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Section A :
Physique théorique.

The geometry of the octet

by

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I. — INTRODUCTION

The adjoint representation of SU (3) plays a specially important role in the physical applications since the electromagnetic and weak currents whose space integrals are the generators of the symmetry belongs to it.

In the physical litterature this representation is usually discussed by using a particular basis namely the Gell-Mann [1] λ-matrices. To evaluate most of the physical quantities which are functions of the matrix elements of the currents on vectors of the adjoint representation, one has to compute expressions involving products of the tensors components f_{ijk} and $\sqrt{3}\, d_{abc}$. These are sines and cosines of angles multiples of 30°.

We think there might be advantages in going beyond this trigonometrical approach by studying the geometry of the 8-dimensional space $\mathbf{R}^8$ of the SU (3) adjoint representation. In our opinion this allows to obtain a deeper understanding of the structure of $\mathbf{R}^8$, a structure much richer than the one of $\mathbf{R}^3$ on which acts the 3-dimensional rotation group or its covering group SU (2). Indeed $\mathbf{R}^8$ is not " isotropical " under the action of SU (3), but contains families of orbits of special directions with peculiar geometrical properties. It is a remarkable fact that the directions in $\mathbf{R}^8$ corresponding to physical quantities (strong and weak hypercharge, electromagnetic charge, weak currents, etc.) all belong to these exceptional families.

We have studied this correspondence [which actually goes beyond SU (3)] in a series of papers [2] containing only a brief summary of our geometrical method and of its results. A detailed accound was given in a preprint reproduced in 1969 in Pisa and Tel-Aviv, whose publication was delayed by our intention to write a more systematic and general treatement of all the SU (n).

We now think that the elementary approach of our preprint might have some interest for physicists and for this reason we have decided to publish it with a few modifications and improvements. We have in particular changed the normalization of the symmetrical ($\vee$) product so that the structure constants of this algebra are now equal to $\sqrt{3}$ times the d-tensor of Gell-Mann.

The first two sections of this paper do not contain new results : formulae equivalent to ours can be found in several articles [3], [4], [5].

We believe however that our derivations are simpler than the ones which explicitely use the coordinates and that they allow an easier extension to more complicated cases.

II. — THE SYMMETRICAL AND ANTISYMMETRICAL SU (n) INVARIANT ALGEBRAS ON $\mathbf{R}^{n^2-1}$

1. The polynomial invariants on $\mathbf{R}^{n^2-1}$

SU (n) is the group of $n \times n$ unitary ($u^{-1} = u^*$), unimodular ($\det u = 1$) matrices. Every $u \in \mathrm{SU}(n)$ can be written in the form

$$u = e^{ix} \tag{II.1}$$

where x is a $n \times n$ Hermitian ($x = x^*$) traceless ($\operatorname{tr} x = 0$) matrix. The set of all x form an ($n^2 - 1$)-dimensional, real vector space, $\mathbf{R}^{n^2-1}$. We define the Euclidean scalar product of two vectors x and $y \in \mathbf{R}^{n^2-1}$. by

$$(x, y) = (y, x) = \frac{1}{2} \operatorname{tr} xy. \tag{II.2}$$

The action of SU (n) on $\mathbf{R}^{n^2-1}$ is given by

$$u : \quad x \mapsto uxu^* \tag{II.3}$$

for every $u \in \mathrm{SU}(n)$ and every $x \in \mathbf{R}^{n^2-1}$. This defines the adjoint representation of SU (n); it is an irreducible representation that we denote by U (u). The transformation (II.3) obviously leaves invariant the scalar product (II.2).

The scalar product is the only SU (n) invariant bilinear function of two vectors. For three vectors x, y, z it can be proved that there are two linearly independent trilinear invariants :

$$\{ x, y, z \} = \frac{\sqrt{n}}{2} \operatorname{tr} \{ x, y \} z = \frac{\sqrt{n}}{2} \operatorname{tr} (xy + yx) z, \tag{II.4}$$

$$[x, y, z] = -\frac{i}{2} \operatorname{tr} [x, y] z = -\frac{i}{2} (xy - yx) z. \tag{II.5}$$

By using the symmetry properties of the trace it can be verified that $\{ x, y, z \}$ and $[x, y, z]$ are completely symmetrical and antisymmetrical respectively in the three vectors.

More general polynomial invariants are polynomials in the traces of product of vectors (*see* e. g. [6]).

2. The invariant algebras

From the existence of only one bilinear form and two linearly independent trilinear forms on $\mathbf{R}^{n^2-1}$ invariant under SU (n) we conclude that there are only two linearly independent algebras on $\mathbf{R}^{n^2-1}$ with SU (n) as automorphism group. One is the Lie algebra whose multiplication law is

$$\text{(II.6)} \qquad x_\wedge y = -\frac{i}{2}[x, y] = -\frac{i}{2}(xy - yx) = -y_\wedge x.$$

Another is the symmetrical algebra whose multiplication law is

$$\text{(II.7)} \quad x_\vee y = \frac{\sqrt{n}}{2}\{ x, y \} - \frac{1}{\sqrt{n}}\operatorname{tr} xy = \frac{\sqrt{n}}{2}(xy + yx) - \frac{2}{\sqrt{n}}(x, y) = y_\vee x.$$

Any algebra on $\mathbf{R}^{n^2-1}$ with SU (2) as automorphism group, is of the form

$$\text{(II.8)} \qquad x_{\mathrm{T}}\, y = \alpha\, x_\vee y + \beta\, x_\wedge y$$

with α and β real. Indeed we have

$$\text{(II.9)} \qquad ux_{\mathrm{T}}\, yu^{-1} = (uxu^{-1})_{\mathrm{T}}\,(uyu^{-1}).$$

The two trilinear invariants can now be written :

$$\text{(II.10)} \qquad \{ x, y, z \} = (x_\vee y, z) = (x, y_\vee z),$$

$$\text{(II.11)} \qquad [x, y, z] = (x_\wedge y, z) = (x, y_\wedge z).$$

3. The operators $f_x = x_\wedge$ and $d_x = x_\vee$

Every $x \in \mathbf{R}^{n^2-1}$ defines a linear mapping x_{T}, $\mathbf{R}^{n^2-1} \xrightarrow{x_{\mathrm{T}}} \mathbf{R}^{n^2-1}$:

$$\text{(II.12)} \qquad x_{\mathrm{T}} : \quad y \mapsto x_{\mathrm{T}}\, y$$

for every $y \in \mathbf{R}^{n^2-1}$. The r. h. s. of (II.12) is defined by (II.8).

We will in particular consider the two mappings :

$$\text{(II.13)} \qquad f_x = x_\wedge \quad \text{(often denoted by ad } x\text{)},$$

$$\text{(II.13')} \qquad d_x = x_\vee.$$

From (II.10), (II.11) it follows that the linear operators d_x, f_x are respectively symmetrical and antisymmetrical, i. e.

$$d_x^{\mathrm{T}} = d_x; \qquad f_x^{\mathrm{T}} = -f_x. \tag{II.14}$$

The adjoint representation U (u) of SU (n), transforms f_x, d_x according to

$$\mathrm{U}(u)\, f_x\, \mathrm{U}(u)^{-1} = f_{uxu^{-1}}, \qquad \mathrm{U}(u)\, d_x\, \mathrm{U}(n)^{-1} = d_{uxu^{-1}}. \tag{II.15}$$

The linear mappings [1] Φ, Φ', $\mathbf{R}^{n^2-1} \xrightarrow{\Phi} \mathbf{R}$, $\Phi(x) = \mathrm{Tr}\, f_x$, $\Phi'(x) = \mathrm{Tr}\, d_x$, commute with the group action on $\mathbf{R}^{n^2-1}$, e. g. For example :

$$\Phi(uxu^{-1}) = \mathrm{Tr}\, \mathrm{U}(u)\, f_x\, \mathrm{U}(u)^{-1} = \mathrm{Tr}\, f_x = \Phi(x).$$

Hence the kernel of Φ [i. e. the set of x such that $\Phi(x) = 0$; it is a vector subspace] is invariant by the group. Since the representation U (u) is irreductible, this kernel must be either $\{0\}$ or the whole space. Since its dimension is $\geq n^2 - 2 > 0$, Ker $\Phi = \mathbf{R}^{n^2-1}$, Ker $= \Phi'$ i. e.

$$\mathrm{Tr}\, f_x = 0 = \mathrm{Tr}\, d_x. \tag{II.16}$$

The first equality (II.16) can also be derived from (II.14). An elementary proof of the second one will be given later.

We begin by establishing a number of relations satisfied by f_x and d_x which depend upon the fact that SU (n) is an automorphism group of the two algebras $\wedge$ and $\vee$. This implies that ad $x = f_x$ is a derivation for both of them, that is

$$f_x(y \wedge z) = (f_x y) \wedge z + y \wedge (f_x z), \tag{II.17}$$

$$f_x(y \vee z) = (f_x y) \vee z + y \vee (f_x z). \tag{II.18}$$

Equation (II.17), which is simply Jacobi's identity, can also be written in the form

$$[f_x, f_y] = f_{x \wedge y} \tag{II.19}$$

which states that the commutator of two derivation is a derivation. Similarly (II.18) is equivalent to

$$[f_x, d_y] = d_{x \wedge y}. \tag{II.20}$$

From (II.18) we obtain two other equations by cyclic permutations of x, y, z. Adding these three equations we get :

$$f_x(y \vee z) + f_y(z \vee x) + f_z(x \vee y) = 0 \tag{II.21}$$

[1] We use tr for the trace of n by n matrices and Tr for the trace of operators on $\mathbf{R}^{n^2-1}$ i. e. $(n^2-1) \times (n^2-1)$ matrices.

or

$$f_x\, d_y + f_y\, d_x = f_{x_\vee y}. \tag{II.22}$$

By transposition of (II.22) we obtain with (II.15) :

$$d_y\, f_x + d_x\, f_y = f_{x_\vee y}. \tag{II.23}$$

When $x = y$ these last two equations become

$$f_x\, d_x = d_x\, f_x = \frac{1}{2} f_{x_\vee x}. \tag{II.24}$$

We can now give another proof of the second equation of (II.16). Indeed since the trace of a commutator vanishes, equation (II.20) states that for any x and y, $d_{x_\wedge y}$ is traceless. On the other hand it follows from the Jacobi identity that the set of all vectors of the form $x_\wedge y$ is an ideal of the Lie algebra. Since SU (n) is simple this ideal coicides with the Lie algebra. This completes the proof of (II.16).

4. Further relations satisfied f_x and d_x

To obtain further relations for the two operators f_x and d_x we evaluate with the help of (II.7) the associator of the $\vee$-algebra, i. e. the vector $x_\vee(y_\vee z) - (x_\vee y)_\vee z$. We find

$$x_\vee(y_\vee z) - (x_\vee y)_\vee z = ny_\wedge(x_\wedge z) - 2\,x\,(y, z) + 2\,z\,(x, y) \tag{II.25}$$

or

$$d_x\, d_y - nf_y\, f_x = d_{x_\vee y} - 2\,x > < y + 2\,(x, y)\,\mathrm{I} \tag{II.26}$$

where the dyadic operator $x > < y$ is defined by

$$x > < y . z = x\,(y, z) \tag{II.27}$$

and I is the $(n^2 - 1)\times(n^2 - 1)$ unit matrix.

By antisymmetrizing equation (II.26) we obtain

$$[d_x, d_y] + nf_{x_\wedge y} = -\,2\,(x > < y - y > < x). \tag{II.28}$$

Setting $x = y$ in (II.26) we get

$$d_x^2 - nf_x^2 = d_{x_\vee x} + 2\,(x, x)\,(\mathrm{I} - \mathrm{P}_x), \tag{II.29}$$

where

$$\mathrm{P}_x = (x, x)^{-1}\, x > < x = \mathrm{P}_x^2 \tag{II.30}$$

is the projection operator on the vector x.

The expressions $\operatorname{Tr} f_x f_y$, $\operatorname{Tr} d_x d_y$, $\operatorname{Tr} f_x d_y$ are bilinear forms which are invariant under the group SU (n) [use (II.15)]. Because of the unicity up to a factor of such an invariant bilinear form, they are proportional to (x, y) defined in (II.2).

For example $\operatorname{Tr} f_x f_y = \lambda (x, y)$ is the Cartan-Killing form. We shall now compute it. Let $| e_A \rangle$ be a basic of orthonormal vectors in $\mathbf{C}^{n^2-1}$, the complexified of $\mathbf{R}^{n^2-1}$:

$$\langle e_A, e_B \rangle = \frac{1}{2} \operatorname{tr} e_A^* e_B = (e_A^*, e_B).$$

We choose $x = y = a$, where a is a traceless Hermitean diagonal matrix, whose diagonal elements are α_i, real, with $\sum_{i=1}^{n} \alpha_i = 0$.

Let e_{ij} be the $n \times n$ matrix whose elements are

$$(e_{ij})_{kl} = \sqrt{2}\, \delta_{ik}\, \delta_{jl};$$

then

$$e_{ji}^* = e_{ij} \qquad \text{and} \qquad (e_{ij}^*, e_{i'j'}) = \delta_{ii'}\, \delta_{jj'}.$$

All diagonal matrices commute so that $a \wedge e_{ii} = f_a e_{ii} = 0$.

Using : (II.6), (II.11) we get,

$$\operatorname{Tr} f_a f_a = \operatorname{Tr} f_a f_a \sum_{A} e_A > < e_A = \operatorname{Tr} f_a f_a \sum_{i \neq j} e_{ij} > < e_{ij} = \sum_{i \neq j} (e_{ji} \wedge a, a \wedge e_{ij})$$

$$= -\frac{1}{4} \sum_{i \neq j} (a_i - a_j)^2 = -\frac{1}{4} \sum_{i, j} (a_i - a_j)^2 = -n (a, a)$$

since

$$\sum_i a_i = 0, \qquad \sum_i a_i^2 = \operatorname{tr} a^2 = 2 (a, a)$$

and therefore

$$\operatorname{Tr} f_x f_y = -n (x, y). \tag{II.31}$$

This equation, with the trace of (II.26), yields :

$$\operatorname{Tr} d_x d_y = (n^2 - 4) (x, y). \tag{II.32}$$

Finally, from (II.24) and (II.16), we get :

$$\operatorname{Tr} f_x d_y = 0. \tag{II.33}$$

Equation (II.32) show that for $n = 2$, $\operatorname{tr} d_x^2 = 0$.

Since $d_x = d_x^T$, d_x^2 is a positive symmetrical operator, and

$$\operatorname{tr} d_x^2 = 0 \quad \Rightarrow \quad d_x = 0.$$

Therefore, for $n = 2$ the $\vee$ algebra is trivial (*see* § II.5).

We recall that the Casimir operator of the adjoint representation of SU (n) :

(II.34) $$C = -\sum_{A} f_{e_A}^2 = c\,\mathrm{I}.$$

is a multiple of the identity operator. From (II.31) we deduce :

(II.34') $$c = (n^2 - 1)^{-1}\,\mathrm{tr}\,C = n.$$

Similarly we remark that $\mathrm{Tr}\, f_x f_y f_z$, $\mathrm{Tr}\, f_n d_y d_z$ are linear combinaison of $[x, y, z]$ and $\{ x, y, z \}$.

Using (II.14), and $\mathrm{Tr\,CAB} = \mathrm{Tr\,ABC} = \mathrm{Tr\,C^T B^T A^T}$ we see that $\mathrm{Tr}\, f_n f_y f_z$ and $\mathrm{Tr}\, f_x d_y d_z$ are completely anti-symmetrical and $\mathrm{Tr}\, f_x f_y d_z$, $\mathrm{Tr}\, d_x d_y d_z$ are completely symmetrical in x, y, z. Hence from (II.19) and (II.31) :

(II.35) $$\mathrm{Tr}\, f_x f_y f_z = -\frac{n}{2}[x, y, z].$$

This equation, together with (II.26) multiplied by f_z, yield;

(II.36) $$\mathrm{Tr}\, d_x d_y f_z = \mathrm{Tr}\, f_x d_y d_z = \frac{n^2 - 4}{2}[x, y, z].$$

The trace of (II.23) multiplied by f_z, yields with (II.31) and (II.10) :

(II.37) $$\mathrm{Tr}\, d_x f_y f_z = \mathrm{Tr}\, f_x f_y d_z = -\frac{n}{2}\{ x, y, z \}.$$

Similarly the trace of (II.29) multiplied by d_x :

(II.38) $$\mathrm{Tr}\, d_x d_y d_z = \frac{n^2 - 12}{2}\{ x, y, z \}.$$

For the trace of the product of a larger number of operators ford we refer to [4] and, for SU (3) only, to [3].

To obtain more relations for the two algebras $\wedge$ and $\vee$ we must make explicit use of the fact that their elements are $n \times n$ matrices which satisfy an algebraic equation of degree n. Before considering the case of SU (3) we will briefly review the well known case of SU (2).

5. The space R^3 of the adjoint representation of SU (2)

As an example we discuss the well known case of $n = 2$. The characteristic equation for a 2×2 traceless matrix is

(II.39) $$x^2 - \mathrm{I}\det x = x^2 - \mathrm{I}\,(x, x) = x \vee x = 0.$$

The symmetrical algebra is thus trivial in this case. Indeed if

$$\varphi(x) = x_\vee x = 0,$$

(II.40) $$0 = \frac{1}{2}(\varphi(x+y) - \varphi(x) - \varphi(y)) = x_\vee y.$$

Hence $d_x = 0$ or [*see* eq. (II.7)] :

(II.41) $$\frac{1}{2}\{x, y\} = (x, y)\,\mathrm{I}.$$

Appart from Jacobi's identity, the only non trivial equation is, in this cas (II.26) :

(II.42) $$-f_x f_y = -x >< y + (x, y)\,\mathrm{I}.$$

Using Jacobi's identity in the form (II.19) we get

(II.43) $$[f_x, f_y] = f_{x_\wedge y} = -(x >< y - y >< x).$$

Equations (II.29) and (II.31) give :

(II.44) $$\begin{cases} -f_x^2 = (x, x)(\mathrm{I} - \mathrm{P}_x) = (x, x)\,\mathrm{P}_x^\perp \\ \quad -\mathrm{Tr}\, f_x^2 = 2(x, x). \end{cases}$$

Hence for a normalized vector, $(x, x) = 1$, *if*$_x$ has the eigenvalues 1, 0, -1 and the Casimir operator C has the eigenvalue 2 :

III. — THE GEOMETRY OF THE SU (3) OCTET

1. Relations for f_x and d_x specific to SU (3)

For $n = 3$, the vector space of the Lie algebra is $\mathbf{R}^8$, whose elements are 3×3 Hermitian traceless matrices. They satisfy the equation

(III.1) $$\varphi(x) = x^3 - \gamma(x)\,x - \mu(x) = 0,$$

where

(III.2) $$\gamma(x) = (x, x) = \frac{1}{2}\mathrm{tr}\, x^2,$$

(III.3) $$\mu(x) = \det x = \frac{1}{3}\mathrm{tr}\, x^3 = \frac{2}{3\sqrt{3}}\,\theta(x),$$

where

(III.3′) $$\theta(x) = (x_\vee x, x).$$

Since x is Hermitean, its three eigenvalues are real i. e.

(III.4) $$4\,\gamma^3(x) \geqq 27\,\mu^2(x) \iff \gamma(x^3) \geqq 0\ (x^3).$$

We notice that equation (III.1) can be rewritten as

(III.5) $$(x_\vee x)_\vee x = x_\vee(x_\vee x) = \gamma(x)\,x = x_\vee x_\vee x.$$

For further reference we also give the fourth order relation

(III.6) $$(x_\vee x)_\vee(x_\vee x) = 2\,\theta(x)\,x - \gamma(x)\,x_\vee x$$

which is a particular cas of equation (II.25).

By polarization of (III.1) we get :

(III.7) $$0 = \varphi(x+y+z) - \varphi(x+y) - \varphi(y+z) - \varphi(x+z) + \varphi(x) + \varphi(y) + \varphi(z)$$

or

(III.8) $$x_\vee(y_\vee z) + y_\vee(z_\vee x) + z_\vee(x_\vee y) = x\,(y, z) + y\,(z, x) + z\,(x, y).$$

In terms of the operators d, equation (III.8) reads :

(III.9) $$\{d_x, d_y\} + d_{x_\vee y} = (\mathrm{I}\,(x, y) + x >< y + y >< x).$$

From (III.9) and (II.26) symmetrized with respect to x and y we obtain

(III.10) $$\{d_x, d_y\} - \{f_x, f_y\} = 2\,(x, y)\,\mathrm{I},$$

(III.11) $$d_{x_\vee y} + \{f_x, f_y\} = x >< y + y >< x - \mathrm{I}\,(x, y).$$

Setting $x = y$ we get

(III.12) $$d_x^2 - f_x^2 = \gamma(x)\,\mathrm{I},$$

(III.13) $$d_{x_\vee x} + 2\,f_x^2 = -\gamma(x)\,(\mathrm{I} - 2\,\mathrm{P}_x).$$

We end this section by recording other useful relations

(III.14) $$\gamma(x_\vee y) + \gamma(x_\wedge y) = \gamma(x)\,\gamma(y),$$

(III.14′) $$(x_\vee x, y_\vee y) + \gamma(x)\,\gamma(y) = 2\,(x, y)^2 + 2\,\gamma(x_\wedge y).$$

They are deduced from (III.12) and (III.13) by multiplying them by $\gamma(y)\,\mathrm{P}_y$ and taking the trace.

In particular if $x = y$ we get for any vector x :

(III.15) $$\gamma(x_\vee x) = \gamma(x)^2$$

which could also directly be obtained from (III.5).

With the use of (III.15), Schwartz inequality applied to (III.14) yields

$$(x, y)^2 + \gamma (x_\wedge y) \leqq \gamma (x) \gamma (y). \tag{III.16}$$

[For SU (2), this is an equality.]

2. The q-vectors

A vector q of $\mathbf{R}_8$ will be called a q-vector if it satisfies one of the following conditions :

$$\text{(i)} \quad q_\vee q + \eta (q) q = 0 \tag{III.17}$$

where $\eta (q)$ is a real number :

$$\text{(ii)} \quad \gamma (q)^3 = \theta (q)^2. \tag{III.18}$$

(iii) The matrix of q has a double eigenvalue.

(iv) The eigenvalues of f_q are proportional to 1, 0, — 1.

Condition (i) shows that the matrix of q has only two distinct eigenvalues and is thus equivalent to (iii) or (ii).

We will now prove that (i) or (ii) or (iii) imply (iv). From (III.5) and (III.6) if follows :

$$\eta (q)^2 = \gamma (q); \qquad \eta (q)^3 = - \theta (q) \tag{III.19}$$

hence

$$\eta (q) = - \frac{\theta (q)}{\gamma (q)}. \tag{III.19'}$$

If we multiply (III.12) by f_q and use twice (II.24) we get

$$- f_q^3 = \frac{3}{4} \gamma (q) f_q \tag{III.20}$$

which implies

$$\left(\frac{-4}{3 \gamma (q)} f_q^2 \right)^2 = - \frac{4}{3 \gamma (q)} f_q^2 \tag{III.21}$$

$\left(- \frac{4}{3} \gamma (q) \right) f_q^2$ is thus a projection operator whose trace is 4, and whose eigenvalues are 0 and 1 (each four times degenerates). Its square roots :

$$\pm \mathrm{Q} (q) = \pm \frac{2 i}{\sqrt{3 \gamma (q)}} f_q \tag{III.21'}$$

which are antisymmetrical have thus the eigenvalues zero (four times degenerate) and ± 1 (each twice degenerate). We have thus proved that (i), or (ii) ou (iii) imply (iv). That (iv) implies the other conditions will be proven at the end of paragraph III.4.

Let $\mathcal{U}_q$ be the centralizer of q, i. e. :

(III.22) $$\mathcal{U}_q = \{ x \in \mathbf{R}_8 : f_q x = q_\wedge x = 0 \},$$

$\mathcal{U}_q$ is a $_\wedge$-algebra which is the Lie algebra of the isotropy group G_q of q (also called the stabilizer or little group of q) :

(III.23) $$G_q = \{ u \in SU(3) : uqu^{-1} = q \}.$$

By diagonalizing the matrix q one verifies that G_q is a U (2) group.

The Lie algebra $\mathcal{U}_q(2)$ is thus a direct sum $\mathcal{U}_q(1) \oplus S\,\mathcal{U}_u(2)$ of a $\mathcal{U}(1)$ and an $S\,\mathcal{U}(2)$ Lie algebra. Hence each q-vector provides a decomposition of $\mathbf{R}^8$ into a direct sum of orthogonal subspaces :

(III.24) $$\mathbf{R}_8 = \mathcal{U}_q \oplus \mathcal{U}_q^\perp = \mathcal{U}_q(1) \oplus S\,\mathcal{U}_q(2) \oplus \mathcal{U}_q^\perp.$$

In terms of the corresponding projection operators :

(III.25) $$I = P_q + P_{S\mathcal{U}_q} + P_{\mathcal{U}_q^\perp}.$$

Equations (III.21) and (III.13) can thus be written :

(III.26) $$-\frac{4}{3\gamma(q)} f_q^2 = Q_{(q)}^2 = P_{\mathcal{U}_q^\perp},$$

(III.26′) $$d_q = \frac{-\theta(q)}{\gamma(q)}\left(-P_q + P_{S\mathcal{U}_q^\perp} - \frac{1}{2}P_{\mathcal{U}_q^\perp}\right).$$

In the following we will call a q-vector positive or negative when $\theta(q)$ is negative or positive respectively : that is $[\gamma(q) = 1]$,

(III.27) $\theta(q) = -1 \iff q$ is a positive normalized q-vectors.

We add some remarks concerning the pairs of normalized positive q-vectors.

THEOREM 1. — *If q_1 and q_2 are normalized positive q-vectors,* $-\frac{1}{2} \leqq (q_1, q_2) \leqq 1.$

Proof. — For such a pair equation (III.14) reads :

(III.28) $$\begin{aligned} 2\gamma(q_1 \wedge q_2) &= -2(q_1, q_2)^2 + (q_1, q_2) + 1 \\ &= -2\left[(q_1, q_2) + \frac{1}{2}\right][(q_1, q_2) - 1] \geqq 0. \end{aligned}$$

COROLLARY 1 :

$$q_1 \wedge q_2 = 0 \iff (q_1, q_2) = -\frac{1}{2} \quad \text{or} \quad (q_1, q_2) = 1, \quad \text{i. e.} \quad q_1 = q_2.$$

THEOREM 2. — *If q_1, q_2 are distinct commuting normalized positive q-vectors, then q_3 defined by $q_3 + q_1 + q_2 = 0$, is also a normalized positive q-vector which commutes with q_1 and q_2.*

Proof. — $(q_3, q_3) = (q_1 + q_2, q_1 + q_2) = 1$ shows that q_3 is normalized. Moreover :

$$\begin{aligned}\theta(q_3) &= (q_3, q_3 \vee q_3) = (q_3, -q_1 - q_2 + 2\, q_1 \vee q_2) \\ &= 1 - 2\,(q_1, q_1 \vee q_2) - 2\,(q_2, q_1 \vee q_2) \\ &= 1 - 2\,(q_1 \vee q_1, q_2) - 2\,(q_2 \vee q_2, q_1) \\ &= 1 + 4\,(q_1, q_2) = -1.\end{aligned} \tag{III.29}$$

This shows that q_3 is a normalized positive q-vector. Furthermore

$$(q_1, q_3) = (q_2, q_3) = -\frac{1}{2}$$

so that q_3 commute with q_1 and q_2.

3. The r-vectors

A r-vector r is a vector whose matrix is singular i. e.

$$\theta(r) = 0. \tag{III.30}$$

We will only consider normalized r-vectors, i. e. $\gamma(r) = 1$.

From (III.6) it follows that $r \vee r$ is a normalized positive q-vector; i. e.

$$r \vee r = q_r \tag{III.31}$$

with

$$q_r \vee q_r = -q_r. \tag{III.32}$$

Equation (III.5) then gives :

$$q_r \vee r = r \vee q_r = r \tag{III.33}$$

which shows that $r \in \mathrm{S}\,\mathfrak{U}_{q_r}(2)$ [use equation (III.26')].

We will now prove the following

LEMMA 1. — *Every 2-plane = two dimensional vector space* P *of* $\mathbf{R}^8$ *contains at least one r-vector.*

Let Γ be the unit circle, $\gamma(x) = 1$, of P. The real valued function on Γ, $\theta(x) = (x \vee x, x)$ is continuous and has the property $\theta(x) = -\theta(-x)$. Hence θ has at least two zeros $\theta(\pm r)$, corresponding to an r-vector.

There are k-planes of $\mathbf{R}_8$ which contain only r-vectors as we will show now for $k = 3$ and 4.

PROPOSITION 1. — *The vectors which belong to the* 3- *and* 4-*dimensional subspaces* $\mathrm{S}\,\mathfrak{U}_q$ *and* $\mathfrak{U}_q^{\perp}$ *defined by a q-vector q, are r-vectors.*

Proof. — Let x be in $S\,\mathcal{U}_q$ or in $\mathcal{U}_q^{\perp}$. From (III.26′) it then follows

(III.34) $$d_q\, x = q_{\vee} x = \lambda\, x$$

with $\lambda = -\dfrac{\theta(q)}{\gamma(q)}$ if $x \in S\,\mathcal{U}_q$ and $\lambda = +\dfrac{1}{2}\dfrac{\theta(q)}{\gamma(q)}$ if $x \in \mathcal{U}_q^{\perp}$.

Using (III.34) and (III.5) we obtain

(III.35) $$\theta(x) = (x, x_{\vee} x) = \frac{1}{\lambda}(q_{\vee} x, x_{\vee} x)$$
$$= \frac{1}{\lambda}(q, x_{\vee} x_{\vee} x) = \frac{\gamma(x)}{\lambda}(q, x) = 0.$$

This shows that x is a r-vector. In accordance with the discussion of section II.2 the r-vectors of $S\,\mathcal{U}_q$ and those of $\mathcal{U}_q^{\perp}$ correspond respectively to the eigenvalues 0 and ± 1 of $Q(q)$: *see* equation (III.21′).

If in equation (III.10) we set $x = r'$, $y = r''$ with $r, r'' \in S\,\mathcal{U}_q$ and we apply the operators to q we get [use equation (III.33)] :

(III.36) $$r', r'' \in S\,\mathcal{U}_q(2) \;\Rightarrow\; r'_{\vee} r'' = (r', r'')\, q.$$

4. The Cartan subalgebras of $S\,\mathcal{U}(3)$

If x is not a q-vector, x and $x_{\vee} x$ span a two-dimensional space $\mathcal{C}_x$; equations (III.5) and (III.6) show that $\mathcal{C}_x$ is a $_{\vee}$-algebra. It is also an Abelian Lie algebra which is the centralizer of x in the Lie algebra [*see* also (III.5) and (III.6 ii). Indeed, if x is not a q-vector $\theta(x)^2 < \gamma(x)^3$; according to (III.4), x has three distinct eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$, with $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Let v be a unitary transformation which diagonalizes x; the set of $u \in SU(3)$ which commute with x is characterized by : $[vuv^{-1}, vxv^{-1}] = 0$. Hence vuv^{-1} is diagonal and since $\det u = 1$, it is a $U(1) \times U(1)$ group, called Cartan subgroup of SU (3), which we denote by C_x. Its Lie algebra is $\mathcal{C}_x$.

We choose for convenience normalized x : $\gamma(x) = 1$. Since x is not a q-vector, $-1 < \theta(x) < 1$.

We shall now prove the following propositions :

(i) $\mathcal{C}_x$ *contains three normalized positive q-vectors*, q_1, q_2, q_3 :

(III.37) $$q_1 = \alpha_i\, x + \beta_i\, x_{\vee} x.$$

Proof. — If $\theta(x) \neq 0$ from the conditions $\gamma(q_i) = 1$, $\theta(q_i) = -1$ we deduce, with the help of (III.5) and (III.6), the equations

(III.38) $$\alpha_i^2 = \beta_i^2 - \beta_i,$$

(III.39) $$(1 - \theta^2(x))\,\alpha_i^3 - \frac{3}{4}\alpha_i - \frac{\theta(x)}{2} = 0,$$

(III.40) $$(1 - \theta^2(x))\,\beta_i^3 - \frac{3}{4}\beta_i - \frac{1}{4} = 0.$$

These three equations have three real solutions (α_i, β_i) when $0 < |\theta(x)| < 1$.

If $\theta(x) = 0$ (i. e. x is a r-vector) a direct computation gives three real roots $(\alpha_i, \beta_i) = (0, 1), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ which also satisfy (III.38) and (III.39, 40). The proposition is thus proved for all

$$\theta(x): \quad -1 < \theta(x) < 1.$$

The solutions (α_i, β_i) satisfy $\sum_i \alpha_i = \sum_i \beta_i = 0$ which imply

(III.41) $$\sum_i q_i = q_1 + q_2 + q_3 = 0.$$

We introduce the following symbols :

(III.42) $$\begin{cases} \varepsilon_{ijk} \begin{cases} = 1 \\ = -1 \end{cases} \text{ if } i, j, k \text{ is an } \begin{cases} \text{even} \\ \text{odd} \end{cases} \text{ permutation of } 1, 2, 3; \\ \varepsilon_{ijk} = 0 \quad \text{otherwise (i. e. two indices at least are equal)} \end{cases}$$

and

(III.42′) $$\eta_{ijk} = (\varepsilon_{ijk})^2$$

and use the summation convention on repeated indices.

One can easily verify that

(III.43) $$(q_i, q_j) = \frac{1}{2}(3\,\delta_{ij} - 1)$$

and

(III.43′) $$q_i \vee q_j = -\,\delta_{ij}\, q_1 - \eta_{ijk}\, q_k.$$

(ii) $\mathcal{C}_x$ *contains six r-vectors* $\pm r_1, \pm r_2, \pm r_3$.

The proof is obtained by verifying that

(III.44) $$r_i = -\frac{1}{2\sqrt{3}}\,\varepsilon_{ijk}\,(q_j - q_k)$$

are r-vectors. They have the following properties :

(III.45) $$(r_i, r_j) = \frac{1}{2}(3\,\delta_{ij} - 1),$$

(III.46) $$(q_i, r_j) = \frac{\sqrt{3}}{2} \sum_k \varepsilon_{ijk},$$

(III.47) $$r_i \vee r_j = \delta_{ij}\, q_j - \eta_{ijk}\, q_k,$$

(III.48) $$r_i \vee q_j = \delta_{ij}\, r_i - \eta_{ijk}\, r_k$$

and therefore

$$\theta(r_i) = (r_i, q_i) = 0.$$

Hence each pair q_i, r_i is an orthonormal basis for $\mathcal{C}_x$. We remark that the six unit r-vectors of $\mathcal{C}_x$ are the vertices of a regular hexagon.

(iii) *The six unit vectors* $\pm r_i \in \mathcal{C}_x$ *are the roots of* $S\mathcal{U}(3)$ *for* $\mathcal{C}_x$.

We begin by recalling the definition of the roots of $S\mathcal{U}(3)$. Let us consider the vector space $\mathbf{C}^8$, obtained by complexifying $\mathbf{R}^8$. The elements of $\mathbf{C}^8$ are 3×3 traceless, complex matrices z. With the Hermitian scalar product :

$$\langle z_1, z_2 \rangle = \frac{1}{2}\operatorname{tr} z_1^* z_2 = (z_1^*, z_2), \tag{III.49}$$

$\mathbf{C}^8$ is a Hilbert space. Notice that $\langle x, y \rangle$ reduces to (x, y) on $\mathbf{R}^8$. The operators d_x and f_x are naturally extended to $\mathbf{C}^8$: d_x is real and symmetric and therefore Hermitian; f_x is real and antisymmetric and therefore antihermitian. For every real $a \in \mathcal{C}_x$, if_a is thus Hermitian and

$$if_a\, \mathcal{C}_x = 0; \qquad if_a\, \mathcal{C}_x^\perp \subset \mathcal{C}_x^\perp. \tag{III.50}$$

Since for all real $a \in \mathcal{C}_x$ the operators if_a form a set of commuting Hermitian operators they have a commun orthonormal basis of eigenvectors. Let $z \in \mathcal{C}_x^\perp$ be such an eigenvector :

$$if_a z = \frac{1}{2}[a, z] = \rho_z(a)\, z, \tag{III.51}$$

where $\rho_z(a)$ is a real function of a, which is a linear form on $\mathcal{C}_x$:

$$\rho_z(\alpha\, a + \beta\, b)\, z = if_{\alpha a + \beta b}\, z = i(\alpha\, f_a + \beta\, f_b)\, z = \alpha\rho_z(a) + \beta\rho_z(b). \tag{III.52}$$

Hence there is a real vector $r_z \in \mathcal{C}_x$ such that

$$\rho_z(a) = (r_z, a). \tag{III.53}$$

The vectors r_z for all z are by definition the roots of $S\mathcal{U}(3)$ for $\mathcal{C}_x$.

To summarize, the roots of the Cartan subalgebra $\mathcal{C}$ are defined by the equation in $\mathbf{C}^8$:

$$\forall\, a \in \mathcal{C}, \quad if(a)\, z \equiv ia_\wedge z = (a, r_z)\, z. \tag{III.54}$$

To compute the roots r_z we choose successively for a the three normalized q-vectors q_1, q_2, q_3 of $\mathcal{C}$. We know from equations (III.21) to (III.26) that the eigenvalues of $if(q_i)$ on $\mathcal{C}^\perp$ are doubly degenerate and equal to $0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$, so that 6 root vectors satisfy :

$$(\pm r_i, q_i) = 0, \qquad (\pm r_i, q_j) = \pm\frac{\sqrt{3}}{2} \quad \text{or} \quad +\frac{\sqrt{3}}{2} \quad \text{for} \quad i \neq j.$$

The comparison of these equations with (III.46) completes the proof of (iii).

We shall give an alternative algebraic proof of (iii), obtained from the characteristic equation of f_x. To establish this equation we set $y = x \vee x$ in equation (III.11), we multiply it by f_x and use (II.24). We thus obtain

$$(\text{III.55}) \qquad (\gamma(x) + 4 f_x^2) f_{x \vee x} + 2\,\theta(x) f_x = 0.$$

We then multiply (III.12) by f_x^2 and use (II.24) twice :

$$(\text{III.56}) \qquad f_{x \vee x}^2 = 4 f_x^2 (\gamma(x) + f_x^2).$$

By eliminating $f_{x \vee x}$ between these two equations we get :

$$(\text{III.57}) \qquad f_x^2 [(\gamma(x) + f_x^2)(\gamma(x) + 4 f_x^2)^2 - \theta(x)^2] \equiv f_x^2 [16 f_x^6 + 24\,\gamma(x) f_x^4 + 9\,\gamma(x)^2 f_x^2 + \gamma(x)^3 - \theta(x)^2] = 0.$$

The equation for f_x^2 obtained by setting equal to zero in (III.57) the expression between square brackets has three roots ≤ 0 which are distinct if $\gamma(x)^3 - \theta(x)^2 \neq 0$ and $\theta(x)^2 \neq 0$. Equation (III.57) is thus the characteristic equation of the antisymmetrical operator f_x which is already known to have zero as a double root when $\theta(x)^2 \neq \gamma(x)^2$.

If we put $\gamma(x) = 1$ and $-1 \leq \theta(x) = \sin\varphi \leq 1$, i. e. $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$, the six eigenvalues of if_x in $\mathcal{C}_x^{\perp}$ are $\cos\left(\frac{\varphi}{3} + k\frac{\pi}{3}\right)$ with $k = 0, 1, 2, 3, 4, 5$. The angles $\frac{\varphi}{3} + k\frac{\pi}{3}$ are the angles between x and the six (unit) r-vectors $\pm r_i \in \mathcal{C}_x$. [Indeed $\theta(x) = 0 \Leftrightarrow \varphi = 0$].

(iv) If $a \in \mathcal{C}_x$, the eigenvalues of d_a on $\mathcal{C}_x^{\perp}$ are (q_i, a) where q_i are the 3 positive q-vectors of $\mathcal{C}_x$.

Proof. — Equation (II.20) shows that for every $b \in \mathcal{C}_x$, d_b commutes with the set of if_a, $a \in \mathcal{C}_x$. Hence the restriction of the d_b's to $\mathcal{C}_x^{\perp}$ form a commuting set of Hermitian operators. We can thus write

$$(\text{III.58}) \qquad d_a z = \lambda_z(a) z = (t_z, a) z.$$

We shall call the vectors t_z the " pseudo-roots " of SU (3) for $\mathcal{C}_x$. If the unit vector a is not a q-vector ($|\theta(a)| \neq 1$), there exist only two unit vectors $b_\pm$, defined up to a sign, such that $b \vee b = a$. We can check that [use (III.5) and (III.6)] :

$$(\text{III.59}) \qquad b_\pm = \frac{1}{\sqrt{2(\pm 1 + \theta(a))}} (a \pm a \vee a),$$

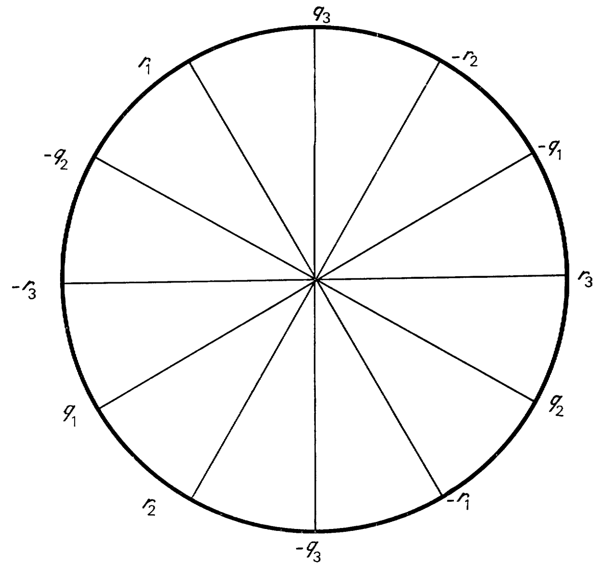

Fig. 1. — The six roots $\pm r_i$ and the three pseudo-roots q_i of a Cartan subalgebra. Together with the $-q_i$, they represent the " 12 h " of the SU (3) clock.

TABLE 1

Eigen values and eigen vectors of the if_a and d_a for the elements a of a Cartan subalgebra $\mathcal{C}$

$\mathbf{C}^8 = \mathcal{C} \oplus \mathcal{C}^{\perp}$	Eigenvalues, eigenvectors in $\mathcal{C}$	Eigenvalues, eigenvectors in $\mathcal{C}^{+}$
if_a	$0 \quad \forall\, x \in \mathcal{C}$	$(r_i, a) \quad z_i$ $(-r_i, a) \quad z_i^*$
d_a	$\pm\sqrt{\gamma(a)}$ $\dfrac{a \pm a \vee a\left(\dfrac{1}{\sqrt{\gamma(a)}}\right)}{\sqrt{2\left(1 \pm \left(\dfrac{\theta(a)}{\sqrt{\gamma(a)^3}}\right)\right)}}$ (*see* note)	$(q_i, a) \quad \alpha z_i + \beta z_i^*$

Note : If $\theta(a) = \pm \gamma(a)^{3/2}$, a is a q-vector q. The eigenvectors of d_q in $\mathcal{C}$ are q and the root r_q such that $q = r_q \vee r_q (-\theta(q))^{1/3}$. The corresponding eigenvalues are $\mp \dfrac{\theta(q)^{\frac{3}{2}}}{\gamma(q)}$.

Using equation (III.13) we find

$$\text{(III.60)} \qquad d_a z = (2(if_b)^2 - I) z = [2(r_z, b)^2 - 1] z = (a, r_z \vee r_z) z = (a, q_{r_z}) z.$$

The last equality is obtained from (III.22) using $\gamma(r_z) = \gamma(b) = 1$, $b \wedge r_z = 0$. Since $r_z \vee r_z$ is a positive q-vector $\in \mathcal{C}_x$ and equation (III.27) shows that (III.60) still holds when a is q-vector, we have thus proved (iv). The q-vectors play for the operator d_a the role of the roots r for the operators if_a; for this reason we shall call the q-vectors « pseudo-roots ».

The results of this section are summarized in figure 1 and Table 1, which gives the eigenvalues of if_a and d_a for $a \in \mathcal{C}$. The explicit values of the eigenvalues of if_a as function of a show that condition (iv) of section III.2 implies the other three.

5. Orbits and strata of SU (3) on $\mathbf{R}^8$ and S_7

In this section we study non linear geometrical properties, mainly the *orbits* and *strata* of SU (3) on the octet space. For the definition of orbits and strata and for general notions on group actions, the reader is refer to Appendix 2.

The necessary and sufficient conditions for two vectors x and y of $\mathbf{R}^8$ to belong to a same orbit of SU (2) [i. e., there exists a $u \in$ SU (3) such that $y = uxu^{-1}$] are

$$\text{(III.61)} \qquad \gamma(x) = \gamma(y), \qquad \theta(x) = \theta(y).$$

Indeed any Hermitian matrix can be diagonalized by a unitary transformation (II.3), with its eigenvalues ordered in decreasing sequence : $\lambda_1 \geqq \lambda_2 \geqq \lambda_3$. Equation (III.61) is equivalent to say that the Hermitian 3×3 matrices x and y have the same eigenvalues.

We can thus establish a one to one correspondence between the orbits of SU (3) on $\mathbf{R}^8$ and the points of a domain $\mathcal{D}$ in the plane θ, γ defined by $\mathcal{D} : \gamma^3 \geqq \theta^2$.

The strata of $\mathbf{R}^8$ can be easily identified with the help of the γ-θ-diagram illustrated in figure 2.

(i) The general stratum is the open dense domain $\gamma^3 > \theta^2$ of $\mathcal{D}$. The isotropy groups of the orbits of this stratum are U (1) $\times$ U (1) groups whose Lie algebra are the two dimensional Abelian Lie algebras discussed in section III.4.

(ii) The q-stratum is the frontier of the domain $\mathcal{D}$, $\gamma^3 = \theta^2$, with the exclusion of the point $\gamma = \theta = 0$. The isotropy groups are conjugated U (2) subgroup of SU (3) whose Lie algebra are the $\mathcal{U}_q(2)$ Lie algebras discussed in section III.2.

(iii) The fixed point O, $\gamma = \theta = 0$ whose isotropy group is SU (3) itself.

In the physical applications we are only interested in the directions of $\mathbf{R}^8$, and not in the lenght of the vectors. We can thus consider the action of SU (3) on the unit sphere $S_7 \subset \mathbf{R}^8$ ($\gamma = 1$) or even on the real projective space PR (7) obtain from S_7 by identification of points diametricaly opposite. The orbits of SU (3) on S_7 can then be put into a one to one correspondence with the points of the segment $\gamma = 1, -1 \leq \theta \leq 1$ (see *fig.* 2).

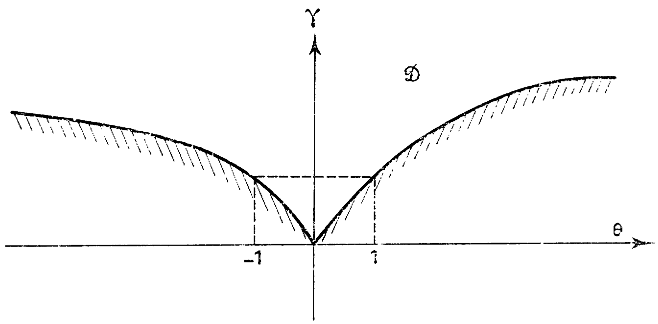

Fig. 2

There are only two strata on S_7 :

(i) The general stratum which contains the orbits corresponding to all the points of the segment except for its two end points.

(ii) The q-stratum which is described by the two end points $\theta = \pm 1$. They correspond to the two orbits of the normalized negative and positive q-vectors respectively.

To study PR (7) we have to identify the orbits with $\pm \theta$. We will therefore denoted the orbits of PR (7) by $|\theta|$. There are three strata :

General stratum $0 < |\theta| < 1$, dimension 7, open dense;

The q-stratum $|\theta| = 1$, one orbit of dimension 4;

The r-stratum $|\theta| = 0$, one orbit (the set of roots) of dimension 6.

It would beinteresting (and useful for physics) to study the orbits of SU_3 on manifolds whose elements are geometrical objects of $\mathbf{R}_8$, for examples : pair of vectors, k-planes, subalgebras.

We leave such a systematic study to the reader; we only make some remarks on this point.

The isotropy group of an ordered pair of vectors x, y is the intersection of the isotropy group of each vector. Hence in the action of SU (3) on $\mathbf{R}^8 \oplus \mathbf{R}^8$, the general stratum corresponds to the trivial isotropy group ($\mathcal{C}_x \cap \mathcal{C}_y = \{ 0 \}$ when $x_\wedge y \neq 0$) and it contains 8-dimensional orbits. The Montgomery and Yang theorem (*see* Appendix 2) tells us that this general stratum is 16-dimensional and therefore contains a 8-parameter family of (8-dimensional) orbits. One can choose for these parameters the algebraically independent invariants constructed out of two vectors : (x, x), (y, y), (x, y), $\{ x, x, x \}$, $\{ x, x, y \}$, $\{ x, y, y \}$, $\{ y, y, y \}$ and one more, for instance $\gamma (x_\wedge y)$, $\gamma (x_\vee y)$ or $(x_\vee x, y_\vee y)$ which are linearly related by equations (III.14) and (III.14′). Any other invariant built from x and y is function of these eight invariants.

For a non ordered pair of vectors x, y, satisfying (III.61), one must include in the isotropy group of the pair the SU (3) transformations which exchange the vectors.

6. The Lie Subalgebras of $S\mathcal{U}(3)$

We have alread found two orbits of $S\mathcal{U}(3)$ Lie algebras, namely : $\mathcal{U}_q(2)$, the centralizers of q-vectors and the Cartan subalgebra $\mathcal{C}_x$, the centralizers of the other elements. We list here for completness all the $S\mathcal{U}(3)$ Lie subalgebras.

(i) 1-dimensional algebras generated by any vector $x \neq 0$.

(ii) 2-dimensional algebras. They are all Abelian and isomorphic to the Cartan algebras defined in equation (III.4). These algebras are all conjugated by the group; i. e. they form one orbit for the group action (indeed they are defined by any root r they contain). There exist no non-Abelian 2-dimensional subalgebras. Indeed if follows from (II.11) that if $x_\wedge y \neq 0$, it is orthogonal to both x and y, so that $x_\wedge y$, x, y span a three dimensional space.

(iii) 3-dimensional subalgebras. They are all isomorphic to $S\mathcal{U}(2)$ and their elements are all r-vectors.

Indeed, let $\mathcal{L}_3$ be such an algebra and e_1, e_2, e_3 be an orthonormal basis of it. Since the centralizers of the elements are $\mathcal{C}$ or $\mathcal{U}_q$, we know that $\mathcal{L}_3$ is not Abelian; so there are two e_i, e. g. e_1, e_2 such that $e_1 {}_\wedge e_2$ is different from zero and orthogonal to e_1, e_2; i. e. :

(III.62) $$e_1 {}_\wedge e_2 = \lambda\, e_3, \qquad \lambda \neq 0.$$

This can be written [*see* (II.11)],

(III.62′) $$[e_1, e_2, e_3] = \lambda$$

we first prove that all elements of $\mathcal{L}_3$ are r-vectors.

Indeed, we call an arbitrary normalized element e_1 and we compute

$$\theta(e_1) = (e_1, e_1 \vee e_1) = \frac{1}{\lambda}(e_1, d_{e_2 \wedge e_3} e_1) \tag{III.62''}$$
$$= \frac{1}{\lambda}[(e_1, f_{e_2} d_{e_3} e_1) - (e_1, d_{e_3} f_{e_2} e_1)]$$
$$= (e_3, d_{e_3} e_1) + (e_1, d_{e_3} e_3)$$
$$= 2(e_3, d_{e_3} e_1).$$

By using $e_i = -\frac{1}{\lambda}(e_3 \wedge e_1)$ we get

$$\theta(e_1) = -\frac{2}{\lambda}(e_3, d_{e_3} f_{e_3} e_2) = -\frac{2}{\lambda}(e_3, f_{e_3} d_{e_3} e_2)$$
$$= \frac{2}{\lambda}(f_{e_3} e_3, d_{e_3} e_2) = 0.$$

[In this computation we have used (II.20) and (II.24).]

Therefore from a given basis we can define three normalized positive q-vectors :

$$q_i = e_i \vee e_i.$$

By using [*see* eq. (III.62″)] :

$$\theta(e_1) = 2(e_3, d_{e_3} e_1) = 2(q_3, e_1),$$

it follows that q_3 is orthogonal to e_1. Since without a change, we could permute 2 and 3 in the previous calculation it also follows that q_2 is orthogonal to e_1. Morever $\theta(e_i) = (q_i, e_i) = 0$, hence

$$(q_i, e_j) = 0 \tag{III.63}$$

that is, for every element r of the algebra $\mathcal{L}_3$ the corresponding q-vector $q = r \vee r$ is orthogonal to all elements of the algebra. Finally we define

$$3k = q_1 + q_2 + q_3, \tag{III.64}$$

and we have

$$\forall e_i \in \mathcal{L}_3, \quad e_i \wedge k = 0; \tag{III.65}$$

indeed, using II.24, we have :

$$q_1 \wedge e_3 = 2e_1 \vee (e_1 \wedge e_3) = -2\lambda e_1 \vee e_2 = -2e_2 \vee (e_2 \wedge e_3) = -q_2 \wedge e_3.$$

We have two cases to consider :

(*a*) $k \neq 0$: Its centralizer contains $\mathcal{L}_3$, so that k is a q-vector. $[\forall r \in \mathcal{L}_3, \gamma(r) = 1 \Rightarrow r \vee r = k]$ and $\mathcal{L}_3 = \mathcal{SU}_k(2)$. Equation (III.14) with $x = e_1$, $y = e_2$ yields $|\lambda| = 1$ [*see* also (III.36)]. By choosing the right orientation of the trihedron e_1, e_2, e_3 we can set $\lambda = 1$. Since the positive unit q-vectors form a single orbit, these are all conjugated by SU (3).

(*b*) $k = 0$: This yields $(q_1, q_2) = -\frac{1}{2}\cdot$ From corollary 1 of (III.2) q_1, q_2, q_3 are the three positive unit q-vectors of a Cartan subalgebra.

Equation (III.14′) yields in this case $|\lambda| = \frac{1}{2}\cdot$ As before we can choose a right handed basis for which $\lambda = \frac{1}{2}\cdot$

All S $\mathcal{U}$ (2) of type (b) are conjugated. This can be proved as follows :

Let e_i and e'_i be right handed orthonormal basis of the two SU (2) algebras. The eight algebra cally independant invariants [listed at the end of (III.5)] made with a pair of vectors e_1, e_2 :

$$(e_i, e_j) = \delta_{ij}, \qquad (q_i, e_j) = 0, \qquad \gamma\,(e_1 \wedge e_2) = \frac{1}{4}$$

have the same value as those made with the pair e'_i, e'_2. Thus these two pairs of vectors are conjugated and the S $\mathcal{U}$ (2) Lie algebras they generate are also conjugated. All these alsebras are conjugated to SO (3), the subgroup of orthogonal (= real unitary) matrices of SU (3). Indeed the SO (3) elements of the SU (3) Lie algebra are the antisymmetrical matrices $x^{\mathrm{T}} = -x \in \mathbf{R}^8$. [Remark that $(x \wedge y)^{\mathrm{T}} = -x^{\mathrm{T}} \wedge y^{\mathrm{T}})$.] A orthogonal basis e_i of this Lie algebra is in Gell'Mann's notation (*see* Appendix 1) $e_1 = \lambda_7$, $e_2 = -\lambda_5$, $e_3 = \lambda_2$.

(iv) 4-dimensional subalgebras. They are conjugated and isomorphic to $\mathcal{U}_q$ (2).

We will prove first that any four-dimensional Lie algebra $\mathcal{L}_4$ has a two dimensional Abelian subalgebra. Let $\mathcal{X} \overset{\perp}{\oplus} \mathcal{Y} = \mathcal{L}_4$ be the decomposition of its vector space into two perpendicular 2-planes. We choose two linearly independent vectors in each 2-planes : $x_1, x_2 \in \mathcal{X}$, $y_1, y_2 \in \mathcal{Y}$ and define $x_1 \wedge x_2 = y$, $y_1 \wedge y_2 = x$. We remark that the direction of y (resp. x) is independent of the choice of x_1, x_2 (resp. y_1, y_2); indeed

$$(\alpha_1 x_1 + \alpha_2 x_2) \wedge (\beta_1 x_1 + \beta_2 x_2) = (\alpha_1 \beta_2 - \alpha_2 \beta_1)\, y$$

and similarly for x. If either x or $y = 0$, *see* below : $\mathcal{L}_4 = \mathcal{A} \overset{\perp}{\oplus} \mathcal{B}$. Otherwise we note [*see* (II.11)] that $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. We choose $x' \perp x$, $x' \in \mathcal{X}$, $y' \perp y$, $y' \in \mathcal{Y}$. Since $x' \wedge y'$ is perpendicular to x' and y', $x' \wedge y' = \alpha\, x + \beta\, y$, with

$$\alpha = (x, x' \wedge y') = (x \wedge x', y') = \lambda\,(y, y') = 0,$$

and similarly

$$\beta = (y, x' \wedge y') = -(y \wedge y', x') = \lambda'\,(x, x') = 0.$$

Hence in all cases we can choose for the decomposition of $\mathcal{L}_4$ into two perpendicular two planes : $\mathcal{L}_4 = \mathcal{A} \overset{\perp}{\oplus} \mathcal{B}$ where $\mathcal{A}$ carries a two-dimensional Abelian Lie algebra generated by x' and y'. Let $b \perp b'$, b, $b' \in \mathcal{B}$.

Then for every $a \in \mathcal{A}$, $b_\wedge a = \beta(a)\, b'$ and $\beta(a)$ is a linear form on $\mathcal{A}$. Therefore there is a vector $q \in \mathcal{A}$ such that $\beta(q) = 0$ and $\mathcal{A} \oplus \{\lambda\, b\} \subset$ centralizer of q. Thus q is a q-vector and $\mathcal{L}_4 = \mathcal{U}_q(2)$.

We finally prove that $S\mathcal{U}(3)$ has no Lie subalgebra with dimension greater that 4. To prove this statement we begin by introducing the notations : $A_\wedge B$ for the vector space spanned by all vectors $a_\wedge b$ with a and b contained in the vector spaces A and B respectively; and $A \perp B$ to denote that A and B are orthogonal, i. e. $(a, b) = 0$, $\forall\, a \in A$, $\forall\, b \in B$. Since $S\mathcal{U}(3)$ is a simple real compact Lie algebra :

$$A \perp B \;\Rightarrow\; A \cap B = \{0\} \tag{III.66}$$

and furthermore if A, B are two perpendicular subalgebras :

$$A_\wedge A \subset A,\quad B_\wedge B \subset B,\quad A \perp B \;\Rightarrow\; A \perp (A_\wedge B) \perp B. \tag{III.67}$$

Indeed, from (II.11),

$$a_1, a_2 \in A,\quad b_1, b_2 \in B \;\Rightarrow\; (a_1, a_2{}_\wedge b_1) = (a_1{}_\wedge a_2, b_1) = 0,$$
$$(a_1{}_\wedge b_1, b_2) = (a_1, b_1{}_\wedge b_2) = 0.$$

We are now ready to prove that if $\mathcal{L}$ is a proper [i. e. $\neq S\mathcal{U}(3)$] Lie algebra of $S\mathcal{U}(3)$, $l = \dim \mathcal{L} \leq 4$:

Proof. — Let us first suppose $l \leq 6$. It follows from lemma 1 (in section III.3) that $\mathcal{L}^\perp$ contains at least one vector x which is not a q-vector. Since $\dim \mathcal{C}_x = 2$, $\dim \mathcal{C}_x \cap \mathcal{L} = 0$ or 1 and then $\dim x_\wedge \mathcal{L} = l$ or $l - 1$. Equation (III.67) applied to the Lie algebra $\{\lambda\, x\}$ and $\mathcal{L}$ implies $\mathcal{L} \perp (\{\lambda\, x\}_\wedge \mathcal{L}) \perp \{\lambda\, x\}$; from (III.66) it follows that $l + l - 1 + 1 \leq 8$ i. e. $l \leq 4$. Consider now the case $l = \dim \mathcal{L} = 7$, i. e. $\mathcal{L}^\perp = \mathcal{U}_x(1)$. If x is not a q-vector the proof used for $l \leq 6$ is still valid. If instead x is a q-vector, $\dim(\mathcal{C}_q \cap \mathcal{L}) = 7 - 3 = 4$. Equations (III.66) and (III.67) would then imply $7 + 4 + 1 \leq 8$ which is absurd. This completes the proof that $l = \dim \mathcal{L} \leq 4$.

To summarize this section, the Lie algebra of SU (3) are :

(i) The one dimensional $\mathcal{U}_x(1)$; an infinite family of conjugation classes, each one defined by the parameter $0 \leq \dfrac{\theta(x)^2}{\gamma(x)^3} \leq 1$.

(ii) The conjugate class of the two dimensional Abelian algebra : i. e. the Cartan subalgebra.

(iii *a*) The conjugate class of $S\mathcal{U}_q(2)$.

(iii *b*) The conjugate class of the Lie algebra of SO (3) $\subset$ SU (3) the subgroup of orthogonal (= real) matrices of SU (3).

(iv) The conjugate class of $\mathcal{U}_q(2)$ i. e., 4-dimensional centralizer of the elements which are q-vector.

7. The Subspaces $\mathcal{V}$ of $\mathbf{R}^8$ generated by sets of non zero vectors

Let $x, y, z, \ldots$ be a finite set of vectors and let $\mathcal{V}_n$ denote the n-dimensional space which is the closure of the set under addition, multiplication by a scalar and under the $\wedge$- and $\vee$-laws. $\mathcal{V}_n$ is therefore both a $\wedge$- and a $\vee$-algebra.

From the results of the previous section it follows that the only $\mathcal{V}'$ s are :

(i) $\mathcal{V}_1$ generated by a q-vector.

(ii) $\mathcal{V}_2 = \mathcal{C}_x$, the 2-dimensional Cartan generated by a vector x which is not a q-vector.

(iii) $\mathcal{V}_4 = \mathcal{U}_q(2)$ generated by two non-commuting vectors whose centralizers have a 1-dimensional intersection which must then be a q-vector.

(iv) $\mathcal{V}_8 = \mathbf{R}^8$ generated by two non-commuting vectors whose centralizers have zero intersections.

We remark that two non-commuting q-vectors q_1 and q_2 generate a $\mathcal{V}_4 = \mathcal{U}_q(2)$ where the positive normalized q-vector q is given by

$$q = ((q_1, q_2) - 1)^{-1} \left(\frac{1}{2} (q_1 + q_2) + q_1 \vee q_2 \right) \tag{III.68}$$

[for a physical example *see* [2 d], eq. (42)].

APPENDIX 1

In this appendix we give some of the relations which are necessary to translate our formulation into the one communly used in physics.

Let λ_i be the elements of an orthonormal basis of $\mathbf{R}^{n^2-1}$:

$$(\lambda_1, \lambda_j) = \delta_{ij}. \tag{A.1}$$

The structure constants of the $\wedge$ and $\vee$ algebra are defined by

$$\lambda_i \wedge \lambda_j = f_{ijk} \lambda_k, \tag{A.2}$$

$$\lambda_i \vee \lambda_j = \sqrt{3}\, d_{ijk} \lambda_k. \tag{A.3}$$

In this appendix we use the convention that repeated indices should be summed from 1 to $n^2 - 1$. Using (A.1), equation (A.2′) and (A.3) are equivalent to

$$f_{ijk} = [\lambda_i, \lambda_j, \lambda_k], \tag{A.2′}$$

$$\sqrt{3}\, d_{ijk} = \{ \lambda_i, \lambda_j, \lambda_k \}. \tag{A.3′}$$

Thus f_{ijk} and d_{ikj} are the coordinates of completely antisymmetrical and symmetrical third rank tensors invariant under SU (3). They are also related to the matrix elements of the f_{λ_i} and d_{λ_i} that we simply denote by f_i and d_i.

Then

$$(\text{A.4}) \qquad (\lambda_i, f_j \lambda_k) = (f_j)_{ik} = f_{ijk} = -(f_j)_{ki},$$

$$(\text{A.5}) \qquad (\lambda_i, d_j \lambda_k) = (d_j)_{ik} = d_{ijk} = \quad (d_j)_{ki}.$$

Gell-Mann [1] chose an explicit realization of the λ_i suggested by physics, with two matrices : λ_8 in the direction of the hypercharge and λ_3 along of the third component of isospin.

However the particle states of a SU (3) octet are eigen vectors (elements of $\mathbf{C}^8$, the complexified of R^8) of the operators f_a for $a \in \mathcal{C}$. It is therefore more convenient to choose an orthonormal basis of $\mathbf{C}^8$ made with the eigenvectors z, z^* introduced in III.4. We give them explicitely as 3 by 3 matrices and in terms of Gell-Mann's.

Let e_{ij} be the matrix with elements :

$$(\text{A.6}) \qquad (e_{ij})_{\alpha\beta} = \delta_{i\alpha}\, \delta_{j\beta}.$$

The Cartan subalgebra $\mathcal{C}_\mathrm{D}$ of diagonal matrices contains the three unit positive q-vectors :

$$(\text{A.7}) \qquad q_i = \frac{1}{\sqrt{3}}\,(\mathrm{I} - 2\, e_{ii})$$

and the six vectors $\pm\, r_i$ with :

$$(\text{A.8}) \qquad \left\{ \begin{array}{l} r_1 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} -1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \\ \qquad\qquad r_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}. \end{array} \right.$$

For all $a \in \mathcal{C}_d$, z_a have in commun the six eigenvectors z_i, z_i^* in $\mathcal{C}_d^{\perp}$:

$$z_1 = \frac{1}{\sqrt{2}}\, e_{23}, \qquad z_2 = \frac{1}{\sqrt{2}}\, e_{31}, \qquad z_3 = \frac{1}{\sqrt{3}}\, e_{12}.$$

The vectors of

$$q_2, \quad r_2, \quad z_1, \quad z_1^*, \quad z_2, \quad z_2^*, \quad z_3, \quad z_3^*$$

form an orthonormal basis of $\mathbf{C}^8$ [with the Hermitian scalar product (III.49)].

Their relation to the Gell-Mann basis and the particle states are given in Table 2.

TABLE 2

Correspondance between 3×3 matrices and particles of the octet

Gell Mann's...	λ_8	λ_3	$\frac{\lambda_1 + i\lambda_2}{\sqrt{2}}$	$\frac{\lambda_1 - i\lambda_2}{\sqrt{2}}$	$\frac{\lambda_6 + i\lambda_7}{\sqrt{2}}$	$\frac{\lambda_6 - i\lambda_7}{\sqrt{2}}$	$\frac{\lambda_4 - i\lambda_5}{\sqrt{2}}$	$\frac{\lambda_4 + i\lambda_5}{\sqrt{2}}$
Our notation...	q_3	r_3	z_3	z_3^*	z_1	z_1^*	z_2	z_2^*
Octet $\left(\frac{1}{2}\right)^+$....	Λ^0	Σ^0	Σ^+	Σ^-	n	Ξ^0	Ξ^-	p^+
Octet 0^-......	η^0	π^0	π^+	π^-	K^0	$\overline{K}^0$	K^-	K^+
Octet 1^-......	φ'^0	ρ^0	ρ^+	ρ^-	K^{*0}	$\overline{K}^{*0}$	K^{*-}	K^{*+}

To respect the symmetry under the Weyl's group, i. e. the permutations of the index values 1, 2, 3, λ_3 should be replaced by $-\lambda_3$ in the Gell-Mann basis.

TABLE 3

Relations between vectors of complexified octet

$$(z_i, z_j) = 0, \qquad (z_i^*, z_j) = \delta_{ij}$$

$$iz_i \wedge z_j = \frac{1}{\sqrt{2}} \varepsilon_{ijk}\, z_k^*, \qquad iz_i \wedge z_j^* = r_i\, \delta_{ij'}, \qquad -iz_i \wedge z_j^* = \frac{1}{\sqrt{2}} \varepsilon_{ijk}\, z_k$$

$$z_i \vee z_i = 0, \qquad z_i \vee z_i^* = q_i$$

$$z_i \vee z_j = \sqrt{\frac{3}{2}}\, z_k^*, \qquad z_i \vee z_j^* = 0 \qquad (i \neq j \neq k \neq i)$$

$$[r_i, z_j, z_k] = 0, \qquad i\,[r_i, z_j, z_k^*] = \delta_{ij}\,\delta_{jk}$$

$$i\,[z_i, z_j, z_k] = \frac{1}{\sqrt{2}} \varepsilon_{ijk} = -\, i\,[z_i^*, z_j^*, z_k^*]$$

$$[z_i, z_j, z_k^*] = 0 = [z_i^*, z_j^*, z_k]$$

$$\{ z_i, z_j, z_k \} = \sqrt{\frac{3}{2}}\, |\, \varepsilon_{ijk}\, | = \{ z_i^*, z_j^*, z_k^* \}$$

$$\{ z_i, z_j, z_k^* \} = 0 = \{ z_i, z_j^*, z_k^* \}.$$

The last four lines give the values of the f and d coefficients in this basis.

TABLE 4

Physical Directions along which SU (3) *is broken*

Third direction of the isospin.....	$t_3 = r_3$
Hypercharge direction............	$y = q_3$
Electric charge direction..........	$q = -q_1$
Cabibbo's : direction of weak currents......	$\begin{cases} c_+ = z_3 \cos\theta + z_2^* \sin\theta \\ c_- = c_+^* = z_3^* \cos\theta + z_2 \sin\theta \end{cases}$
direction of weak hypercharge.	$c = 2\, c_+ {}_\wedge c_-$

Table 3 contains some mathematical relation which are useful in the applications of the formalism developped in this paper to physical problems. Table 4, summarizes the definitions of the directions along which SU (3) is broken. We finally list the relations between the vectors introduced in Table 4 :

$$q = \frac{1}{2}(y + \sqrt{3}\, t_3),$$

which is the Gell-Mann and Nishijima relation;

$$c = c_i {}_\vee c_i, \qquad c_3 = c_1 {}_\wedge c_2,$$
$$(y, c) = 1 - \frac{3}{2}\sin^2\theta$$

which were given by Cabibbo;

$$y_\vee q = y + q, \qquad c_\vee q = c + q,$$
$$2\,(1 - (y, c))\, q + 2\, y_\vee c + y + c = 0,$$

which were given in reference [2 *d*].

APPENDIX 2

In this appendix we give the necessary definitions and some results concerning the action of a group G on a set M.

An *action of* G *on* M is given by a homomorphism

$$\mathrm{G} \xrightarrow{f} \text{Permutation group on M.}$$

If Ker f is trivial, G acts *effectively* on M.

We will use latin letters for the elements of G, greek letters for those of M, and when there is no ambiguity we will use $x.\alpha$ instead of $f(x)[\alpha]$; e. g. $xy.\alpha = x.(y.\alpha)$ expresses the group law in the action.

An *orbit of* G is the set of $x.\alpha$ for all $x \in G$ and a fixed α. M is thus partitionned into orbits by the action of G. Two orbits E and E′ of G are said to be of the same type if there is a bijective (one-to-one onto) mapping $E \xrightarrow{\varphi} E'$ commuting with the group action i. e. for every $x \in G$, $\varphi \circ f(x) = f'(x) \circ \varphi$.

Given $\alpha \in M$, the elements $x \in G$ which leave α fixed : $x.\alpha = \alpha$, form a subgroup G_α of G, which is called the *isotropy group*, or little group of α. If α and β belong to the same orbit there is at least one $g \in G$ such that $\beta = g.\alpha$ so $G_\beta = g\, G_\alpha\, g^{-1}$, i. e. the *isotropy group of the points of an orbit are all conjugated.*

One can prove that if the isotropy groups of two points of M are conjugated, the orbit are of the same type (for this use as prototypes for orbits of isotropy groups conjugated to H, the cosets g H of H, the group acting by left translation $x.g\,H = x\,g.\,H$).

Thus if α and $\beta \in H$ have conjugated isotropy groups, even though they may not belong to the same orbit, they belong however to orbits of the same type.

We call *stratum* [7] of M the set of all points with the same isotropy group up to a conjugation. A stratum is thus the union of all orbits of a given type.

Many theorem exist for the differentiable actions of Lie groups on manifolds M. We use only one in this paper. If G is compact, there is one stratum on M which is open dense [8]. We will call it the generic stratum.

APPENDIX 3

Part II was written for SU (n) and part III for SU (3) only. We give here some hints for the generalization to SU (n) of the definitions and properties of r and q-vectors using the same notations of section II. The characteristic equation of a vector $x \in R^{n^2-1}$:

$$x^n - \gamma_2(x)\, x^{n-2} - \gamma_3(x)\, x^{n-3} \ldots - \gamma_n(x) = x^n - \sum_{k=2}^{n} \gamma_k\, x^{n-k} = 0$$

where the coefficients $\gamma_k(x)$ satisfy :

$$\gamma_k(x) = \frac{1}{k}\operatorname{tr}\left\{ x^k - \sum_{l=2}^{k-2} \gamma_l(x)\, x^{k-l} \right\};$$

$$\gamma_2(x) = \gamma(x) = \frac{1}{2}\operatorname{tr} x^2; \qquad \gamma_3(x) = \frac{1}{3}\operatorname{tr} x^3 = \frac{2}{3\sqrt{3}}\,\theta(x).$$

A unit r-vector is defined by

$$\gamma_2(r) = 1, \qquad \gamma_l(r) = 0, \qquad l > 2,$$

i. e. the eigenvalues of r are 1, -1 and 0 which appears $(n-2)$ times.

To each r-vector there corresponds for $n > 2$ a q-vector

$$r_\vee r = \sqrt{n-2}\, q.$$

The unit positive q-vectors satisfy :

$$q_\vee q = \frac{n-4}{\sqrt{n-2}} q, \qquad n > 2.$$

Given a maximal Abelian plane $\mathcal{C}_x$ i. e. a Cartan subalgebra of SU (n) (its dimension is $n-1$), the roots are its units r-vectors and the pseudoroots the positive unit q-vectors, i. e.

$$\forall\ a \in \mathcal{C}_x : \quad if_{a \wedge} z = (r_z, a) \neq z;$$
$$d_{a \vee} z = (r_{z \vee} r_z, a)\, z = \sqrt{n-2}\,(q_z, a)\, z.$$

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