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Section A :
Physique théorique.

On unitary irreducible representations of semidirect products with nilpotent normal subgroup

by

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ABSTRACT. — Unitary irreducible representations (u. i. r.'s) of semidirect products $\mathcal{G} = \mathcal{R} . \mathcal{N}$, with nilpotent $\mathcal{N}$ and regular semisimple action of $\mathcal{G}$ on $\mathcal{N}$, are investigated. Given a point in an orbit of $\hat{\mathcal{N}}$ and a u. i. r. of its stabilizer $\mathcal{S} . \mathcal{N}$ (which characterizes any u. i. r. of $\mathcal{G}$), it is shown that the ε-u. i. r. M of the little group $\mathcal{S}$, which yields M' has a real-valued multiplier ε, and is therefore a true u. i. r. of $\mathcal{S}$ or of a twofold covering group of $\mathcal{S}$.

This result is then applied to the 15-dimensional group containing Lorentz transformations on space-time, and position and momentum operators verifying the Heisenberg relation. The spectrum of the squared mass operator in these u. i. r.'s is exhibited; and they are related to u. i. r.'s of the conformal group by means of Wigner-Inonü contraction.

INTRODUCTION

All unitary irreducible representations (u. i. r.'s) of groups which are semidirect products with abelian normal subgroup, are known to be induced representations, provided some topological restrictions hold, since Mackey's papers on induced representations ([1], [2]). When one drops the abelian subgroup hypothesis, projective representations generally appear [3].

A question which arises is whether this hypothesis can be replaced by a less restrictive one, but which keeps projective representations away.

One generalization is to take a semidirect product $\mathcal{R}.\mathcal{N}$ where $\mathcal{N}$ is nilpotent. In the present paper it is shown that projective representations may appear with this hypothesis; but if an algebraic condition is imposed, projective representations are limited to either true representations or two-valued ones. (This happens, for instance, if $\mathcal{R}$ is semisimple.)

To establish this result we make use of Dixmier-Kirillov's results on nilpotent groups ([4], [5]); the mathematical background which is used is exposed in section 1.

In section 2 the main result is proved. The proof is based on induction upon the dimension of $\mathcal{N}$ and involves properties of the characteristic ideals of the central descending series of the Lie Algebra of $\mathcal{N}$.

Section 3 contains an extension of Kirillov's classification to groups of the form $\mathcal{R}.\mathcal{N}$; there is also a discussion of the hypotheses and hints for possible generalizations.

In section 4, the general results are used to obtain all u. i. r.'s of a 15-dimensional group $\mathcal{G} = \mathrm{SL}(2, \mathbf{C}).\mathcal{N}$, where $\mathcal{N}$ is a 9-dimensional nilpotent group. The center I of $\mathcal{G}$ is one-dimensional, and $\mathcal{N}/\mathrm{I}$ splits into two Minkowski spaces, on which SL (2, **C**) acts in the usual way. Each 4-dimensional abelian subgroup is isomorphic to a Minkowski space, but two vectors belonging to a different Minkowski space each, do not commute (unless they are orthogonal) and their bracket lies in I.

The choice of $\mathcal{G}$ as an example is due to the fact that it has a physical significance : $\mathcal{G}$ is a kinematical group which acts on states; its Lie algebra is the smallest one containing both position and momentum operators along with Lorentz transformations. In fact $\mathcal{G}$ is obtained if, given the group of active covariance $\mathcal{L}$, one defines four infiniterimal generators for position, satisfying the Heisenberg relation; the complete duality between position and momentum appears in $\mathcal{G}$, while it does not appear if position operators are defined as an integral of a spectral measure over some domain [6]. Faithful u. i. r.'s of $\mathcal{G}$ are determined, according to the general results, by a character of the center I and a u. i. r. of SL (2, **C**). The spectrum of the squared mass operator $p_\mu p^\mu$ is then the whole real line; thus no discrete mass splitting exists for faithful u. i. r.'s, as predicted by Jost-Segal's no-go theorem [7].

On the other hand, for unfaithful u. i. r.'s, for which the center I is trivially represented (corresponding to the classical limit $\hbar \to 0$ of the Heisenberg relation $[p, q] = \hbar$), the squared mass operator has its spectrum concentrated in one point.

Finally, since $\mathcal{G}$ is a Wigner-Ionü contraction of the conformal group $\mathcal{C}$ (special conformal transformations contracting to position operators), the relation between u. i. r.'s of $\mathcal{G}$ and $\mathcal{C}$ through contraction is sketched in the end of section 4.

1. THE MATHEMATICAL BACKGROUND

The mathematical background we shall make use of, concerns essentially two topics; the theory of u. i. r. 's of nilpotent groups on one hand, the theory of projective representations of groups extensions on the other.

The theory of u. i. r.'s of nilpotent groups was developed by Dixmier and Kirillov ([4], [5]); we summarize it as follows :

Given a nilpotent group $\mathcal{N}$, connected, simply connected, there is a one-to-one correspondance between $\mathcal{N}$ and its Lie Algebra $\mathfrak{n}$. Let $\mathfrak{n}'$ be the dual vector space of the real vector space underlying $\mathfrak{n}$. One can define on $\mathfrak{n}'$ the contragredient of the adjoint representation of $\mathcal{N}$, which we shall call R. The action of $\mathcal{N}$ on $\mathfrak{n}'$ through R defines a partition of $\mathfrak{n}'$ into orbits : two elements x, y of $\mathfrak{n}'$ belong to the same orbit if there exists g in $\mathcal{N}$ such that $x = \mathrm{R}(g).y$.

The group dual $\hat{\mathcal{N}}$, i. e. the set of equivalence classes of u. i. r.'s. of $\mathcal{N}$ (distinct from the linear dual $\mathfrak{n}'$!) is mapped on the set of orbits of $\mathfrak{n}'$ under R, the mapping being one-to-one. This mapping is given by Kirillov's classification :

THEOREM 1. — *Let h be in $\mathfrak{n}'$, and let $\mathfrak{n}_h$ be a subalgebra of $\mathfrak{n}$ of maximal dimension such that $[\mathfrak{n}_h, \mathfrak{n}_h]$ belongs to the kernel of h. Let $\mathcal{N}_h$ be the corresponding subgroup and χ_h the character of $\mathcal{N}_h$, defined by $\chi_h(\exp \mathrm{X}) = \exp i(\mathrm{X}, h)$ for X in $\mathcal{N}_h$. Then, the representation of $\mathcal{N}$ induced by χ_h is irreductible, and two such u. i. r.'s are equivalent if and only if the corresponding elements h belong to the same orbit of $\mathfrak{n}'$ under* R; *all u. i. r.'s of $\mathcal{N}$ are obtained in this way.*

We shall also state another result, due to Pukansky [8], which is essential for establishing the general results by a method based on the dimension of $\mathcal{N}$.

THEOREM 2. — *Let $\mathcal{N}$ have one-dimensional center $\mathfrak{Z}$. Then every u. i. r. of $\mathcal{N}$, the restriction of which to the center is nontrivial, is induced by a u. i. r. of a suitably chosen subgroup $\mathcal{N}'$, such that the codimension of $\mathcal{N}'$ in $\mathcal{N}$ is one and the codimension of $\mathfrak{Z}$ in the center $\mathfrak{Z}'$ of $\mathcal{N}'$ is strictly positive.*

REMARK. — All along this paper when speaking of a nilpotent group we shall assume it to be connected and simply connected.

The general theory of projective representations of group extensions has been developped by Mackey [3]. We shall give here the main results, under a simplified form, which suffices for our purpose.

Let $\mathcal{G}$ be a locally compact group, $\mathcal{N}$ a closed invariant subgroup and $\hat{\mathcal{N}}$ the set of equivalent classes of u. i. r.'s of $\mathcal{N}$. Let $\mathrm{L} \to \mathrm{L}^g$ be the action of $\mathcal{G}$ on $\hat{\mathcal{N}}$, defined by

$$L^g(n) = L(gng^{-1}).$$

for $g \in \mathcal{G}$, $n \in \mathcal{N}$, $L \in \hat{\mathcal{N}}$.

We shall assume that this action of $\mathcal{G}$ on $\hat{\mathcal{N}}$ is regular with respect to the Borel structure of $\hat{\mathcal{N}}$: that is, if a measure class in $\hat{\mathcal{N}}$ is invariant under the action of $\mathcal{G}$, it corresponds to an orbit $\hat{\mathcal{N}}$ under $\mathcal{G}$. Let, for any L in $\hat{\mathcal{N}}$, $\mathcal{S}'_{L'}$ denote the stabilizer of L, i. e. the subgroup of $\mathcal{G}$ such that L^s and L are equivalent for any s in $\mathcal{S}'_{L'}$. Clearly, $\mathcal{S}'_{L'}$ contains $\mathcal{N}$; $\mathcal{S}'_{L'}$ and $\mathcal{S}'_{L'}$ are conjugated subgroups if L and L′ belong to the same orbit.

THEOREM 3. — *There is an one-to-one correspondance, between equivalence classes of u. i. r.'s of $\mathcal{G}$ and couples $(\mathcal{O}, M)$ where $\mathcal{O}$ is an orbit of $\hat{\mathcal{N}}$, and M a u. i. r. of the stabilizer $\mathcal{S}$ of some point in $\mathcal{O}$, The u. i. r. of $\mathcal{G}$ corresponding to $(\mathcal{O}, M)$ is induced by* M.

THEOREM 4. — *Let $\mathcal{G}$, $\mathcal{N}$, $\mathcal{O}$, $\mathcal{S}'$, be as above, and* L *a point in $\mathcal{O}$. Then there exists a projective representation $\tilde{L}$ of $\mathcal{S}'$, with multiplier τ [i. e. $\tilde{L}_{xy} = \tau(x, y)\tilde{L}_x\tilde{L}_y$], such that $\tilde{L}(n) = L(n)$ for every n in $\mathcal{N}$. τ may be chosen to be of the form $1/\omega \circ f$, where ω is a multiplier of $\mathcal{S}'/\mathcal{N}$ and if is the canonical homomorphism from $\mathcal{S}' \times \mathcal{S}'$ to $\mathcal{S}'/\mathcal{N} \times \mathcal{S}'/\mathcal{N}$. If τ is so chosen, ω is uniquely determined by* L *up to a factor of the form $\rho(x).\rho(y).\rho(xy)^{-1}$ (called a trivial multiplier).*

THEOREM 5. — *Let* T *be a projective u. i. r. of $\mathcal{S}'/\mathcal{N}$ with multiplier ω, and* T′ *the projective u. i. r. of $\mathcal{S}'$, with multiplier $\omega \circ f$, canonically obtained from* T. *Then the mapping $T \to \tilde{L} \otimes T'$ sets up, an one-to-one correspondance (up to equivalence), between the set of all projective u. i. r.'s of $\mathcal{S}'/\mathcal{N}$ with multiplier ω, and the set of all u. i. r.'s of $\mathcal{S}'$, the restriction of which on $\mathcal{N}$ is a multiple of* L.

It is well known that if $\mathcal{G}$ is a semidirect product and $\mathcal{N}$ is abelian, the multiplier ω drops and one obtains ordinary representations for $\mathcal{S}'/\mathcal{N}$. Our purpose is to show what happens if $\mathcal{N}$ is nilpotent.

2. THE MAIN THEOREM

From now on we shall denote by $\mathcal{G} = \mathcal{R}.\mathcal{N}$ a semi direct product such that

(*a*) $\mathcal{N}$ is nilpotent with Lie Algebra $\mathfrak{n}$.

(*b*) $\mathcal{R}$ acts regularly on $\mathcal{N}$.

(*c*) The restriction φ of the adjoint representation to the subspace $\mathfrak{n}$ and to the subgroup $\mathcal{R}$ is semisimple.

The stabilizer $\mathcal{S}'$ of an u. i. r. L of $\mathcal{N}$ is of the form $\mathcal{S}.\mathcal{N}$, where $\mathcal{S} = \mathcal{S}' \cap \mathcal{R}$ is the corresponding « little group ».

We want to show that in this case we have a stronger result than Theorem 4 :

THEOREM 6. — *For any orbit $\mathcal{O}$ of $\hat{\mathcal{N}}$ under $\mathcal{R}$ and any $L \in O$, there is an ε-u. i. r. $\tilde{L}$ of $\mathcal{S}'$, such that :* (i) $\tilde{L}(n) = L(n)$ *for* $n \in \mathcal{N}$, *and* (ii) $\tilde{L}(x)\tilde{L}(y) = \varepsilon(x, y)\tilde{L}(x, y)$ *for* $x, y \in \mathcal{S}$. *The multiplier ε can take only the two values* ± 1.

REMARK 1. — From what precedes we see that ε is uniquely determined by $\mathcal{S}$, up to a trivial multiplier. If ε is two-valued, it defines a central extension $\bar{\mathcal{S}}$ of $\mathcal{S}$ by $\mathbf{Z}_2$, and $\tilde{L}$ can be considered as a true u. i. r. of $\bar{\mathcal{S}}$.

REMARK 2. — Condition (*c*) is automatically fullfilled if $\mathcal{R}$ is semi-simple, but if $\mathcal{R}$ has a non discrete center, the condition may not be fullfilled as we shall see further in a example.

The proof will be based on induction over the dimension of $\mathcal{N}$. Clearly the theorem holds for one-dimensional $\mathcal{N}$, which is abelian. We shall thus assume that it holds for any $\mathcal{N}'$ of lower dimension than $\mathcal{N}$.

Proof. — Let the group law in $\mathcal{G}$ be

$$(n, r)(n', r') = (n.r(n'), rr').$$

Let L be in $\mathcal{O}$ and $\mathcal{S}$ be the subgroup of $\mathcal{R}$ such that L^s and L are equivalent for any s in $\mathcal{S}$, that is, for any s in $\mathcal{S}$, any n in $\mathcal{N}$:

$$L(S(n)) = K_s L(n) K_s^{-1}. \tag{2.1}$$

We shall show that the operators K_s, determined up to a scalar factor, can be chosen in a way that the mapping

$$(n, s) \rightarrow L(n) K_s \tag{2.2}$$

be an ε-u. i. r. of $\mathcal{S}.\mathcal{N}$, which will be taken as $\tilde{L}$. We shall first prove :

LEMMA 1. — *Let $\mathcal{S}$, $\mathcal{N}$, L defined as above. If $\tilde{L}$ can be defined whenever L is a faithful u. i. r. of $\mathcal{N}$, then $\tilde{L}$ can be defined for every* L.

Proof. — Let $\mathcal{N}_0$ be the subgroup of $\mathcal{N}$ on which L is constant. From (2.1) we see that $\mathcal{S}$ leaves $\mathcal{N}_0$ invariant. Putting $\mathcal{N}' = \mathcal{N}/\mathcal{N}_0$, and p the canonical projection from $\mathcal{N}$ to $\mathcal{N}'$, we define L′ on $\mathcal{N}'$, such that $L = L' \circ p$. But p can be extended trivially to the projection $\bar{p}$ from $\mathcal{S}.\mathcal{N}$ to $\mathcal{S}.\mathcal{N}'$ by

$$\bar{p}(S, n) = (S.pn).$$

Then, since L′ is faithful, $\tilde{L}'$ is defined, and $\tilde{L}$ is defined by $\tilde{L} = \tilde{L}' \circ \bar{p}$, which proves the lemma.

From now on we shall thus assume that L is faithful, which means that $\mathcal{N}$ has one-dimensional center $\mathcal{Z}$, and that the restriction of L to $\mathcal{Z}$ is nontrivial. We shall now describe more presicely the structure of $\mathcal{N}$, in order to use induction over the dimension.

The center $\mathfrak{z}$ of $\mathfrak{n}$ coïncides with the last ideal of the central descending series. Let $\mathfrak{n}$ be the next-to-the-last one and $\mathfrak{b}$ its centralizer; let $\mathcal{A}$ and $\mathcal{B}$ be the corresponding subgroups. $\mathcal{A}$ and $\mathcal{B}$ are invariant under all automorphisms of $\mathcal{N}$. The Jacobi identity shows that $\mathfrak{b}$ contains the first derived ideal of $\mathfrak{n}$, since

$$(2.3) \qquad [\mathfrak{a}, [\mathfrak{n}, \mathfrak{n}]] \subset [\mathfrak{n}, [\mathfrak{a}, \mathfrak{n}]] = [\mathfrak{n}, \mathfrak{z}] = 0.$$

Neglecting the trivial case $\mathfrak{z} = \mathfrak{n}$, we see that we have either

$$\bar{\mathfrak{z}} = [\mathfrak{n}, \mathfrak{n}] \qquad \text{or} \qquad \mathfrak{a} \subset [\mathfrak{n}, \mathfrak{n}].$$

These two cases shall be treated distinctly.

A. $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$. — Such a group is an immediate generalization of the three dimensional non abelian nilpotent group. It is a central extention of $\mathcal{Z}$ by $\mathbf{R}^{2k}$ (considered as an abelian group) and it is determined by a skew-symmetric bilinear form from $\mathbf{R}^{2k}$ into $\mathcal{Z}$. The center being one dimensional, the bilinear form is non degenerate. Writing the group law in $\mathcal{N}$,

$$(2.3) \qquad (a; z)(a'; z') = \left(a + a'; z + z' + \frac{1}{2} a \wedge a'\right)$$

one sees that $\varphi(\mathcal{R})$ must be a subgroup of $\mathrm{Sp}(k, \mathbf{R}) \times \mathbf{R}^*$; the action of an element (A, λ) of $\mathcal{R}$ is

$$(\mathrm{A}, \lambda) : \quad (a; z) \to (\mathrm{A}\, a; \lambda\, z)$$

obeying

$$(2.4) \qquad \mathrm{A}\, a \wedge \mathrm{A}\, a' = \lambda\, a \wedge a'.$$

REMARK. — We have implicitedly supposed here that the complementary of $\mathcal{Z}$ which is invariant by $\mathcal{R}$ is the set $(a; 0)$.

By choosing a canonical basis in $\mathbf{R}^{2k} = \mathbf{R}^k \oplus \mathbf{R}^k$, one can express the symplectic structure in a more analytical way. One has

$$(x, y) \wedge (x', y') = x.y' - y.x'$$

for x and y in $\mathbf{R}^k$, the dot standing for the scalar product in $\mathbf{R}^k$.

Now, a faithful representation L is determined by a non zero character χ of $\mathcal{Z}$ and is induced by the u. i. r. of the abelian subgroup $\{(0, y, z)\}$:

$$(0, y; z) \to \exp i \chi z,$$

L can thus be written

$$\text{(2.5)} \qquad L(x, y; z) f(t) = \exp i\chi\left(z + \frac{1}{2} x.y + t.y\right) f(t+x)$$

and s is the subgroup of $\mathcal{R}$ determined by the condition $\lambda = 1$, that is, s is in Sp $(k, \mathbf{R})$.

To extend L to Sp $(k, \mathbf{R})$ we first introduce the following unitary operators :

$$\text{(2.6)} \qquad \begin{cases} J_S f(t) = \exp(-\chi^{tSt/2}) f(t), \\ V_A f(t) = |\det A|^{-1/2} f(A^{-1} t), \\ \mathcal{F} f(t) = \displaystyle\int_{\mathbf{R}^k} e^{i\chi u.t} f(u)\, du, \end{cases}$$

where S and A are $k \times k$ real matrices, S symmetric, A invertible, and $du = \left|\frac{\chi}{2\pi}\right|^{\frac{k}{2}} du_1 \dots du_k$.

Then for almost every element g of Sp $(k, \mathbf{R})$ one can write :

$$\text{(2.7)} \qquad \tilde{L}(g) = \rho(g) J_{CA^{-1}} V_A \mathcal{F} J_{-A^{-1}B} \mathcal{F}^{-1} \qquad \text{with} \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where ρ is some complex-valued function of modulus 1.

We leave it to the reader to check that $\tilde{L}$ satisfies (2.1).

D. Shale [9] has shown that $\tilde{L}$ is a two-valued ε-u. i. r. of Sp $(k, \mathbf{R}).\mathcal{N}$.

We must point out, however, that there are subgroups of Sp $(k, \mathbf{R})$, on which the restriction of ε is a trivial multiplier, hence the restriction of $\tilde{L}$ yields a one-valued u. i. r. [10]. One can see from (2.6) that GL $(k, \mathbf{R})$ is such a subgroup. This is also the case for U $(k - k', k')$; we refer to [11] for a realization of the faithful u. i. r's of U $(k).\mathcal{N}$ on a space of holomorphic functions.

B. $\mathfrak{a} \subset [\mathfrak{n}, \mathfrak{n}]$. — We have the sequence of decreasing ideals

$$\mathfrak{n} \supset \tilde{\mathfrak{b}} \supset [\mathfrak{n}, \mathfrak{n}] \supset \mathfrak{a} \supset \mathfrak{z}$$

then $\mathcal{X} = \mathcal{N}/\mathcal{B}$ is an abelian group, and the group law in $\mathcal{N}$ can be written

$$(b; x)(b'; x') = (b.\mathrm{Adx}(b').Q(x, x'); x + x').$$

where Q is a polynomial expression of the coordinates of x, satisfying

$$Q(x, x').Q(x + x', x'') = \mathrm{Adx}\, Q(x', x'').Q(x, x' + x''),$$

$$Q(x, -x) = Q(x, 0) = Q(0, x) = 1$$

and Adx is the element of Aut $\mathcal{B}$ determined by

$$(\mathrm{Adx}.b;\, 0) = (1;\, x)\ (b;\, 0)\ (1;\, -x).$$

We observe that if Adx (a) is known for every a in $\mathcal{A}$, x is well determined. Let now L be a faithful u. i. r. of $\mathcal{N}$. From theorem 2 one can easily deduce (by alternately applying theorem 2 and projecting the subgroup of codimension one to a suitably chosen factor group with one-dimensional center again) that L is induced by a u. i. r. U of $\mathcal{B}$ which has the same restriction on $\mathcal{Z}$ as L has. Conversely, if U′ of $\mathcal{B}$ induces on $\mathcal{N}$ a u. i. r. equivalent to L, U′ is equivalent to the representation $b \to \mathrm{U}\,(\mathrm{Adx}.\,b)$ for some x in $\mathcal{X}$. We shall use the following realization for L :

$$\mathrm{L}\,(b;\, x)\, f\,(t) = \mathrm{U}\,(\mathrm{Adt}\,(b).\,\mathrm{Q}\,(t,\, x))\, f\,(t + x),$$

f being a square integrable function from $\mathcal{X}$ to the carrier space of U.

Let now $(\Phi,\, \Theta,\, \Psi)$ be the automorphism of $\mathcal{N}$ defined by

$$(b;\, x) \to (\Theta\,(b).\,\Psi\Phi\,(x);\, \Phi\,(x)).$$

The group law shows that Θ must be in Aut $\mathcal{B}$ and Φ in GL $(\mathfrak{x})$ ($\mathfrak{x}$ being considered as a real vector space), and one must also have :

$$\text{(2.8 } a) \qquad \mathrm{Ad}\,\Phi\,(x)\,(b) = \Psi\Phi\,(x)^{-1}.\,\Theta\ \mathrm{Adx}\ \Theta^{-1}\,(b).\,\Psi\Phi\,(x),$$

$$\text{(2.8 } b) \qquad \begin{aligned} &\Psi\Phi\,(x).\,\mathrm{Ad}\,\Phi\,(x)\,[\Psi\Phi\,(x')].\,\mathrm{Q}\,(\Phi\,(x),\, \Phi\,(x')) \\ &\quad = \Theta\ \mathrm{Q}\,(x,\, x').\,\Psi\Phi\,(x + x'). \end{aligned}$$

Let now $\varphi\,(\mathcal{R})$ be a subgroup of Aut $\mathcal{N}$ satisfying the conditions of the theorem. Because of the semisimplicity condition we know that there is a complementary subspace $\mathfrak{a}'$ of $\mathfrak{z}$ in $\mathfrak{a}$, invariant by $\mathcal{R}$. Let $\mathcal{A}'$ be the corresponding abelian subgroup. We shall choose U in a manner that $\mathcal{A}'$ belongs to the kernel of U. Then, if $(\Phi,\, \Theta,\, \Psi)$ is in $\mathcal{S}$, Θ belongs to the subgroup of Aut $\mathcal{B}$ which leaves U invariant, and since dim $\mathcal{B}$ is strictly lower than dim $\mathcal{N}$, one can assume

$$\mathrm{U}\,(\Theta\, b) = \tilde{\mathrm{U}}_{\Theta}.\,\mathrm{U}\,(b).\,(\tilde{\mathrm{U}}_{\Theta})^{-1}$$

such that

$$\tilde{\mathrm{U}}_{\Theta\Theta'} = \pm\, \tilde{\mathrm{U}}_{\Theta}.\,\tilde{\mathrm{U}}_{\Theta'}.$$

Let now $(\Phi,\, \Theta,\, \Psi)$ be in $\varphi\,(\mathcal{S})$ and express that $\mathrm{L}^{(\Phi,\Theta,\Psi)}$ is equivalent to L. We have, using (2.8) :

$$\begin{aligned} &\mathrm{L}^{(\Phi,\,\Theta,\,\Psi)}\,(b;\, x)\, f(t) \\ &\quad = \mathrm{L}\,(\Theta\, b.\,\Psi\Phi\,(x))\, f(t) = \mathrm{U}\,(\mathrm{Adt}\,(\Theta\, b).\,\mathrm{Adt}\,\Psi\Phi\,(x).\,\mathrm{Q}\,(t,\, \Phi\, x))\, f(t + \Phi\, x) \\ &\quad = \mathrm{U}\,(\Psi\,(t)^{-1}.\ \Theta\,[\mathrm{Ad}\,\Phi^{-1}\, t\,(b).\,\mathrm{Q}\,(\Phi^{-1}\, t,\, x)].\,\Psi\,(t + \Phi\, x)\, f(t + \Phi\, x) \\ &\quad = \mathrm{K}\,(\Phi,\, \Theta,\, \Psi)\,\mathrm{L}\,(b;\, x)\,\mathrm{K}\,(\Phi,\, \Theta,\, \Psi)^{-1}\, f\,(t), \end{aligned}$$

with

$$(2.9)\qquad \mathrm{K}(\Phi, \Theta, \Psi) f(t) = |\det \Phi|^{-1/2}.\mathrm{U}(\Psi(t))^{-1}.\tilde{\mathrm{U}}_{\Theta}.f(\Phi^{-1} t).$$

It is then quite easy to check that

$$\mathrm{K}(\Phi, \Theta, \Psi).\mathrm{K}(\Phi', \Theta', \Psi') = \pm \mathrm{K}(\Phi\Phi', \Theta\Theta', (\Theta\Psi' \Phi)^{-1}.\Psi)$$

which is effectively the group law in Aut $\mathcal{N}$, hence in $\varphi(\mathcal{S})$. Then the u. i. r. $\tilde{\mathrm{L}}$ is given by

$$(2.10)\qquad \tilde{\mathrm{L}}(n, (\Phi, \Theta, \Psi)) = \mathrm{L}(n).\mathrm{K}(\Phi, \Theta, \Psi).$$

REMARK 3. — Using the semisimplicity condition on $\mathcal{S}$, one could annihilate Ψ as well. On the other hand, for trivial Ψ, the assumption that Θ leaves U invariant can be dropped. One sees thus that the theorem is true for a larger family of groups. However, when neither Ψ is trivial, nor does Θ leave U invariant, a non trivial multiplier appears and $\tilde{\mathrm{L}}$ is then a true projective representation. That is why the semisimplicity conditions are needed for the general case.

REMARK 4. — On exhaustive study is still to be done about the conditions, under which one obtains one-valued u. i. r.'s only. Restrictions may be imposed either to $\mathcal{N}$ or to $\mathcal{R}$. Here are some hints :

(*a*) If one wants to exclude case A as a final step of the induction which starts in case B, by imposing conditions on $\mathcal{N}$ only, the list of the acceptable nilpotent groups would be quite poor.

(*b*) To have a result concerning all nilpotent groups, one must take $\mathcal{R}$ such that it contains a twofold covering of $\varphi(\mathcal{S})$ for every $\mathcal{S}$ which gives rise to a twofold ε-u. i. r. This condition depends on L, hence on $\mathcal{N}$. Is there an $\mathcal{N}$-independant condition, weaker than $\mathcal{R}$ reductive with simply connected maximal semisimple subgroup? We leave this problem open.

3. RESULTS AND REMARKS

From what precedes one can extend Kirillov's classification to groups of the form $\mathcal{R}.\mathcal{N}$, since a point in $\hat{\mathcal{N}}$ is an orbit in $\mathfrak{n}'$ (under the action of $\mathcal{N}$), we thus have

THEOREM 7. — *Let $\mathcal{G} = \mathcal{R}.\mathcal{N}$ as in theorem 6 ; given h in $\mathcal{N}'$, let $\mathcal{N}_h$ and χ_h be as in theorem 1, L_h be the induced representation by χ_h, and $\mathcal{S}_h.\mathcal{N}$ be the stabilizer of L_h in $\mathcal{G}$. Then the representation induced on $\mathcal{G}$ by the u. i. r. $\tilde{\mathrm{L}}_h \otimes \mathrm{T}$ of $\mathcal{S}_h.\mathcal{N}$, where T is any representation of $\mathcal{S}_h.\mathcal{N}$, trivial on $\mathcal{N}$, is irreducible.*

Two such representations corresponding to (h, T) *and* (h′, T′) *are equivalent if and only if h and h′ belong to the same orbit of* T′ *under* $\mathcal{G}$ *and if* T *and* T′ *are equivalent. All u. i. r.'s of* $\mathcal{G}$ *are obtained in this way.*

Discussing the hypotheses of theorem 6, one first observes that the condition of regular action is essential : as a matter of fact, the theory of representations of group extensions elaborated by G. Mackey, based on the properties of systems of imprimitivity, requires a Borel set preserving condition to give full results. Otherwise, counter-examples are known, given by Mackey himself.

On the other hand, the condition of semisimple action of $\mathcal{R}$ on $\mathcal{N}$, simply drops if $\mathcal{N}$ is taken to be abelian. In this sense, theorems 6 and 7 are more restrictive than a full generalization of Mackey's classical result on abelian subgroups to nilpotent ones. We must point out that it is not a necessary condition, as it appears in the proof of the theorem (*see* Remark 3); and it covers a few more cases than the mere assumption of $\mathcal{R}$ being semisimple.

However, even if this condition can be slightly generalized (for example the theorem holds for suitably chosen subgroups of $\mathcal{R}$) one cannot get rid of it, as can be shown by the following example :

Example. — Let $\mathcal{G} = \mathcal{R}.\mathcal{N}$ be defined as follows : E_i being an Euclidian vector space ($i = 1, 2$) the underlying space of $\mathcal{N}$ is $E_1^2 \oplus E_2^2 \oplus \mathbf{R}^3$ with the group law :

$$(x_0, x_1, x_2; y_0, y_1, y_2; z)(x'_0, x'_1, x'_2; y'_0, y'_1, y'_2; z')$$
$$= (x_0 + x'_0, x_1 + x'_1, x_2 + x'_2; y_0 + y'_0, y_1 + y'_1,$$
$$y_2 + y'_2 ; z + z' + x_0.y'_0 + x_1.y'_1 + x_2.y'_2),$$

with x_i, y_i in E_i and x_0, y_0, z in $\mathbf{R}$.

Let $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$, where $\mathcal{R}_i$ is the connected subgroup of GL (E_i), i. e. the set of invertible matrices with positive determinant. We shall write for u_i in $\mathcal{R}_i$,

$$\lambda_i = \text{Log det } u_i. \tag{3.1}$$

Let the action of $\mathcal{R}$ on $\mathcal{N}$ be

$$(u_1, u_2).(x_0, x_1, x_2; y_0, y_1, y_2; z).(u_1, u_2)^{-1}$$
$$= (x_0, u_1 x_1, u_2 x_2; y_0, u_1^{*-1} y_1, u_2^{*-1} y_2; z + \lambda_1 x_0 + \lambda_2 y_0).$$

Let L be the u. i. r. of $\mathcal{N}$ described in section 2, case A,

$$L(x; y; z) f(t) = \exp i\left(z + \frac{1}{2}x.y + t.y\right) f(t + x),$$

with the obvious notation $x = (x_0, x_1, x_2)$, etc.

The operator $\tilde{L}(u_1, u_2)$ such that

$$\tilde{L}(u_1, u_2)\,\tilde{L}(x; y; z)\,\tilde{L}(u_1, u_2)^{-1} = L^{(u_1, u_2)}(x; y; z)$$

turns to be

$$\tilde{L}(u_1, u_2) f(t_0, t_1, t_2)$$
$$= \rho(u) \exp\left(-\frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 - i\lambda_1 t_0\right) f(t_0 + \lambda_2, u_1^{-1} t_1, u_2^{-1} t_2)$$

where $\rho(u)$ is a complex factor of modulus one, depending on (u_1, u_2).

It is easy then to find that

$$\tilde{L}(u_1, u_2)\,\tilde{L}(u'_1, u'_2) = \rho(u).\rho(u').\rho(uu')^{-1} \exp(-i\lambda'_1 \lambda_2)\,\tilde{L}(u_1 u'_1, u_2 u'_2)$$

and there is no choice of ρ which annihilates the multiplier $\exp(-i\lambda'_1 \lambda_2)$. $\tilde{L}$ turns thus to be a true projective representation.

On this example one can remark that $\mathcal{G}$ is a subgroup of the group $\mathcal{G}'$, obtained by suppression of (3.1), taking thus λ_i and u_i independant. We have $\mathcal{G}' = \mathcal{R}.\mathcal{N}'$, where $\mathcal{N}'$ is the nilpotent subgroup obtained by $u_1 = u_2 = 1$. Clearly, $\mathcal{G}'$ verifies theorem 6, but $\mathcal{G}$ does not. On the other hand, considering the subgroup $\mathcal{G}''$ of $\mathcal{G}$ obtained by $x_2 = y_2 = 0$ one checks that the results of theorem 6 hold; taking $\mathcal{G}'' = \mathcal{R}''.\mathcal{N}''$, where $\mathcal{N}''$ is obtained by taking u_2 scalar and $u_1 = 1$, the relevant u. i. r.'s of $\mathcal{N}''$ are of the form

$$L''(\lambda_2; x; y; z) f(t)$$
$$= \exp i\left(z + t.y - \frac{1}{2}\lambda_2 y_0 + \frac{1}{2}x.y + \chi(\lambda_2 - x_0)\right) f(t_0 + x_0 + \lambda_2, t_1 + x_1)$$

and the multiplier disappears from $\tilde{L}''$, as one easily checks.

From this example one sees that, though semisimple automorphisms and nilpotent ones (which yield a larger nilpotent group) behave well if they act independently, complications arise if they are linked together. We point out that this situation is characteristic of solvable groups.

4. AN EXAMPLE

Let $\mathfrak{g}$ be the Lie algebra generated by the operators $M_{\nu\mu} = -M_{\mu\nu}$, X_μ, P_μ, I ($\mu, \nu = 0, 1, 2, 3$) with the commutation relations

(4.1) $$[M_{\mu\nu}, M_{\lambda\rho}] = -g_{\mu\lambda} M_{\nu\rho} + g_{\mu\rho} M_{\nu\lambda} + g_{\nu\lambda} M_{\mu\rho} - g_{\nu\rho} M_{\mu\lambda};$$

(4.2) $$\begin{cases} [M_{\mu\nu}, P_\rho] = g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu, \\ [M_{\mu\nu}, X_\rho] = g_{\nu\rho} X_\mu - g_{\mu o} X_\nu; \end{cases}$$

(4.3) $$[P_\mu, X_\nu] = g_{\mu\nu} I;$$

all other brackets being zero; $g_{\mu\nu}$ denotes the symmetric tensor of the Minkowski metric $(+, -, -, -)$.

The simply connected corresponding Lie group is $G = SL(2, \mathbf{C}).\mathcal{N}$, where $\mathcal{N}$ is a nine-dimensional nilpotent subgroup; the group law is given by

$$(\Lambda, p, x, t)(\Lambda', p', x', t') = (\Lambda\Lambda', p + \Lambda p', x + \Lambda x', t + t' + \langle p, \Lambda x' \rangle)$$

with $t \in \mathbf{R}$, $\Lambda \in SL(2, \mathbf{C})$; p and x are four-vectors and $\langle p, x \rangle$ denotes their Minkowski scalar product, SL (2, **C**) acts in the ordinary way on the Minkowski space.

Before determining the u. i. r.'s of G, let us remark that it contains the universal covering $\bar{\mathfrak{P}}$ of the Poincaré group; its Lie algebra is the smallest one containing both position and momentum operators along with homogeneous Lorentz transformations.

As we shall show below, $\mathfrak{g}$ can be obtained as a Wigner Inonü contraction of the conformal Lie Algebra, $\mathfrak{so}\ (4,2)$.

We shall now apply the preceding general results to determine the u. i. r.'s of G; we know that the method of sec. 2 yields all u. i. r.'s.

The dual $\hat{\mathcal{N}}$ is the union of two sets; the first is isomorphic to $\mathbf{R} - \{0\}$, and contains all faithful u. i. r.'s of $\mathcal{N}$, which are in a one-to-one correspondance with the set of nontrivial characters of the center I. The second is isomorphic to $\mathbf{R}^8$ and contains all u. i. r.'s of $\mathcal{N}/I$, which is isomorphic to $\mathbf{R}^8$, too. We shall examine them separately.

A. *Faithful u. i. r.'s.* — Let $(p, t) \to \exp i\lambda t$ be the inducing character; the representation L_λ of N is given by

$$L_\lambda(p, x, t) f(\xi) = \exp i\lambda\left(t - \left\langle p, \xi - \frac{1}{2}x \right\rangle\right) f(\xi + x) \tag{4.4}$$

with $f \in L^2(\mathbf{R}^4)$.

The stabilizer for non-zero λ is the whole group; to extend L_λ to SL (2, **C**) we put $\tilde{L}_\lambda(\Lambda) f(\xi) = f(\Lambda^{-1}\xi)$ and the Lorentz invariance of the scalar product proves that $\tilde{L}_\lambda$ is indeed a representation.

Finally, if S is an arbitrary u. i. r. of SL (2, **C**) with carrier space $\mathcal{H}$, all faithful u. i. r.'s of G are of the form

$$\begin{aligned} &U_{s,\lambda}(\Lambda, p, x, t) f(\xi) \\ &\quad = \exp i\lambda\left(t - \left\langle p, \xi - \frac{1}{2}x \right\rangle\right) S(\Lambda) f(\Lambda^{-1}(x + \xi)). \end{aligned} \tag{4.5}$$

where $f \in \mathcal{H} \otimes L^2(\mathbf{R}^4)$.

We remark that the infinitesimal generators are, in this realization

$$(4.6)\qquad \begin{cases} P_\mu = -i\lambda\xi_\mu; & X_\mu = \dfrac{\partial}{\partial\xi^\mu} = \partial_\mu; \\ M_{\mu\nu} = +\xi_\mu\partial_\nu - \xi_\nu\partial_\mu + S_{\mu\nu}; & I = i\lambda; \end{cases}$$

$U_{s,\lambda}$ can be considered as acting on the momentum space. It is easy to observe that its restriction to a Poincaré subgroup is highly reducible, since the squared mass, $P_\mu P^\mu$ can take all real values.

B. *Unfaithful u. i. r.'s.* — The factor group $\mathcal{N}/I$ being Abelian, we start from a character (ξ, q) of $\mathcal{N}$ such that

$$(p, x, t) \to \exp i(\langle p, q\rangle + \langle x, \xi\rangle).$$

SL (2, **C**) acts on the right on $\hat{\mathbf{R}}^8$ by

$$\Lambda : \quad (\xi, q) \to (\Lambda^{-1}\xi . \Lambda^{-1} q).$$

We have to determine the orbits of $\hat{\mathbf{R}}^8$; we shall suppose that ξ and g are linearly independant, since otherwise there exists a 4-dimensional abelian subgroup which is represented trivially, and the quotient of such a u. i. r. by its kernel will yield a u. i. r. of $\bar{\mathcal{T}}$.

Separating $\hat{\mathbf{R}}^8$ into orbits, we find that a proper orbit is a 5-dimensional surface determined by three real constants A, B, C :

$$\xi_\mu\xi^\mu = A; \qquad q^\mu\xi_\mu = B; \qquad q_\mu q^\mu = C$$

and, in addition, if $A \geq 0$ (resp. $C \geq 0$) by sign ξ_0 (resp. sign q_0).

To find the stabilizer of such an orbit, we observe that if $\Lambda\xi = \xi$, $\Lambda q = q$ every linear combination of ξ and q is also invariant. The restriction of the Lorentz scalar product to the (ξ, q) plane yields a bilinear form, the signature of which we shall call the signature of the plane. Let u, v, be two orthogonal vectors in this plane : if one of them is timelike the signature is $(+, -)$; if one is lightlike the signature is $(0, -)$ and the bilinear form is degenerate; if both are specelike the signature is $(-, -)$. Planes with same signature have isomorphic stabilizers.

To effect the calculations is detail, we first change the labelling of the orbits. Let $D = AC - B^2$.

D can be considered as a $(+, -, -)$ quadratic form on $\mathbf{R}^3$, and its sign characterizes the (ξ, q) plane; for if one applies any linear mapping φ on the (ξ, q) plane, D is multiplied by $(\det\varphi)^2$; in fact GL (2, **R**) acts canonically on $\mathbf{R}^3$ like $\mathbf{R}^+ \times SO(2, 1)$.

We shall thus determine a proper orbit by A, B, D and, eventually, sign ξ_0, sign q_0.

Let us first consider a $(+, -)$ plane, which covers all cases $A > 0$, $C > 0$; a spatial rotation brings ξ and q in the $(0, 3)$ plane. The subgroup of SL (2, **C**) which leaves invariant every point in this plane is the subgroup U (1), such that

$$\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} : \quad x_1 + ix_2 \quad \rightarrow \quad \bar{a}^2 (x_1 + ix_2) \qquad \text{with} \quad a\,\bar{a} = 1.$$

A rapid calculation shows that for

$$\xi_1 = \xi_2 = q_1 = q_2 = 0, \qquad D = -(\xi_0 q_3 - \xi_3 q_0)^2;$$

D is always negative, and inferior to AC.

According to the values of A, C there are several ways of choosing a characteristic point (Ξ, Q) in the orbit. One can then factorize SL (2, **C**) by $\Lambda = \gamma . \tau_{\xi, q}$, such that $\gamma \in U(1)$ and $(\Lambda^{-1}\,\Xi,\ \Lambda^{-1}\,Q) = (\xi, q)$. If Ω denotes an orbit, there exists an invariant measure

$$d\mu = |\,\xi_0\,|^{-1}\,|\,\xi_0\, q_3 - q_0\, \xi_3\,|^{-1}\, d^3 \xi\, dq_1\, dq_2.$$

Then, according to Mackey's results, to every character χ of U (1) and to every choice of Ω, corresponds a u. i. r. of G of the form

(4.7) $U^{\chi, \Omega} (\Lambda, p, x, t) f (\xi, q)$

$$= \chi (\tau_{\xi, q} . \Lambda . \tau^{-1}_{\Lambda^{-1}\xi} . \Lambda^{-1}) \exp i \langle p, q \rangle \exp i \langle \xi, x \rangle f (\Lambda^{-1} \xi, \Lambda^{-1} q),$$

with $f \in L^2 (\Omega; d\mu)$.

If the (ξ, q) plane is spacelike, we can identify it with the (1, 2) plane. In this case, for $\xi_0 = q_0 = \xi_3 = q_3$, one finds $D = (\xi_1 q_2 - \xi_2 q_1)^2$; thus D is strictly positive in this case. The stabilizer of any such orbit is the multiplicative group of real numbers $\mathbf{R}^*$, acting like

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : \quad \begin{aligned} p_0 &\rightarrow \frac{1}{2} (a^2 + a^{-2})\, p_0 - \frac{1}{2} (a^2 - a^{-2})\, p_3, \\ p_3 &\rightarrow \frac{1}{2} (a^{-2} - a^2)\, p_0 + \frac{1}{2} (a^2 + a^{-2})\, p_3. \end{aligned}$$

The characters χ of $\mathbf{R}^*$ being of the form $\chi (a) = \varepsilon^n\, |\, a\,|^{i\lambda}$, with $\varepsilon = \operatorname{sign} a$, $n = 0$ or $n = 1$, and λ real, the expression of the u. i. r. $V^{\chi, \Omega}$ is the same as (4.7).

Finally, if the (ξ, q) plane contains a spacelike vector V and a lightlike vector V, mutually orthogonal, we see that

$$\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2 = 0$$

hence D is also zero. Fixing, for example

$$\xi_0 - \xi_3 = q_0 - q_3 = \xi_1 = q_1 = 0,$$

we obtain as little group $\mathbf{R} \times \mathbf{Z}_2$, formed by the set of matrices

$$(\varepsilon, \alpha) = \begin{pmatrix} \varepsilon & \varepsilon\alpha \\ 0 & \varepsilon \end{pmatrix}, \qquad \varepsilon = \pm 1,$$

which act on $\mathbf{R}^4$ like

$$(\varepsilon, \alpha) : \begin{pmatrix} p_0 + p_3 \\ p_2 \\ p_0 - p_3 \\ p_1 \end{pmatrix} \to \begin{pmatrix} p_0 + p_3 + 2\alpha p_1 + \alpha^2 (p_0 - p_3) \\ p_2 \\ p_0 - p_3 \\ p_1 + \alpha (p_0 - p_1) \end{pmatrix}.$$

The characters of this abelian group are of the form $\chi(\varepsilon, \alpha) = \varepsilon^n e^{i\lambda\alpha}$ with λ real and $n = 0, 1$. With the convenient choice of the section $\tau_{\xi, q}$, we obtain again the formula (4.7) for $U^{\chi,\Omega}$ with Ω belonging to the subset verifying $D = 0$.

These results are summarized in the following table :

<table>
<tr><th>Stabilizer</th><th colspan="2">Range of A, B, C, sign ξ_0, sign q_0</th></tr>
<tr><td rowspan="3">U (1)</td><td rowspan="3">$B^2 > AC$</td><td>$A < 0, \quad C < 0, \quad B \neq 0$</td></tr>
<tr><td>$A \geq 0, \quad C < 0,$ sign $\xi_0 = \pm 1$
$A < 0, \quad C \geq 0,$ sign $q_0 = \pm 1$</td></tr>
<tr><td>$A \geq 0, \quad C \geq 0, \quad B \neq 0,$ sign $\xi_0 .$ sign $q_0 =$ sign B</td></tr>
<tr><td rowspan="2">$\mathbf{Z}_2 \times \mathbf{R}$</td><td rowspan="2">$B^2 = AC$</td><td>$A \leq 0, \quad C > 0$</td></tr>
<tr><td>$A = B = 0 > C$ sign $\xi_0 = \pm 1$
$A < 0 = B = C$ sign $q_0 = \pm 1$</td></tr>
<tr><td>$\mathbf{R}^*$</td><td>$AC > B^2$</td><td>$A < 0, \quad C < 0$</td></tr>
</table>

Here we have determined all u. i. r.'s of G. There is one series of faithful representations and three series of proper unfaithful ones, and, in addition, the whole series of representations of $\overline{\mathfrak{L}}$. Each series is divided into may subseries, two u. i. r.'s of the same series corresponding to the same little group.

The results of sections 2 and 3 help to determine the faithful u. i. r.'s only, since the unfaithful ones are available by the classical results of Mackey.

We shall end with some remarks on this example. We shall first compare the reduction on $\overline{\mathfrak{L}}$ of faithful and unfaithful u. i. r.'s of G.

Let, in (4.5), S be the identity representation of SL (2, $\mathbb{C}$) and $\lambda = 1$. The restriction U of $U_{s,\lambda}$, on $\overline{\mathfrak{L}}$ is highly reducible : U is the direct integral over all values, positive and negative, of the squared mass, $\xi_\mu \xi^\mu$, (since translations are dropped) of the spinless u. i. r.'s of $\overline{\mathfrak{L}}$. On the other hand, taking a proper unfaithful u. i. r., say, $A = -C = 1$; $B = 0$, we see that the mass is fixed. The restriction on $\overline{\mathfrak{L}}$ is a direct sum over all spins, as easily checked by a technique quite similar to the technique of reducing the tensor product of two u. i. r.'s of $\overline{\mathfrak{L}}$.

Another fact we shall notice is that G is a Wigner-Inonü contraction of the conformal group SO (4.2).

Let $M_{\mu\nu}$, K_μ, Q_μ, D be the generators of $\mathfrak{so}$ (4.2) with the commutation relations

$$(4.2)\quad [M_{\mu\nu}, K_\rho] = g_{\nu\rho} K_\mu - g_{\mu\rho} K_\nu; \qquad [M_{\mu\nu}, Q_\rho] = g_{\nu\rho} Q_\mu - g_{\mu\rho} Q_\nu,$$

$$(4.8)\qquad [D, K_\mu] = K_\mu; \qquad [D, Q_\mu] = Q_\mu,$$

$$(4.9)\qquad [K_\mu, Q_\nu] = 2 (g_{\mu\nu} D - M_{\mu\nu})$$

and (4.1).

Putting

$$X_\mu = -\alpha Q_\mu, \qquad P_\mu = -\frac{1}{2}\alpha K_\mu, \qquad I = \alpha^2 D,$$

one obtains

$$(4.10)\qquad [I, X_\mu] = \alpha^2 X_\mu; \qquad [I, P_\mu] = \alpha^2 P_\mu,$$

$$(4.11)\qquad [P_\mu, X_\nu] = g_{\mu\nu} I - \alpha^2 M_{\mu\nu}.$$

For non zero α these relations together with (4.1), (4.2) give a Lei algebra isomorphic to $\mathfrak{so}$ (4.2). Bur for $\alpha = 0$, one obtains $\mathfrak{g}$.

We shall briefly sketch how this contraction is reflected in representations. Let $\mathfrak{g}_\alpha$ be the algebra determined by (4.1), (4.2); (4.10), (4.11), and $\mathfrak{g}_0 = \mathfrak{g}$. We consider induced u. i. r.'s of SO (4.2),

$U^{\alpha}_{S,\lambda}$, (the little group being the Weyl group), generated by $M_{\mu\nu}$, P_μ, I. The expression of the generators in such a representation are :

$$(4.12)\quad \begin{cases} M_{\mu\nu} = \xi_\mu\, \partial_\nu - \xi_\nu\, \partial_\mu + S_{\mu\nu}, \\ X_\mu = \partial_\mu, \\ I = -\,\alpha^2\, \xi_\mu\, \partial^\mu + i\,\lambda - 2\,\alpha^2, \\ P_\mu = -\,\alpha^2 \left[\xi^\rho\, S_{\rho\mu} + \dfrac{1}{2}\xi^\rho\, \xi_\rho\, \partial_\mu - \xi_\mu\, \xi^\rho\, \partial_\rho - 2\,\xi_\mu \right] - i\,\lambda\xi_\mu \end{cases}$$

where $S_{\mu\nu}$ and $i\,\lambda$ are the generators of the inducing representation (which has as kernel the 4-dimensional ideal $\{ P_\mu \}$).

For $\alpha = 0$ one obtains relations (4.6). It appears thus that $U^{\alpha}_{S,\lambda}$ contracts to $U_{S,\lambda}$, as $\mathfrak{g}_\alpha$ contracts to $\mathfrak{g}$, while the little groups undergo the same contraction. We leave open the further examination of the contraction of SO (4.2) to G.

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