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Section A :
Physique théorique.

On the significance of electromagnetic potentials in the Quantum Theory

by

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ABSTRACT. — A model for the Aharanow-Bohm effet can be given as follows. Introduce Aharanow-Bohm fields, which are special Maxwell fields (as specified in section 3). These fields exhibit the following properties :

1° They account for the experimentally observable phase change $\Delta\varphi = \frac{\Delta S}{\hbar}$ which is produced by the Aharanow-Bohm effect.

2° Despite the missing Lorentz force law, the Aharanow-Bohm effect can ultimately still be described in terms of some " force law " on a more abstract level however. This is due to the specifically topological character of the Aharanow-Bohm fields.

3° The Aharanow-Bohm fields exhibit a " quantization property " in virtue of which the Aharanow-Bohm effect is characterized as a quantum effect.

1. INTRODUCTION

The Aharanow-Bohm effect (hereafter referred to as AB effect) as described in our section 2, displays two essential quantum mechanical features. The first is concerned with the question of whether or not the vector potential $\vec{A}$, which satisfies the equation $\vec{B} = \text{curl}\,\vec{A}$ ($\vec{B}$: magnetic flux density) must still be regarded as mathematical auxiliary when entering into Quantum Mechanics, i. e. as a device in making calculations, or whether it becomes a real physical field. This question has been discussed by different authors [1], [2], [3], [7],

[8]. The second aspect of the AB effect exhibits that the force concept gradually fades away in Quantum Mechanics. Instead of forces, one deals with the way interactions change the wavelengh of the waves.

In fact, AB showed that one of the results of Quantum theory is that there are physical effects on charged particles in regions in which the electromagnetic field is nonexistent, but where $\vec{A}$ is nonzero. Hence, the key problems related to the AB effect are the following :

1. The problem of localization and action-at-a-distance and
2. The meaning of the force concept in quantum mechanics.

Since the notion of field avoids the idea of action-at-a-distance in the description of electromagnetic phenomena, it turns out that the interaction of a charged particle with the electromagnetic field must be a local one (i. e. the field can operate only where the charge is). Therefore, in the description of this interaction, only those fields which are nonzero in the region to which the charged particles are confined can account for observable physical effects. The AB effect is actually such that the local formulation is possible only with the aid of the potentials $A_\mu (x^\mu)$, since $F_{\mu\nu} (x^\mu) = 0$. This problem of localization has been discussed extensively by AB [4] and we therefore will not enter into this question here.

In this paper we are mainly concerned with the problem of investigating the mechanism that, for quantum mechanics, replaces the Lorentz force law

$$\vec{K} = e \vec{E} + \frac{e}{c} \left[\vec{v}, \vec{B} \right]. \tag{1}$$

It turns out that the law that determines the behaviour of quantum mechanical particles is given in terms of the phase difference

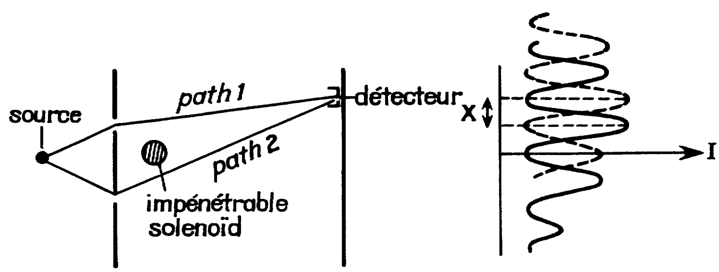
$$\frac{S_1 - S_2}{\hbar} = \left(\frac{e}{\hbar c} \right) \oint \vec{A} d\vec{x}. \tag{2}$$

Such a law will exist, even though there are no magnetic forces acting in the places where the particle beam passes. Otherwise stated, formula (2) determines the interference pattern of the AB arrangement (refer to our section 2), i. e. characterizes how the notion in a field-free region of the charged particles is changed by the electromagnetic field.

Although it is the physical law (2) that accounts for the AB effect, it will be shown in this paper that a straight-forward model for the AB effect ultimately still describes this effect in terms of some « force law » on a more abstract level, (refer to section 4). That is, it turns out that, with recourse to a topological framework, the AB-effect may be characterized as a « homotopy effect ». The purpose of this article is, therefore, to provide a consistent topological model for this effect in terms of some appropriate fields, the AB fields.

2. A DESCRIPTION OF THE AB-EFFECT

The AB effect can be described as follows. Suppose an arrangement of an impenetrable solenoid behind a wall and between two slits as shown in the adjacent figure. The situation with and without current



through the solenoid is the following : If there is no current, clearly $\vec{B} = \vec{A} = 0$ and those electrons emitted from a source, which arrive at the detector by either path 1 or 2 create the familiar pattern of intensity I at the backstop. If the current is turned on, one has $\vec{B} \neq 0$ inside the solenoid, however $\vec{B} = 0$ and $\vec{A} \neq 0$ outside. This corresponds to shifting the entire interference pattern by a constant amount x. The wave function, when there is no field in the solenoid, is composed of two parts ψ_1^0 and ψ_2^0 corresponding to the paths 1 and 2. When the field is on, the solution to Schrödinger's equation, in the presence of $\vec{A}$, is

$$\psi(x) = \psi_1^0 \, e^{ie\int_{\text{path 1}} \vec{A}\, d\vec{x}} + \psi_2^0 \, e^{ie\int_{\text{path 2}} \vec{A}\, d\vec{x}}. \tag{3}$$

Because the integral $\int_{\text{path 1}} \vec{A}\, d\vec{x}$ of $\vec{A}$ along different paths is different, the interference pattern depends on $\vec{A}$, in particular on $\oint \vec{A}\, d\vec{x}$, i. e. the total flux $\Phi = \int \vec{B}\, d\vec{f}$. Finally, the Hamiltonian which corresponds to the AB effect is given by

$$H = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m}. \tag{4}$$

Remark 1. — On account of the scalar potential φ a different experimental arrangement was proposed by AB to demonstrate the effect of this potential [1]. In our subsequent definition of an AB field we therefore refer to the 4-potential $A_\mu(x^\mu) = (\varphi, \vec{A})$, i. e. the field tensor $F_{\mu\nu}(x^\mu)$. That is, we define the Maxwell-two-form as

$$\overset{2}{\omega} = E_i\, dx^i \wedge dx^0 + {}^*H_{ij}\, dx^i \wedge dx^j, \tag{5}$$

but restrict our later discussion to $x^0 = \text{const.}$ hyperplanes.

3. A TOPOLOGICAL MODEL FOR THE AB EFFECT

A possible mathematical model for the AB effect consists of

(*a*) A configuration space corresponding to the dynamical Aharanow-Bohm system.

(*b*) A topological Aharanow-Bohm field (AB field) on the cotangent bundle $\tilde{M} = T^*M$ to M (differentiable n-manifold).

(*c*) The canonical structure on T^*M (T^*M is simply connected).

On account of remark 1 and from physical arguments we introduce a physical AB field (refer to remark) 2 as a Maxwell field in the following sense. Let $\overset{1}{\omega} = A_\mu\, dx^\mu \in \overset{\circ}{F}{}^1(N')$, where $A_\mu(x^\mu) = (\varphi, \vec{A})$ and $c_1 \in \overset{\circ}{C}_1(N')$ $[\overset{\circ}{F}{}^p(N')$ and $\overset{\circ}{C}_p(N')$, $p = 0, 1, 2, \ldots$, denote the groups of closed p-forms and p-cycles respectively on the dynamical manifold N', which is specified below$]$. Then we can define an AB field as follows :

DEFINITION. — An AB field is given in terms of the pairing (6) and the conditions (7) and (8) as follows :

$$(6)\quad (\overset{1}{\omega}, c_1), \qquad \overset{1}{\omega} = \sum_\mu A_\mu . dx^\mu \in F^1(\mathbf{R}^2), \qquad \mu = 0, 1, 2, 3;$$

$$(7)\quad \overset{2}{\omega} = \sum_{\mu,\nu} F_{\mu\nu}(x)\, dx^\mu\, dx^\nu \begin{cases} = 0 & \text{if } {}^*H_{ij}\, dx^i\, dx^j \in F^2(\mathbf{R}^2 - D_r) \\ \neq 0 & \text{if supp } ({}^*H_{ij}\, dx^i\, dx^j) \subset D_r \end{cases} \quad \text{where } x^0 = \text{const.}$$

(D_r denotes a disk of radius r);

$$(8)\quad \overset{2}{\omega} = d\overset{1}{\omega} \qquad (d : F^1 \to F^2 \text{ denotes the exterior derivative}).$$

This amounts to confining ourselves to the case of the AB effect with a vector potential only, that is, where (7) reduces to (9) $\vec{B} = 0$ and (10) $\vec{A} \neq 0$ in the manifold $N' = \mathbf{R}^2 - D_r$. That is, we consider the physical space $\mathbf{R}^3$ with the cylindrical solenoid removed, i. e. the multiply connected region $(\mathbf{R}^2 - D_r) \times \mathbf{R}$ where the charged particles travel. More precisely : We drop the z-direction (along the axis of the solenoid) and the space of interest is the plane minus a disk D_r.

Remark 2. — We call (6) the *physical* AB field. The corresponding *topological* AB field on T* M is supposed to be consistent with the requirements (7) and (8) and is related to the physical fields as follows : If $\chi : \tilde{M} \to \mathbf{R}^2 - D_r$ is any smooth function on $\tilde{M} = T^* M$ with values in $\mathbf{R}^2 - D_r$ then we have :

$$(6') \qquad \left(\tilde{\overset{1}{\omega}}, \tilde{c}_1\right); \qquad \tilde{\overset{1}{\omega}} = \sum_k \tilde{A}_k \, d\tilde{x}^k, \qquad k = 1, 2, 3,$$

where

$$(11) \qquad \tilde{\overset{1}{\omega}} = \chi^* \overset{1}{\omega}$$

and

$$(12) \qquad c_1 = \chi_* \tilde{c}_1 ;$$

and

$$\chi^* : \quad F^p (\mathbf{R}^2 - \{ 0 \}) \to F^p (\tilde{M})$$

$$\chi_* : \quad C_p \left(T^+ \circ M \, C_p (T^* M) = \tilde{M}\right) \to C_p (\mathbf{R}^2 - \{ 0 \}),$$

denote the mappings which are induced by χ.

The reason for introducing the AB-field as a pairing of a one form and a one-cycle is the following :

c_1 may be written formally as a linear combination $c_1 = \lambda_1 \sigma_1 + \lambda_2 \sigma_2$ where $\sigma_{1,2} = \{ \varphi : s_{1,2} \to M, \varphi \in C^0 \}$; $s_{1,2}$ denote Euclidean simplices. Then, with each σ_i, $i = 1, 2$, is associated the wave function (3), more precisely

$$(13) \qquad \psi_i = \psi_i^0 (x) \, e^{ie \int_{\text{path}\, i} \vec{A} \, d\vec{x}} = \psi_i^0 (x) \, e^{ie \int_{\sigma_i^1} \overset{1}{\omega}}.$$

Since the integral of $\vec{A}$ along different paths σ_i^1 is dependent on these paths these paths obviously do contribute to a description of the AB effect.

4. PROPERTIES OF THE AB FIELDS

PROPERTY 1. — Let $H_1 (N')$ and $H^1 (N')$ be the first homology and cohomology groups of N' respectively. By virtue of the Rham's first theorem there exists a nondegenerate bilinear mapping

$$(14) \qquad \begin{cases} \beta : \quad H^1 (N') \times H_1 (N') \to \mathbf{R} \\ \qquad\qquad \left(\overset{1}{\omega}, c_1\right) \to \displaystyle\int_{c_1} \overset{1}{\omega} = \frac{\Delta S}{\hbar} = \Delta\varphi, \end{cases}$$

which maps the physical AB field $(\overset{1}{\omega}, c_1)$ into the real number $\int_{c_1} \overset{1}{\omega} = \frac{\Delta S}{\hbar} = \frac{S_1 - S_2}{\hbar}$, i. e. the map (14) accounts for the phase change $\Delta\varphi = \frac{\Delta S}{\hbar}$ which is produced by the AB effect.

Proof. — Let $\frac{\Delta S}{\hbar} = \Delta\varphi \in \mathbf{R}$ and suppose to each one-cycle c_i is assigned a number per (c_i), the period of c_i :

$$\text{per} : \quad c_i \in \mathring{C}_1 (N') \to \text{per}(c_i) = \Delta\varphi.$$

Then de Rham's second theorem yields the existence of a closed one form $\overset{1}{\omega}$ on N' which has the assigned periods, i. e.

$$\int_{c_i^1} \overset{1}{\omega} = \text{per}(c_i), \qquad \forall\, c_i. \tag{14'}$$

From physical arguments one infers the law : The phase of the wave along any trajectory is changed by the presence of a magnetic field by an amount equal to the integral of the vector potential; therefore :

$$\Delta\varphi = \text{const.} \oint A_k \, dx^k$$

where

$$\overset{1}{\omega} = \vec{A}\, d\vec{x} \quad \text{in (14')}.$$

Remark 3. — Obviously, formula (14) does not hold for any Maxwell field, since, in general, $d\overset{1}{\omega} = \overset{2}{\omega} \neq 0$.

Property 2. — *Gauge transformation of AB fields.* — The freedom of gauge transformations yields the following property of AB fields : Two AB fields (i. e. their "cohomologous components" $\overset{1}{\omega}$ and $\overset{1}{\omega}{}'$) which differ by the gradient of some zero-form $f \in F^0 (N')$, are representatives of the same cohomology class, which means :

$$\overset{1}{\omega}, \quad \overset{1}{\omega}{}' \in \{ \omega \} \in H^1 (N') \tag{15}$$

[refer to the remarks (4) and (5) below].

Proof. — By definition of an AB field (8) one has

$$d\overset{1}{\omega} = 0 \quad \text{i. e. } \overset{1}{\omega} \in \mathring{F}^1 (N') \tag{16}$$

(vector space of closed one forms). Therefore the gauge transformation

$$\omega' = \omega + df \qquad (A'_k = A_k + \partial_k f), \tag{17}$$

is a necessary and sufficient condition for the relationship (15) to hold.

Remark 4. — Within our framework, the AB fields, as defined before, ensure the gauge invariance of the integral (14). Indeed

$$(18)\quad \int_c \omega' = \int_c \tilde{\omega} + \int df \Rightarrow \int_c \omega' = \int \omega \Leftrightarrow \oint A_k\, dx^k \text{ invariant}$$

$\left(\text{according to Stoke's Theorem : } \int_c df = \int_{\partial c} f = 0\right).$

Remark 5. — The formula (15) and (17) obviously do not hold for any Maxwell field, since, in general, $\overset{2}{\omega} = \Sigma\, F_{\mu\nu}\, dx^\mu\, dx^\nu \neq 0$. Therefore, the relationship (15) is just as good a definition of an AB field.

PROPERTY 3 OF AB FIELDS. — A topological AB field $(\overset{\sim}{\overset{1}{\omega}}, \tilde{c}_1)$ (more precisely : its " cohomologous component " $\overset{\sim}{\overset{1}{\omega}}$) can be related to the « action » $\tilde{S} : \tilde{M} \to \mathbf{R}$ by means of the formula

$$(19)\qquad \overset{\sim}{\overset{1}{\omega}} = \tilde{A}_k\, d\tilde{x}^k = d\tilde{S} \quad (\text{in Hartree units } h = c = e = 1).$$

Proof by quotation. — Since $\tilde{M}$ is simply connected, it follows that the Poincaré group of $\tilde{M}$ must vanish, i. e.

$$(20)\qquad \Pi_1(\tilde{M}) = 0.$$

Formula (20) implies a natural isomorphism to exist :

$$(21)\qquad i_* : \Pi_1(\tilde{M}) = 0 \to H_1(\tilde{M}) = H^1(\tilde{M}) = 0.$$

Since $\overset{\sim}{\overset{1}{\omega}} \in H^1(\tilde{M})$ it follows : $\overset{\sim}{\overset{1}{\omega}}$ is exact. On the other hand, the physical quantity $\vec{B} = \text{curl}\, \vec{A}$ vanishes identically, thus $\vec{A}$ may be written as the gradient of some scalar function : $\vec{A} = \frac{\hbar c}{e}\, \text{grad}\, \varphi$ which is just (19).

Finally, AB fields display the following important.

PROPERTY 4 (« Quantization property »). — A pairing $(\overset{\sim}{\overset{1}{\omega}}, \tilde{c}_1)$ represents an AB field if and only if

$$(22)\qquad \overset{\sim}{\overset{1}{\omega}} = \frac{\hbar}{2i}\left(\frac{d\bar{\psi}}{\bar{\psi}} - \frac{d\psi}{\psi}\right) = d\tilde{S}$$

where

$$\begin{cases} \psi = \psi_0\, e^{\frac{iS}{\hbar}}, \\ \bar{\psi} = \psi_0\, e^{-\frac{iS}{\hbar}}. \end{cases}$$

The proof is straightforward (elementary computation).

Remark 6. — In the connected physical space $\mathbf{R}^2 - \{0\}$ we have : $\overset{1}{\omega} = \sum_k A_k\, dx^k$ is closed but not exact, i. e.

$$\overset{1}{\omega} = dS + \omega' \tag{22'}$$

(ω' is not exact) since $\tilde{\omega} = \chi^* \omega = d\tilde{S}$ and $d\tilde{S} = \chi^* dS$ this implies $\chi^* \omega' = 0$, i. e. $\overset{1}{\omega}' \in \operatorname{Ker} \chi^*$, but $\overset{1}{\omega} \notin \operatorname{Ker} \chi^*$ (Ker χ^* denotes the kernel of the map χ^*).

4. A TOPOLOGICAL FORCE CONCEPT ASSOCIATED WITH AB FIELDS

In this section we are going to discuss the properties 1 through 4, of AB fields. The most relevant ones are properties 1 and 3, in particuclar, property 3 accounts for an abstract force concept. This can be seen by comparison with mechanics.

Consider a force field $\omega = \sum_i F_i\, dx^i$ which is required to be conservative. Which are the corresponding properties to be imposed upon the underlying geometry M ? That is, which properties of M imply $\omega = - d\varphi \left(\text{or equivalently } F_i = - \dfrac{\partial \varphi}{\partial x_i} \right)$? It turns out [5] that the corresponding topological force field $(\overset{1}{\omega}, c_1)$, $\overset{1}{\omega} \in dF^0(M)$ (vector space of exact 1-forms), $c_1 \in \mathring{C}_1(M)$, is completely specified if

$$\left\{ \begin{array}{l} \Pi_1(M) = 0 = H_1(M) = H^1(M) \\ \text{(M denotes any arcwise connected manifold).} \end{array} \right. \tag{23}$$

This amounts to saying that the following is true :

$$\Pi_1(M) = 0 \;\Rightarrow\; \overset{1}{\omega} = - d\varphi. \tag{24}$$

Condition (24) accounts for the fact that all de Rham periods $\int_c \omega$ of ω vanish, which is just the elementary property that $\varphi(x) = - \int_{x^0}^{x} \omega$ be independent of the path joining x_0 to x, or equivalently $\oint_\gamma \omega = 0$ for all closed curves γ which are homotopic to zero, i. e. deformable to a point.

Likewise, the properties 1 and 3 of an AB field display the same structure as (23). Therefore we conclude that the AB effect may be interpreted as a « homotopy effect » in terms of an AB field $\left(\overset{1}{\tilde{\omega}}, \tilde{c}_1\right)$ and by virtue of the relationship

$$(25) \qquad \Pi_1(\tilde{\mathrm{M}}) = 0 \quad \Rightarrow \quad \overset{1}{\tilde{\omega}} = \tilde{\mathrm{A}}_k\, d\tilde{x}^k = d\tilde{\mathrm{S}}.$$

The implication (25) characterizes the AB effect in terms of some « force mechanism ». That is, if we interpret an AB field as being derived from geometry in the sense that the properties of the topology $\tilde{\mathrm{M}}$ imply the properties of the field, then this geometry must display the property (25).

The quantization property 4 of AB fields characterizes the AB effect as a quantum effect.

This property [formula (22)] settles another controversial feature of the AB effect. Concerning its interpretation L. Janossy claims in his paper [8], that the AB effect should not be considered as a quantum-mechanical effect. This has been refuted by Bohm and Phillipidis [7] on the grounds of the validity of the formulae

$$(26) \qquad \rho = \tilde{\psi}\,\psi \quad \text{(hydrodynamical density distribution)}$$

and

$$(27) \qquad \alpha = \frac{\hbar}{m}\, d\mathrm{S} - \frac{e}{mc}\,.\,\mathrm{A}_k\, dx^k \in \mathrm{F}^1\left(\mathbf{R}^2 \setminus \{ 0 \}\right)$$

[α stands for the velocity field

$$(28) \qquad \vec{v} = \frac{\hbar}{m}\,.\,\mathrm{grad}\ \mathrm{S} - \frac{e}{mc}\,.\,\vec{\mathrm{A}}\Big].$$

A straight forward analysis of relationship (27) shows, that the properties 3 and 4 of AB fields, in particular formula (22), strongly support the arguments of Bohms and Phillipidis. To begin with, suppose $\left(\overset{1}{\omega}, c_1\right)$ to be a physical AB field, i. e. $\overset{1}{\omega} = \mathrm{A}_k\, dx^k$. The corresponding topological field $\left(\overset{1}{\tilde{\omega}}, \tilde{c}_1\right)$ then enjoys the properties (19) and (22). Suppose now the converse to be true, i. e. let $\left(\overset{1}{\tilde{\omega}}, \tilde{c}_1\right)$ be a pairing on $\mathrm{T}^*\mathrm{M}$, subject to the conditions (19) and (22) for $\overset{1}{\tilde{\omega}}$. Then, according to remark 6, the preimage of (22) is given by

$$(22') \qquad \omega' = \omega - d\mathrm{S} \quad \text{(in Atomic units).}$$

That is, α is (up to a sign) just the one-form (22). Otherwise stated, on the physical space $\mathbf{R}^2 - \{ 0 \}$, α is related to the physical AB field by virtue of (27), (22') and

$$(29) \qquad d\alpha = d\overset{1}{\omega} = 0.$$

This again emphasizes the quantum mechanical character of the AB effect. That is, the non-classical relationship (29) (refer to remark 7 below) supplements our description of the AB effect as a quantum effect.

Remark 7. — Physically, the relationship (29) states that AB fields do not give rise to a distribution of vortices, which, for classical Maxwell fields, is given as

$$d\alpha = -\frac{e}{mc}\,{}^*\mathrm{H}_{ij}\,dx^i\,dx^j \neq 0$$

$$\left[\text{equivalently : curl}\,\vec{v} = -\frac{e}{mc}\,\vec{\mathrm{H}} \neq 0 \text{ if } x = (\vec{x}, t) \in \operatorname{Supp} \rho\,(\vec{x}, t)\right].$$

5. THE TOPOLOGICAL ASPECT OF THE QUANTUMTHEORETICAL AB EFFECT

The aim of this section is to provide a « quantization mechanism » for the eigenvalues E of the Hamiltonian (4) in terms of the quantization property 4 of AB fields and by working with appropriate topological conditions. It turns out that the quantized values of E are those for which the Bohr-Sommerfeld quantum conditions

$$\text{(30)} \qquad \mathrm{J} = \int \bar{p}\,dq = nh \quad (h : \text{Planck's constant})$$

holds, where

$$\text{(31)} \qquad \bar{p} = p + \frac{e}{c}\,\mathrm{A} \quad (p : \text{canonical momentum coordinate}).$$

The AB effect may be described as a quantum effect in a suitable geometric language as follows. Let

$$\text{(32)} \qquad \mathrm{T}^*\,\mathrm{M} = \{ (p, q) \mid q \in \mathrm{M},\ p \in \mathrm{T}^*_q\,\mathrm{M} \}$$

be the cotangent bundle over the configuration manifold of the dynamical AB system. The smooth map Π, called the projection of the cotangent bundle, is displayed as

$$\text{(33)} \qquad \Pi : \mathrm{T}^*\,\mathrm{M} \to \mathrm{M} \qquad \text{by} \quad (q, p) \to q.$$

The canonical structure on $\mathrm{T}^*\,\mathrm{M}$ is defined in terms of the one form

$$\text{(34)} \qquad \theta = p\,dq,$$

which gives rise to the distinguished closed 2-form $\Omega = d\theta$ of maximal rank, the fundamental form. Let $\psi : \mathrm{M} \to \mathrm{T}^*\,\mathrm{M}$ be a C^k map, $k \geq 1$, which is defined by $\psi^*(\Omega) = 0$. This yields $\psi^* \Omega = \psi^* d\theta = d(\psi^* \theta) = 0$. A sufficient condition for this relationship to hold is

(35) $$\psi^*(\theta) = dS, \quad \text{where} \quad S \in F^0(M).$$

On the other hand, on account of the electromagnetic potential $\overset{1}{\omega} = A_k \, dx^k$, there is a diffeomorphism

(36) $$\begin{cases} \varphi : \quad T^* M \to T^* M, \qquad p \to p - \frac{e}{c} A, \\ q \to q \left(\frac{\partial (\bar{q}_1 \ldots \bar{q}_n, \bar{p}_1 \ldots \bar{p}_n)}{\partial (q_1 \ldots q_n, p_1 \ldots p_n)} = 1 \right). \end{cases}$$

The transformation (36) induces an automorphism φ^* of the space $F^p(T^* M)$ of p-forms ($p = 0, 1, \ldots$). Now suppose the « action » $\bar{S} : M \to \mathbf{R}$ and the corresponding 1-form

(37) $$d\bar{S} : \quad M \to T^* M : \quad q \to (q, d_q \bar{S}) = d\bar{S}(q)$$

to be related to the undashed quantities S and dS by means of the formula

(38) $$\varphi^* d\bar{S} = dS.$$

Then the following lemma holds :

Lemma 1. — *The Bohr-Sommerfeld quantum conditions* (30) *are adiabatically invariant* [9], *that is*

(39) $$\int_{\varphi_* \tilde{c}_1} d\bar{S} = \int_{\tilde{c}_1} dS = n \hbar.$$

Moreover, the 1-*form* (37) *is related to the* AB *field component* $\overset{\sim}{\overset{1}{\omega}}$ *by means of the formula*

(40) $$d\bar{S} = \psi^* \overset{\sim}{\overset{1}{\omega}} + dS.$$

Proof. — Let $(\overset{\sim}{\overset{1}{\omega}}, \tilde{c}_1)$ be a topological AB field of the type (6′), then $\int_{\varphi_* \tilde{c}_1} d\bar{S} = \int_{\tilde{c}_1} \varphi^* d\bar{S} = \int_{\tilde{c}_1} dS$ where $\tilde{c}_1$ is a one-chain, $S \in F^0(M)$ and $\varphi_* : C_1(\tilde{M}) \to C_1(\tilde{M})$. Furthermore, since φ^* is 1-1, we have

$$\psi^*(\bar{\theta}) = \overset{-1}{\varphi}{}^* dS = d\bar{S}, \qquad \text{where} \quad \bar{\theta} = \bar{p}\, dq, \quad \bar{p} = p + \frac{e}{c} A(q),$$

therefore

$$\begin{aligned} d\bar{S} - dS &= \psi^*(\bar{\theta}) - \psi^*(\theta) \\ &= \psi^* \left[\left(p + \frac{e}{c} A \right) dq - p\, dq \right] = \psi^* \left(\frac{e}{c} \tilde{A}\, d\tilde{q} \right) = \psi^* \overset{\sim}{\overset{1}{\omega}}. \end{aligned}$$

The « action » $\overline{S}$ is a solution of the Hamilton-Jacobi differential equation of particles in an electromagnetic field, i. e.

(40) $$H \circ d\overline{S} = E,$$

where

(41) $$H : \quad T^* M \rightarrow \mathbf{R}$$

denotes the Hamiltonian Following [6] quantization rules which are associated with the AB effect may be obtained by imposing topological conditions on the level surfaces of H by virtue of the following

LEMMA 2. — *The quantummechanical constraint manifolds which correspond to the* AB *effect are given by*

(42) $$d\overline{S}\,(M) \subset T^*_E\, M.$$

($T^*_E\, M$ denote the energy level surfaces at the fixed energy E.)

Proof. — By definition

(43) $$d\overline{S}\,(M) = \{ (q, d_q \overline{S}) \} = \{ (q, p) \in T^* M : H \circ d\overline{S}\,(q) = E \},$$

clearly satisfies the inclusion (42).

We now summarize our results as follows

THEOREM. — *The* AB *effect may be described in terms of the following topological conditions : There exist* AB *fields* $(\overset{1}{\tilde{\omega}}, \tilde{c}_1)$ *which satisfy the following properties :*

(1) $H^1 (N') \times H_1 (N') \rightarrow \mathbf{R} : (\overset{1}{\omega}, c_1) \rightarrow \int_{c_1} \overset{1}{\omega} = \frac{\Delta S}{\hbar};$

(2) $\overset{1}{\omega}, \overset{1}{\omega}{}' \in \{ \omega \} \in H^1 (N') \Leftrightarrow \overset{1}{\omega}{}' = \overset{1}{\omega} + df$ (*gauge condition*);

(3) $\overset{1}{\tilde{\omega}} = \frac{e}{c} \tilde{A}_k \, d\tilde{q}_k = d\tilde{S};$

(4) $d\tilde{S} = \frac{\hbar}{2i} \left(\frac{d\overline{\psi}}{\overline{\psi}} - \frac{d\psi}{\psi} \right)$ (*quantization condition*).

These fields describe the AB *effect as a quantum effect by virtue of the following statements :*

(*a*) *There exist functions which correspond to the quantization condition* (4) *:*

(44) $$\psi = \psi_0 \exp \left(\frac{i\,\overline{S}}{\hbar} \right) \in F^0 (\Pi\, (T^*_E\, M))$$

(*b*) *The Bohr-Sommerfeld rules are associated with the constraint manifolds such that.*

(45) $$d\bar{S}\,(\Pi\,(T_E^*\,M)) \subset T_E^*\,M$$

such that.

(*c*) *The adiabatic invariance condition* $\int_{\varphi_*\tilde{c}} d\bar{S} = \int_{\tilde{c}} dS = n\,\hbar$ *holds.*

Here $\tilde{c}$ *and* $\varphi_*\,\tilde{c}$ *denote the topological components of the corresponding* AB *fields.*

Discussion. — Relationship (43) in conjunction with the formula $\psi^*(\bar{\theta}) = d\bar{S}$ is equivalent to the Hamilton-Jacobi equation (40), which in local coordinates, reads $H\left(\bar{q}_1 \ldots \bar{q}_n, \frac{\partial\bar{S}}{\partial\bar{q}_1} \ldots \frac{\partial\bar{S}}{\partial\bar{q}_n}\right) = E$. The aforementioned two Lemmata account for the characteristic features of quantum mechanics. Otherwise stated : The constraint manifolds (43) in combination with the Bohr-Sommerfeld rules (30) (i. e. 39) determine in the totality of all possible energy level surfaces, those which can be implemented quantummechanically, i. e. those which correspond to an AB system. This interrelation between quantum theory and the method of the Hamiltonian action function $\bar{S}$ is achieved by expressing the Bohr Sommerfeld phase integrals (30) through the function $\bar{S}$, i. e. in terms of the quantities $\int d\bar{S} = n\,\hbar = \int \frac{\partial\bar{S}}{\partial q_k}\,dq_k$.

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