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Section A :
Physique théorique.

On the existence of a geometrical interpretation of spinors of the various pseudo-euclidean spaces of dimension 3 and 4 by means of real, irreducible tensors of rank *p*

by

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ABSTRACT. — A geometrical interpretation of spinors is possible in the case of the groups SO (R, 3), SO (R, 4) and SO $(3, 1)^0$, in terms of real, irreducible tensors of the lowest rank $p = 3$, 6 and 2, respectively, but not in the cases SO $(2, 1)^0$ and SO $(2, 2)^0$. Thus the Minkowski-space is distinguished from the other 3 or 4 dimensional spaces, by the fact that is admits a geometrical interpretation of spinors by means of tensors of the lowest rank $p = 2$. In this way, we make precise a conjecture stated in E. Cartan's, *Theory of Spinors*, p. 132, and prove it in this form.

1. INTRODUCTION

From a geometrical point of view spinors are rather abstract quantities, because they belong to the covering group, which has not a simple geometrical meaning. Therefore we would like to replace spinors in an invariant way by more concrete quantities, i. e. tensors, which have an obvious geometrical meaning.

2. DEFINITION OF TENSORS AND SPINORS

We restrict ourselves to the following cases of geometrical invariance groups (1) :

(1) $G_0 = SO(R, 3)$, $G = SU(2)$:
$\sigma_{1,0} = \text{id}$, $W = R^3$, $V = C^2$;

(2) $G_0 = SO(2, 1)^0$, $G = SL(R, 2)$:
$\sigma_{1,0} = \text{id}$, $W = R^3$, $V = C^2$;

(3) $G_0 = SO(R, 4)$, $G = SU(2) \times SU(2)$:
$\sigma_{1,0} = \text{pr}_1$, $W = R^4$, $V = C^2$;

(4) $G_0 = SO(3, 1)^0$, $G = SL(C, 2)$:
$\sigma_{1,0} = \text{id}$, $W = R^4$, $V = C^2$;

(5) $G_0 = SO(2, 2)^0$, $G = SL(R, 2) \times SL(R, 2)$:
$\sigma_{1,0} = \text{pr}_1$, $W = R^4$, $V = C^2$;

wherein $SO(k, l)^0$ is the 1-component of the (real) pseudo-Euclidean rotation group $O(k, l)$ acting on the (real) pseudo-Euclidean space W of dimension $k + l$ and signature $k - l$.

$\sigma_{1,0}$ is the spinor representation of G_0, pr_1 means that the first factor of G operates identically on V. The elements of V are called " (semi) spinors of the first kind ".

The connection between tensors and spinors can be illustrated by the following two diagrams :

$$\begin{array}{ccc} G & \xrightarrow{\sigma_{1,0}} & GL(V) \\ \downarrow{\scriptstyle \pi_0} & & \\ G_0 & \xrightarrow{\tau_p} & GL(T_p(W)) \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\sigma_{1,0}(g)} & V \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ T_p(W) & \xrightarrow{\tau_p(\pi_0(g))} & T_p(W) \end{array}$$

Diagram 1 Diagram 2

Herein π_0 is the double covering map; GL is the real general linear group. $T_p(W)$ is the p^{th} component of the tensor algebra of W. τ_p is the p^{th} tensor product of the identical representation $\tau_1 = \text{id}$. The elements of $T_p(W)$ are called " real tensors of rank p ".

COMMENT 1. — The complex orthogonal group $SO(C, n)$, $n > 2$ has a twofold universal covering group : $\text{Spin}(C, n)$, i. e. the mapping

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(1) For the definition of spinors in the general case $SO(k, l)^0$ *see* e. g. Cartan [1], Atiyah-Bott-Shapiro [2], Chevalley [3]. In the cases of $SO(R, 2)$, $SO(1, 1)^0$ and $SO(R, 1)$ the definition of spinors is rather arbitrary.

π_0^* : Spin (C, n) $\rightarrow$ SO (C, n) is a double covering. The groups G_0 listed above are subgroups of SO (C, n). The corresponding groups G are given by $G = \pi_0^{*-1}(G_0)$. π_0 is the restriction of π_0^* on G.

COMMENT 2. — G is the universal covering group of G_0 in the cases SO (R, 3), SO (R, 4), SO $(3, 1)^0$, but the universal covering groups of SO $(2, 1)^0$ and SO $(2, 2)^0$ are infinite-fold.

COMMENT 3. — $\sigma_{1,0}$ is the simplest non-trivial representation of G.

3. THE CONCEPT OF GEOMETRICAL INTERPRETATION OF SPINORS

We look for (in general non linear) mappings $\varphi : V \rightarrow T_p(W)$, called " geometrical interpretations (GI) of spinors by means of real, irreducible tensors of rank p " with the following four properties :

(GI 1) φ is continuous;

(GI 2) φ is G-invariant, i. e. the diagram (2) commutes, $\forall\, g \in G$;

(GI 3) $\varphi^{-1}(\varphi(v)) = \{ v, -v \}$, $\forall\, v \in V$;

(GI 4) $\varphi(V) \subset T' \subset T_p(W)$, where T' is an irreducible, G-invariant linear subspace of $T_p(W)$.

Remark. — Because of (GI 2) and $\pi_0(1) = \pi_0(-1) = 1$ (= neutral element of any group) it follows $\varphi(v) = \varphi(-v)$. Therefore

$$\varphi^{-1}(\varphi(v)) \supset \{ v, -v \}.$$

So, since φ cannot be injective, (GI 3) is the most we can require.

4. CARTAN-PENROSE'S FLAG AS A SPECIAL EXAMPLE

In the case $G_0 = SO(3, 1)^0$ = 1-component of the Lorentz group, E. Cartan ([1], p. 131) has given explicitly such a $\varphi : V \rightarrow T_p(W)$ with $p = 2$, *see* formula (7 *c*). In the physical literature (*cf.* R. Penrose, [4], p. 151) this $\varphi(v) \in T_2(W)$ is called the " flag " corresponding to the spinor $v \in V$. In this case W is the Minkowski space and $\varphi(v)$ is, geometrically, a real null-plane tangent to the null cone of the Minkowski space. The tangent line is called the " flag-pole ". T' satisfying (GI 4) is the space of the skew symmetric second rank tensors.

Remark. — This T' is irreducible, but not absolutely irreducible; i. e. the complexification T'* of T' is not irreducible.

5. THEOREM ON THE EXISTENCE OF A GEOMETRICAL INTERPRETATION OF SPINORS

Concerning the other cases SO (R, 3), SO (R, 1), SO $(2, 1)^0$, SO $(2, 2)^0$, E. Cartan ([1], p. 132) has stated :

" An interpretation of this sort in terms of a *real* image is possible only in the space of special relativity, but not in real Euclidean four-dimensional space ".

In contrast to this we state the following

THEOREM 5.1. — *The lowest p, for which a* $\varphi : V \to T_p(W)$ *satisfying* (GI 1), (GI 2), (GI 3), (GI 4), *exists, is*

$p =$	3	6	2	*in the cases*
	SO (R, 3)	SO (R, 4)	SO $(3, 1)^0$	

respectively, whereas in the cases SO $(2, 1)^0$ *and* SO $(2, 2)^0$ *no such a* $\varphi : V \to T_p(W)$ *exists.*

Remark 1. — Admitting *complex* tensors E. Cartan ([1], p. 93 and 106) has proved for all cases SO $(k, l)^0$ that there exists a $\varphi : V \to T^*_\nu(W)$ with $2\nu = k + l$ or $2\nu + 1 = k + l$, respectively.

Remark 2. — For the cases SO $(2, 1)^0$ and SO $(2, 2)^0$, there exists a $\varphi : V^R \to T_2(W)$, where V^R is the space of all *real* spinors. ($V^{R*} = V$), which is here an invariant subspace of V.

The proof of the theorem will be given in the appendix.

6. APPLICATION TO PHYSICS

It is an old question why the world is a 4-dimensional metrical continuum with signature $(- - - +)$. To put light on it, all mathematical features peculiar to Minkowski space should be investigated.

If it is true, that everything in the world is made of spinors $v \in V$ (*cf.* R. Penrose, 1971; W. Heisenberg, 1962; C. F. von Weizsacker, 1958), and if it was an accustomed mode of thought in classical physics to represent everything by tensors $t \in T_p(W)$ of the lowest rank p, then if follows from theorem 5.1 that $p = 2$ and W is the Minkowski space [2].

[2] We point out, however, that the spin representation is a projective representation (as ordinarily required in Quantum Mechanics) while the tensor representation is not.

7. EXPLICIT REPRESENTATIONS OF SPINORS $v \in V$ BY REAL, IRREDUCIBLE TENSORS $t = \varphi(v)$ OF RANK p

Let us choose the covering map π_0 in such a way that the fundamental formulae connecting vectors x with spinors X of the second rank are, in case of (3) :

(*a*) SO (R, 3) :

$$x^i = \sigma^i_{AB} X^{AB}, \quad \text{with} \quad \sigma^1_{AB} = \sigma_1, \quad \sigma^2_{AB} = -i\,1, \quad \sigma^3_{AB} = \sigma_3;$$

(*b*) SO (R, 4) :

$$x^\mu = \sigma^\mu_{AB'} X^{AB'}, \quad \text{with} \quad \sigma^1_{AB'} = \sigma_1, \quad \sigma^2_{AB'} = -\sigma_2, \quad \sigma^3_{AB'} = \sigma_3, \quad \sigma^4_{AB'} = i\,1;$$

wherein the first factor of $G = SU_2 \times SU_2$ acts on the unprimed indices A and the second factor of G acts on the primed indices B′ and

(*c*) SO $(3, 1)^0$:

$$x^\mu = \sigma^\mu_{A\dot{B}} X^{A\dot{B}}, \quad \text{with} \quad \sigma^1_{A\dot{B}} = -\sigma_1, \quad \sigma^2_{A\dot{B}} = \sigma_2, \quad \sigma^3_{A\dot{B}} = -\sigma_3, \quad \sigma^4_{A\dot{B}} = -1$$

wherein G = SL (C, 2) acts on the unprimed indices identically, and it acts on the dotted indices by the complex-conjugate transformation. σ_i are the Pauli matrices.

Under these assumptions the tensor $t = \varphi(v)$ of rank p corresponding to the spinor $v \in V$ can be chosen as follows (4) :

(*a*) SO (R, 3), $p = 3$:

$$t^{ijk} = \sigma^i_{AB}\,\sigma^j_{CD}\,\sigma^k_{EF}\,\psi^{ABCDEF},$$

where

$$\psi^{ABCDEF} = t_0^{ijk}\,\eta_i^{AB}\,\eta_j^{CD}\,\eta_k^{EF},$$

with t_0^{ijk} given in [8 C (*a*)] and

$$\eta_1^{AB} = (\varepsilon^{CB} v^{C*})\,v^A + (\varepsilon^{CA} v^{C*})\,v^B,$$

$$\eta_2^{AB} = i v^A v^B + i\,(\varepsilon^{AC} v^{C*})\,(\varepsilon^{BD} v^{D*}),$$

$$\eta_3^{AB} = v^A v^B - (\varepsilon^{AC} v^{C*})\,(\varepsilon^{BD} v^{D*}), \qquad \text{with} \quad \varepsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ (5).$$

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(3) The indices run as follows : A, B, C, ... = 1, 2; $i, k, \ldots = 1, 2, 3$; $\mu, \nu, \ldots = 1, 2, 3, 4$. Einstein's sum rule is used.

(4) The following formulae are a particular choice among an infinite set of possibilities.

(5) The tensor t^{ijk} can be visualized e. g. by an orthonormal 3-frame (because its fix group is trivial, *see* below). — v^{A*} transforms as $v_A = \varepsilon_{BA} v^B$, with $\varepsilon_{AB} = \varepsilon^{AB}$.

(*b*) SO (R, 4), $p = 6$:

$$t^{\mu\nu\lambda\rho\sigma\tau} = i\, \sigma^{\mu}_{AG'}\, \sigma^{\nu}_{BH'}\, \sigma^{\lambda}_{CK'}\, \sigma^{\rho}_{DL'}\, \sigma^{\sigma}_{EM'}\, \sigma^{\tau}_{FN'}\, \psi^{ABCDEF}\, \varepsilon^{G'H'}\, \varepsilon^{K'L'}\, \varepsilon^{M'N'}$$

with the same ψ^{ABCDEF} as in case (*a*).

(*c*) SO $(3, 1)^0$, $p = 2$:

$$t^{\mu\nu} = \sigma^{\mu}_{A\dot{B}}\, \sigma^{\nu}_{C\dot{D}}\, \zeta^{A\dot{B}C\dot{D}},$$

where

$$\zeta^{A\dot{B}C\dot{D}} = v^{A}\, v^{C}\, \varepsilon^{\dot{B}\dot{D}} + \varepsilon^{AC}\, v^{\dot{B}}\, v^{\dot{D}},$$

with

$$\varepsilon^{\dot{A}\dot{B}} = \varepsilon^{AB}, \qquad v^{\dot{A}} = v^{A*}.$$

That these formulae are invariant (GI 2) is obvious. That they are invertible (GI 3) and that the tensor t is real and irreducible (GI 4) is a consequence of the proof given in the appendix.

8. APPENDIX : PROOF OF THE THEOREM 5.1

A. General idea of the proof

Suppose $t_0 = \varphi(v_0)$, $v_0 \in V$, $t_0 \in T'$. Then by (GI 2) (invariance), $\varphi(v)$ is defined for all $v \in V$ belonging to the same orbit as v_0 : $\varphi(gv_0) = \pi_0(g).t_0$.

This definition is unique if $FG(v_0) \subset FG(t_0)$ where $FG(v_0)$ and $FG(t_0)$ are fix groups of v_0 and t_0, respectively :

$$FG(v_0) = \{ g \in G \mid gv_0 = v_0 \},$$
$$FG(t_0) = \{ g \in G \mid \pi_0(g)\, t_0 = t_0 \}.$$

If $\varphi : V \to T'$ would be injective, we would have : $FG(v_0) \supset FG(t_0)$. But φ satisfies (GI 3) and therefore

$$FG(t_0) = FG'(v_0) = \{ g \in G \mid gv_0 = \pm v \}, \tag{1}$$

In both cases SO (R, 3) and SO (R, 4), (1) is also a sufficient condition for the existence of φ, because the orbits of $V \simeq C^2$ are characterized by a real number λ, $0 \leq \lambda < \infty$, where $\lambda = v_1 v_1^* + v_2 v_2^*$. Therefore, if we choose an arbitrary continuous, strictly monotonic function $f : R \to R$ with $f(0) = 0$, there is just one $\varphi : V \to T'$ given by $\varphi(\lambda\, v_0) = f(\lambda)$. x_0 satisfying (GI 1), (GI 2), (GI 3), (GI 4).

B. Two lemmas

LEMMA 1. — *All irreducible,* real *representations of the groups* SO (R, 3), SO $(2, 1)^0$, SO (R, 4), SO $(2, 2)^0$ *are absolutely irreducible i. e. their complexifications are also irreducible.*

Remark. — For the group SO (3, 1)^0, the lemma is not true.

Proof. — According to Freudenthal [5], Theorem 55.8, all real irreducible representations of a Lie group are absolutely irreducible, if and only if all irreducible complex representations of this group are virtually real. A representation is called virtually real if it is equivalent to a real representation. All irreducible representations of SL (R, 2) and of SL (R, 2) $\times$ SL (R, 2) are virtually real. Therefore the lemma is true for SO (2, 1)^0 and for SO (2, 2)^0.

According to Freudenthal [5], Theorem 57.3, under certain assumptions, an irreducible complex representation is virtually real if and only if ε defined in 57.2.6 has the value $+1$ (ε is a sign, i. e. $\varepsilon = \pm 1$).

The assumptions 57.2 of this Theorem are fulfilled for the groups SO (R, 3), SO (R, 4), SU (2), SU (2) $\times$ SU (2), because all their irreducible representations are-self-contravalent (6).

If the tensor product of two irreducible representations is again irreducible, ε behaves multiplicatively.

For SO (R, 3), the lemma is well known. Therefore, for all irreducible representations of SO (R, 3), $\varepsilon = +1$. The other representations of SU (2) (= spin representations) are not absolutely irreducible. Therefore for them, $\varepsilon = -1$. All irreducible representations of SO (R, 4) are tensor products of two representations of SO (R, 3) [then : $\varepsilon = (+1).(+1)$] or of two spin representations of SU (2) [then : $\varepsilon = (-1)(-1)$]. Hence in any case, $\varepsilon = +1$. Therefore the lemma holds true also for SO (R, 4).

LEMMA 2. — *Let* D *be a complex irreducible representation space of the group* G. *Let* A *and* B *be real,* G-*invariant, irreducible representation spaces of* G, *embedded in* D :

$$A \underset{G}{\subset} D, \qquad B \underset{G}{\subset} D, \qquad A^* = D, \qquad B^* = D.$$

Then it follows $A \underset{G}{\cong} B$ *i. e. the two representation spaces* A *and* B *are real equivalent.*

Proof. — We have dim A = dim B. Let $e_i : i = 1, \ldots, n$ be a base of A and e'_i; $i = 1, \ldots, n$ be a base of B. Both are also a base of D. In these bases, let the representation be M (g) and M$'$ (g) = PM (g) P^{-1}, $g \in$ G, respectively, where M (g) and M$'$ (g) are real $n.n$-matrices.

(6) A representation D (g) is called self-contravalent if it is equivalent to the contragradient representation D $(g)^{-1\,\mathrm{tr}}$. For SU (2), and therefore for SO (R, 3), this is well known. For SU (2) $\times$ SU (2), and therefore for SO (R, 4), it can be verified immediately, because every irreducible representation of SU (2) $\times$ SU (2) is a tensor product of two irreducible representations of SU (2).

Let $P = Q + i S$, where Q and S are real matrices. Then we calculate : $(S^{-1} Q) M = M (S^{-1} Q)$. Because $M(g)$ is absolutely irreducible, by Schur's lemma it follows : $S^{-1} Q = \lambda 1$. Therefore D is a complex multiple of the real matrix Q, therefore $M'(g) = QM(g) Q^{-1}$, i. e. the representation spaces A and B are real equivalent.

C. **Proof of the theorem**

(*a*) *Case* SO (R, 3) :

We define

$$\begin{aligned} FG(t) &= \{ d \in SO(R, 3) \mid d.t = t \}, && t \in T' \subset T_p(W), \\ FG'(v) &= \{ d \in SO(R, 3) \mid \pi_0^{-1}(d)\, v = \pm v \}, && v \in V, \end{aligned}$$

as the fix groups of t and respectively.

By (GI 2), (GI 3), follows that

$$FG'(v) = FG(\varphi(v)), \qquad \forall\, v \in V$$

is a necessary condition for the existence of the mapping φ. We have

$$FG'(v) = \{ 1 \}, \qquad \forall\, v \in V, \qquad v \neq 0,$$

where 1 is the neutral element of SO (R, 3). But $FG'(0) = SO(R, 3)$. We have to investigate all invariant, irreducible, linear subspaces T' of $T_p(W)$; $p = 0, 1, 2$ and we have to show that for no $t \in T'$, $FG(t) = \{ 1 \}$.

Equivalent T' must be taken only once. Therefore the elements of T' are the scalars [with fix group SO (R, 3)], the vector [i. e. $T' = T_1(W) = W$ with fix group SO (R, 2)], or the traceless symmetric tensors of the second rank, i. e. $T' \subset T_2(W) = W \otimes W$, the components of which we denote by $t^{ij} = t^{ji}$; $j = 1, 2, 3$. t^{ij} can be reduced to diagonal from by application of an element $d \in SO(R, 3)$: $d.t = t'$, where $t = (t^{ij})$, $t' = (t'^{ij})$ with $t'^{ij} = 0$ for $i \neq j$ and $FG(t) \cong FG(t')$. Furthermore $d_0.t' = t'$ with

$$d'_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in FG(t') \subset SO(R, 3).$$

is valid. Therefore $FG(t') \neq \{ 1 \}$ and $FG(t) \neq \{ 1 \}$. Thus we have proved : $p > 2$.

Now, we prove $p = 3$:

For $t_0 \in T_3(W) = W \otimes W \otimes W$ with the non-vanishing components

$$t_0^{111} = -3, \qquad t_0^{112} = t_0^{121} = t_0^{211} = -1, \qquad t_0^{122} = t_0^{212} = t_0^{221} = 2,$$
$$t_0^{133} = t_0^{313} = t_0^{331} = 1, \qquad t_0^{233} = t_0^{323} = t_0^{332} = 1, \qquad t_0 \text{ is real,}$$

totally symmetric and traceless (i. e. irreducible) and we have $FG(t_0) = \{1\}$. This can be shown in the following way : Let us define

$$\rho_0 : \quad T_3(W) \to T_2(W)$$
$$(t^{ijk}) \mapsto (y^{kl}) = \left(\sum_{i,j=1}^{3} t^{ijk} t^{ijl} \right)$$

then we have $\rho_0(d.t) = d(\rho_0(t))$, $\forall\, d \in SO(R, 3)$, $t \in T_3(W)$ (invariance of ρ_0).

So we have : $FG(t) \subset FG(\rho_0(t))$. Let us look at $y = (y^{kl}) \in GL(W)$ as an element of GL (W). Then all eigenvalues of y are different. Therefore the eigendirections of y are uniquely determined. Therefore $FG(\rho_0(t_0))$ can consist of four elements only : identity and three rotations by the angle π with the three eigen directions as axes. But the rotations will not t_0 let fiw. Therefore $FG(t_0) = \{1\}$.

(*b*) *Case* SO (R, 4) :

In this case $G = SU(2) \times SU(2)$ acts on $\tilde{V} : \underset{C}{\cong} V$ by$g\ \tilde{v} = pr_2(g)\,\tilde{v}$, $\tilde{v} \in \tilde{V}$, $g \in G$. Then we have also a natural action of G on $T_p(V \otimes \tilde{V})$, on $S_m(V) \otimes S_n(\tilde{V})$, etc. $S_m(V)$ means the m^{th} component of the symmetric algebra of V. It is well known (*see* e. g. E. Cartan [1], p. 129) that there exists a continuous, bijective, linear, G-invariant mapping, given by (7 *b*), :

$$\chi_0 : \quad T_1^*(W) \to V \otimes \tilde{V} \qquad \text{i. e.} \quad T_1^*(W) \underset{G}{\cong} V \otimes \tilde{V},$$

which is the spinor representation of complex vectors.

The image $H = \chi_0(T_1(W))$ is an R-linear, G-invariant subspace of $V \otimes \tilde{V}$. G acts, canonically, on $T_p(H)$. Therefore χ_0 induces a continuous, bijective, linear, G-invariant mapping

$$(2) \qquad \chi : \quad T_p(W) \to T_p(H) \underset{G}{\subset} T_p(V \otimes \tilde{V})$$

which is the representation of real tensors by spinors of double rank. We have to show that in the cases $p = 0, 1, 2, 3, 4, 5$, there exists no $\varphi : V \to T_p(W)$. Let us define :

$$FG(h) = \{ g \in G \mid gh = h \}, \qquad h \in T_p(H)$$

and

$$FG'(v) = \{ g \in G \mid gv = \pm v \}, \qquad v \in V$$

as the fix groups of h and v, respectively. We find

$$FG'(v) = \{1, -1\} \times SU(2), \qquad \forall\, v \in V, \qquad v \neq 0.$$

Then, by (1), we have as a necessary condition for the existence of φ : $FG'(v) = FG(\chi(\varphi(v)))$, $\forall\, v \in V$. Let T′ be any G-invariant, G-irreducible linear subspace of any such $T_p(W)$ and define $H' = \chi(T')$ for which

$$(3) \qquad H'^* \underset{G}{\subset} T_p(V \otimes \tilde{V})$$

is valid. Then we have to show, that there exists no $h \in H'$ with

$$(4) \qquad FG(h) = \{1, -1\} \times SU(2).$$

By lemma 1, H′* is irreducible. Therefore for suitable m, n, holds :

$$(5) \qquad H'^* \underset{G}{\cong} S_m(V) \otimes S_n(\tilde{V})$$

because the right hand side of (5) is a complete system of representants of the equivalence classes of all irreducible, complex representation spaces of G. If there is an $h \in H'$ fulfilling (4), there must be $n = 0$ in (5) and it follows : $m \leqq 5$ and $m =$ even.

So, we have either :

$$H'^* \underset{G}{\cong} S_2(V) \qquad \text{or} \qquad H'^* \underset{G}{\cong} S_4(V)$$

and it remains to show that there exist no $h \in H'$ with

$$(6) \qquad \{g_1 \in SU(2) \mid g_1 h = h\} = \{1, -1\}.$$

It well known that $S_2(V) \underset{SU(2)}{\cong} T_1^*(R^3)$ [*cf.* (7 *a*)], where SU (2) acts on $T_1^*(R^3)$ by means of the covering mapping $SU(2) \to SO(R, 3)$. In the same way, we have $S_4(V) \underset{SU(2)}{\cong} (S_2(R^3) - R)^*$ where " $S_2(R^3) - R$ " mean the traceless symmetric second rank tensors. By lemma 2 it follows that we have either

$$H' \underset{SU(2)}{\cong} T_1(R^3) \qquad \text{or} \qquad H' \underset{SU(2)}{\cong} (S_2(R^3) - R) \underset{SU(2)}{\subset} T_2(R^3).$$

In case (a), we have already shown that there exists no $h \in T_p(R^3)$ ($p = 1, 2$) fulfilling (6).

Now, we have to show the existence of a $\varphi : V \to T_6(W)$. We have by (2) : $T_6(W) \underset{G}{\cong} T_6(H)$ contains a G-invariant, irreducible, R-linear subspace H′ with

$$H'^* \underset{G}{\cong} S_6(V) \underset{SU(2)}{\cong} S_3^*(R^3) \underset{SU(2)}{\subset} T_3^*(R^3).$$

So, by lemma 2, we have :

$$H' \underset{SU(2)}{\cong} S_3(R^3) \underset{SU(2)}{\subset} T_3(R^3),$$

and we know from the case SO (R, 3) that there exists a $h = h_0 \in H'$ fulfilling (6) and therefore fulfilling (4).

(*c*) *Case* SO $(3, 1)^0$:

The proof is already given by E. Cartan [1] (*cf.* chap. 4).

(*d*) *Case* SO $(2, 1)^0$:

Here is G = SL (R, 2). Let $T' \underset{G}{\subset} T_p(W)$, $0 \leqq p < \infty$, be any G-invariant, R-linear, irreducible subspace of $T_p(W)$:

$$V^R = R^2 \underset{G}{\subset} C^2 = V$$

is a G-invariant subset of V. In the representation theory of SL (R, 2), it is well known that

$$T' \underset{G}{\cong} S_n(V^R) \tag{7}$$

because the right hand side of (7) is for $n = 0, 1, 2, \ldots$ a complete system of representants of real, irreducible, G-invariant representation spaces of G. Because T' is also a representation space of G_0, n must be even. Suppose that $\varphi : V \to S_n(V^R)$ is a geometrical representation, the fix groups of v and $\varphi(v)$ must be identical or more precisely :

$$\{ g \in G \mid gv = \pm v \} = \{ g \in G \mid \pi_0(g)\varphi(v) = \varphi(v) \}, \qquad \forall v \in V. \tag{8}$$

Using coordinates for $v = v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the fix group of the left hand side consists of the set of matrices :

$$g = \begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix}, \qquad \beta \in R. \tag{9}$$

Denote the components of $\psi = \varphi(v_0) \in S_n(V^R)$ by $\psi^{A_1 \ldots A_n}$; $A_i = 1, 2$ where $\psi^{A_1 \ldots A_n}$ is totally symmetric in its indices. If ψ is fix under all transformations (9), it follows that $\psi^{A_1 \ldots A_n} = 0$, exept that $\psi^{111 \ldots 1} \neq 0$. $\psi^{111 \ldots 1} \in R$. The fix group of λv_0, $\lambda \in C$ is the same as the fix group of v_0. Therefore $\varphi(\lambda v_0) = f(\lambda)\psi$ with $f(\lambda) \in R$. It follows that $f : C \to R$ would be injective and continuous. But such an f does not exist.

(*e*) *Case* SO $(2, 2)^0$:

In this case $G = SL(R, 2) \times SL(R, 2)$ and $T' \underset{G}{\cong} S_n(V^R) \otimes S_m(V^R)$ because the right hand side is for $n, m = 0, 2, 1, \ldots$ a complete system of representants of real, irreducible, G-invariant representation spaces of G [G acts on $S_n(V^R)$ by pr_1, and on $S_m(V^R)$ by pr_2]. The condition for the fix groups is formally the same as (8).

The fix group of $v_0 = \begin{pmatrix}1\\0\end{pmatrix}$ is $\left\{\begin{pmatrix}\pm 1 & \beta \\ 0 & \pm 1\end{pmatrix}\right\} \times SL(R, 2)$. It follows : $m = 0$. Because T′ is a representation space of G_0, n is even and the problem is reduced to the case (*d*) of $SO(2, 1)^0$.

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