $t = \infty$.

On the level of low order perturbation theory

$$(\theta, S'(\infty, -\infty)\psi) \tag{5.3}$$

is free from infrared divergencies.

The phase $\tilde{\Phi}_p(t)$ predicted by this formalism seems to be appropriate to cancel Coulomb phases (which occur if $Z > 1$ and diverge logarithmically as $t \to \pm\infty$).

The matrix elements (5.3) define a scattering operator S :

$$S : \mathcal{H}_{sc}^{(Z)} \to \mathcal{H}_{sc}^{(Z)},$$

where

$$\mathcal{H}_{sc}^{(Z)} = \int^{\oplus} d^3 p_1 \dots d^3 p_Z \, \mathcal{H}_{sc}(p_1, \dots, p_Z) \tag{5.4}$$

and $\mathcal{H}_{sc}(p_1, \dots, p_Z)$ is the IDPS (*see* section 2.1) corresponding to the product reference vector

$$\exp \frac{1}{\sqrt{2}} \left\{ b^* \left(\sum_{i=1}^{Z} w_{p_i} \right) - b \left(\sum_{i=1}^{Z} w_{p_i} \right) \right\} \varphi_0, \tag{5.5}$$

where

$$w_p(k) = \lambda \, | k |^{-1/2} (| k | - (k, \nabla E(p)))^{-1} \mathfrak{z}(k) \tag{5.6}$$

and $p_1, \dots, p_Z$ are the momenta of the asymptotic electrons.

Unfortunately it seems to be extremely difficult to justify this nice formalism on the level of mathematical rigor. Even on the charge one sector $\mathcal{H}^{(1)}$ it does not seem that :

$\Omega(t)$	converges strongly as $t \to \pm \infty$ or that
$S'(t, -t)$	converges weakly as $t \to \pm \infty$ to some limit
$S'(\infty, -\infty)$	different from 0 and 1.

In our opinion there are more arguments in favour of conjectures α or β of section 4.3 than in favour of the formalism described above.

The lacking steps in a proof of conjecture α are technical difficulties : We do not know whether the function $E(p)$ and the vectors $C_p(t)\,\Omega_p(x)$ are sufficiently smooth in p. These smoothness properties are predicted by perturbation theory. In the Faddeev-Kulish formalism however the difficulties are much less localized.

It is hard to imagine that one can solve the collision theory problem by just guessing the correct modified free time evolution $e^{i H_{as}(t) t}$ without solving first certain dynamical problems such as the existence and properties of DES.

Formally of course the approaches of section 4.3 and the approach described above (in its restriction to $\mathcal{H}^{(1)}$) agree. They predict the same scattering states and scattering amplitudes.

Furthermore on the level of perturbation theory for the calculation of scattering amplitudes all these approaches *coincide.*

We think therefore that the proposals of Faddeev and Kulish are a good starting point for a rigorous perturbation theory which is free from infrared divergencies. At this point it is important to remark that the coupling constant λ and the energy function $E(p)$ entering into the definition of $V_{as}(t)$ are the *renormalized, physical* quantities rather than the bare ones $\left[E(p) \neq \frac{p^2}{2M} ! \right]$.

In Nelson's model the bare and the renormalized coupling constants coincide. But the situation is different in Quantum Electrodynamics.

5.2. Outlook

In this section we want to add some remarks on chapter 4 and we mention some interesting problems.

(1) The only rigorous results concerning the scattering theory of our model are contained in section 4.4 and in this section the only result which is free from unproven conjectures reads :

Let $\{ F_\pm^c(\Delta) \}$ denote the spectral projections of the unitary group $e^{i(t H_\pm^c - x P_\pm^c)}$ defined in lemma 4.9 (which describes the time evolution of the charge).

Let Q_- be any projection in $\pi_{\mathcal{H}^{(1)}}(\overline{\Delta_-(V)})''$.

We prepare our system at $t = -\infty$ as follows :

The state of the system at $t = -\infty$ is described by a vector θ in the subspace $F^c_-(\Delta)\, Q_-\, \mathcal{H}^{(1)} \subset \mathcal{H}^{(1)}$, where Δ is some Borel set and $\|\theta\| = 1$.

The vector θ prescribes initial conditions for our system at $t = -\infty$: The charge is in the region Δ of energy momentum space and the configuration of bosons belongs to the subspace $Q_-\, F^c_-(\Delta)\, \mathcal{H}^{(1)}$.

From these initial conditions and the distributions

$$(5.7) \qquad \{ (\theta,\, d^4 F^c_+(p, p^0)\, A_+ \theta) \mid A_+ \in \overline{\Delta_+(V)} \}$$

we can calculate the cross sections for the scattering of the charge and the bosons.

Since this is the only rigorous result and since the scattering isomorphism s constructed in section 4.4, (3), (4.42) is not likely to be unitarily implementable on $\mathcal{H}^{(1)}$ (if there is *non trivial* scattering) one might conclude that the cross section approach of section 4.4 is the appropriate approach to a collision theory in models without a massgap and one particle shells of charged particles. Actually the results of section 4.4 (that are independent of conjectures α or β) can be extended to the spaces $\mathcal{H}^{(Z)}$ and they can also be derived in an axiomatic frame, [16]. One might conclude moreover that even asymptotically (for $t = \pm\infty$) the particle structure of the charge [in the sense of definitions α and β of section 4.4, (4)] is questionable. Whatever this means, a " quasi particle structure " [e. g. in the sense of $PPI_{\pm\infty}$, section 4.4, (4)] of the charge for large times should be contained in any *realistic* model.

(2) We have not made use of the knowledge of the DES in the limit $\sigma = 0$ (elaborated in chapters 2 and 3) for the derivation of the rigorous results in sections 4.1 and 4.4. But it is clear that the properties of the collision theory in our model and the ones of the DES must be related to each other as indicated by proposition 4.10, the consequences of conjectures α and β and section 4.4, (5).

It is a challenging problem to derive additional conditions on the spectrum of H which if combined with our knowledge of DES imply that the spectrum of $(H^c_\pm, P^c_\pm)$ contains a one particle shell $\{ (p, p^0) \mid |p| < \rho_0,\, p^0 = E(p) \}$.

A proof of the existence of such a one particle shell can be essentially reduced to either one of the following conditions :

— Conjecture α of section 4.3 [which holds if $E(p)$ and $C_p(t)\, V_p(x)\, \psi_p(x)$ are sufficiently smooth in p).

— Conjecture β of section 4.3 [which would essentially follow from a bound : $(\theta_{\sigma_k,\pm}(h),\, N_b\, \theta_{\sigma_k,\pm}(h)) \leq$ const., uniformly in k, where h is C^∞ and supp $h \subset M_{\rho_0}$].

— The set $\overline{M}_{\rho_0}$ is contained in $\sigma_{\text{a.c.}}(P_+) \neq \emptyset$. The representation $\pi_{\mathcal{H}_p^+}$ of $\overline{\Delta_+(V)}$ is unitarily equivalent to $\pi_{\mathcal{H}_p^{(1)}}(\gamma_p(\overline{\Delta(V)}))$, where γ_p is a *automorphism of $\overline{\Delta(V)}$ such that

$$\gamma_p(V(g)) = V(g) \quad \text{for all } g \text{ in } \mathcal{S}_r,$$

for almost all p in $\overline{M}_{\rho_0}$.

(3) Unless we are able to prove the existence of a one particle shell in spec $((H_+^c, P_+^c) \restriction \mathcal{H}^{(1)})$ there are no results that suggests that a generalized Haag-Ruelle theory in the limit $\sigma = 0$ can be established on the spaces $\mathcal{H}^{(Z)}$, $Z > 1$.

In a formal manner one can easily show that there is a long range attraction between the dressed electrons (surrounded by the clouds of soft bosons) :

$$V_{p,p'}(k) = \ker(V_p^* V_{p'}^* H_I V_{p'} V_p)$$
$$= \frac{\lambda^2 \mathfrak{I}(k)}{|k|(|k| - (k, \nabla E(p)))} + \frac{\lambda^2 \mathfrak{I}(k)}{|k|(|k| - (k, \nabla E(p')))}$$

of the type of the Coulomb force. Therefore a collision theory on $\mathcal{H}^{(Z)}$ in the spirit of sections 4.2 and 4.3 will meet the additional difficulty of Coulomb phases, $(Z > 1)$.

(4) From section 4.4, (3) we know that the operators H_+^c, P_+^c can be constructed whenever there exists a C* algebra $\overline{\Delta_+(V)}$ of asymptotic boson observables such that the space-time translations determine a *automorphism group $\tau_{t,x}$ of $\overline{\Delta_+(V)}$.

Hence the existence of $\overline{\Delta_+(V)}$ yields a decomposition of the dynamics e^{itH} in a dynamics for the asymptotic bosons and in one for the asymptotic charge.

An LSZ asymptotic condition for the boson field is sufficient for the existence of $\overline{\Delta_+(V)}$.

Therefore, in a general theory without a massgap and without mass shells for the charged particles, the LSZ asymptotic condition for the *observable* fields (the *boson* fields, but *not* for the charged fields) rather than a Haag-Ruelle theory seems to be adequate for the description of collision processes. The assumption that spec $(H_\pm^c, P_\pm^c)$ contains one particle shells replaces the usual spectrum assumption of the Haag-Ruelle theory. *See* [16] for an elaboration of these concepts. This concludes chapter 5.

Appendix 1

PROOF OF LEMMA 3.1

Let $\Omega(p) = \frac{p^2}{2M}$. We recall the definitions of $T_{\sigma,p}$, $t(\sigma, p)$ and $t(\sigma, x)$:

$$T_{\sigma,p} = H_{\sigma,p} - \frac{p^2}{2M}, \qquad t(\sigma, p) = \inf \operatorname{spec} T_{\sigma,p} = E(\sigma, p) - \frac{p^2}{2M}.$$

For an arbitrary unit vector $\mathbf{e}$ in $\mathbf{R}^3$ we put

$$t(\sigma, x) = t(\sigma, x.\mathbf{e})$$

and we define $t'(\sigma, x) = \frac{\partial}{\partial x} t(\sigma, x)$, for all $\sigma \geqq 0$.

We now prove:

(i) There is a sequence $\{\sigma_l\}_{l=0}^{\infty}$ converging to 0 such that for almost all x in $[-\rho_0, \rho_0]$ $\lim_{l\to\infty} t'(\sigma_l, x) \equiv \hat{t}(x)$ exists.

The function $\hat{t}(x)$ is monotonically decreasing and

$$\hat{t}(x) = t'(x) \equiv t'(\sigma = 0, x)$$

(except at possibly countably many points of discontinuity in $[-\rho_0, \rho_0]$).

(ii) The function $-t'(x)$ is differentiable in x, for almost all x in $[-\rho_0, \rho_0]$. Its derivative is a finite measure. For almost all x in $[-\rho_0, \rho_0]$ there is a constant $C(x) < \infty$ such that

$$|t'(y) - t'(x)| \leqq C(x)|y - x|, \quad \text{for all } y, \qquad |y - x| \leqq 1.$$

Remark. — (i) and (ii) hold simultaneously for almost all x in $[-\rho_0, \rho_0]$ (since the union of two null sets is a null set).

Proof of (i). — Let N be an arbitrary finite natural number and $N < \rho_0 + 1$.

Since $t'(\sigma, x)$ is monotonically decreasing and finite for each $x \in \mathbf{R}$, there is a constant C_N such that

$$|t'(\sigma, x)| \leqq C_N < \infty, \quad \text{for all } \sigma > 0, \quad x \text{ in } [-N, N]$$

Therefore one can find by Cantor's diagonal procedure a sequence $\{\sigma_l\}_{l=0}^{\infty}$ converging to 0 such that

$$\left| t'\left(\sigma_l, \frac{m}{l}N\right) - \hat{t}\left(\frac{m}{l}N\right)\right| \leqq \frac{1}{l} \tag{A 1.1}$$

for all m in $\mathbf{Z}$ and $-l \leqq m \leqq l$ and some $\hat{t}(.)$.

Up to now $\hat{t}(x)$ is defined on the set $\{ x = r\mathrm{N} \}$, where r is a rational number in $[-1, 1]$, $\hat{t}(x)$ is monotonically decreasing on this set and $|\hat{t}(x)| \leq \mathrm{C_N}$, for all x in $[-\mathrm{N}, \mathrm{N}]$.

Thus there is a interpolation which extends $\hat{t}(x)$ onto the whole interval $[-\mathrm{N}, \mathrm{N}]$:

If y is in $[-\mathrm{N}, \mathrm{N}]$, choose an increasing sequence $\{ y_n \}_{n=1}^{\infty} \subset \{ x = r\mathrm{N} \}$ such that $y_n \uparrow y$, as $n \to \infty$; set

$$\hat{t}(y) = \lim_{n \to \infty} \hat{t}(y_n) = \inf_n \hat{t}(y_n).$$

Obviously $\hat{t}(x)$ is monotonically decreasing on $[-\mathrm{N}, \mathrm{N}]$.

If $\mathrm{Var}(f)$ denotes the total variation of f on $[-\mathrm{N}, \mathrm{N}]$ then :

$$\text{(A 1.2)} \qquad \mathrm{Var}(t'(\sigma, .)) \leq 2.\mathrm{C_N} \qquad \text{and} \qquad \mathrm{Var}(\hat{t}) \leq 2.\mathrm{C_N}.$$

A simple theorem of real analysis tells us now that (A 1.1) and (A 1.2) imply that $t'(\sigma_l, x) \to \hat{t}(x)$ for almost all x in $[-\mathrm{N}, \mathrm{N}]$.

Using Cantor's diagonal procedure we can choose $\{ \sigma_l \}_{l=0}^{\infty}$ such that $t'(\sigma_l, x) \to \hat{t}(x)$, for almost all x in $[-\rho_0, \rho_0]$.

But

$$t(\sigma_l, x) - t(\sigma_l, 0) = \int_0^x dy\, t'(\sigma_l, y),$$

$$t(\sigma_l, x) - t(\sigma_l, 0) \to t(x) - t(0), \quad \text{for all } x,$$

$$t'(\sigma_l, x) \to \hat{t}(x) \quad \text{almost everywhere, as } l \to \infty.$$

We choose $\mathrm{N} \geq x$. Then

$$\operatorname*{essup}_{|y| \leq |x|} |\hat{t}(y)| \leq \mathrm{C_N} < \infty.$$

Applying Lebesgue's dominated convergence theorem we get :

$$\text{(A.1 3)} \qquad t(x) - t(0) = \lim_{l \to \infty} \int_0^x dy\, t'(\sigma_l, y)$$

$$= \int_0^x dy \left(\lim_{l \to \infty} t'(\sigma_l, y) \right) = \int_0^x dy\, \hat{t}(y).$$

Hence $t'(y) = \hat{t}(y)$, for all y in $\mathbf{R}$.

(i) is proven.

Proof of (ii). — An important theorem of real analysis tells us that a monotonic function is differentiable almost everywhere, (the derivative is a measure which is the sum of an absolutely continuous, a singular and a discrete measure; *see* Riesz-Nagy [30]). The rest of (ii) follows from the definition of differentiability and from the boundedness of $\hat{t}(x)$ on $[-N, N]$.

Q. E. D.

COROLLARY. — *If* $\Omega(p) = \frac{p^2}{2M}$ *then for all* p *in a rotation invariant set* $\mathcal{E}$, *where* $\{q/|q| \leq \rho_0\} \setminus \mathcal{E}$ *is af measure* 0, *we have :*

$\nabla E(p)$ *exists and* $\nabla E(p) = \lim_{l \to \infty} \nabla_p E(\sigma_l, p)$, *for some sequence* $\{\sigma_l\}_{l=0}^{\infty}$ *converging to* 0.

$\nabla E(q)$ *is Lipshitz at* $q = p$ *in* $\mathcal{E}$, *i. e. there is a constant* $C(p) < \infty$ *such that*

$$|\nabla E(q) - \nabla E(p)| \leq C(p)|q - p| \qquad (\textit{for all } q,\ |q - p| \leq 1).$$

Proof. — Let $\mathbf{e}$ be a unit vector in $\mathbf{R}^3$ and put

$$E(\sigma, x) = E(\sigma, x.\mathbf{e}) = t(\sigma, x) + \frac{x^2}{2M}.$$

Obviously (i) and (ii) hold for $E(\sigma, x)$, as well.

The rotational invariance of $\mathcal{E}$ follows from the rotational invariance of $E(\sigma, p)$ $(\sigma \geq 0)$ and from (i) and (ii).

Q. E. D.

APPENDIX 2

PROOF OF THEOREM 4.3

Since we do not really know wether there is a good chance that theorem 4.3 admits a generalization to the limiting case $\sigma = 0$, we present a proof which is obviously restricted to the case $\sigma > 0$ which is however simple.

We recall a few facts which are important for our proof :

(*a*) The function $(\psi_0, Q_{\sigma,p}\, \psi_0)$ is strictly positive on

$$M_{\rho_0(\lambda)} = \{p/|p| < \rho_0(\lambda)\}$$

and holomorphic in some complex neighbourhood of $M_{\rho_0(\lambda)}$ (*see* lemma 1.6, and II, *a*, chap. 3).

If h is some C^∞ function and $\operatorname{supp} h \subset M_{\rho_0(\lambda)}$ we may therefore replace $\theta_{\sigma,t}(h \mid x)$ by

$$\chi_{\sigma,t}(h \mid x) = \theta_{\sigma,t}(h (\psi_0, Q_{\sigma,p}\,\psi_0)^{-1/2} \mid x).$$

(*b*) The function $E(\sigma, p)$ is holomorphic in p in some complex neighbourhood of $M_{\rho_0(\lambda)}$ (*see* lemma 1.6).

(*c*) Therefore the kernel $w_{\sigma,p}(k)$ is C^∞ in $[p, k]$ in $M_{\rho_0(\lambda)} \times \mathbf{R}^3$.

DEFINITION. — Let $p = (p^1, p^2, p^3)$ a vector in $\mathbf{R}^3$, and define

$$D_p^m = \frac{\partial^{|m|}}{(\partial p^1)^{m_1} (\partial p^2)^{m_2} (\partial p^3)^{m_3}}, \qquad m = (m_1, m_2, m_3), \qquad |m| = \sum_{i=1}^{3} m_i.$$

It follows from the definition (4.12) of $w_{\sigma,p}(k)$ and (*b*) that

$$\int d^3 k \mid D_p^m D_k^l w_{\sigma,p}(k) \mid < \infty,$$

for all $|m| < \infty$, $|l| < \infty$ and all p in $M_{\rho_0(\lambda)}$.

(*d*) $\Omega_{\sigma,p} = \psi_1(\sigma, p)$ in $\mathcal{F}_B$ is the unique groundstate of $H_{\sigma,p}$ obtained in theorem 1.4. Since $\sigma > 0$, $\Omega_{\sigma,p}$ is C^∞ in p in $M_{\rho_0(\lambda)}$ in the sense of $\mathcal{F}_B$-valued functions of p and so is $\Omega_{\sigma,p}(x) = e^{-i P_B x}\, \Omega_{\sigma,p}$.

Using the equations

$$\text{(A 2.1)} \quad \Omega_{\sigma,p}(x) = e^{-i P_B x} \int_\Gamma d\zeta\, R_{\sigma,p}(\zeta)\, \psi_0 \left\| \int_\Gamma d\zeta\, R_{\sigma,p}(\zeta)\, \psi_0 \right\|^{-1}$$

[*see* equation (1.24)] and

$$\text{(A 2.2)} \qquad \frac{\partial}{\partial p^i} R_{\sigma,p}(\zeta) = R_{\sigma,p}(\zeta) \left(\frac{\partial}{\partial p^i} \Omega \right) (p - P_B)\, R_{\sigma,p}(\zeta)$$

one can easily calculate $D_p^m\, \Omega_{\sigma,p}(x)$ explicitly. By a straightforward generalization of lemma 1.5 (*see* lemma 1.6 and II, *a*, chap. 3), $D_p^m\, \Omega_{\sigma,p}(x)$ is shown to be an entire vector for N_B, (for all $|m| < \infty$ and all p in $M_{\rho_0(\lambda)}$).

But then the equation

$$\text{(A 2.3)} \qquad C_{\sigma,p} = \exp - i\, \Pi\, (w_{\sigma,p})$$

and (*c*) imply that $e^{-it H_{0B}}\, C_{\sigma,p}\, e^{it H_{0B}}\, \Omega_{\sigma,p}(x)$ is strongly C^∞ in $p \in M_{\rho_0(\lambda)}$.

DEFINITION :

$$\text{(A 2.4)} \quad G_\sigma(p, p' \mid t, x) = \left(C_{\sigma,p}\, e^{it H_{0B}}\, \Omega_{\sigma,p}(x),\, C_{\sigma,p'}\, e^{it H_{0B}}\, \Omega_{\sigma,p'}(x) \right)_{\mathcal{F}_B}.$$

On the basis of (A 2.1), (A 2.3), (A 2.4) and (c) one can calculate explicitly arbitrary derivatives of the C^∞ function $G_\sigma(p, p' \mid t, x)$ in p and p' and prove that for all p and p' in an arbitrary fixed compact set $K \subset M_{\rho_0(\lambda)}$ there is a constant $C(K, m, m')$ such that

(A 2.5) $$| D_p^m D_{p'}^{m'} G_\sigma(p, p' \mid t, x) | \leq C(K, m, m').$$

Thus if h is a C^∞ function and supp $h \subseteq K$, $\| \chi_{\sigma,t}(h \mid x) \|_{\mathcal{F}_B}^2$ is integrable in x :

(A 2.6) $$\int d^3 x \, \| \chi_{\sigma,t}(h \mid x) \|_{\mathcal{F}_B}^2 = (2\pi)^{-3} \int d^3 x \int d^3 p \, d^3 p' \, e^{ix(p'-p)} \times e^{it(E(\sigma,p) - E(\sigma,p'))} \overline{h(p)} \, h(p') \, G_\sigma(p, p' \mid t, x) < \infty$$

for all $|t| < \infty$, which establishes theorem 4.3, (i).

We now want to show that $e^{iH(\sigma)t} \chi_{\sigma,t}(h)$ converges strongly as $t \to \pm\infty$,

(A 2.7) $$\begin{aligned} \| e^{iH(\sigma)t} \chi_{\sigma,t}(h) - e^{iH(\sigma)t'} \chi_{\sigma,t'}(h) \|^2 &= \| \chi_{\sigma,t}(h) \|^2 + \| \chi_{\sigma,t'}(h) \|^2 - 2 \operatorname{Re} (\chi_{\sigma,t}(h), e^{iH(\sigma)(t'-t)} \chi_{\sigma,t'}(h)) \\ &= \| \chi_{\sigma,t}(h) \|^2 - \| \chi_{\sigma,t'}(h) \|^2 \\ &\quad - 2 \operatorname{Re} \int_{t'}^{t} ds \left(\frac{\partial}{\partial s} e^{iH(\sigma)s} \chi_{\sigma,s}(h), e^{iH(\sigma)t'} \chi_{\sigma,t'}(h) \right). \end{aligned}$$

We observe that it is sufficient to show that :

1° $\| \chi_{\sigma,t}(h) \|^2$ converges as $t \to \pm\infty$, and that

2° $\int_{t'}^{t} ds \left(\frac{\partial}{\partial s} e^{iH(\sigma)s} \chi_{\sigma,s}(h), e^{iH(\sigma)t'} \chi_{\sigma,t'}(h) \right)$ tends to 0 as $t \to \pm\infty$, $t' \to \pm\infty$.

Proof of 1° :

DEFINITION :

$$\sqrt{2} \, u_{\sigma,p,p'}(k) = w_{\sigma,p}(k) - w_{\sigma,p'}(k).$$

With this definition

(A 2.8) $$\begin{aligned} C^*_{\sigma,p} C_{\sigma,p} &= \exp i \Pi (\sqrt{2} \, u_{\sigma,p,p'}) \\ &= \exp [- B^*(u_{\sigma,p,p'})] \\ &\quad \times \exp [B(u_{\sigma,p,p'})] \exp \left[- \frac{1}{2} \| u_{\sigma,p,p'} \|_2^2 \right]. \end{aligned}$$

Since $e^{-itH_{0B}}\Omega_{\sigma,p}(x)$ is an entire vector for N_B for all p in supp h we can use a power series expansion for the right hand side of (A 2.8) in order to get an explicit expression for $G_\sigma(p, p' \mid t, x)$:

$$(A\,2.9)\quad G_\sigma(p, p' \mid t, x) = \sum_{m,n=0}^{\infty} \frac{1}{n!}\frac{(-1)^m}{m!}\int \prod_{i=1}^{m} d^3 k_i\, u_{\sigma,p,p'}(k_i) \times e^{i(|k_i|t - k_i.x)} \prod_{j=1}^{n} d^3 k'_j\, u_{\sigma,p,p'}(k'_j)\, e^{-i(|k'_j|t - k'_j.x)} \times \exp\left[-\frac{1}{2}\| u_{\sigma,p,p'} \|_2^2\right] \times \left(\prod_{i=1}^{m} B(k_i)\,\Omega_{\sigma,p}, \prod_{j=1}^{n} B(k'_j)\,\Omega_{\sigma,p'} \right).$$

Using lemma 1.5 we conclude that the sums over m and n converge uniformly in $p \in \text{supp}\, h$, $p' \in \text{supp}\, h$.

We now put the expression (A 2.9) into (A 2.6). It is then obviously permitted to exchange the x- and the p- and p'-integrations. Integration over x and p' yields :

$$(A\,2.10)\quad \| \chi_{\sigma,t}(h) \|^2_{\mathcal{H}^{(1)}} = \sum_{m,n=0}^{\infty} \frac{1}{n!}\frac{(-1)^m}{m!}\int d^3 p \times \left\{ \prod_{i=1}^{m} d^3 k_i\, e^{-i|k_i|t}\, u_{\sigma,p,p'}(k_i) \times \prod_{j=1}^{n} d^3 k'_j\, e^{i|k'_j|t}\, u_{\sigma,p,p'}(k'_j)\, \overline{h(p)}\, h(p') \times \exp\left[-\frac{1}{2}\| u_{\sigma,p,p'} \|_2^2\right] e^{it(E(\sigma,p)-E(\sigma,p'))} \times \left(\prod_{i=1}^{m} B(k_i)\,\Omega_{\sigma,p}, \prod_{j=1}^{n} B(k'_j)\,\Omega_{\sigma,p'} \right) \right\}_{p'=p+k'_\Sigma - k_\Sigma}.$$

Using again lemma 1.5 we get estimates on the " correlation-functions "

$$\left(\prod_{i=1}^{m} B(k_i)\,\Omega_{\sigma,p}, \prod_{j=1}^{n} B(k'_j)\,\Omega_{\sigma,p'} \right)$$

which are uniform in $p \in \text{supp}\, h$ and $p' \in \text{supp}\, h$.

We conclude that the sum on the R. H. S. of (A 2.10) converges absolutely and *uniformly in t*, provided supp $h \subseteq K \subset M_{\zeta_0(\lambda)}$.

Therefore we may interchange the summation over m and n and the limit $t \to \pm \infty$.

An easy application of the Riemann-Lebesgue lemma completes the proof of 1°.

Remark. — Obviously the estimates given above yield

$$\lim_{t \to \pm\infty} \| \chi_{\sigma, t}(h) \|^2_{\mathcal{H}^{(1)}}$$
$$= \int d^3 p \left\{ \overline{h(p)}\, h(p') \exp\left[-\frac{1}{2} \| u_{\sigma, p, p'} \|_2^2 \right] (\Omega_{\sigma, p}, \Omega_{\sigma, p'}) \right\}_{p=p'}$$
$$= \int d^3 p \, | h(p) |^2 \, \| \Omega_{\sigma, p} \|^2 = \| h \|_2^2 .$$

It is straightforward to extend the arguments and estimates given above in order to show that

$$\lim_{t \to \infty} (\chi_{\sigma, t}(h), A_t \chi_{\sigma, t}(h')) = \int d^3 p \, \overline{h(p)}\, h'(p) \, (C_{\sigma, p} \psi_0, A_B C_{\sigma, p} \psi_0)_{\mathcal{F}_B}$$

for all unitary operators A in $\overline{\Delta_b(S_0)}$ which are of the form $\exp i [b^*(f) + b(f)]$ and $f \in \mathcal{S}_0(\mathbf{R}^3)$.

Since these operators generate $\overline{\Delta_b(S_0)}$, theorem 4.3, (4.14) is proven. The intertwining relations (4.15) and (4.16) follow immediately from lemma 4.2, (ii) and theorem 4.3, (i), (ii).

2° The integral $\int_{t'}^{t} ds \left(\frac{\partial}{\partial s} e^{iH(\sigma)s} \chi_{\sigma, s}(h), \; e^{iH(\sigma)t'} \chi_{\sigma, t'}(h) \right)$ tends to 0 as t and t' tend to $\pm \infty$.

Proof :

$$\text{(A 2.11)} \quad \int_{t'}^{t} ds \left(\frac{\partial}{\partial s} e^{iH(\sigma)s} \chi_{\sigma, s}(h), \; e^{iH(\sigma)t'} \chi_{\sigma, t'}(h) \right)$$
$$= - \| \chi_{\sigma, t'}(h) \|^2 + (e^{iH(\sigma)t} \chi_{\sigma, t}(h), \; e^{iH(\sigma)t'} \chi_{\sigma, t'}(h)).$$

Let

$$\chi_M^{\sigma, t}(h \mid x) = \sum_{m=0}^{M} \frac{(-i)^m}{m!} \int d^3 p \, h(p) \, e^{i(xp - tE(\sigma, p))} \, I_p^* \, e^{-itH_{0B}}$$
$$\times (H(w_{\sigma, p}))^m \, e^{itH_{0B}} \, \Omega_{\sigma, p}(x).$$

Repeating arguments similar to the ones given in 1° we easily convince ourselves that $\chi_{\sigma, t}^{M}(h)$ converges strongly to $\chi_{\sigma, t}(h)$, *uniformly in* $t \in \mathbf{R}$.

Hence it follows from (A 2.11) that it is enough to show that

$$\int_{t'}^{t} ds \left(\frac{\partial}{\partial s} e^{iH(\sigma)s} \chi_{\sigma, s}^{M}(h), \; e^{iH(\sigma)t'} \chi_{\sigma, t'}(h) \right)$$

tends to 0 as t and t' tend to $\pm \infty$, for all $M < \infty$.

This however holds if e. g.

$$\left\| \frac{\partial}{\partial s} e^{i\,\mathrm{H}(\sigma)s} \chi^{\mathrm{M}}_{\sigma,s}(h) \right\| \leq \frac{\mathrm{C_M}}{(1+|s|)^{1+\varepsilon}} \tag{A 2.12}$$

for some $\varepsilon > 0$ and a constant $\mathrm{C_M} < \infty$ and for all $\mathrm{M} < \infty$.

Handling with domain problems in a similar way as it is done in [21] we get :

$$\left\| \frac{\partial}{\partial s} e^{i\,\mathrm{H}(\sigma)s} \chi^{\mathrm{M}}_{\sigma,s}(h) \right\|^2$$
$$= \int d^3x \left\| \sum_{m=0}^{\mathrm{M}-1} \frac{(-i)^m}{m!} \int d^3p\, e^{-i(xp+t\mathrm{E}(\sigma,p))} \times h(p)\, \mathrm{F}_{\sigma,p}(t,x)\, e^{-it\mathrm{H}_{0\mathrm{B}}} (\Pi(w_{\sigma,p}))^m e^{it\mathrm{H}_{0\mathrm{B}}} \Omega_{\sigma,p}(x) \right\|^2,$$

where

$$\mathrm{F}_{\sigma,p}(t,x) = \lambda \int d^3k\, |k|^{-1/2} g_\sigma(k)\, w_{\sigma,p}(k) \times \{ \cos(kx)\cos(|k|t) - \sin(kx)\sin(|k|t) \}.$$

Again we may interchange the integrations over x and over the momenta p and p'.

If $f(m, n, m_1, n_1, j)$ is an arbitrary function of the integers m, n, m_1, n_1, and j we define

$\mathrm{up}_{\mathrm{M}-1} f = \sup_{\alpha \leq \mathrm{M}-1} |f(m, n, m_1, n_1, j)|$, and α stands for m, n, m_1, n_1, j.

We then arrive at the following estimate

$$\left\| \frac{\partial}{\partial s} e^{i\,\mathrm{H}(\sigma)s} \chi^{\mathrm{M}}_{\sigma,s}(h) \right\|^2$$
$$\leq \text{const.} \times \sup{}_{\mathrm{M}-1} \times \left[\int \cdots \int d^3p\, d^3k\, d^3 l \times \left\{ \prod_{\mathrm{I}=1}^{m} d^5 k_{\mathrm{I}}\, w_{\sigma,p}(k_{\mathrm{I}})\, e^{-i|k_{\mathrm{I}}|s} \times \prod_{\mathrm{J}=1}^{n} d^3 k'_{\mathrm{J}}\, w_{\sigma,p}(k'_{\mathrm{J}})\, e^{i|k'_{\mathrm{J}}|s} \| w_{\sigma,p} \|_2^{2m} \times \| w_{\sigma,p'} \|_2^{2n_1} (w_{\sigma,p}, w_{\sigma,p'})^j \times h(p)\, h(p')\, e^{is(\mathrm{E}(\sigma,p)-\mathrm{E}(\sigma,p'))} \right.\right.$$

$$\times \left(\prod_{I=1}^{m} B(k_I)\, \Omega_{\sigma,p} \prod_{J=1}^{n} B(k'_J)\, \Omega_{\sigma,p'} \right)$$

$$\times w_{\sigma,p}(k)\, g_\sigma(k) \,|\, k\,|^{-1/2}$$

$$\left. \times e^{\pm i |k| s}\, w_{\sigma,p'}(l)\, g_\sigma(l) \,|\, l\,|^{-1/2}\, e^{\pm i |l| s} \right\}_{p' = p + k'_\Sigma - k_\Sigma \pm k \pm l} \Bigg].$$

But

$$\Bigg\{ h(p') \prod_{J=1}^{n} w_{\sigma,p'}(k'_J)\, \|\, w_{\sigma,p'} \|_2^{2 n_1}\, (w_{\sigma,p}, w_{\sigma,p'})^j\, w_{\sigma,p'}(l)$$

$$\times w_{\sigma,p}(k)\, g_\sigma(k) \,|\, k\,|^{-1/2}$$

$$\left. \times \left(\prod_{I=1}^{m} B(k_I)\, \Omega_{\sigma,p},\ \prod_{J=1}^{n} B(k'_J)\, \Omega_{\sigma,p'} \right) \right\}_{p' = p + k'_\Sigma - k_\Sigma \pm k \pm l}$$

is C^∞ in k if k is in supp g_σ and $(p + k'_\Sigma - k_\Sigma \pm k \pm l)$ is in supp h.

Furthermore $g_\sigma(k) = 0$ if $|\, k\,| \leq \sigma$ and

$$\frac{\partial E(\sigma, p + k'_\Sigma - k_\Sigma \pm k \pm l)}{\partial \,|\, k\,|} < 1 = \frac{\partial}{\partial \,|\, k\,|} \,|\, k\,|$$

if k is in supp g_σ and $(p + k'_\Sigma - k_\Sigma \pm k \pm l)$ is in supp h.

Therefore the techniques of Hepp, [20], imply that

$$\left\| \frac{\partial}{\partial s} e^{i H(\sigma) s}\, \chi^{M}_{\sigma,s}(h) \right\|^2 \leq \frac{C_{M,N}}{(1 + |\, s\,|)^N}$$

for some $C_{M,N} < \infty$ and all $M < \infty$ and $N < \infty$.

Q. E. D.

Theorem 4.3 is proven.

Appendix 3

PROOFS OF LEMMATA 4.4 AND 4.5

H'_p is the Hamiltonian obtained in chapter 3, corollary 3.3 which is s. a. on $\mathcal{F}_B$ and has a unique groundstate θ_p in $\mathcal{F}_B$ corresponding to the eigenvalue $E(p) = \inf \operatorname{spec} H'_p = \inf \operatorname{spec} H_p$, for all p in $\mathcal{E}$.

We first show :

A 3.1 : *Suppose that* $\operatorname{Im} \zeta \neq 0$ *or* $\operatorname{Re} \zeta < E(0)$.

Then $(\zeta - H'_q)^{-1}$ *is weakly measurable and bounded in* q *(for all* q *in* $\overline{M}_{\rho_0} = \{ q / | q | \leq \rho_0 \}$*). If* $\nabla E(q)$ *is continuous at* $q = p$ *then* $(\zeta - H'_q)^{-1}$ *is continuous in norm at* $q = p$.

Proof. — Step 1°. Some preliminaries :

(*a*) Suppose that B is a densely defined symmetric operator and A is a s. a. operator which is bounded below and suppose that in the sense of quadratic forms :

$$\pm B \leq a (A + b), \qquad \text{for some} \quad b < \infty \quad \text{and} \quad a \in [0, 1).$$

Then if Re ζ is sufficiently small

$$(\zeta - A - B)^{-1} = (\zeta - A)^{-1} \left\{ I + \sum_{n=1}^{\infty} (B (\zeta - A)^{-1})^n \right\}$$

converges in norm to the resolvent of a s. a. operator which is bounded below and is the unique Friedrichs extension of the form $A + B$. (The hypothesis that B is an opertor is not necessary.)

(*b*) If $A + B_\varepsilon$ is s. a. and bounded below for all $\varepsilon \in [0. 1)$ and $B_{\varepsilon = 0} = 0$ and if $\pm B_\varepsilon \leq \varepsilon (A + b)$ for some fixed $b < \infty$ and all $\varepsilon \in [0, 1)$ then

$$(\zeta - A - B_\varepsilon)^{-1} \xrightarrow{n} (\zeta - A)^{-1}$$

as $\varepsilon \downarrow 0$, provided $\zeta \notin \text{spec } A$.

(*c*) Let ρ be an arbitrary positive real number and

$$\rho_0 (\lambda) \leq \rho < \rho_0.$$

Because of lemma 3.1 and property (v) of $E(\sigma, p)$ [section 1.2, (1.23)] there is a $\sigma(\rho) > 0$ such that

$$\inf_{|k| \geq \lambda} (E(\sigma, p - k) + |k| - E(\sigma, p)) \geq \Delta (p, \lambda) > 0$$

for all $\lambda > 0$, all p in $M_\rho = \{ q / | q | < \rho \}$ and all $\sigma \in [0, \sigma(\rho)]$, and $| \nabla_p E(\sigma, p) | < 1$ for all p in M_ρ and all $\sigma \in [0, \sigma(\rho)]$.

Let $\mathcal{E}_\rho = \mathcal{E} \cap M_\rho$.

We now use (*a*) and (*b*) to prove A 3.1 :

Step 2°. We define :

$$(\text{A } 3.1) \quad \begin{cases} w_{\sigma, p, \delta} (k) = \lambda \, \mathfrak{I}_\delta (k) \, | k |^{-1/2} (| k | - (k, \nabla E(\sigma, p)))^{-1} \, g_\sigma (k), \\ \mathfrak{I}_\delta (k) = \begin{cases} 1, & 0 \leq | k | \leq \delta \\ 0, & | k | \geq 2 \delta \end{cases}, \quad 0 \leq \mathfrak{I}_\delta (k) \leq 1, \quad \mathfrak{I}_\delta \text{ is } C^\infty, \\ \mathfrak{I}_1 (k) = \mathfrak{I} (k) \qquad [\textit{see} \text{ theorem 3.2, (ii)}]; \end{cases}$$

(A 3.2) $\quad w_{\sigma,p}(k) = w_{\sigma,p,1}(k), \qquad \mathbf{w}_{\sigma,p,\delta}(k) = k w_{\sigma,p,\delta}(k),$

$\chi_{\rho,R}$ is the characteristic function of $\{ k \in \mathbf{R}^3 \mid \rho \leq |k| \leq R \}$,

$$\text{(A 3.3)} \quad \begin{cases} H'_{\sigma,p}(\delta) = s\text{-}\lim_{\rho \downarrow 0} \exp - i\Pi(w_{\sigma,p,\delta}\chi_{\rho,R}) \\ \qquad\qquad \times H_{\sigma,p,\rho,\infty} \exp i\Pi(w_{\sigma,p,\delta}\chi_{\rho,R}), \\ H'_{\sigma,p} = H'_{\sigma,p} \quad (\delta = 1), \qquad H'_{0,p} = H'_p \qquad \text{(for all } p \text{ in } \mathcal{E}). \end{cases}$$

We now show that $H'_{\sigma,p}(\delta)$ is s. a. and bounded from below for some sufficiently small $\delta \leq 1$ and all $\sigma \in [0, \sigma(\rho)]$.

Then

$$H'_{\sigma,p} = \exp - i\Pi(w_{\sigma,p} - w_{\sigma,p,\delta}) H'_{\sigma,p}(\delta) \exp i\Pi(w_{\sigma,p} - w_{\sigma,p,\delta})$$

has obviously the same properties.

Since $g_\sigma(k) \, |k|^{-1/2} \chi_{0,R}(k)$ is in $L^2(\mathbf{R}^3)$,

$$H^1_p(g_\sigma \chi_{0,R}) = \Omega(p - P_B) + H_{0B} + \lambda\Phi(g_\sigma |k|^{-1/2}\chi_{0,R}) + E_1(\chi_{0,R})$$

is s. a. and bounded below on $\mathcal{F}_B$ (Kato's theorem).

We assume now that $2 < R < \infty$ and $\delta \leq 1$.

We have shown in corollary 3.2 that for all $R < \infty$,

$$\begin{aligned} H'_{\sigma,q,R}(\delta) &= \exp - i\Pi(w_{\sigma,q,\delta}) H^1_q(g_\sigma \chi_{0,R}) \exp i\Pi(w_{\sigma,q,\delta}) \\ &= H^1_q(g_\sigma\chi_{0,R}) + \frac{1}{2M}\Big\{ \Big(P_B - q, \Phi(\mathbf{w}_{\sigma,q,\delta}) \\ &\qquad\qquad - \frac{1}{2}(w_{\sigma,q,\delta}, \mathbf{w}_{\sigma,q,\delta}) \Big) + \text{h. c.} \Big\} \\ &\quad + \frac{1}{2M}(\Phi(\mathbf{w}_{\sigma,q,\delta}) - \frac{1}{2}(w_{\sigma,q,\delta}, \mathbf{w}_{\sigma,q,\delta}))^2 \\ &\quad - \Phi(|k| w_{\sigma,q,\delta}) + \frac{1}{2}\big\| |k|^{1/2} w_{\sigma,q,\delta} \big\|_2^2 \\ &\quad - \lambda(g_\sigma |k|^{-1/2}\chi_{0,R}, w_{\sigma,q,\delta}). \end{aligned}$$

DEFINITION :

$$V_{\sigma,q}(\delta) = H'_{\sigma,q,R}(\delta) - H^1_q(g_\sigma\chi_{0,R}).$$

Since $R > 2$, $V_{\sigma,q}(\delta)$ is independent of R.

Observations. — The vector $(w_{\sigma,q,\delta}, \mathbf{w}_{\sigma,q,\delta})$ in $\mathbf{R}^3$, $\big\| |k|^{1/2} w_{\sigma,q,\delta} \big\|_2^2$ and

$$(g_\sigma |k|^{-1/2}\chi_{0,R}, w_{\sigma,q,\delta}) = (g_\sigma |k|^{-1/2}, w_{\sigma,q,\delta})$$

are continuous functions of q at each point p where $\nabla_q E(\sigma, q)$ is continuous (for all $\delta \leq 1, \sigma \in [0, \sigma(\rho)]$). The vector valued kernel $|k|^{-1/2} \mathbf{w}_{\sigma,q,\delta}$ and the kernel $|k|^{1/2} w_{\sigma,q,\delta}$ are continuous in q in the L^2-norm at each point p where $\nabla_q E(\sigma, q)$ is continuous (for all $\delta \leq 1, \sigma \in [0, \sigma(\rho)]$).

Furthermore

$$\text{(A 3.4)} \qquad \left\| \, |k|^{-1/2} (\mathbf{w}_{\sigma,q,\delta} - \mathbf{w}_{\sigma,p,\delta}) \right\|_2 \leq C_{\sigma,p} \, |p - q|$$

and $\left\| \, |k|^{1/2} (w_{\sigma,q,\delta} - w_{\sigma,p,\delta}) \right\|_2 \leq D_{\sigma,p} \, |p - q|$ for certain constants $C_{\sigma,p}$ and $D_{\sigma,p}$ and all p in $\mathcal{E}_\rho$.

Obviously $V_{\sigma,q}(\delta)$ is a densely defined symmetric operator for all $\delta \leq 1$, $\sigma \in [0, \sigma(\rho)]$, $q \in \mathcal{E}_\rho$.

From theorem 1.1′ we know that

$$(\zeta - H_q^1 (g_\sigma \chi_{0,R}))^{-1} \xrightarrow{s} (\zeta - H_{\sigma,q})^{-1} \quad \text{as } R \to \infty \quad \text{for all } \sigma \in [0, \sigma(\rho)].$$

Using Nelson's canonical transformation ((1.12), II, *a*, [28]) and certain " first order estimates " (*see* II, *a* and [28]) one easily shows that

$$\pm V_{\sigma,q}(\delta) \leq a(\delta) (H_q^1 (g_\sigma \chi_{0,R}) + b(\delta)),$$

and because of Rosen's principle of cutoff independence [32] :

$$\pm V_{\sigma,q}(\delta) \leq a(\delta) (H_{\sigma,q} + b(\delta)),$$

where $b(\delta)$ and $a(\delta)$ are independent of R and σ, and $a(\delta)$ depends continuously on

$$\sup_{q, \sigma \in [0, \sigma(\rho)]} \left\| \, |k|^{-1/2} \mathbf{w}_{\sigma,q,\delta} \right\|_2 < \infty$$

and

$$\sup_{q, \sigma \in [0, \sigma(\rho)]} \left\| \, |k|^{1/2} w_{\sigma,q,\delta} \right\|_2 < \infty, \qquad q \in \mathcal{E}_\rho.$$

There is therefore a $\delta \leq 1$ such that $a(\delta) < 1$.

We now apply step 1° (*a*) and conclude that $H'_{\sigma,q}(\delta)$ is s. a. and bounded below on $\mathcal{F}_B$ for all $\sigma \in [0, \sigma(\rho)]$ and all q in $\mathcal{E}_\rho$.

Hence $H'_{\sigma,q}$ has the same properties.

Using now (A 3.4) we get immediately

$$\text{(A 3.5)} \quad \pm (V_{\sigma,q}(1) - V_{\sigma,p}(1)) \leq K_{\sigma,p} \, |p - q| \, (H_p^1 (g_\sigma \chi_{0,R}) + G_{\sigma,p})$$

for some finite constants $K_{\sigma,p}$ and $G_{\sigma,p}$ which are independent of R and for all $\sigma \in [0, \sigma(\rho)]$ and p in $\mathcal{E}_\rho$.

By cutoff independence (A 3.5) still holds in the limit $R = \infty$.

From II, *a* we infer that $(\zeta - H_{\sigma,q})^{-1}$ is continuous in q in norm for all $\sigma \in [0, \sigma(\rho)]$.

Combining this with (A 3.5) and step 1° (*b*) we conclude $(\zeta - H'_{\sigma,q})^{-1}$ is continuous in q in norm at each point p where $\nabla_q E(\sigma, q)$ is continuous, provided $\zeta \notin \operatorname{spec} H_{\sigma,p}$.

Step 3°. Suppose that $\{\sigma_l\}_{l=0}^{\infty}$ is chosen such as in lemma 3.1 [such that $\nabla_p E(\sigma_l, p) \to \nabla E(p)$ as $l \to \infty$ for all p in $\mathcal{E}_\rho$] and let $\{\sigma_l\}_{l=0}^{\infty} \subset [0, \sigma(\rho)]$.

Then $(\zeta - H'_{\sigma_l,p})^{-1} \xrightarrow{n} (\zeta - H'_p)^{-1}$ as $l \to \infty$ for all p in $\mathcal{E}_\rho$, provided $\zeta \notin \operatorname{spec} H_p$.

Proof. — Our assumptions on $\{\sigma_l\}_{l=0}^{\infty}$ and on p imply that $|k|^{-1/2}\mathbf{w}_{\sigma_l,p}$ and $|k|^{1/2} w_{\sigma_l,p}$ converge in the L^2-norm to $|k|^{-1/2}\mathbf{w}_p$, $|k|^{1/2} w_p$, respectively and $(w_{\sigma_l,p}, \mathbf{w}_{\sigma_l,p})$, $\left\| |k|^{1/2} w_{\sigma_l,p} \right\|_2^2, \ldots$ converge to $(w_p, \mathbf{w}_p)$, $\left\| |k|^{1/2} w_p \right\|_2^2, \ldots$, respectively, as $l \to \infty$.

Hence, given an $\varepsilon > 0$, there is an $l_0(\varepsilon) < \infty$ such that

$$\text{(A 3.6)} \qquad \pm (V_{\sigma_l,p}(1) - V_{0,p}(1)) \leq \varepsilon (H_p + b)$$

for all $l \geq l_0(\varepsilon)$.

From II, *a* and the proof of lemma 4.1 we know that

$$(\zeta - H_{\sigma_l,p})^{-1} \xrightarrow{n} (\zeta - H_p)^{-1} \qquad \text{as} \quad l \to \infty.$$

Combining this fact with (A 3.6) and step 1° (*b*) we conclude

$$\text{(A 3.7)} \qquad (\zeta - H'_{\sigma_l,p})^{-1} \xrightarrow{n} (\zeta - H'_p)^{-1} \qquad \text{as} \quad l \to \infty$$

for all p in $\mathcal{E}_\rho$, provided $\zeta \notin \operatorname{spec} H'_p = \operatorname{spec} H_p$.

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Step 4°. We assume that $\operatorname{Im} \zeta \neq 0$ or $\operatorname{Re} \zeta < E(0)$. Then $(\zeta - H'_{\sigma_l,p})^{-1}$ is bounded uniformly in l and p, $|p| \leq \rho$, since

$$\inf \operatorname{spec} H'_{\sigma_l,p} \geq E(\sigma_l, 0) \geq E(0).$$

For all $l < \infty$ $\nabla_p E(\sigma_l, p)$ is C^∞ in $p \in M_\rho$ and $(\zeta - H'_{\sigma_l,p})^{-1}$ is continuous in norm in p, $|p| < \rho$. Because of step 3° we know that

$$(\zeta - H'_{\sigma_l,p})^{-1} \xrightarrow{n} (\zeta - H'_p)^{-1}$$

as $l \to \infty$ for all p in $\mathcal{E}_\rho$, i. e. almost everywhere in M_ρ. $(\zeta - H'_p)^{-1}$ is bounded uniformly in p, $|p| \leq \rho$, by assumption on ζ. Since $\rho < \rho_0$ was arbitrary, we get : The operator $(\zeta - H'_p)^{-1}$ is weakly measurable in p in M_{ρ_0} and continuous in p in norm wherever $\nabla E(p)$ is continuous. This completes the proof of (A 3.1).

We can now prove *lemma* 4.4 :

Define

$$g_n(\lambda) = \begin{cases} 1, & \lambda \leq \frac{1}{n}, \\ 0, & \lambda \geq \frac{2}{n}, \end{cases} \qquad 0 \leq g_n(\lambda) \leq 1,$$

g_n is C^∞ and chosen such that

$$g_n \downarrow g = \begin{cases} 1, & \lambda \leq 0, \\ 0, & \lambda > 0. \end{cases}$$

For all $\sigma \geq 0$ and all $n < \infty$ $g_n(H'_{\sigma,q} - E(\sigma, q))$ can be approximated in norm by polynomials in $(\zeta - H'_{\sigma,q})^{-1}$, where $\zeta \in \mathbf{R}$ and

$$\zeta < \inf_{\sigma \geq 0} E(\sigma, q) = E(q).$$

Therefore $g_n(H'_{\sigma,q} - E(\sigma, q))$ is continuous in norm in q on $\{ | q | < \rho \}$, for all $\sigma \in (0, \sigma(\rho)]$ and all $n < \infty$.

If the sequence $\{ \sigma_l \}_{l=0}^{\infty}$ is chosen such as in lemma 3.1 and $\{ \sigma_l \}_{l=0}^{\infty} \subset [0, \sigma(\rho)]$ and if $n < \infty$ then

$$g_n(H'_{\sigma_l,p} - E(\sigma_l, p)) \to g_n(H'_p - E(p))$$

as $l \to \infty$, in norm for all p in $\mathcal{E}_\rho$.

This follows from the definition of g_n, A 3.1 and the fact that $E(\sigma_l, p) \downarrow E(p)$ as $l \to \infty$ for all p.

Therefore, since $\rho < \rho_0$ was arbitrary, $g_n(H'_p - E(p))$ is weakly measurable in $p \in M_{\rho_0}$ and norm continuous in p for all p in $\mathcal{E}$.

For all p in $\mathcal{E}$ we have by definition of Q_p :

$$Q_p \psi = g(H'_p - E(p)) \psi = \left(\inf_n g_n(H'_p - E(p))\right) \psi$$
$$= s\text{-}\lim_{n \to \infty} g_n(H'_p - E(p)) \psi.$$

Thus $Q_p \psi$ is weakly measurable in p and, since $\| Q_p \psi \| \leq \| \psi \|$ for all p, $| p | \leq \rho_0$, $Q_p \psi$ is strongly integrable on each compact set $K \subseteq \overline{M}_{\rho_0}$.

It has been shown in section 3.2 and II, *b* that $e^{-t\hat{H}_p}$ is positive ergodic (for all $t > 0$) on the cone $\mathfrak{K}$ defined in chapter 3, section 3.2, C.

If p is in $\mathcal{E}$ then Ω_p is in $\mathfrak{K}$ and $\exp i\,\Pi(w_p)\,\psi_0$ is in $\mathfrak{K}$. There is therefore a $t_0 > 0$ such that

$$0 < \left(\exp i\,\Pi(w_p)\,\psi_0,\ e^{-t\hat{\mathrm{H}}_p}\,\Omega_p\right) = \left(\exp i\,\Pi(w_p)\,\psi_0,\ \Omega_p\right) e^{-t_0 \mathrm{E}(p)}$$
$$= (\psi_0,\ \theta_p)\, e^{-t_0 \mathrm{E}(p)}.$$

Thus $(\psi_0, \theta_p) > 0$ for all p in $\mathcal{E}$ and hence $(\psi_0, \mathrm{Q}_p\,\psi_0) > 0$ for all p, $|p| \leq \rho_0$.

But this completes the proof of lemma 4.4.

Remark. — An analogous result could still be proven if the eigenvalue $\mathrm{E}(p)$ of H'_p were degenerate. This might be the case in some more difficult models such as nonrelativistic.

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We now want to turn to the proof of *lemma* 4.5 :

A 3.2 : *The operators* $e^{i\Phi_p(t)}\,\mathrm{C}_p(t)\,\mathrm{V}_p(x)$ *are weakly measurable in* $p \in \mathrm{M}_{\rho_0}$.

Proof. — We have defined $\Phi_{\sigma,p}(t)$, $\mathrm{C}_{\sigma,p}(t)$ and $\mathrm{V}_{\sigma,p}(x)$ in chapter 4, section 4.1, (4.12). We choose the sequence $\{\sigma_l\}_{l=0}^{\infty}$ such as in lemma 3.1 and $\{\sigma_l\}_{l=0}^{\infty}[0, \sigma(\rho)]$.

From what we have done in the proof of A 3.1 it follows that the sequence $e^{i\Phi_{\sigma_l,p}(t)}\,\mathrm{C}_{\sigma_l,p}(t)\,\mathrm{V}_{\sigma_l,p}(x)$ converges strongly to $e^{i\Phi_p(t)}\,\mathrm{C}_p(t)\,\mathrm{V}_p(x)$ as $l \to \infty$ for all p in $\mathcal{E}_\rho$.

[This is easily shown on the domain $\mathrm{D}(\mathrm{N_B})$ by use of Duhamel's formula :

$$e^{\mathrm{A}t} - e^{\mathrm{B}t} = \int_0^t ds\, e^{\mathrm{A}(t-s)}\,(\mathrm{A} - \mathrm{B})\, e^{\mathrm{B}s}.$$

Since $\mathrm{D}(\mathrm{N_B})$ is dense in $\mathcal{F}_\mathrm{B}$ the same is true on $\mathcal{F}_\mathrm{B}$.]

From (4.12) it follows directly that $e^{i\Phi_{\sigma_l,p}(t)}\,\mathrm{C}_{\sigma_l,p}(t)\,\mathrm{V}_{\sigma_l,p}(x)$ is strongly continuous in $p \in \mathrm{M}_\rho$, $(\sigma_l \leq \sigma(\rho))$. Since $\rho < \rho_0$ was arbitrary it follows that $e^{i\Phi_p(t)}\,\mathrm{C}_p(t)\,\mathrm{V}_p(x)$ is weakly measurable in $p \in \mathrm{M}_{\rho_0}$.

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Since $e^{i\Phi_p(t)}\,\mathrm{C}_p(t)\,\mathrm{V}_p(x)$ is a unitary operator for all p in $\mathcal{E}$ $e^{i\Phi_p(t)}\,\mathrm{C}_p(t)\,\mathrm{V}_p(x)$ is strongly integrable in p on each compact set $\mathrm{K} \subseteq \overline{\mathrm{M}}_{\rho_0}$,

Thus we have proven lemma 4.5.

This concludes appendix 3.

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