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Section A :
Physique théorique.

On an inverse scattering problem with an energy-dependent potential (*)

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ABSTRACT. — The inverse scattering problem is considered for the radial s-wave Schrödinger equation with the energy-dependent potential $V^{+}(E, x) = U(x) + 2\sqrt{E}\,Q(x)$, in the case of real potential $U(x)$ and $Q(x)$, when there is no bound state. To each element $(U(x), Q(x))$ of a large class $\mathcal{C}'$ of pairs of potentials is associated a function $S^{+}(k)\,(k \in \mathbf{R})$ called the " characteristic function " : $S^{+}(k)\,(k \geqq 0)$ represents the scattering " matrix " and $S^{+}(k)\,(k < 0)$ plays the role of a parameter in the solution of the inverse problem. It is proved that there exists at most one pair $(U(x), Q(x))$ in $\mathcal{C}'$ which admits a given function $S^{+}(k)\,(k \in \mathbf{R})$ as its characteristic function, and, under very general assumptions on $S^{+}(k)\,(k \in \mathbf{R})$, this pair is explicity constructed. The method of solution used generalizes the one given by Marchenko in the case $Q(x) = 0$. The results obtained also allow us to solve an inverse scattering problem associated with the radial s-wave Klein-Gordon equation of zero mass with a static potential — in this case the characteristic function $S^{+}(k)\,(k \in \mathbf{R})$ for $k \geqq 0$ represents the scattering " matrix " for a particle and for $k \leqq 0$ represents the scattering " matrix " for the correspondent antiparticle —.

RÉSUMÉ. — On étudie le problème inverse de la diffusion associé à l'équation de Schrödinger radiale pour l'onde s avec le potentiel dépendant de l'énergie $V^{+}(E, x) = U(x) + 2\sqrt{E}\,Q(x)$, dans le cas où les

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potentiels U (x) et Q (x) sont réels et où il n'y a pas d'état lié. A tout élément (U (x), Q (x)) d'une classe très générale $\mathcal{C}'$ de couples de potentiels on associe une fonction $S^+(k)$ $(k \in \mathbf{R})$ appelée « fonction caractéristique » : $S^+(k)$ $(k \geq 0)$ représente la « matrice » de diffusion et $S^+(k)$ $(k < 0)$ joue le rôle d'un paramètre dans la résolution du problème inverse. On montre qu'il existe au plus un couple (U (x), Q (x)) de $\mathcal{C}'$ qui admet une fonction donnée $S^+(k)$ $(k \in \mathbf{R})$ pour fonction caractéristique, et, pour des hypothèses très générales sur $S^+(k)$ $(k \in \mathbf{R})$, on construit explicitement ce couple. La méthode de résolution utilisée généralise celle donnée par Marchenko dans le cas Q $(x) = 0$. Les résultats obtenus permettent de résoudre également un problème inverse associé à l'équation de Klein-Gordon radiale pour l'onde s, de masse nulle et avec un potentiel statique — dans ce cas, la fonction caractéristique $S^+(k)$ $(k \in \mathbf{R})$ représente pour $k \geq 0$ la « matrice » de diffusion pour une particule et représente pour $k \leq 0$ la « matrice » de diffusion pour l'antiparticule correspondante —.

1. Introduction and preliminary results

In this paper we are interested in the inverse scattering problem for the radial s-wave Schrödinger equation

$$y^{+\prime\prime} + [\mathrm{E} - \mathrm{V}^+(\mathrm{E}, x)]\, y^+ = 0 \qquad (x \geq 0), \tag{1.1}$$

with the energy-dependent potential

$$\mathrm{V}^+(\mathrm{E}, x) = \mathrm{U}(x) + 2\sqrt{\mathrm{E}}\, \mathrm{Q}(x) \qquad (x \geq 0,\ \mathrm{E} \in \mathbf{C}), \tag{1.2}$$

where $\sqrt{\mathrm{E}}$ is defined as

$$\sqrt{\mathrm{E}} = -\,|\mathrm{E}|^{1/2}\, e^{\frac{i}{2}\operatorname{Arg}\mathrm{E}} \qquad (0 < \operatorname{Arg}\mathrm{E} \leq 2\pi). \tag{1.3}$$

This problem is of interest not only for its own sake, but also because there are other inverse problems in physics which can be reduced to it. In particular we shall apply our results to the solution of an inverse problem associated with the radial s-wave Klein-Gordon equation of zero mass with a static potential. Other applications will be discussed in a forthcoming publication.

It has been shown [1] that this inverse problem can be reduced to the solution of two coupled integral equations ([1], formula (5.17)) with a coupling condition ([1], formula (5.20)). But these equations have not been solved in general, and it has not been proved in general that the potentials constructed from these equations — in the case where they can be solved — reproduce the input data of the inverse problem. These difficulties are not surprising for complex U (x) and Q (x) since they already arise for complex U (x) and for Q $(x) = 0$. Here, we propose to overcome them for real U (x) and Q (x), when there is no bound state. This paper is an abridged version of part of an unpublished work [2] to which we refer for more details.

Before describing more explicitly our inverse problem, we give some preliminary results which will be useful in studying it. It is useful to consider the equations

$$y^{\pm\prime\prime} + [k^2 - \mathrm{V}^{\pm}(k, x)]\, y^{\pm} = 0 \qquad (x \geqq 0), \tag{1.4}$$

$$\mathrm{V}^{\pm}(k, x) = \mathrm{U}(x) \pm 2\, k\, \mathrm{Q}(x) \qquad (x \geqq 0,\ k \in \mathbf{C}); \tag{1.5}$$

if we set $k = \sqrt{\mathrm{E}}$ $(\mathrm{E} \in \mathbf{C})$, we see that for the index " + " these formulas reduce to (1.1) and (1.2). The Jost solution $f^{\pm}(k, x)$ $(\mathrm{Im}\, k \leqq 0)$ of (1.4)-(1.5) is defined as the solution satisfying the condition

$$\lim_{x \to \infty} e^{ikx} f^{\pm}(k, x) = 1.$$

The Jost function $f^{\pm}(k)$ is defined by the formula $f^{\pm}(k) = f^{\pm}(k, 0)$. Let $\mathcal{C}'$ be the set of pairs of real potentials (U (x), Q (x)) which satisfy assumptions A_1, A_2 and A_3 :

Assumption A_1 : U (x) *is continuous for* $x \geqq 0$ *and* $x\,\mathrm{U}(x)$ *is integrable.*

Assumption A_2 : Q (x) *is continuously differentiable for* $x \geqq 0$, *and* Q (x) *and* $x\,\mathrm{Q}'(x)$ *are integrable.*

Assumption A_3 : $f^{+}(k) \neq 0$ *if* $\mathrm{Im}\, k < 0$ *or* $k = 0$.

Assumption A_3 expresses the hypothesis that there is no bound state. Let $\mathrm{S}^{\pm}(k)$ $(k \in \mathbf{R})$ be the function defined as

$$\mathrm{S}^{\pm}(k) = \frac{f^{\pm}(k)}{f^{\mp}(-k)} \qquad (k \in \mathbf{R}). \tag{1.6}$$

The function $\mathrm{S}^{+}(k)$ $(k \geqq 0)$ represents the scattering " matrix " of the system. The function $\mathrm{S}^{+}(k)$ $(k \in \mathbf{R})$ though not known experimentally for $k < 0$ is still useful in studying the inverse problem. We call it the " characteristic function " associated with the pair (U (x), Q (x)) belonging to $\mathcal{C}'$. In the two following lemmas, we give the principal properties of the functions $f^{\pm}(k)$ and $\mathrm{S}^{\pm}(k)$. We set

$$\mathrm{F}_0^{\mp} = \exp\left(\mp i \int_0^{\infty} \mathrm{Q}(t)\, dt\right).$$

Lemma 1 :

(P_1) $f^{\pm}(k)$ *is continuous for* $\mathrm{Im}\, k \leqq 0$ *and analytic for* $\mathrm{Im}\, k < 0$;

(1.7) (P_2) $\lim_{|k| \to \infty} f^{\pm}(k) = \mathrm{F}_0^{\mp}$ $\qquad (\mathrm{Im}\, k \leqq 0)$;

(1.8) (P_3) $f^+(0) = f^-(0)$;

(1.9) (P_4) $\overline{f^{\pm}(k)} = f^{\mp}(-\bar{k}) \qquad (\operatorname{Im} k \leqq 0)$;

(1.10) (P_5) $f^{\pm}(k) \neq 0 \qquad (\operatorname{Im} k \leqq 0)$;

(P_6) *there exists an integrable function* $a^{\pm}(t)$ *such that*

$$f^{\pm}(k) = F_0^{\mp} + \int_0^{\infty} a^{\pm}(t)\, e^{-ikt}\, dt \qquad (\operatorname{Im} k \leqq 0), \tag{1.11}$$

$$\overline{a^+(t)} = a^-(t). \tag{1.12}$$

LEMMA 2 :

(Q_1) $S^{\pm}(k)$ *is continuous for* $k \in \mathbf{R}$;

(1.13) (Q_2) $|S^{\pm}(k)| = S^{\pm}(0) = 1$;

(1.14) (Q_3) $S^{\pm}(\infty) = S^{\pm}(-\infty) = (F_0^{\mp})^2$;

(1.15) (Q_4) $S^{\pm}(-k) = [S^{\mp}(k)]^{-1} = \overline{S^{\mp}(k)}$;

(Q_5) *there exists an integrable function* $s^{\pm}(t)$ *such that*

$$S^{\pm}(k) = (F_0^{\mp})^2 + \int_{-\infty}^{\infty} s^{\pm}(t)\, e^{-ikt}\, dt \qquad (k \in \mathbf{R}), \tag{1.16}$$

$$\overline{s^+(t)} = s^-(t); \tag{1.17}$$

(1.18) (Q_6) $\arg S^{\pm}(k) \big|_{-\infty}^{\infty} = 0$,

$\arg S^{\pm}(k)$ *being defined as a continuous function, equal for each k to an argument of the complex number* $S^{\pm}(k)$.

These lemmas follow easily from [1]. (Q_6) expresses the Levinson theorem in our case. On the other hand, it has been proved in [1] that the Jost solution $f^{\pm}(k, x)$ of (1.4) is generated by two functions $F^{\pm}(x)$ and $A^{\pm}(x, t)$:

$$f^{\pm}(k, x) = F^{\pm}(x)\, e^{-ikx} + \int_x^{\infty} A^{\pm}(x, t)\, e^{-ikt}\, dt \qquad (\operatorname{Im} k \leqq 0,\ x \geqq 0), \tag{1.19}$$

where

$$F^{\pm}(x) = \exp\left(\mp i \int_x^{\infty} Q(t)\, dt\right) \qquad (x \geqq 0), \tag{1.20}$$

and where $A^{\pm}(x, t)$ is a function such that

$$U(x) = \frac{f^{\pm}(x)}{F^{\pm}(x)} \qquad (x \geqq 0), \tag{1.21}$$

$$f^{\pm}(x) = F^{\pm\prime\prime}(x) - 2\frac{d}{dx} A^{\pm}(x, x) + 2\frac{F^{\pm\prime}(x)}{F^{\pm}(x)} A^{\pm}(x, x) \qquad (x \geqq 0). \tag{1.22}$$

Furthermore, there exist two coupled integral equations ([1], formula (5.17)) connecting $A^+(x, t)$, $A^-(x, t)$, $F^+(x)$ and $F^-(x)$ with $s^+(t)$ and $s^-(t)$. $A^+(x, t)$ and $A^-(x, t)$, $F^+(x)$ and $F^-(x)$, $s^+(t)$ and $s^-(t)$ being complex conjugate with our assumptions, these integral equations are also complex conjugate and are equivalent to the following integral equation

$$(1.23)\quad \begin{cases} A^+(x, t) = \overline{F^+(x)}\, s^+(x+t) + \displaystyle\int_x^\infty s^+(u+t)\, \overline{A^+(x, u)}\, du, \\ \qquad\qquad (t \geqq x \geqq 0). \end{cases}$$

This equation will play the role in our study of the Marchenko equation (*see* [3], chapter III) in the case $Q(x) = 0$.

Our inverse problem is the construction of the pairs $(U(x), Q(x))$ of $\mathcal{C}'$ which admit a given function $S^+(k)$ $(k \geqq 0)$ as its scattering " matrix ". In fact, more conveniently, we are given a function $S^+(k)$ $(k \in \mathbf{R})$, the part $S^+(k)$ $(k < 0)$ playing the role of a parameter, and we look for the pairs $(U(x), Q(x))$ of $\mathcal{C}'$ whose associated characteristic function is precisely $S^+(k)$ $(k \in \mathbf{R})$. For the present time, we make the following assumptions on the input function $S^+(k)$ $(k \in \mathbf{R})$:

Assumption I_1. — *The function* $S^+(k)$ $(k \in \mathbf{R})$ *satisfies the following conditions :*

(1.24) (Q'_2) $|S^+(k)| = S^+(0) = 1$;

(Q'_5) *there exists an integrable function* $s^+(t)$ *and a complex number* F_0^+ *such that*

$$(1.25)\qquad S^+(k) = (F_0^+)^2 + \int_{-\infty}^{\infty} s^+(t)\, e^{-ikt}\, dt$$

[*note that,* $S^+(k)$ *being known,* $(F_0^+)^2$ *is uniquely determined but not* F_0^+];

(1.26) (Q'_6) $\arg S^+(k)\,\big|_{-\infty}^{\infty} = 0$.

Assumption I_2. — *The fonction* $s^-(t)$ *is continuously differentiable for* $t \geqq 0$, *and the function* $t s^{+\prime}(t)$ $(t \geqq 0)$ *is integrable.*

Note that I_1 and I_2 must be necessarily satified for a function $S^+(k)$ $(k \in \mathbf{R})$ to be the characteristic function of a pair $(U(x), Q(x))$ belonging to $\mathcal{C}'$. As a consequence of I_1 and I_2, we have

$$(1.27)\qquad |s^+(t)| \leqq \sigma_0(t),$$

where

$$(1.28)\qquad \sigma_0(t) = \int_t^\infty |s^{+\prime}(u)|\, du \qquad (t \geqq 0);$$

clearly, the functions $\sigma_0(t)$ and $t\,\sigma_0^2(t)$ are integrable. We also have

$$(1.29)\qquad s^-(t) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} [S^+(k) - (F_0^+)^2]\, e^{ikt}\, dk \qquad (t > 0).$$

The starting point of our method of solution of the inverse problem is the investigation of the integral equation (1.23). In section 2, we prove our main theorem, and using it, we show in section 3, that, $F^+(x)$ being fixed, the equation (1.23) has a unique solution $A^+(x, t)$. In section 4, we make the dependence of $A^+(x, t)$ on $F^+(x)$ explicit. To determine $F^+(x)$, we use a differential equation [formula (4.6)] instead of the coupling condition introduced in [1] ([1], formula (5.20)) (we explain the reason for this at the end of section 7). So we can define in section 6 a pair of potentials $(U(x), Q(x))$. In section 8, we prove, with the additional assumptions I'_2 and I_3, that this pair belongs to $\mathcal{C}'$ and admit the input function $S^+(k)$ $(k \in \mathbf{R})$ as its characteristic function. The principal results of this paper are theorems 7 and 8. These results are comparable with the classic ones obtained by Marchenko for real $U(x)$ and for $Q(x) = 0$. They allow us in section 9 to solve the inverse problem for the radial s-wave Klein-Gordon equation of zero mass with a static potential (theorems 9 and 10).

2. The main theorem

The following theorem will be useful to solve the equation (1.23). It is similar to a well known result for real $U(x)$ and for $Q(x) = 0$ ([4], p. 85).

THEOREM 1. — *Let a function* $S^+(k)$ $(k \in \mathbf{R})$ *satisfy assumption* I_1. *Let* $S^-(k)$ $(k \in \mathbf{R})$ *and* F_0^- *be the function and the number defined as*

$$(2.1)\qquad S^-(k) = [S^+(-k)]^{-1}, \qquad F_0^- = [F_0^+]^{-1}.$$

Then :

1. *The functions* $S^+(k)$ *and* $S^-(k)$ *satisfy the conditions* (Q_1) *to* (Q_6) *and the numbers* F_0^+ *et* F_0^- *are complex conjugate and of magnitude unity.*

2. *There exist two functions* $f^+(k)$ *and* $f^-(k)$ $(\operatorname{Im} k \leq 0)$ *satisfying the conditions* (P_1) *to* (P_6) *and the relation* (1.6);

3. *If* $f_1^+(k)$ *and* $f_1^-(k)$ $(\operatorname{Im} k \leq 0)$ *are two bounded functions satisfying the conditions* (P_1), (P_4) *and the relation*

$$(2.2)\qquad f_1^\pm(k) = S^\pm(k)\, f_1^\mp(-k) \qquad (k \in \mathbf{R}),$$

then, there exists a real constant K *such that*

$$(2.3)\qquad f_1^\pm(k) = K\, f^\pm(k) \qquad (\operatorname{Im} k \leq 0);$$

in particular, $K = 1$ *if* $f_1^+(k)$ *satisfies also* (P_2).

The first part of the theorem is obvious. To prove the second part, we consider the function $\operatorname{Log} \frac{S^{\pm}(k)}{(F_0^{\mp})^2}$, defined as the continuous function, equal for each k to one value of the logarithm of the complex number $\frac{S^{\pm}(k)}{(F_0^{\mp})^2}$, which goes to zero as $k \to \pm\infty$. Because of the conditions (Q_5) and (Q_6), there exist two integrable functions $\gamma^{+}(t)$ and $\gamma^{-}(t)$ such that

$$(2.4) \qquad \operatorname{Log} \frac{S^{\pm}(k)}{(F_0^{\mp})^2} = \int_0^{\infty} (\gamma^{\pm}(t)\, e^{-ikt} - \gamma^{\pm}(t)\, e^{ikt})\, dt \qquad (k \in \mathbf{R});$$

to show this, we use the Wiener-Levy theorem under a slightly more general form (*see* [2]) than the classic ones ([5], p. 63; [4], p. 102). It is then not difficult to see that the functions $f^{+}(k)$ et $f^{-}(k)$ defined as

$$(2.5) \qquad f^{\pm}(k) = F_0^{\mp} \exp\left(\int_0^{\infty} \gamma^{\pm}(t)\, e^{-ikt}\, dt\right) \qquad (\operatorname{Im} k \leqq 0),$$

satisfy the conditions (P_1) to (P_6) and the relation (1.6). To prove the last part of the theorem, we consider the function $\frac{f_1^{\pm}(k)}{f^{\pm}(k)}$. It is bounded and continuous for $\operatorname{Im} k \leqq 0$, analytic for $\operatorname{Im} k < 0$ and real for $k \in \mathbf{R}$. By the Schwarz reflection principle it has a bounded analytic continuation into the entire complex plane. Therefore, by the Liouville theorem, this function is constant. Hence the statement 3.

3. Existence and uniqueness of the solution $A^{+}(x, t)$ of the equation (1.23) for fixed $F^{+}(x)$

Having fixed $F^{+}(x)$, we seek the solutions $A^{+}(x, t)$ of (1.23) which are, for each fixed $x \geqq 0$, continuous and integrable in t for $t \geqq x$. To this end we consider, for $t_0 \geqq 0$, the following equation

$$(3.1) \qquad y(t) = f(t) + \int_{t_0}^{\infty} s^{+}(t+u)\, \overline{y(u)}\, du,$$

in which $f(t)$ (the data) and $y(t)$ (the unknown) belong to $L^1(t_0, \infty)$, the space of classes of functions integrable for $t \geqq t_0$ (it is known that in this space two integrable functions almost everywhere equal are represented by the same element). In what follows $L^1(t_0, \infty)$ will be considered as a real vector space. $L^1(t_0, \infty)$ is a Banach space with the norm

$$(3.2) \qquad \| y(t) \| = \int_{t_0}^{\infty} | y(t) |\, dt.$$

Let M be the linear operator in $L^1(t_0, \infty)$ defined as

$$M(y(t)) = \int_{t_0}^{\infty} s^+(t+u)\,\overline{y(u)}\,du. \tag{3.3}$$

With the help of a Frechet-Kolmogorov theorem ([6], p. 275) one easily prove that the operator M is compact. Therefore, by the Fredholm Alternative theorem, the equation (3.1) has a unique solution if the associated homogeneous equation has only the trivial solution. Let us consider this homogeneous equation

$$y(t) = \int_{t_0}^{\infty} s^+(t+u)\,\overline{y(u)}\,du \qquad (t \geq t_0). \tag{3.4}$$

From (3.4) we obtain

$$\int_{t_0}^{\infty} \overline{y(t)}\,y(t)\,dt = \int_{t_0}^{\infty} \overline{y(t)}\,dt \int_{t_0}^{\infty} s^+(t+u)\,\overline{y(u)}\,du. \tag{3.5}$$

Setting

$$\tilde{y}(k) = \int_{t_0}^{\infty} y(t)\,e^{-ikt}\,dt \qquad (\operatorname{Im} k \leq 0), \tag{3.6}$$

we see from theorems on Fourier transforms that (3.6) can be written in the form

$$\int_{-\infty}^{\infty} \overline{\tilde{y}(k)}\,\tilde{y}(k)\,dk = \int_{-\infty}^{\infty} [\overline{\tilde{y}(k)}]^2\,S^+(k)\,dk. \tag{3.7}$$

Since we clearly have $\overline{\tilde{y}(k)}\,\tilde{y}(k) = |[\overline{\tilde{y}(k)}]^2\,S^+(k)|$, (3.7) implies that $\overline{\tilde{y}(k)}\,\tilde{y}(k) = [\overline{\tilde{y}(k)}]^2\,S^+(k)$. Hence

$$\tilde{y}(k) = S^+(k)\,\overline{\tilde{y}(k)} \qquad (k \in \mathbf{R}). \tag{3.8}$$

It is clear, if we set $f_1^+(k) = \tilde{y}(k)$ and $f_1^-(k) = \overline{\tilde{y}(-\bar{k})}$ $(\operatorname{Im} k \leq 0)$, that the bounded functions $f_1^+(k)$ and $f_1^-(k)$ satisfy (P_1), (P_1) and (2.2). By theorem 1, there exists a constant K such that

$$\tilde{y}(k) = K\,f^+(k) \qquad (\operatorname{Im} k \leq 0). \tag{3.9}$$

K is equal to zero since, as $|k|$ goes to infinity, $\tilde{y}(k)$ goes to zero and $f^+(k)$ goes to $F_0^+ \neq 0$. As a consequence, (3.4) has only the trivial solution and (3.1) has a unique solution for each $f(t)$ belonging to $L^1(t_0, \infty)$. It is now easy to deduce from this the following theorem :

THEOREM 2. — *For fixed* $F^+(x)$, *equation* (1.23) *has a unique solution* $A^+(x, t)$ *in the space of functions of* (x, t) $(t \geq x \geq 0)$ *which are, for fixed* $x \geq 0$, *continuous and integrable in* t *for* $t \geq x$.

4. The functions $\alpha^{\pm}(x, t)$ and $\beta^{\pm}(x, t)$ and a useful differential equation

We propose to make the dependence of $A^+(x, t)$ on $F^+(x)$ explicit. Let $a_1^+(x, t)$ be the solution of the equation (1.23) corresponding to $F^+(x) = 1$ and $a_2^+(x, t)$ be the one corresponding to $F^+(x) = -i$. Let $\alpha^+(x, t)$ and $\beta^-(x, t)$ be the functions defined as

$$(4.1)\quad \begin{cases} \alpha^+(x, t) = \dfrac{a_1^+(x, t) - i a_2^+(x, t)}{2} \qquad \beta^-(x, t) = \dfrac{a_1^+(x, t) + i a_2^+(x,t)}{2} \\ \qquad\qquad (t \geqq x \geqq 0), \end{cases}$$

and let $\alpha^-(x, t)$ and $\beta^+(x, t)$ be the complex conjugate functions. We easily obtain the following theorem:

THEOREM 2 *bis*. — *If* $A^+(x, t)$ *is the solution of equation* (1.23) *and* $A^-(x, t)$ *is the complex conjugate function, then*

$$(4.2)\quad A^{\pm}(x, t) = F^{\mp}(x)\, \alpha^{\pm}(x, t) + F^{\pm}(x)\, \beta^{\mp}(x, t) \qquad (t \geqq x \geqq 0),$$

where $F^-(x)$ *is defined as the complex conjugate function of* $F^+(x)$.

$\psi_x^{\pm}(x, t)$ and $\varphi_x^{\pm}(x, t)$ $(t \geqq x \geqq a)$ being the functions defined in [1] we obviously have

$$(4.3)\quad \alpha^{\pm}(x, t) = \psi_x^{\pm}(x, t), \qquad \beta^{\pm}(x, t) = \varphi_x^{\pm}(x, t) \qquad (t \geqq x \geqq a).$$

One can show that the functions $\alpha^{\pm}(x, t)$ and $\beta^{\pm}(x, t)$ are continuously differentiable for $t \geqq x \geqq 0$ and that

$$(4.4)\quad \begin{cases} |\alpha^{\pm}(x, t)| \leqq C\, \sigma_0(x + t), \qquad |\beta^{\pm}(x, t)| \leqq C\, \sigma_0(x + t) \\ \qquad\qquad (t \geqq x \geqq 0), \end{cases}$$

$$(4.5)\quad \begin{cases} \left|\dfrac{d\alpha^{\pm}}{dx}(x, x)\right| \leqq C\,[\sigma_0^2(2x) + |s^{+\prime}(2x)|], \\ \left|\dfrac{d\beta^{\pm}}{dx}(x, x)\right| \leqq C\,[\sigma_0^2(2x) + |s^{+\prime}(2x)|] \\ \qquad\qquad (x \geqq 0), \end{cases}$$

where C is a general positive constant. For this, one first investigate the functions $a_1^+(x, t)$ and $a_2^+(x, t)$ by using arguments similar to those used in ([3], chapter V).

In the sequel, we shall need the following result:

THEOREM 3. — *The differential equation*

$$(4.6\,a)\quad \begin{aligned} z' = {} & 2i\,\alpha^+(x, x)\, e^{iz} - 2i\,\alpha^-(x, x)\, e^{-iz} \\ & - 2i\,\beta^+(x, x) + 2i\,\beta^-(x, x) \qquad (x \geqq 0), \end{aligned}$$

$$(4.6\ b) \qquad z(\infty) = 0,$$

has a unique solution in the space of real functions differentiable for $x \geqq 0$.

The right hand side of (4.6 *a*) being written $f(x, z)$, it is clear from (4.4) that there exists a positive constant k such that, for every $x \geqq 0$, and for every z, z_1 and z_2 real, we have

$$(4.7) \qquad |f(x, z)| \leqq k\, \sigma_0(2x),$$

$$(4.8) \qquad |f(x, z_2) - f(x, z_1)| \leqq k\, \sigma_0(2x)\, |z_2 - z_1|.$$

Using these bounds, we easily prove that the sequence of real functions $z_n(x)$ defined by

$$(4.9) \qquad z_{n+1}(x) = \int_\infty^x f(t, z_n(t))\, dt, \qquad z_0(x) = 0 \qquad (x \geqq 0,\ n \in \mathbf{N}),$$

converges to a function which is a solution of (4.6), and that this solution is unique.

5. The uniqueness theorem

We propose to prove the following theorem :

THEOREM 4. — *If there exists a pair of potentials belonging to* $\mathcal{C}'$ *which has a given function* $S^+(k)$ $(k \in \mathbf{R})$ *as its characteristic function, then it is unique and it is given by relations* (5.4) *and* (1.21).

Let us first recall that if a function $S^+(k)$ $(k \in \mathbf{R})$ is the characteristic function of a pair $(U(x), Q(x))$ of $\mathcal{C}'$, it satisfies the conditions I_1 and I_2, so that the results of sections 2 to 4 hold. In particular, since the functions $F^+(x)$ and $A^+(x, t)$ which generate the Jost solution satisfy (1.23), the relation (4.2) holds. We use this relation to write the equation

$$(5.1) \qquad f^+(x)\, F^-(x) = F^+(x)\, f^-(x) \qquad (x \geqq 0),$$

which follows from (1.21). Setting $y(x) = [F^+(x)]^2$, we find

$$(5.2) \qquad \frac{d}{dx}\left(\frac{y'}{y} + 2\, y\, \alpha^-(x, x) - \frac{2}{y}\, \alpha^+(x, x) + 2\, \beta^+(x, x) - 2\, \beta^-(x, x)\right) = 0 \qquad (x \geqq 0).$$

It is easy to deduce from (5.2) the equation

$$(5.3) \qquad \frac{y'}{y} + 2\, y\, \alpha^-(x, x) - \frac{2}{y}\, \alpha^+(x, x) + 2\, \beta^+(x, x) - 2\, \beta^-(x, x) = 0 \qquad (x \geqq 0).$$

Let us set

$$z(x) = 2\int_x^\infty Q(y)\,dy \qquad (x \geqq 0). \tag{5.4}$$

We have $y(x) = \exp(-iz(x))$. Inserting this expression in (5.3), we find that $z(x)$ is the solution of the differential equation (4.6). Theorem 4 follows then from theorems 2 and 3, and from relations (5.4) and (1.21).

6. Determination of $Q(x)$, $F^+(x)$ and $U(x)$

In this section, we make again the assumptions I_1 and I_2 on the input function $S^+(k)$ $(k \in \mathbf{R})$. The results of section 5 pave the way to define $U(x)$ and $Q(x)$. $z(x)$ being the unique solution of (4.6), $Q(x)$ will be defined as

$$Q(x) = -\frac{z'(x)}{2} \qquad (x \geqq 0). \tag{6.1}$$

Clearly $Q(x)$ is real, continuously differentiable for $x \geqq 0$ and satisfies the bounds

$$|Q(x)| \leqq C\,\sigma_0(2x) \qquad (x \geqq 0), \tag{6.2}$$

$$|Q'(x)| \leqq C\,[\sigma_0^2(2x) + |s^{+\prime}(2x)|] \qquad (x \geqq 0). \tag{6.3}$$

(6.2) and (6.3) show that the functions $Q(x)$ and $x\,Q'(x)$ are integrable. $F^{\pm}(x)$ is defined from $Q(x)$ by (1.20). One can prove that the function $f^{\pm}(x)$ defined by (1.22) have a meaning. Starting from (4.6) we easily find the relation (5.1), and we can therefore define $U(x)$ by (1.21). $U(x)$ is real, continuous for $x \geqq 0$ and satisfies the bound

$$|U(x)| < C\,[\sigma_0^2(2x) + |s^{+\prime}(2x)|] \qquad (x \geqq 0), \tag{6.4}$$

so that $U(x)$ is integrable for $x \geqq 0$. Hence the following theorem :

THEOREM 5. — *The functions* $U(x)$ *and* $Q(x)$ $(x \geqq 0)$ *defined by* (1.21) *ond* (6.1) *are real and satisfy assumptions* A_1 *and* A_2.

7. The Jost solution associated with $(U(x), Q(x))$

We make the additional assumption I_2' :

ASSUMPTION I_2'. — *The function* $s^+(t)$ *is twice continuously differentiable for* $t \geqq 0$, *and the function* $ts^{+\prime\prime}(t)$ *is integrable.*

We will prove the following theorem :

THEOREM 6. — *The function* $f^{\pm}(k, x)$ *defined by the relation* (1.19) *in which* $F^{\pm}(x)$ *is defined in section* 6 *and* $A^{+}(x, t)$ *is the solution of* (1.23), *is the Jost solution of the differential equation* (1.4) *for* $U(x)$ *and* $Q(x)$ *defined in section* 6.

One can prove, with the assumptions I_1, I_2 and I'_2, that the function

$$(7.1)\quad \begin{cases} a^{\pm}(x, t) = \dfrac{\partial^2 A^{\pm}}{\partial x^2}(x, t) - \dfrac{\partial^2 A^{\pm}}{\partial t^2}(x, t) \pm 2\,i\,Q(x)\dfrac{\partial A^{\pm}}{\partial t}(x, t) \\ \qquad (t \geqq x \geqq 0) \end{cases}$$

has a meaning and is continuous with respect to (x, t) for $t \geqq x \geqq 0$ and integrable in t for $t \geqq x$. Applying the operator

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} + 2\,i\,Q(x)\frac{\partial}{\partial t}$$

to both sides of (1.23), we find, by means of differentiation under the integral sign and of integration by parts, that $a^{+}(x, t)$ is the solution of the equation obtained by replacing $F^{+}(x)$ by $f^{+}(x)$ in (1.23). We therefore have by theorem 2 *bis* :

$$(7.2)\quad a^{\pm}(x, t) = f^{\mp}(x)\,\alpha^{\pm}(x, t) + f^{\pm}(x)\,\beta^{\mp}(x, t) \qquad (t \geqq x \geqq 0).$$

Recalling (1.21) and (4.2) we find the partial differential equation

$$(7.3)\qquad a^{\pm}(x, t) = U(x)\,A^{\pm}(x, t) \qquad (t \geqq x \geqq 0).$$

On the other hand, one can prove without difficulty (*see* [2]) that

$$(7.4)\qquad |A^{\pm}(x, t)| \leqq C\,\sigma_0(x + t) \qquad (t \geqq x \geqq 0),$$

$$(7.5)\quad \lim_{N \to \infty}\left(\sup_{t+x \geqq N}\left|\frac{\partial A^{\pm}}{\partial x}(x, t)\right|\right) = \lim_{N \to \infty}\left(\sup_{t+x \geqq N}\left|\frac{\partial A^{\pm}}{\partial t}(x, t)\right|\right) = 0.$$

Using equations (7.3) and (1.22) and conditions (7.4) and (7.5), one easily obtain theorem 6 [for this one can apply theorem (4.2) of [1]]. It follows from this theorem that the functions $f^{+}(0, x)$ and $f^{-}(0, x)$ are equal, i. e.

$$(7.6)\quad F^{+}(x) + \int_x^{\infty} A^{+}(x, t)\,dt = F^{-}(x) + \int_x^{\infty} A^{-}(x, t)\,dt \qquad (x \geqq 0).$$

Using (4.2), we obtain from (7.6)

$$(7.7)\qquad [F^{+}(x)]^2\left(1 - \int_x^{\infty}[\alpha^{-}(x, t) - \beta^{-}(x, t)]\,dt\right)$$
$$= 1 - \int_x^{\infty}[\alpha^{+}(x, t) - \beta^{+}(x, t)]\,dt \qquad (x \geqq 0).$$

(7.7) allows us to determine $F^{-}(x)$ for the values of x which do not cancel the second factor of the left hand side. This is clearly true at least for x sufficiently large. In general we do not known the position of zeros of this quantity. For this reason, we have used the differential equation (4.6) to determine $F^{+}(x)$ $(x \geqq 0)$ and not the coupling condition (7.6) [identical with the formula (5.20) in [1]] used in [1].

8. The characteric function associated with (U (x), Q (x)). The existence theorem

By theorem 6, the Jost function $\tilde{f}^{\pm}(k)$ associated with the pair $(U(x), Q(x))$ defined in section 6, is

$$(8.1) \qquad \tilde{f}^{\pm}(k) = f^{\pm}(k, 0) = F^{\pm}(0) + \int_0^{\infty} A^{\pm}(0, t)\, e^{-ikt}\, dt \qquad (\operatorname{Im} k \leqq 0),$$

where, from (4.2),

$$(8.2) \qquad A^{\pm}(0, t) = F^{\mp}(0)\, \alpha^{\pm}(0, t) + F^{\pm}(0)\, \beta^{\mp}(0, t) \qquad (t \geqq 0).$$

On the other hand, we know by theorem 1 that there exist two functions $f^{+}(k)$ and $f^{-}(k)$ which satisfy the relation (1.6) — $S^{+}(k)$ $(k \in \mathbf{R})$ being the input function — and which can be written in the form (1.11). Using the properties of Fourier transforms we can easily show from (1.6), (1.11) and (1.25) that

$$(8.3) \qquad a^{+}(t) = \overline{F_0^{+}}\, s^{+}(t) + \int_0^{\infty} s^{+}(u + t)\, \overline{a^{+}(u)}\, du \qquad (t \geqq 0).$$

We see on (8.3) that $a^{+}(t)$ is the solution of the equation deduced from (1.23) by putting $x = 0$ and by replacing $F^{+}(0)$ by F_0^{+}. It follows from section 4 that

$$(8.4) \qquad a^{\pm}(t) = F_0^{\mp}\, \alpha^{\pm}(0, t) + F_0^{\pm}\, \beta^{\mp}(0, t) \qquad (t \geqq 0).$$

We make an additional assumption :

ASSUMPTION I_3. — *If $z(x)$ is the solution of the differential equation* (4.6) *and if* $(F_0^{+})^2$ *is the number defined in assumption* I_1 (Q'_5), *then*

$$(8.5) \qquad \exp(-iz(0)) = (F_0^{+})^2.$$

We remark that

1. I_3 must necessarily be satisfied for a function $S^{+}(k)$ $(k \in \mathbf{R})$ to be the characteristic function of a pair of potentials belonging to $\mathcal{C}'$.

2. Assumption I_3 is not very restrictive : using the fact that $f^{+}(0)$ is equal to $f^{-}(0)$ and $\tilde{f}^{+}(0)$ is equal to $\tilde{f}^{-}(0)$ it is easy to see that a

necessary condition for I_3 to be satisfied is

$$(8.6) \qquad 1 - \int_0^\infty [\alpha^-(0, t) - \beta^-(0, t)]\, dt \neq 0.$$

Because of assumption I_3, we can choose F_0^+ in such a way that $F_0^+ = F^+(0)$. The formulas (8.2) and (8.4) show that $A^\pm(0, t)$ and $a^\pm(t)$ are equal. Hence

$$(8.7) \qquad \tilde{f}^\pm(k) = f^\pm(k) \qquad (\operatorname{Im} k \leqq 0).$$

Theorem 1 shows that the pair $(U(x), Q(x))$ defined in section 6 satisfies A_3 and that the input function $S^+(k)$ $(k \in \mathbf{R})$ is the characteric function associated with $(U(x), Q(x))$. To recapitulate, we have obtained the following results :

THEOREM 7. — *A necessary condition for a function* $S^+(k)$ $(k \in \mathbf{R})$ *to be the characteristic function associated with a pair of potentials belonging to* $\mathcal{C}'$, *is that conditions* I_1, I_2 *and* I_3 *be satisfied. If there exists a pair* $(U(x), Q(x))$ *of* $\mathcal{C}'$ *which reproduces* $S^+(k)$ $(k \in \mathbf{R})$, *it is unique and given by formulas* (6.1) *and* (1.21).

THEOREM 8. — *A sufficient condition for a function* $S^+(k)$ $(k \in \mathbf{R})$ *to be the characteristic function associated with a pair of potentials belonging to* $\mathcal{C}'$, *is that conditions* I_1, I_2, I'_2 *and* I_3 *be satisfied. There exists a unique pair* $(U(x), Q(x))$ *of* $\mathcal{C}'$ *which reproduces* $S^+(k)$ $(k \in \mathbf{R})$. *It is given by formulas* (6.1) *and* (1.21).

We remark that :

a. we have not tried to give a necessary and sufficient condition for a function $S^+(k)$ $(k \in \mathbf{R})$ to be the characteristic function of a pair $(U(x), Q(x))$ of $\mathcal{C}'$. So it is possible that theorem 8 remains valid if assumption I'_2 is not satisfied;

b. we have assumed for convenience that the potentials were continuous for $x = 0$; this condition is probably not absolutely necessary for our study;

c. our assumptions concerned the characteristic function $S^+(k)$ $(k \in \mathbf{R})$ and not the scattering " matrix " $S^+(k)$ $(k \geqq 0)$. To find the number of solutions to the inverse problem, one would like to know how to construct the functions $S^+(k)$ $(k \in \mathbf{R})$ which satisfy assumptions I_1, I_2, I'_2 and I_3, and which continue a given function $S^+(k)$ $(k \geqq 0)$.

To end this study, let us consider a particular case. If $S^+(k)$ $(k \in \mathbf{R})$ is the characteristic function of a pair $(U(x), Q(x))$ of $\mathcal{C}'$ such that $Q(x) = 0$, we obviously have $(F_0^+)^2 = 1$ and $S^+(k) = [S^+(-k)]^{-1}$ $(k \in \mathbf{R})$. Conversely, if we are given a function $S^+(k)$ $(k \in \mathbf{R})$ satisfying assumptions I_1, I_2 and I'_2 and the above conditions, it is easy to see that I_3

is necessarily satisfied and that $Q(x) = 0$. So, we find again the results that Marchenko obtained for real $U(x)$ and for $Q(x) = 0$.

Our method of solution of this inverse problem can be used to solve other inverse problems in which a differential equation of the same type as (1.4)-(1.5) appears. In particular, in the following section we solve an inverse problem for the Klein-Gordon equation of zero mass.

9. The inverse problem for the radial *s*-wave Klein–Gordon equation of zero mass with a static potential

With the additional condition

$$U(x) = -Q^2(x) \qquad (x \geqq 0), \tag{9.1}$$

the formulas (1.4) and (1.5) for the index " $+$ ", represent, for $k \geqq 0$, the radial *s*-wave Klein-Gordon equation for a particle of zero mass and of energy k with a static potential $Q(x)$; (1.4) and (1.5) for the index " $-$ " describe the correspondent antiparticle. Let $\mathcal{C}'_1$ be the set of potentials $Q(x)$ which satisfy assumptions A_2 and A_3; if $Q(x)$ belongs to $\mathcal{C}'_1$, $(U(x), Q(x))$ — with $U(x)$ defined by (9.1) — belongs to $\mathcal{C}'$. Let us note that assumption A_3, which expresses the hypothesis that there is no bound state, is not physically restrictive here. The characteristic functions $S^+(k)$ $(k \in \mathbf{R})$ associated with $(U(x), Q(x))$ is here physically observable and will be called the scattering " matrix " : for $k \geqq 0$, it represents the scattering " matrix " for the particle, and for $k \leqq 0$ it represents the scattering " matrix " for the antiparticle.

The inverse problem that we consider now is the construction of the potentials $Q(x)$ belonging to $\mathcal{C}'_1$ which admit a given function $S^+(k)$ $(k \in \mathbf{R})$ as its scattering " matrix ". This inverse problem has already been studied (references can be found in [1]) by extending the Gel'fand-Levitan method. It has been reduced to the solution of an integral equation in which a certain spectral function appears. However, the conditions that the input function $S^+(k)$ $(k \in \mathbf{R})$ has to satisfy in order that this equation has a solution are still not known. Theorems 7 and 8 enable us to solve this inverse problem completely. Let us call H_4 the following assumption :

ASSUMPTION I_4. — *If $z(x)$ is the solution of the differential equation* (4.6) *qnd if $f^+(x)$ is the function defined by* (1.22), *then*

$$4 f^+(x) + [z'(x)]^4 F^+(x) = 0 \qquad (x \geqq 0). \tag{9.2}$$

We easily obtain the following results :

THEOREM 9. — *A necessary condition for a function $S^+(k)$ $(k \in \mathbf{R})$ to be the scattering " matrix " of a potential belonging to $\mathcal{C}'_1$ is that condi-*

tions I_1, I_2, I_3 *and* I_4 *be satisfied. If there exists a potential* Q (x) *of* $\mathcal{C}'_1$ *which reproduces* $S^{+}(k)$ $(k \in \mathbf{R})$, *it is unique and given by formula* (6.1).

THEOREM 10. — *A sufficient condition for a function* $S^{+}(k)$ $(k \in \mathbf{R})$ *to be the scattering " matrix " of a potential belonging to* $\mathcal{C}'_1$ *is that conditions* I_1, I_2, I'_2, I_3 *and* I_4 *be satisfied. There exists a unique potential* Q (x) *of* $\mathcal{C}'_1$ *which reproduces* $S^{+}(k)$ $(k \in \mathbf{R})$. *It is given by formula* (6.1).

Let us note that in the case where Q (x) is a superposition of exponential potentials, H. Cornille [7] has studied a slightly different inverse problem which is the reconstruction of Q (x) from the S-matrix discontinuities in the complex k-plane. His study is also based on an extension of the Marchenko formalism and is as well valid for the Klein-Gordon equation of mass m.

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