ANNALES DE L'I. H. P., SECTION A

LUIS BEL

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Annales de l'I. H. P., section A, tome 18, nº 1 (1973), p. 57-75
<http://www.numdam.org/item?id=AIHPA_1973__18_1_57_0>

Ann. Inst. Henri Poincaré,
Vol. XVIII, n° 1, 1973, p. 57-76.

Section A :
Physique théorique.

Hamiltonian Poincaré Invariant Systems

by

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ABSTRACT. — In a preceding paper we proposed a new definition of Hamiltonian Poincaré Invariant Systems (H. P. I. S.). The main idea of this definition is to keep Poincaré transformations canonical and to require that the Poisson Brackets of relative physical positions vanish. In this paper we generalize to any number of particles one of our results. Namely, that for each H. P. I. S. the Induced System at the Center of Mass is lagrangian. We obtain also for $N = 2$ the general equations whose solutions would give all H. P. I. S.

INTRODUCTION

The Predictive [1] Relativistic Mechanics (P. R. M.) of N point-like structureless particles has been developed recently. In P. R. M., a system of N point-like particles is described by a Poincaré Invariant System (P. I. S.), i. e., an ordinary second order system of differential equations with definite invariance properties. Two equivalent formalisms have been used to write down these equations. In the Time-Symmetric formalism a P. I. S. is described by a newtonian-like system

[1] The word Predictive is here used in contradistinction to the word deterministic to emphasize that the physical systems being considered have a finite number of degrees of freedom.

of differential equations

$$\frac{dx_a^i}{dt} = v_a^i; \qquad \frac{dv_a^i}{dt} = \mu_a^i(x_b^j, v_c^k) \qquad (i, j, k = 1, 2, 3;\ a, b, c = 1, 2, \ldots, N)$$

where the μ_a^i's are restricted to satisfy an appropriate system of non linear partial differential equations (D. G. Currie [1], R. N. Hill [2], [3], L. Bel [4], [5]). In the Manifestly Covariant formalism the same physical system is described by a system of differential equations in Minkowski Space-Time :

$$\frac{dx_a^\alpha}{d\tau} = u_a^\alpha; \qquad \frac{du_a^\alpha}{d\tau} = \xi_a^\alpha(x_b^\beta, u_c^\gamma) \qquad (\alpha, \beta, \gamma = 0, 1, 2, 3)$$

where again the ξ_a^α's are restricted to satisfy appropriate equations (Ph. Droz-Vincent [6], L. Bel [5]). Both formalisms have advantages and inconveniences. In this paper we shall use the Time-Symmetric formalism only.

One of the unsolved problems of P. R. M. is to give a convenient definition of Hamiltonian P. I. S. (H. P. I. S.). The first and natural definition that was considered led to trivial systems only, i. e., to non interacting particles (D. G. Currie-T. F. Jordan-E. C. G. Sudarshan [7], J. T. Cannon-T. F. Jordan [8], H. Leutwyler [9], R. N. Hill [3]). The root of the difficulty was first identified by E. H. Kerner [10], and clearly exhibited by R. N. Hill [3], as lying in the assumption that the x_a^i's, i. e., the physical positions, were assumed to have vanishing Poisson-Brackets (P. B.) among themselves. Several new definitions which permit to circumvent this « No Interaction » theorem have already been considered. We may mention those of R. N. Hill-E. H. Kerner [11], L. Bel ([5], [12]) and C. Frondsdal [13]. But we believe it is safe to say that the status of the problem is still confusing and that general agreement on a convenient definition of H. P. I. S. has not yet been reached.

The main idea in the definition proposed in reference [12] was to keep Poincaré transformations canonical while dropping the commutation of the x_a^i's and requiring only that the P. B.'s of the relative positions among themselves vanish :

$$[x_a^i - x_b^i,\ x_c^j - x_d^j] = 0.$$

This paper deals again with the implications of this definition. We generalize to any N and simplify the proof of the theorem of reference [12]. Namely, that for each H. P. I. S. the Induced System at the Center of Mass in lagrangian. We establish for N = 2, the general equations whose solutions would give all H. P. I. S. in the sense of our definition, and we prove that, at the Center of Mass System, the Center of Mass lies on the line joining the positions of the two particles and that the motion of the system is plane motion.

1. POINCARÉ INVARIANT SYSTEMS

Let $\mathfrak{L}$ be the Lie algebra of the Poincaré group and let H, P_i, J_i, K be a basis of P such that (2) :

$$(1) \qquad [P_i, H] = 0, \qquad [J_i, H] = 0, \qquad [K_i, H] = P_i;$$

$$(2) \qquad \begin{cases} [P_i, P_j] = 0, & [J_i, P_j] = \eta_{ijk} P^k, \quad [J_i, J_j] = \eta_{ijk} J^k, \\ [K_i, P_j] = \delta_{ij} H, & [K_i, J_j] = \eta_{ijk} K^k, \quad [K_i, K_j] = -\eta_{ijk} J^k. \end{cases}$$

Let E_3 be the euclidean 3-dimensional space and $\hat{E}_6$ the fiber tangent bundle of all vectors $\vec{v}$ tangent to E_3 and with norm less than 1 $(\vec{v}^2 < 1)$. We shall note V_{6N}, and call the co-phase space, the 6 N-dimensional manifold : $V_{6N} = (\hat{E}_6)^N$. For each cartesian system of coordinates of E_3, trivially extended to V_{6N}, each point A of V_{6N} will have coordinates (x^i_a, v^j_b) (3), the x^i_a's being the coordinates of the projection of A into the basis of V_{6N}, i. e., $(E_3)^N$.

Let $\mathcal{L}(V_{6N})$ be the infinite dimensional Lie algebra of all vector fields of V_{6N}. If $\vec{\Xi}_1$ and $\vec{\Xi}_2$ belong to $\mathcal{L}(V_{6N})$, their Lie-Bracket (L. B.) is defined by

$$(3) \qquad \left[\vec{\Xi}_1, \vec{\Xi}_2\right]^A = \Xi^B_1 \partial_B \Xi^A_2 - \Xi^B_2 \partial_B \Xi^A_1 \qquad \left(\partial_B \equiv \frac{\partial}{\partial y^B}\right)$$

where y^A (A, B, ... = 1, 2, ..., 6 N) is an arbitrary coordinate system of V_{6N} (5).

DEFINITION 1. — A Poincaré Invariant System (P. I. S.), with 6 N degrees of freedom, is an homomorphism ψ of $\mathfrak{L}$ into the infinite dimensional Lie algebra $\mathcal{L}(V_{6N})$ such that

$$(4) \qquad \mathcal{L}(\vec{P}_j)\, x^i_a = -\delta^i_j, \qquad \mathcal{L}(\vec{J}_j)\, x^i_a = \eta_{j\cdot k}^{\ i}\, x^k_a,$$

$$(5) \qquad \mathcal{L}(\vec{H})\, x^i_a = -v^i_a,$$

$$(6) \qquad \mathcal{L}(\vec{K}_j)\, x^i_a = -v^i_a\, x_{aj}.$$

(2) $i, j, k = 1, 2, 3$. The summation convention will be used throughout. The latin indices will be raised or lowered without change of sign. η_{ijk} is the Levi-Civitta symbol. The speed of light in vacuum is taken equal to 1.

(3) $a, b, c, \ldots = 1, 2, \ldots, N$.

(4) The summation convention is also used for the indices A, B, C, ...

(5) A possible choice of y^A is $y^i = x^i_1, \ldots, y^{3N-3+i} = x^i_N$,
$$y^{3N+i} = v^i_1, \quad \ldots, \quad y^{6N-3+i} = v^i_N.$$

$\vec{H}$, $\vec{P}_i$, $\vec{J}_i$, $\vec{K}_i$, being the images by ψ of H, P_i, J_i, K_i, and $\mathcal{L}$ being the Lie derivative operator acting on functions, i. e., if $\vec{\Xi} \in \mathcal{L}(V_{6N})$ and Φ is a function

$$(7) \qquad \mathcal{L}\left(\vec{\Xi}\right)\Phi = \Xi^A \partial_A \Phi.$$

When necessary we shall refer to (1) and (2) to remind the Lie Bracket relations of the vector fields $\vec{H}$, $\vec{P}_i$, $\vec{J}_i$, $\vec{K}_i$.

Let us define μ^i_a as

$$(8) \qquad \mathcal{L}(H)\, v^i_a \equiv -\mu^i_a.$$

From (1), (4), (5), (6), and from

$$(9) \qquad \mathcal{L}\left(\vec{\Xi}_1\right)\mathcal{L}\left(\vec{\Xi}_2\right) - \mathcal{L}\left(\vec{\Xi}_2\right)\mathcal{L}\left(\vec{\Xi}_1\right) = \mathcal{L}\left(\left[\vec{\Xi}_1, \vec{\Xi}_2\right]\right)$$

we obtain

$$(10) \qquad \begin{cases} \mathcal{L}\left(\vec{P}_j\right) v^i_a = 0, \qquad \mathcal{L}\left(\vec{J}_j\right) v^i_a = \eta_{j.k}^{\ i}\, v^k_a, \\ \mathcal{L}\left(\vec{K}_j\right) v^i_a = \delta^i_j - \mu^i_a\, v_{aj} - v^i_a\, v_{aj}. \end{cases}$$

Therefore, the vector fields $\vec{H}$, $\vec{P}_i$, $\vec{J}_i$, $\vec{K}_i$ will be known as soon as we know the functions μ^i_a.

These functions are restricted by equations (2) to be solutions of the following system of differential equations ([6]) :

$$(11) \qquad \begin{cases} \left[\varepsilon_a \dfrac{\partial \mu^i_b}{\partial x^j_a} = 0; \qquad \eta^s_{.kj}\left(x^k_a \dfrac{\partial \mu^i_b}{\partial x^s_a} + v^k_a \dfrac{\partial \mu^i_b}{\partial v^s_a}\right) = \eta^i_{.kj}\, \mu^k_b;\right] \\ v^s_a\,(x^j_a - x^j_b)\dfrac{\partial \mu^i_b}{\partial x^s_a} + [v^s_a\, v_{aj} + \mu^s_a\,(x_{aj} - x_{bj}) - \varepsilon_a\, \delta^s_r]\dfrac{\partial \mu^i_b}{\partial v^s_a} \\ = 2\,\mu^i_b\, v_{bj} + v^i_b\, \mu_{bj} \end{cases}$$

where $\varepsilon_a = 1$.

For each solution of these equations, the system of second order ordinary differential equations

$$(12) \qquad \frac{dx^i_a}{dt} = v^i_a, \qquad \frac{dv^i_a}{dt} = \mu^i_a\,(x^i_b, v^k_c)$$

can be interpreted as the equations of motion of N point-like structureless particles in P. R. M. ([1], [2], [3], [4], [5]).

([6]) The summation convention is used also for the indices *a*, *b*, *c*.

2. HAMILTONIAN P. I. S.

a. Let σ be a sympletic form on V_{6N}, i. e., a 2-form (second rank covariant skew symmetric tensor $\sigma_{AB} = -\sigma_{AB}$) with det $\sigma \neq 0$ and such that (7) :

(13) $$d\sigma = 0 \qquad \text{or} \qquad \partial_A \sigma_{BC} + \partial_B \sigma_{CA} + \partial_C \sigma_{AB} = 0.$$

The following definitions and results of symplectic geometry will be used below ([14], [15], [16]).

DARBOUX THEOREM. — *For any given point of* V_{6N} *there always exists a local coordinate system, whose domain contains this point, such that* σ *takes the form* (8) :

(14) $$\sigma = dy^\alpha \wedge dy_{N+\alpha}$$

where $\alpha = 1, \ldots, N$. *Any system of coordinates for which* σ *takes this form is called a canonical system of coordinates.* y^α *and* $y_{N+\alpha}$ *are said to be canonical mates.*

The P. B. of two functions Φ and Ξ with respect to σ is by definition the function

(15) $$[\Phi, \Xi] = -\sigma^{-1AB}\,\partial_A \Phi\, \partial_B \Xi,$$

σ^{-1} being defined by $\sigma^{-1AC}\sigma_{CB} = \delta^A_B$. Using canonical coordinates the explicit expression for (15) is the usual one, namely :

(16) $$[\Phi, \Xi] = \frac{\partial \Phi}{\partial y^\alpha}\frac{\partial \Xi}{\partial y_{N+\alpha}} - \frac{\partial \Xi}{\partial y^\alpha}\frac{\partial \Phi}{\partial y_{N+\alpha}}.$$

The P. B. is anticommutative :

(17) $$[\Phi, \Xi] = -[\Xi, \Phi]$$

and as a direct consequence of (13) we have for any three functions

(18) $$[\Phi_1, [\Phi_2, \Phi_3]] + [\Phi_2, [\Phi_3\, \Phi_1]] + [\Phi_3, [\Phi_1, \Phi_2]] = 0.$$

These equations being known as the Jacobi identities.

(7) In general, for any p-form Ω (covariant completely skew symmetric tensor of rank p), the exterior differential operator d is defined by

$$(d\Omega)_{A_0 A_1 \ldots A_p} = \frac{1}{p\,!}\,\delta^{CB_1\ldots B_p}_{A_0 A_1 \ldots A_p}\,\partial_C\, \Omega_{B_1\ldots B_p}$$

where δ is the kronecker tensor. The operator d has the property $d^2 = 0$.

(8) In general if Σ is a p-form and Ω a q-form, their exterior product is the $q+p$-form

$$(\Sigma \wedge \Omega)_{A_1\ldots A_{p+q}} = \frac{1}{p\,!\,q\,!}\,\delta^{B_1\ldots B_p C_1 \ldots\ldots C_q}_{A_1\ldots A_p A_{p+1}\ldots A_{p+q}}\,\Sigma_{B_1\ldots B_p}\,\Omega_{C_1\ldots C_q}.$$

If Ξ is a function of several other functions Ψ_ν, then

$$(19) \qquad [\Phi, \Xi] = \frac{\partial \Xi}{\partial \Psi_\nu}[\Phi, \Psi_\nu], \quad \text{with summation on } \nu.$$

LEMMA. — *Given* N *independent functions* y^α *such that*

$$(20) \qquad [y^\alpha, y^\beta] = 0.$$

then there always exists N *canonical mates* $y_{N+\alpha}$, *i. e.*, N *new functions such that* $(y^\alpha, y_{N+\alpha})$ *is a coordinate system of* V_{6N} *and such that* σ *takes the form* (14). *These functions* $y_{N+\alpha}$ *are only defined up to the transformation*

$$y'_{N+\alpha} = y_{N+\alpha} + \frac{\partial S(y^\beta)}{\partial y^\alpha}.$$

Let us consider a transformation of V_{6N},

$$(21) \qquad y^{A'} = F^{A'}(y^B); \qquad y^B = G^B(y^{A'}).$$

This transformation is by definition a canonical transformation if it leaves invariant σ. Or more explicitly if

$$(22) \qquad \sigma_{A'B'}(y^{C'}) \equiv \partial_{A'} G^M \partial_{B'} G^N \sigma_{MN}[G^R(y^{S'})] = \sigma_{AB}(y^{C'})$$

with $A' = A, B' = B$.

For $\vec{\Xi}$ to be the generator of a one parameter group of canonical transformations the necessary and sufficient condition is that [9] :

$$(23) \quad \mathcal{L}(\vec{\Xi})\sigma = 0 \qquad \text{or} \qquad \Xi^C \partial_C \sigma_{AB} + \sigma_{CB} \partial_A \Xi^C + \sigma_{AC} \partial_B \Xi^C = 0.$$

If $\vec{\Xi}$ satisfies equation (23), then there exists a function Ξ such that

$$(24) \qquad i(\vec{\Xi})\sigma = -d\Xi \qquad \text{or} \qquad \Xi^A \sigma_{AB} = -\partial_B \Xi.$$

Obviously, this function Ξ is defined only up to an additive constant. Formula (24) can also be read from right to left, i. e., for each function Ξ, the vector $\vec{\Xi}$ whose components are solutions of (24) satisfies (23).

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[9] Given a p-form Ω and a vector field $\vec{\Xi}$ a useful formula to know is

$$\mathcal{L}(\vec{\Xi})\Omega = i(\vec{\Xi})d\Omega + di(\vec{\Xi})\Omega$$

where $i(\vec{\Xi})$ is the interior product operator defined by

$$\left[i(\vec{\Xi})\Omega\right]_{A_2 \ldots A_p} = \Xi^B \Omega_{BA_2 \ldots A_p}.$$

From this compact formula for the Lie derivarive of a p-form, and from $d^2 = 0$ it is trivial to see that $\mathcal{L}(\vec{\Xi})d\Omega = d\mathcal{L}(\vec{\Xi})\Omega$.

If $\vec{\Xi}$ satisfies (23) and Φ is any function then

$$\mathcal{L}\left(\vec{\Xi}\right)\Phi = [\Xi, \Phi]. \tag{25}$$

If $\vec{\Xi}_1$ and $\vec{\Xi}_2$ satisfy (23) then

$$\mathcal{L}\left(\left[\vec{\Xi}_1, \vec{\Xi}_2\right]\right)\sigma = 0 \quad \text{and} \quad i\left(\left[\vec{\Xi}_1, \vec{\Xi}_2\right]\right)\sigma = -d[\Xi_1, \Xi_2]. \tag{26}$$

b. For obvious reasons, which include preparation towards a quantized theory or relativistic statistical mechanics, P. R. M. will have to be developed by giving an appropriate definition of Hamiltonian P. I. S. (H. P. I. S.). A natural definition would be the following :

An H. P. I. S. is a P. I. S. such that there exists a symplectic form σ satisfying the following conditions

$$\mathcal{L}\left(\vec{\mathrm{H}}\right)\sigma = 0, \quad \mathcal{L}\left(\vec{\mathrm{P}}_i\right)\sigma = 0, \quad \mathcal{L}\left(\vec{\mathrm{J}}_i\right)\sigma = 0, \quad \mathcal{L}\left(\vec{\mathrm{K}}_i\right)\sigma = 0; \tag{27}$$

$$\sigma \wedge dx_1^1 \wedge dx_1^2 \wedge dx_1^3 \wedge \ldots \wedge dx_{\mathrm{N}}^1 \wedge dx_{\mathrm{N}}^2 \wedge dx_{\mathrm{N}}^3 = 0. \tag{28}$$

Conditions (27) express the Poincaré invariance of σ. In other words, they express that the Poincaré transformations generated by $\vec{\mathrm{H}}$, $\vec{\mathrm{P}}_i$, $\vec{\mathrm{J}}_i$ and $\vec{\mathrm{K}}_i$ are canonical transformations. Conditions (28) are equivalent to saying that the P. B. of the x_a^i's among themselves vanish :

$$[x_a^i, x_b^j] = 0. \tag{29}$$

Unfortunately, as has been proven by many people ([7], [8], [9], [3]), the only P. I. S.'s for which a symplectic form σ can be found satisfying (27) and (28) are those with $\mu_a^i = 0$, i. e., the free, non-interacting, particle systems.

c. It becomes then necessary to find a new definition of H. P. I. S. The first new one to be proposed by T. N. Hill-E. H. Kerner [11] is just, in our opinion, a boundary condition to be required, if possible, for infinite inter-particle separation; the second one proposed by L. Bel [5] does not convey enough information to be useful either. We still believe that the definition we proposed in [12] does convey enough information and this paper is essentially devoted to develop the consequences of this definition.

Definition. — An H. P. I. S. is a P. I. S. for which there exists a symplectic form satisfying the following two sets of conditions

$$\mathcal{L}\left(\vec{\mathrm{H}}\right)\sigma = 0, \quad \mathcal{L}\left(\vec{\mathrm{P}}_i\right)\sigma = 0, \quad \mathcal{L}\left(\vec{\mathrm{J}}_i\right)\sigma = 0, \quad \mathcal{L}\left(\vec{\mathrm{K}}_i\right)\sigma = 0; \tag{30}$$

$$[x_a^i - x_b^i,\ x_c^j - x_d^j] = 0 \quad \text{or} \quad [x_a^i - x_b^i,\ x_a^j - x_d^j] = 0. \tag{31}$$

The choice of the numerical value of a, in the second writing, being, of course, arbitrary.

The idea is, thus, to keep the Poincaré transformations as canonical transformations and to maintain only part of the restrictions contained in (29).

Remark 1. — We may notice that in the 1-dimensional space case, for two particles, the only P. B. (31) to be considered is

$$[x_1 - x_2,\ x_1 - x_2] \equiv 0$$

which vanishes identically. Therefore, in this case, the second set of conditions (31) are not restrictions at all. This case which is a very degenerate one, has been extensively considered by R. N. Hill ([3], [17]).

d. Among the first consequences which can be derived from the Definition above, we may mention the following : using (24) we can associate to each of the vector fields $\vec{H}$, $\vec{P}_i$, $\vec{J}_i$, $\vec{K}_i$, a corresponding function H, P_i, J_i, K_i such that the P. B. of these functions are exactly those of (1) and (2), i. e., (1) and (2) without the neutral elements which would be in general present because of the arbitrariness in the additive constants mentioned above. Thus, we shall refer again to (1) and (2) to remind the P. B. relations of the functions H, P_i, J_i, K_i.

From (4), (5), (6), (25) and (30) it follows that

$$(32)\qquad [x^i_a, P_j] = \delta^i_j, \qquad [x^i_a, J_j] = \eta^{\ i}_{.jk}\, x^k_a;$$

$$(33)\qquad [x^i_a, H] = v^i_a;$$

$$(34)\qquad [x^i_a, K_j] = v^i_a\, x_{aj};$$

H, P_i, J_i will be, of course, interpreted as the Total Energy, the Total Momentum, and the total Angular Momentum. And

$$(35)\qquad R^i = H^{-1} K^i$$

will be interpreted as the coordinates of the Center of Mass.

We shall assume that

$$x^i_{(\varkappa)} = x^i_1 - x^i_{(\varkappa)}, \qquad v^i_{(\varkappa)} = v^i_1 - v^i_{(\varkappa)} \qquad (\varkappa, \lambda = 2, \ldots, N)$$

together with P_j and R^i is an admissible coordinate system of V_{6N}. It might be possible to prove this assumption. But were it not, it would have to be introduced if we want, as it is the case, the dynamical systems we are considering to have a newtonian limit.

3. THE INDUCED LAGRANGIAN SYSTEM

a. Let us consider an H. P. I. S. From (31) and (32) we have

$$[x^i_{(\varkappa)}, x^j_{(\lambda)}] = 0, \qquad [x^i_{(\varkappa)}, P_j] = 0 \tag{36}$$

where

$$x^i_{(\varkappa)} = x^i_1 - x^i_{(\varkappa)} \qquad (\varkappa, \lambda = 2, \ldots, N). \tag{37}$$

According to the Lemma of paragraph 2 *a*, there exists canonical mates of these 2 N coordinates, $x^i_{(\varkappa)}$, P_j, that we shall note $\pi^{(\varkappa)}_i$ and Q^j. In terms of these 6 N coordinates, σ will take the form

$$\sigma = dx^i_{(\varkappa)} \wedge d\pi^{(\varkappa)}_i + dQ^i \wedge dP_i. \tag{38}$$

From

$$i(\vec{P}_j)\,\sigma = -\,dP_j \tag{39}$$

we obtain

$$\mathcal{L}(\vec{P})\,\pi^{(\varkappa)}_i = 0, \qquad \mathcal{L}(P_j)\,Q^i = -\,\delta^i_j. \tag{40}$$

On the other hand, from the third equation (30) and from the fact that $\pi^{(\varkappa)}_i$ and Q^i are defined only up to a transformation

$$\pi'^{(\varkappa)}_i = \pi^{(\varkappa)}_i + \frac{\partial S}{\partial x^i_{(\varkappa)}}(x^j_{(\lambda)}, P_k); \qquad Q'^i = Q^i + \frac{\partial S}{\partial P_i}(x^j_{(\lambda)}, P_k), \tag{41}$$

it can be seen that no generality is lost by assuming that

$$\mathcal{L}(\vec{J}_j)\,\pi^{(\varkappa)}_i = \eta_{ji.}{}^k\,\pi^{(\varkappa)}_k, \qquad \mathcal{L}(\vec{J}_j)\,Q^i = \eta_{j.k}{}^i\,Q^k. \tag{42}$$

The first set of equations (40) express that $\pi^{(\varkappa)}_i$ are functions of $x^i_{(\varkappa)}$ and v^i_a only.

b. From (12) we have

$$\frac{dx^i_{(\varkappa)}}{dt} = v^i_{(\varkappa)} \equiv v^i_1 - v^i_{(\varkappa)}, \qquad \frac{dv^i_{(\varkappa)}}{dt} = \mu^i_{(\varkappa)} \equiv \mu^i_1 - {}^i_{(\varkappa)} \tag{43}$$

where $\mu^i_{(\varkappa)}$ as the μ^i_a's, are, as a consequence of the first set of equations (11), functions of $x^i_{(\varkappa)}$ and v^i_a and may therefore be considered as functions of $x^i_{(\varkappa)}$, $v^i_{(\varkappa)}$ and P_j. But since from (1) we know that the P_j's are constants of motion it follows that the solutions of (12) with initial conditions such that $P_i = 0$ will satisfy the system differential equations

$$\frac{dx^i_{(\chi)}}{dt} = v^i_{(\chi)}, \qquad \frac{dv^i_{(\chi)}}{dt} = \mu^{*i}_{(\chi)} \tag{44}$$

where the $\mu^{*i}_{(\chi)}$'s are the functions of $x^i_{(\chi)}$ and $v^i_{(\chi)}$ obtained from the $\mu^i_{(\chi)}$'s by putting in these $P_i = 0$. We shall call the system (44). The Induced System at the Center of Mass.

d. We shall prove now the following theorem :

THEOREM. — *For each H. P. I. S. the Induced System at Center of Mass is Lagrangian.*

To prove that system (44) is Lagrangian we have to prove that there exists a symplectic form σ^* :

$$\sigma^* = dx^i_{(\chi)} \wedge d\pi^{*(\chi)}_i \tag{45}$$

where the $\pi^{*(\chi)}_i$ are functions of $x^i_{(\chi)}$ and $v^i_{(\chi)}$, and such that

$$\mathcal{L}\left(\vec{H}^*\right)\sigma^* = 0 \tag{46}$$

where $\vec{H}^*$ is the vector field of $V_{6(N-1)}$, the cophase space of the system, defined by

$$\mathcal{L}\left(\vec{H}^*\right)x^i_{(\chi)} = -\,v^i_{(\chi)}, \qquad \mathcal{L}\left(\vec{H}^*\right)v^i_{(\chi)} = -\,\mu^{*i}_{(\chi)}. \tag{47}$$

Let us prove that equations (46) are satisfied if we define $\pi^{*(\chi)}_i$ as the functions obtained from the $\pi^{(\chi)}_i$ of (38), which can be considered as functions of $x^i_{(\chi)}$, $v^i_{(\chi)}$ and P_j, by putting in these $P_j = 0$, i. e., by defining σ^* as the symplectic form induced by σ on any of the hyper-surfaces Σ_0 with equations

$$\Sigma_0 : \quad R^i = C^i \text{ (consts.)}, \qquad P_j = 0. \tag{48}$$

Let us consider, in general, the family of hyper-surfaces with equations

$$\Sigma : \quad R^i = C^i, \qquad P_j = P_j \text{ (consts.)}. \tag{49}$$

The vector field $\vec{H}$ can be written as

$$\vec{H} = \vec{H}_L + \vec{H}_T \tag{50}$$

where $\vec{H}_L$ is tangent, at each point, to one of the hyper-surface Σ and H_T is transversal. Let us write, on the other hand, the symplectic form (38) as

$$\sigma \equiv \sigma_L + \sigma_T \tag{51}$$

with

$$\sigma_L \equiv dx^i_{(x)} \wedge d\pi_i^{(x)}, \qquad \sigma_T \equiv dQ^i \wedge dP_i. \tag{52}$$

From the first set of equations (30), we have

$$\mathcal{L}(\vec{H})\sigma = \mathcal{L}(\vec{H}_L)\sigma_L + \mathcal{L}(\vec{H}_T)\sigma_L + \mathcal{L}(\vec{H}_L)\sigma_T + \mathcal{L}(\vec{H}_T)\sigma_T = 0. \tag{53}$$

But from

$$\mathcal{L}(\vec{H})P_i = 0, \qquad \mathcal{L}(\vec{H})R^i = H^{-1}P^i \tag{54}$$

it follows that on Σ_0 :

$$\vec{H}_{T\Sigma_0} = 0, \qquad \vec{H}_{L\Sigma_0} = \vec{H}^*. \tag{55}$$

On the other hand

$$\sigma_{L\Sigma_0} = \sigma^*. \tag{56}$$

Therefore, considering equation (53) on Σ_0, we obtain

$$\mathcal{L}(\vec{H}^*)\sigma^* + \{\mathcal{L}(\vec{H}^*)dQ^i\}_{\Sigma_0} \wedge dP_i = 0. \tag{57}$$

And since the first term does not contain any term with a dP_i product, both terms must be null. Formula (46), and thus our theorem, have been proved.

Let us consider now more precisely the hypersurface Σ^* with equations

$$\Sigma^* : \quad R^i = 0, \qquad P_i = 0. \tag{58}$$

From

$$\mathcal{L}(\vec{J}_j)R^i = \eta_{j.k}^{\ i}R^k, \qquad \mathcal{L}(\vec{J}_i)P_i = \eta_{ji.}^{\ \ k}P_k \tag{59}$$

it follows, using similar notations as before, that on Σ^* :

$$\vec{J}_{jT\Sigma^*} = 0, \qquad \vec{J}_{jL\Sigma^*} = \vec{J}_{j*} \tag{60}$$

where $\vec{J}_j^*$ is the vector field of $V_{6(N-1)}$, identified to Σ^*, given by the relations

$$\mathcal{L}(\vec{J}_i^*)x^i_{(x)} = \eta_{j.k}^{\ i}x^k_{(x)}, \qquad \mathcal{L}(\vec{J}_j^*)v^i_{(x)} = \eta_{j.k}^{\ i}v^k_{(x)}. \tag{61}$$

By similar arguments to those used before, it is easy to prove that

$$\mathcal{L}(\vec{J}_j^*)\sigma^* = 0. \tag{62}$$

Let H^* and J_j^* be the Energy and the Angular Momentum of the Induced System (44). Since H, the Total Energy of the original system, and H* are given by

$$(63)\qquad i(\vec{H})\,\sigma = -\,dH, \qquad i(\vec{H}^*)\,\sigma^* = -\,dH^*$$

using the decompositions (51) and (52), together with (55) and (56), we obtain

$$(64)\qquad i(\vec{H}^*)\,\sigma^* + i(\vec{H}^*)\,\sigma_{T\Sigma^*} = -\,d_L H_{\Sigma^*} - \{ d_T H \}_{\Sigma^*}$$

where

$$(65)\qquad d_L \equiv \frac{\partial}{\partial x^i_{(\chi)}} dx^i_{(\chi)} + \frac{\partial}{\partial v^i_{(\chi)}} dv^i_{(\chi)}, \qquad d_T \equiv \frac{\partial}{\partial R^i} dR^i + \frac{\partial}{\partial P_i} dP_i$$

Therefore, up to an additive constant :

$$(66)\qquad H^* = H_{\Sigma^*}.$$

And similarly, we could prove that

$$(67)\qquad \vec{J}^*_j = J_{j\Sigma^*}.$$

4. GENERAL EQUATIONS OF H. P. I. S. FOR N = 2

a. From now on we restrict ourselves to the simplest case where N, the number of particles involved in the system, is 2. The problem of constructing H. P. I. S. is equivalent in this case to the problem of obtaining ten functions H, P_i, J_i, K_i and six functions x^i_a $(a = 1, 2)$ of the 12 variables x^i, Q^i, π_i and P_i such that :

1. $x^i_1 - x^i_2 = x^i$.

2. The Poisson brackets (16) of these functions among themselves satisfy the commutation relations (1) and (2).

3. The Poisson brackets of x^i_a with P_j, J_j and K_j satisfy the commutation relations (32) and (34), where in the latter the v^i_a's are defined by (33).

In fact, let us assume that we have found such functions x^i_a and H, P_i, J_i, K_i satisfying the conditions 1, 2 and 3 above. To the latter, we can associate, using formula (24) and the symplectic form

$$(68)\qquad \sigma = dx^i \wedge d\pi_i + dQ^i \wedge dP_i,$$

ten vector fields $\vec{H}$, $\vec{J}_i$, $\vec{P}_i$, $\vec{K}_i$ who by construction will satisfy equations (1), (2), (4), (5), (6), and (30). On the other hand equations (31) will be satisfied as a consequence of condition 1 above, and the particular form (68) of σ. Therefore, all the conditions of paragraphs 1 and 2 involved in the Definition of H. P. I. S. will be satisfied.

Were it necessary to construct the functions μ^i_a's of (12) they could be obtained from

$$\mu^i_a = [x^i_a, [x^i_a, H]] \tag{69}$$

by writting these P. B. in terms of x^i_a and v^i_a.

b. From (68) we have

$$\left\{ \begin{array}{llll} [x^i, x^j] = 0, & [x^i, \pi_j] = \delta^i_j, & [x^i, Q^j] = 0, & [x^i, P_j] = 0; \\ [Q^i, Q^j] = 0, & [Q^i, P_j] = \delta^i_j, & [\pi_i, P_j] = 0, & [P_i, P_j] = 0. \end{array} \right. \tag{70}$$

From (42) and from similar equations, satisfied by x^i and P_j, we obtain that the functions J_i are

$$J_i = \eta_{ijk} (x^j \pi^k + Q^j P^k). \tag{71}$$

From $[P_i, H] = 0$, we obtain

$$\frac{\partial H}{\partial Q^i} = 0 \tag{72}$$

and from $[K_i, P_j] = \delta_{ij} H$ we get

$$K^i = H (Q^i + f^i) \tag{73}$$

where the f^i's are independent of Q^j :

$$\frac{\partial f^i}{\partial Q^j} = 0. \tag{74}$$

From $[K_i, H] = P_i$ and (73) we obtain

$$\frac{\partial H}{\partial P^i} = H^{-1} P_i - [f_i, H] \tag{75}$$

and from $[K_i, K_j] = - \eta_{ijk} J^k$, (73) and (71) we get

$$\frac{\partial f_j}{\partial P^i} - \frac{\partial f_i}{\partial P^j} = - [f_i, f_j] + H^{-2} (f_i P_j - f_j P_i - S_{ij}) \tag{76}$$

where

$$S_{ij} = x_i \pi_j - x_j \pi_i. \tag{77}$$

Equations $[J_i, H] = 0$ tell us that H is a scalar function and this can be satisfied, equivalently, by considering H as a function of the scalars :

$$\left\{ \begin{array}{lll} x = (x_i x^i)^{1/2}, & \pi = (\pi_i \pi^i)^{1/2}, & P = (P_i P^i)^{1/2}, \\ (x \pi) = x^i \pi_i, & (x P) = x^i P_i, & (\pi P) = \pi_i P^i. \end{array} \right. \tag{78}$$

Equations $[J_i, K_i] = \eta_{ijk} K^k$, taking into account (73), give

$$[J_i, f^j] = \eta_{i\cdot k}^{\ j} f^k \tag{79}$$

and these together with (74) tell us that f^j are the components of a vector independent of Q^i. Or equivalently, that f^i may be written as

$$f^i = f_{(1)}\, x^i + f_{(2)}\, \pi^i + f_{(3)}\, P^i \tag{80}$$

where $f_{(j)}$ are functions of the six scalars (78).

c. From (32) we get

$$x^i_a = Q^i + \rho^i_a \tag{81}$$

with

$$\frac{\partial \rho^i_a}{\partial Q^j} = 0; \qquad [J_j, \rho^i_a] = \eta_j{}^i{}_{.k}\, \rho^k_a, \tag{82}$$

equations that tell us that ρ^i_a are the components of vectors, independent of Q^i. To meet condition 1 above, we must have

$$\rho^i_1 - \rho^i_2 = X^i. \tag{83}$$

From (34), where the v^i_a's are defined by (33), we obtain, taking into account (73) and (81) :

$$\frac{\partial f_j}{\partial P^i} - \frac{\partial \rho_{ai}}{\partial P^j} + [\rho_{ai}, f_j] = H^{-1}\,(H^{-1}\, P_i + [\rho_{ai} - f_i, H])\,(\rho_{aj} - f_j). \tag{84}$$

These equations have to hold for $a = 1$ and 2.

Using equation (76), equation (84) with $a = 2$ can be written

$$\frac{\partial \alpha_j}{\partial P^i} = [\alpha_j, f_i] - H^{-1}\,[\alpha_j, H]\,\alpha_i - H^{-2}\,(f_j\, P_i - f_i\, P_j + \alpha_i\, P_j - S_{ji}) \tag{85}$$

where

$$\alpha_j \equiv \rho_{2j} - f_j. \tag{86}$$

And substracting equation (84) with $a = 2$ from the same equation with $a = 1$, we obtain

$$\frac{\partial f_j}{\partial \pi_i} = H^{-2}\, P_i\, x_j + H^{-1} \left\{ \frac{\partial H}{\partial \pi^i}(\alpha_j + x_j) + [\alpha_i, H]\, x_j \right\}. \tag{87}$$

We may summarize the preceding results as follows :

To construct H. P. I. S. with $N = 2$ we need to obtain functions H, f_i and α_j of x^i, π_j and P_k, solutions of equations (75), (76), (85) and (87). f^i must have the form (80); α^i must have similarly, as follows from (79), (82) and (86), the form

$$\alpha^j = \alpha_{(1)}\, x^j + \alpha_{(2)}\, \pi^j + \alpha_{(3)}\, P^j \tag{88}$$

and H, $f_{(i)}$ and $\alpha_{(j)}$ must be functions of the six scalars (78). Thus we have, all together, seven unknown functions of six variables.

5. THE INTEGRABILITY CONDITIONS

a. Obvious necessary conditions which could follow from equations (75), (76), (85) and (87) would be the conditions expressing the integrability conditions :

$$(89)\quad \begin{cases} \dfrac{\partial}{\partial P^j}\left(\dfrac{\partial H}{\partial P^i}\right) - \dfrac{\partial}{\partial P^i}\left(\dfrac{\partial H}{\partial P^j}\right) = 0, \\ \dfrac{\partial}{\partial P^k}\left(\dfrac{\partial f_j}{\partial P^i} - \dfrac{\partial f_i}{\partial P^j}\right) + \dfrac{\partial}{\partial P^j}\left(\dfrac{\partial f_i}{\partial P^k} - \dfrac{\partial f_k}{\partial P^i}\right) + \dfrac{\partial}{\partial P^i}\left(\dfrac{\partial f_k}{\partial P^j} - \dfrac{\partial f_j}{\partial P^k}\right) = 0, \\ \dfrac{\partial}{\partial P^k}\left(\dfrac{\partial \alpha_j}{\partial P^i}\right) - \dfrac{\partial}{\partial P^i}\left(\dfrac{\partial \alpha_j}{\partial P^k}\right) = 0, \\ \dfrac{\partial}{\partial \pi^i}\left(\dfrac{\partial f_j}{\partial P^k} - \dfrac{\partial f_k}{\partial P^j}\right) - \dfrac{\partial}{\partial P^k}\left(\dfrac{\partial f_j}{\partial \pi^i}\right) + \dfrac{\partial}{\partial P^j}\left(\dfrac{\partial P_k}{\partial \pi^i}\right) = 0. \end{cases}$$

We have in hand all the elements to effectively calculate these integrability conditions. Instead, the integrability conditions

$$\frac{\partial}{\partial \pi^k}\left(\frac{\partial f_j}{\partial \pi^i}\right) - \frac{\partial}{\partial \pi^i}\left(\frac{\partial f_j}{\partial \pi^k}\right) = 0$$

can not be calculated because none of the general equations say anything about the derivatives of α_j or H with respect to π^i, and therefore, they could not give rise to new necessary conditions.

b. It would be extremely difficult to work out equations (89) without using an appropriate technique. This technique exists though : the exterior calculus for *p*-forms with values in a Lie Algebra. We present below the main results which will be needed, using a notation adapted to the future use of these results.

Let Ω and Σ be respectively a *p*-form and a *q*-form

$$(90)\quad \begin{cases} \Omega = \dfrac{1}{p\,!}\,\Omega_{i_1\ldots i_p}\, dP^{i_1} \wedge \ldots \wedge dP^{i_p}, \\ \Sigma = \dfrac{1}{q\,!}\,\Omega_{j_1\ldots j_q}\, dP^{j_1} \wedge \ldots \wedge dP^{j_q} \end{cases}$$

where $\Omega_{i_1\ldots i_p}$, $\Sigma_{j_1\ldots j_q}$ are functions of x^i, π_j, Q^i, P_j, a coordinate system of a symplectic manifold. Therefore, we know how to calculate the P. B. of any pair of these functions. We define the mixed exterior product $[\Omega, \Sigma]$ as the $p + q$-form with components :

$$(91)\qquad [\Omega, \Sigma]_{k_1\ldots k_{p+q}} = \frac{1}{p\,!\,q\,!}\,\delta^{i_1\ldots i_p j_1\ldots j_q}_{k_1\ldots k_{p+q}}\,[\Omega_{i_1\ldots i_p}, \Sigma_{j_1\ldots j_q}].$$

This mixed exterior product has the following properties :

(92) $$[\Omega_1 + \Omega_2, \Sigma] = [\Omega_1, \Sigma] + [\Omega_2, \Sigma],$$

Ω_1 and Ω_2 being both p-forms.

(93) $$[\Sigma, \Omega] = (-1)^{pq+1} [\Omega, \Sigma],$$

(94) $$[\Omega, \Sigma \wedge \Phi] = [\Omega, \Sigma] \wedge \Phi + (-1)^{pq} \Sigma \wedge [\Omega, \Phi]$$

where $\wedge$ is the usual exterior product symbol and Φ is a r-form

(95) $$\Phi = \frac{1}{r!} \Phi_{k_1 \dots k_r} dP^{k_1} \wedge \dots \wedge dP^{k_r}.$$

(96) $$(-1)^{pr} [\Omega, [\Sigma, \Phi]] + (-1)^{qp} [\Sigma, [\Phi, \Omega]] + (-1)^{rq} [\Phi, [\Omega, \Sigma]] = 0.$$

And finally,

(97) $$d[\Omega, \Sigma] = [d\Omega, \Sigma] + (-1)^p [\Omega, d\Sigma]$$

where d is the exterior differential operator.

c. Using this formalism, equations (75), (76) and (85) can be written

(98) $$dH = H^{-1} P - [f, H],$$

(99) $$df = -\frac{1}{2}[f, f] + H^{-2} (f \wedge P - S),$$

(100) $$d\alpha_j = [\alpha_j, f] - H^{-1} [\alpha_j, H] \alpha + H^{-2} (f_j P - P_j f + P_j \alpha - S_j),$$

(101) $$\frac{\partial f}{\partial \pi^i} = H^{-2} P_i x + H^{-1} \left\{ \frac{\partial H}{\partial \pi^i} (\alpha + x) + [\alpha_i, H] x \right\}$$

where

(102) $$\begin{cases} f \equiv f_i \, dP^i, \quad P \equiv P_i \, dP^i, \quad \alpha \equiv \alpha_i \, dP^i. \quad x \equiv x_i \, dP^i, \\ S \equiv \frac{1}{2} S_{ij} \, dP^i \wedge dP^j, \quad S_j \equiv S_{jk} \, dP^k. \end{cases}$$

The conditions expressing that H is a scalar are

(103) $$[S, H] + dH \wedge P = 0$$

and the conditions expressing that f_i are the components of a vector are

(104) $$[S, f] - df \wedge P = 0$$

or

(105) $$[S, f_i] + df_i \wedge P + f \wedge dP_i = 0$$

or

$$[S_l, f] + \frac{\partial}{\partial P^l}(P \wedge f) - P_l\, df = 0. \tag{106}$$

α and α_i satisfy the same equations.

The Integrability conditions (89) are

$$d\,(dH) = 0, \qquad d\,(df) = 0, \tag{107}$$

$$d\,(d\alpha_j) = 0, \qquad d\left(\frac{\partial f}{\partial \pi^i}\right) - \frac{\partial}{\partial \pi^i}(df) = 0. \tag{108}$$

Let us work out, as an example, the first set of integrability conditions (107). From (98) we get

$$d\,(dH) = -\,H^{-2}\, dH \wedge P - [df, H] + [f, dH] \tag{109}$$

and using again (98), (99) and (97), we obtain

$$\begin{aligned} d(dH) = &-H^{-2}(H^{-1}P - [f, H]) \wedge P \\ &- \left[-\frac{1}{2}[f, f] + H^{-2}(f \wedge P - S), H\right] + [f, H^{-1}P - [f, H]]. \end{aligned} \tag{110}$$

Using now (92), (93) and (94), we get

$$d\,(dH) = H^{-2}\,\{\,[S, H] - [f, H] \wedge P\,\} + \frac{1}{2}[[f, f], H] - [f, [f, H]]. \tag{111}$$

But from (93) and (96) it follows that the second line is zero. Therefore, the integrability conditions we are looking for are equivalent to

$$[S, H] - [f, H] \wedge P = 0. \tag{112}$$

But these conditions are a consequence of (98) and (103). Thus, assuming that H is a function of the six scalars (78), the first set of integrability conditions (107) are satisfied as a consequence of equations (98) and (99).

A similar calculation, but very lengthy in the case of the first set of conditions (108), proves that the remaining integrability conditions (107) and (108) are consequences of (98) to (105) (or the equivalent equations to this last one).

Our conclusion is, then, that no more information is contained in the Definition of H. P. I. S. for N = 2 than that information contained in equations (98) to (104) [together with the analogous equation to (104) for α].

6. THE CENTER OF MASS FORMULA

a. Out of the equations (75), (76), (85) and (87), only the last one gives information about the functions H, f_i and α_j themselves taken at $P_i = 0$. Let us make more explicit this information. Using (80), (88) and the scalar character of H, $f_{(i)}$ and $\alpha_{(i)}$ we obtain

$$f^*_{(2)} = \alpha^*_{(2)} = 0; \tag{113}$$

$$\begin{cases} H^* \dfrac{\partial f^*_{(1)}}{\partial (x\,\pi)} = v^*_{(1)} (2\,\alpha^*_{(1)} + 1) + [\alpha^*_{(1)}, H^*], \\ H^*\,\pi^{-1} \dfrac{\partial f^*_{(1)}}{\partial \pi} = v^*_{(2)} (2\,\alpha^*_{(1)} + 1) \end{cases} \tag{114}$$

where we have written

$$[x_i, H^*] = v^*_{(1)}\, x_i + v^*_{(2)}\, \pi_i \tag{115}$$

and the P. B. are those that correspond to the induced symplectic form σ^* defined in paragraph 2.

b. From (35) and (73) we have

$$R^i = Q^i + f^i \tag{116}$$

and from (81), with $a = 2$, and from (86), we obtain

$$R^i = x^i_2 - \alpha^i. \tag{117}$$

And taking this formula for $P_i = 0$, from (113), we get

$$R^*_i = -\,\alpha^*_{(1)}\, x_{1i} + (1 + \alpha^*_{(1)})\, x_{2i}. \tag{118}$$

Equation (118) tells us that, for $P_i = 0$, the Center of Mass lies on the straight line joining the positions of both particles.

From (67) and (71), it follows that

$$J^*_i = \eta_{ijk}\, x^j\, \pi^k \tag{119}$$

and therefore

$$J^*_i\, x^i = 0. \tag{120}$$

Since by a space translation we can always make $R^*_i = 0$ and then x^i_1, x^i_2 and x^i are colinear, from (120) we see that in this frame of reference

$$J_i\, x^i_1 = 0, \qquad J_i\, x^i_2 = 0 \tag{121}$$

meaning that both orbits lie on a plane. This conclusion is obviously invariant by space translations, and therefore general for frames of reference with $P_i = 0$, even though (121) are not.

c. It is clear that if the two particles being considered are identical, which implies, in particular, that they have the same mass, from (118) it follows that in this case $\alpha^*_{(1)} = \frac{1}{2}\cdot$ Substitution of this particular value in (114) gives then

$$\frac{\partial f^*_{(1)}}{\partial (x\,\pi)} = 0, \qquad \frac{\partial f^*_{(1)}}{\partial \pi} = 0 \tag{122}$$

and therefore, $f^*_{(1)}$ is a function of x only. But since from (41) and (73) it follows that $f^*_{(1)}$ has been defined only up to a transformation

$$f^*_{(1)} = f^*_{(1)} + \lambda\,(x)$$

where λ is an arbitrary function of x, we conclude that no generality is lost by assuming $f^*_{(1)} = 0$. H^* is then, in this case, the only function which remains arbitrary to specify a particular interaction. The choice of H^*, which lies beyond the scope of this paper, will have to be made on physical grounds.

Knowing $\alpha^*_{(1)}$, $f^*_{(1)}$ and H^* equations (98), (99) and (100) would give, at least, by formal power series expansions with respect to the variables P^i, the functions $\alpha_{(i)}$, $f_{(j)}$ and H.

REFERENCES

[1] D. G. Currie, *Phys. Rev.*, t. 142, 1966, p. 817.
[2] R. N. Hill, *J. Math. Phys.*, t. 8, 1967, p. 201.
[3] R. N. Hill, *J. Math. Phys.*, t. 8, 1967, p. 1756.
[4] L. Bel, *Ann. Inst. H. Poincaré*, t. 12, 1970, p. 307.
[5] L. Bel, *Ann. Inst. H. Poincaré*, t, 14, 1971, p. 189.
[6] Ph. Droz-Vincent, *Phys. Scripta*, t. 2, 1970, p. 129.
[7] D. G. Currie, T. F. Jordan et E. C. G. Sudarshan, *Rev. Mod. Phys.*, t. 35, 1963, p. 350.
[8] J. T. Cannon et T. F. Jordan, *J. Math. Phys.*, 5, 1964, p. 299.
[9] H. Leutwyler, *Nuovo Cimento*, t. 37, 1965, p. 556.
[10] E. H. Kerner, *J. Math. Phys.*, t. 6, 1965, p. 1218.
[11] R. N. Hill et E. H. Kerner, *Phys. Rev. Lett.*, t. 17, 1966, p. 1156.
[12] L. Bel, *C. R. Acad. Sc.*, Paris, t. 273, série A, 1971, p. 405.
[13] Ch. Fronsdal, *Phys. Rev.*, D, t. 4, 1971, p. 1689.
[14] C. Godbillon, *Géométrie différentielle et Mécanique analytique*, Hermann, Paris, 1969.
[15] J. M. Souriau, *Structure des systèmes dynamiques*, Dunod, Paris, 1970.
[16] R. Abraham, *Foundations of Mechanics*, Benjamin, 1970.
[17] R. N. Hill, *J. Math. Phys.*, t. 11, 1970, p. 1918.

(*Manuscrit reçu le* 30 *novembre* 1972.)