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Section A :
Physique théorique.

The principle of minimal electromagnetic interaction in terms of pure geometry

by

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ABSTRACT. — A geometric version of the principle of minimal electromagnetic interaction can be given in terms of the following:

THEOREM. — *Let* P (M^n) *be a differentiable principal fibre bundle over the base manifold* M^n *with structural group* G. *Suppose the properties of* P (M^n) *to entail the existence of the Yang-Mills fields, i. e. consider these fields to be derived from the geometry* P (M^n) (*in the sense of Misner and Wheeler* [1]). *Suppose furthermore these geometrical Yang-Mills fields to be coupled with some matter field. Then the geometric components of the former determine completely the symmetry group which belongs to this interaction : the holonomy group of the connection of* P (M^n).

INTRODUCTION

The principle of minimal electromagnetic interaction states that interactions between fields of the electromagnetic type, A_μ (i. e. also the Yang-Mills potentials B_μ) and matter fields must be always " current type " interactions. That is, the replacement of the differential operator ∂_μ by $\partial_\mu - i A_\mu$, when acting on the matter field ψ, leads automatically to this principle.

The aim of this paper is to describe such interactions by means of an appropriate geometry since the knowledge of the geometric structure of field theory might be essential for a deeper understanding of such a theory. This has already been recognized by Misner and Wheeler [1], who pointed out that the laws of nature are described partly in terms of pure geometry, partly by fields added to geometry. To extend Einstein's geometrical description of gravitation a purely geometrical description of all laws of nature would be conceivable. Otherwise stated: Particles and fields other than gravitation would have to be considered as derived from geometry as well. Therefore, a clarification of the concept of a (classical) field, which is derived from some geometry, is necessary.

We define a (classical) field as being derived from geometry, if its properties are related to the properties of the geometry M (M constitutes some differentiable manifold). More precisely: The properties of the geometry M imply the properties of the field. Particularly, the existence or nonexistence of some field may result from some geometrical properties of M. Consequently, a (classical) field which is derived from geometry may essentially be characterized by the specification of the conjunction of data such as: Π_1 (M), the fundamental group of M, the kth homotopy group of M, Π_k (M), the cohomology- or homology-groups, H_1 (M), H^1 (M), ... or by one of these properties of M alone. It appears therefore natural to characterize (classical) fields that are derived from geometry in terms of pairings

$$(1)\quad (\omega, c);\quad \left.\begin{array}{ll} \omega \in F^p(M): & \text{vector space of } p\text{-forms on M} \\ c \in C_p(M): & \text{vector space of } p\text{-chains on M} \end{array}\right\} p = 0, 1, 2, \ldots,$$

c and ω denote by definition the homologous and cohomologous field component of (1) respectively. This means: Two p-chains c'_p and c''_p which differ by a boundary:

$$(2)\qquad c'_p - c''_p = \partial c_{p+1};\qquad \partial:\ C_{p+1}(M) \to C_p(M)$$

and which are called homologous, i. e. $c'_p \sim c''_p$, represent the same field component of (1), which, by abuse of language, is called homologous. Likewise, the cohomologous field component of (1) is provided by the dual concept of cohomologous forms

$$(3)\qquad \overset{p'}{\omega} \sim \overset{p''}{\omega}:\ \overset{p'}{\omega} = \overset{p''}{\omega} + d\overset{p-1}{\omega};\qquad d:\ F^p(M) \to F^{p+1}(M).$$

That is, forms which differ by a differential $d\omega$ define the same field. The general field-expression (1) subsumes the following cases:

(1′ a) (ω, c), $\omega \in \mathring{F}^p(M)$ (vector space of closed forms),
$c \in \mathring{C}_p(M)$ (vector space of closed chains),

(1′ b) (ω, c), $\omega \notin \mathring{F}^p(M)$, $c \in \mathring{C}_p$,

(1′ c) (ω, c), $\omega \in \mathring{F}^p(M)$, $c \notin \mathring{C}_p(M)$, and

(1′ d) (ω, c), $\omega \notin \mathring{F}^p(M)$, $c \notin \mathring{C}_p$.

Discussion. — Formula (1′ a) states that homologous (cohomologous) field components will be elements of the homology and cohomology classes of $H_p(M)$ and $H^p(M)$ respectively (refer to remark 14). Formulae (1′ *b*) and (1′ *c*) state that the properties of the geometry in terms of $H_p(M)$, $H^p(M)$, etc. determine the properties of the field only through one type of its components, through the homologous field component, in the case (1′ *b*), or the cohomologous component in (1′ *c*). An example for (1′ *b*) is provided by the Yang-Mills field [formula (11)], since $\omega \notin \mathring{F}^p(M)$. Clearly, a specification of the field (11) in terms of $H_1(M)$, $H_1(M)$, etc. is only possible by means of $c \in \dot{C}_1(M)$. Thus the cohomologous field components of the Yang-Mills fields (11) and (13) must be discussed in terms of the connection- and curvature form of some appropriate geometry M, as stressed in our subsequent remarks 11 and 12 and formulae (18)-(19). Obviously, the geometry cannot " leave its prints " on fields of the type (1′ *d*) in terms of homology- or cohomology properties. Nevertheless, a geometrical characterization of these fields is possible by other means. An example which accounts for this is a field which derives from a nonorientable manifold [4]. It turns out that such a field is characterized by a twisted exterior form $\boldsymbol{\omega} \in \mathbf{F}^p(M)$ (for further details, refer to [4]).

Remark 1. — On account of the properties of de Rham currents, further specifications on fields of the type (1) are available as is displayed by the following example. Consider a field (ω, c) which describes a charged particle of mass m endowed with spin S [11]. With the physical Ampère-current j through the circle $c^1 \in \dot{C}_1(M)$,

$$j = \frac{e\,S}{2\pi m r^2} \qquad \begin{pmatrix} e : \text{electric point charge} \\ r : \text{radius of } c^1 \end{pmatrix}$$

can be associated a mathematical current which is defined by the same cycle to be given by

$$\left.\begin{aligned} c(\omega) &= \int_{c^1} \omega \\ &= \int_{c^1} \frac{e\,S}{2\pi m r^2}\left[\sum a_i\, dx^i\right] \end{aligned}\right\} \begin{cases} \mathbf{j} \in \mathbf{F}^0(M) & \text{(the vectorspace of twisted 0-forms),} \\ \omega \in F^1(M), \\ \omega = \mathbf{j}\,\varphi. \end{cases}$$

That is, an electrically charged particle with spin may be represented by a field which is given by a twisted de Rham 2-current. (This example will be discussed more explicitely elsewhere.)

Remark 2. — The field concept as exhibited by (1) is subject to the constraint that its cohomologous component be consistent with the following classification :

a. Scalarfields are 0-forms, i. e. $\omega = \psi \in F^0(M)$.

b. Tensorfields are given by the local representation [i. e. with respect to a local chart (U, φ), $\varphi = (x^1, \ldots, x^n)$: local coordinates of U on M] :

$$\omega = \sum a_{i_1 \ldots i_p}(x)\, dx^{i_1} \ldots dx^{i_p}. \tag{4}$$

c. Spinfields may be constructed as follows [2] : Let S (M) be the module of spinors over M. Then there exists a module-isomorphism $i : \mathrm{S}(\mathrm{M}) \leftarrow \mathrm{F}^p(\mathrm{M})$, which assigns to each homogeneous p-form a spinor

$$\psi = \sum_{p=0}^{4} \frac{1}{p\,!} \gamma^{i_1} \ldots \gamma^{i_p} a_{i_1 \ldots i_p} = \sum i\overset{p}{\omega} \tag{5}$$

that is

$$\psi = i\,\omega, \qquad \text{where} \quad \omega = \sum \overset{p}{\omega} \tag{6}$$

constitutes the inhomogeneous form (4).

γ^{i_p} are the anti-Hermitean 4×4 complex Dirac matrices ($i = 0, 1, 2, 3$) which satisfy the commutation rule

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2\, g^{ij}\, \mathrm{I}.$$

Thus the field concept (1) generalizes slightly the concept of conventional field.

Remark 3. — The concept of field associated with some geometry can be extended to quantized fields. In this case, the coefficients of the local representation (4) become operators in Hilbert space ([3], [4]), i. e. the cohomologous field component of (1) becomes a quantized differential form.

A first illustration of the concept of field associated with some geometry is the following : Let $\vec{\mathrm{F}} = (\mathrm{F}_i)$ be a force field and require this field to be conservative. Which are the corresponding properties to be imposed upon the geometry M ? That is, which properties of M imply

$$\mathrm{F}_i = -\frac{\partial \varphi}{\partial x_i}\ ? \tag{7}$$

In terms of our field concept (1) property (7) simply reads :

$$\omega = -d\varphi, \qquad \text{where} \quad \omega = \sum \mathrm{F}_i\, dx^i \in \mathrm{F}^1(\mathrm{M}), \quad \varphi \in \mathrm{F}^0(\mathrm{M}). \tag{8}$$

To begin with, assume ω to be closed, i. e. $\omega \in \mathring{\mathrm{F}}^1(\mathrm{M})$. Then our problem reduces to finding the conditions to which M must be subject in order to yield $\omega \in d\mathrm{F}^0(\mathrm{M})$. The corresponding geometrical constraint is obviously given by

(9) $$\Pi_1(\mathrm{M}) = 0 \qquad \begin{bmatrix} \Pi_1(\mathrm{M}) \text{ denotes the Poincaré group} \\ \mathrm{M} : \text{pathwise connected} \end{bmatrix}$$

and since there exists a natural homomorphism

(10) $$h^* : \quad \Pi_1(\mathrm{M}) \to \mathrm{H}_1(\mathrm{M})$$

this entails (11) $\mathrm{H}_1(\mathrm{M}) = 0$, i. e. the first de Rham group $\mathrm{H}^1(\mathrm{M})$ must vanish. According to de Rham's first theorem, condition (11) expresses that all periods $\int_{\mathrm{C}} \omega$ of $\omega \in \mathring{\mathrm{F}}^1(\mathrm{M})$ vanish. This corresponds to the elementary fact that $\varphi(x) = -\int_{x_0}^{x} \omega$ be independent of the path joining x_0 to x, or equivalently $\oint_{\gamma} \omega = 0$, $\forall\, \gamma$ which are homotopic to zero, i. e. $\forall\, \gamma \in \Pi_1(\mathrm{M}) = 0$.

To summarize: The force field (ω, c), which may be regarded as being derived from the geometry M, is conservative [i. e. satisfies (8)] if the following holds:

$$\omega \in d\mathrm{F}^0(\mathrm{M}), \quad c \in \mathring{\mathrm{C}}_1(\mathrm{M}) : \text{ The vector space of one-cycles,}$$
$$\text{i. e. } \Pi_1(\mathrm{M}) = \mathrm{H}_1(\mathrm{M}) = \mathrm{H}^1(\mathrm{M}) = 0.$$

REMARK 4. — The aforementioned conditions require that the equation $d\omega = 0$ entails that, for any loop $\gamma \in \Pi_1(\mathrm{M})$ which can be shrunken to a point in M a zero form exists, such that $\omega = d\varphi$.

REMARK 5. — Statement (10) is also available by means of Stoke's Theorem:

$$\int_{c_1 = \partial c_2} \omega = \int_{\mathrm{C}_2} d\omega = 0.$$

That is, let $\mathring{\mathrm{C}}_2(\mathrm{M})$ be the vector space of 2-cycles, then there $\exists\, c_2 \in \mathring{\mathrm{C}}_2(\mathrm{M})$:

$$c_2 : \quad \mathrm{I} \times \mathrm{I} \to \mathrm{M}, \qquad \mathrm{I} = [0, 1],$$

where

$$c_2(0, t) = x_0 = c_1(0) = c_1(1),$$
$$c_2(s, 0) = c_2(s, 1) = c_1(s)$$

and

$$c_1 : \quad \mathrm{I} \to \mathrm{M} \quad \text{is a loop.}$$

Remark 6. — This rather sophisticated discussion of a conservative force field (ω, c) does not, in fact, provide any new information. However, as we shall see in the case of the Yang-Mills field [5], this formalism is very powerful.

Consider now Yang-Mills potentials and fields to be given by

$$(\overset{1}{\omega}, c_1), \tag{11}$$

where

$$\overset{1}{\omega} = \sum \mathrm{B}_\mu \, dx^\mu \in \mathrm{F}^1 (\mathrm{M}), \tag{12}$$

$c_1 \in \ddot{\mathrm{C}}_1 (\mathrm{M})$ a closed 1-chain (1-cycle) to be specified subsequently;

$$(\overset{2}{\omega}, c_2), \tag{13}$$

where

$$\overset{2}{\omega} = \sum_{\mu, \nu} \mathrm{F}_{\mu\nu} \, dx^\mu \, dx^\nu \tag{14}$$

and

$$\mathrm{F}_{\mu\nu} = \frac{\partial \mathrm{B}_\mu}{\partial x^{\prime}} - \frac{\partial \mathrm{B}_\nu}{\partial x^\mu} + \begin{Bmatrix} \text{linear combinations} \\ \text{of the B-potentials} \end{Bmatrix}. \tag{15}$$

There exists an appropriate geometry M such that:

a. The properties of M imply the properties of the fields (11) and (13).

b. The Yang-Mills fields which are derived from the geometry M interact with some matter field ψ_α which transforms according to

$$\overline{\psi}_\alpha (p) = e^{-i\Lambda(p)} \, \psi_\alpha (p). \tag{16}$$

c. The symmetry group which is associated with such a " minimal electromagnetic interaction " is induced by these Yang-Mills fields as will be displayed by our Theorem below.

A geometry M which fulfills (*a*)-(*c*) is given in terms of a principal fibre bundle $\mathrm{P}(\mathrm{M}^n)$ over the base space M^n (the case $n = 4$ constitutes a curved space-time manifold). With this principal bundle is associated a connexion, i. e. a Lie algebra valued connexion 1-form $\tilde{\omega} \in \mathrm{F}^1 (\mathrm{P}(\mathrm{M}^n))$ by means of the canonical correspondence

$$\begin{cases} \mathrm{T}_p (\pi^{-1}(x)) \to \mathfrak{g}(\mathrm{G}) & [\pi : \mathrm{P}(\mathrm{M}^n) \to \mathrm{M}^n \text{ (projection map)} \ldots], \\ \qquad \tilde{\mathrm{X}} \to \tilde{\omega}(\tilde{\mathrm{X}}) = \hat{\mathrm{X}} \end{cases} \tag{17}$$

where $\mathrm{T}_p(\pi^{-1}(x))$ denotes the tangent space at $p \in \pi^{-1}(x)$, the fibre over $x \in \mathrm{M}^n$, and $\mathfrak{g}(\mathrm{G})$ the Lie algebra of the structural group G. Likewise, the curvature form $\Omega = \nabla \tilde{\omega}$ (∇ denotes the covariant differential) is a horizontal g-valued 2-form, associated with the connexion of P (M^n). Therefore, a first geometrical characterization of the Yang-Mills fields may be obtained in terms of the following formulae [6]:

$$\tilde{\overset{1}{\omega}} = \sum B^\alpha E_\alpha, \tag{18}$$

$$\overset{2}{\omega} = \Omega = \sum F^\alpha E_\alpha, \tag{19}$$

$\{ E_\alpha \}$ constitutes a basis of g.

This yields that the bundle connexion is the source of the gauge vector field B_μ. The gauge tensorfield has as source the bundle curvature which is the result of nonintegrability of the bundle connexion, i. e.

$$[\nabla_\nu, \nabla_\mu] = \sum_\alpha F^\alpha_{\mu\nu} E_\alpha, \qquad \nabla_\mu = \nabla_{\frac{\partial}{\partial x^\mu}}. \tag{21}$$

Thus the existence of Yang-Mills fields within this framework is infered from the property that P (M^n) is endowed with a connexion, i. e. the corresponding curvature.

To summarise: A geometrical description of gauge vector- and tensor-fields in terms of the connexion and curvature respectively of some internal space [i. e. the fibres $\pi^{-1}(x)$ over each point $x \in M^n$ may be equipped with a Hilbert space structure] is obtained along the same lines as Einstein's geometrical description of the external field in terms of the curvature of the external space. The Yang-Mills approach interrelates geometry and physics in a fashion, bringing it in close relation to general relativity theory and regards connexions as fields also [7].

Next, one has the following:

THEOREM. — *Let* P (M^n) *be a principal fibre bundle whose properties entail the existence of the fields* (11) *and* (13). *Suppose these fields to interact with some matter field* ψ_α. *Then the homologous field components* $c^1_i \in \hat{C}_1 (M^n)$ *of the Yang-Mills field* (11) *determine completely the interaction symmetry group.*

Proof. — Let P (M^n) be the principal bundle over M^n and $\pi^{-1}(x_0) = F_{x_0}$ the fibre over x_0 endowed with Hilbert space-structure. Then the following assignment holds (refer to remark 7 below):

$$\left\{ \begin{array}{l} c^1_i = \gamma_i \to \tau_{\gamma_i}, \qquad \gamma_i \in \Pi_1 (M^n, x_0) \\ [\Pi_1 (M^n, x_0) : \text{Poincaré group at } x_0 \text{ of } M^n]. \end{array} \right. \tag{22}$$

That is, with each loop γ_i is associated the parallel displacement

$$\tau_{\gamma_i}: \quad \pi^{-1}(x_0) \to \pi^{-1}(x_0) \tag{23}$$

which constitutes a diffeomorphism (automorphism) of the fibre F_{x_0}, such that

$$\begin{cases} \tau_{\gamma_i}(pg) = \tau_{\gamma_i}(p) \cdot g, \\ g \in G, \text{ the structural group of } P(M^n), \quad p \in P(M^n). \end{cases} \tag{24}$$

Denote by $\{\tau_{\gamma_i}\}$ the set of all automorphisms associated with the homologous field components c_i^1 (the index i runs through the corresponding homology class, according to remark 7 below). By virtue of the multiplication in the set of all loops,

$$\gamma_3(t) = (\gamma_1 \star \gamma_2)(t) = \begin{cases} \gamma_1(2t): & 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t-1): & \frac{1}{2} \leq t \leq 1, \end{cases}$$

clearly

$$\tau_{\gamma_1 \star \gamma_2} = \tau_{\gamma_2} \circ \tau_{\gamma_1}. \tag{25}$$

Since moreover the inverse $\tau_{\gamma-1} = \tau_\gamma^{-1}$ is associated with the reverse γ^{-1} of the loop γ, where $\gamma^{-1}(t) = \gamma(1-t)$, the set of diffeomorphisms $\{\tau_{\gamma_i}\}$ constitutes a group, called the holonomy group at x_0 and is denoted by Φ_{x_0}. Now, according to Lemma 1 below there exists an injective mapping

$$\Phi_x \xrightarrow{\text{into}} \Phi_p \subset G, \qquad x \in M^n, \tag{26}$$

Φ_p is then referred to as the holonomy group with reference point at $p \in P(M^n)$. This subgroup of the structural group G can be given a straightforward characterization (refer to the subsequent Lemma 2). Suppose now $p \in P(M^n)$, $\pi(p) = x_0 \in M^n$, then there exists a unique horizontal lift $\tilde{\gamma}$ of $\gamma : [0, 1] \to M^n$, beginning at $p \in \pi^{-1}(x_0)$. If $p \sim pg$, $g \in G$, i. e. p is joined to pg by the horizontal curve $\tilde{\gamma}$ (Lemma 2), then $\pi \circ \tilde{\gamma} = \gamma \in \Pi_1(M^n)$ [$\Pi_1(M^n)$ constitutes the fundamental group of the base space]. This amounts to saying that p and $R_g\, p$ (R_g : right translation associated with $g \in G$) belong to the same fibre F_x, $\forall\, g \in G$. In virtue of the uniqueness of the horizontal lift $\tilde{\gamma}$, clearly $\tilde{\gamma}$ must be the solution curve to the vector field $\tilde{X}_\mu$ passing through p, i. e. by means of formula (18) this vector field must be of the form

$$\tilde{\gamma} \to \tilde{X}_\mu(p) = \frac{\partial}{\partial x^\mu} - B_\mu^\xi\, E_\sigma \tag{27}$$

and

$$[E_\rho, E_\sigma] = c^\alpha_{\rho\sigma} E_\alpha, \tag{28}$$

where $\{ E_\rho \}$ span the (restricted) holonomy group (refer to our subsequent remarks 10 and 12).

$\tilde{X}_\mu$ constitutes the horizontal lift of the vector field $X_\mu = \dfrac{\partial}{\partial x^\mu}$ (whose integral curve is $\gamma : [0, 1] \to M^n$) that is $X_\mu (\pi (p)) = d\pi (p) . \tilde{X}_\mu$. Otherwise stated : The gauge-covariant derivatives

$$\begin{aligned} \nabla_\mu \psi_\alpha &= \left(\frac{\partial}{\partial x^\mu} - \tilde{\omega}^\beta_{\alpha\mu} \right) \psi_\beta \\ &= \left(\frac{\partial}{\partial x^\mu} - B^\rho_\mu E^\beta_\rho \right) \psi_\beta \end{aligned} \tag{29}$$

that are associated with the internal holonomy group $\Phi_{x_0} \subset G$ and which are to be identified with the horizontal lift (27), account for the interaction between the Yang-Mills field and the matter field (16). Therefore Φ_{x_0} characterizes completely such an interaction, which achieves the proof.

The proof of the aforementioned theorem is based upon the following two Lemmata :

Lemma 1. — *Let Φ_x be the holonomy group at $x \in M^n$. Then there exists an injective homomorphism*

$$i : \quad \Phi_x \to \Phi_p \subset G \qquad [G : \textit{structural group of } P (M^n)]. \tag{26}$$

Proof :

$$(\forall\ \tau_\gamma \in \Phi_x) (\exists\ g \in G) : \quad \tau_\gamma (p) = R_g\, p$$

$$(R_g \text{ denotes the right translation associated with } g \in G)$$

$$\therefore \quad (\tau_{\gamma_1} \circ \tau_{\gamma_2}) (p) = \tau_{\gamma_1} (R_{g_2}\, p) = R_{g_2}\, \tau_{\gamma_1} (p) = R_{g_1 g_2}\, p,$$

$$i : \quad \tau_{\gamma_1} \circ \tau_{\gamma_2} \to g_1\, g_2 \in G$$

which homomorphism is seen to be injective.

Lemma 2 [12]. — *Let p and $pg \in P (M^n)$, $g \in G$, be joined by a horizontal curve in $P (M^n)$ (symbolically : $p \sim pg$). Define*

$$\Phi_p = \{ g \in G : p \sim pg,\ p,\ pg \in \pi^{-1} (x) \}. \tag{30}$$

Then Φ_p is a subgroup of G.

Proof. — Let

$$g, g' \in \Phi_p \quad \Rightarrow \quad g^{-1} g' \in \Phi_p$$

since there exist horizontal curves such that

$$\left.\begin{array}{l} p \sim pg^{-1} \\ p \sim pg'^{-1} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} pg \sim p, \\ pg \sim pg'^{-1}\, g \end{array}\right.$$

as g operates on these curves.

This yields the required result $p \sim pg'^{-1}\, g$, since " $\sim$ " is obviously an equivalence relation.

REMARKS :

7. The index i in formula (22) runs through some homology class $\in H_1(M^n)$. According to remark 1 one has $c_i^1 \sim c_j^1$ which means that c_i^1 and c_j^1 belong to the same Yang-Mills field.

8. Within our framework where the Yang-Mills fields are interpreted as being derived from the geometry P (M^n) it turns out that not the structural group [for instance SU (n)] of P (M^n) itself but only its holonomy-subgroup Φ_x (or Φ_p) takes over the role of the internal symmetry.

9. An important feature of our approach is that the symmetry originates from the geometry. This is a natural consequence of the fact that the fields themselves are regarded as being derived from geometry.

10. By virtue of the Ambrose-Singer Theorem [8] it turns out that our theorem could be better specified by means of the Yang-Mills field (13). In fact, the cohomologous component $\overset{2}{\omega}$ can be related to a diffeomorphism τ_γ in terms of the formula

$$\tau_\gamma = I + \frac{1}{2}\, \Omega_{\mu\nu}\, dx^\mu\, dx^\nu. \tag{31}$$

11. Since parallel displacement $\tau_\gamma : \pi^{-1}(x) \to \pi^{-1}(y)$, $x, y \in M^n$, is associated with a given connection, i. e. a connection form $\tilde{\omega} \in F^1(P(M^n))$, the internal symmetry is implicitly determined, apart from the homologous field components, also by the cohomologous ones.

Discussion of the Theorem. — According to remark 10, i. e. formula (31), clearly the interaction is described in terms of the curvature property $\Omega \in F^2(P(M^n))$ of the bundle. This is similar to Einstein's approach of relativity where forces and interactions manifest themselves through the curvature properties of geometry. Our Theorem yields the following

COROLLARY. — *The necessary conditions for a minimal electromagnetic interaction to be " adiabatically switched off " are given by*

$$\Phi_{x_0} = \{ I \}, \tag{a}$$

I *denotes the identity transformation, and*

(*b*) $$B^{\mu}.E_{\mu} \in \mathrm{Ker}\,(d\pi),$$

the kernel of the Fréchet-differential $d\pi\,(p)$, $p \in P\,(M^n)$.

REMARK 12. — The aforementioned conditions (*a*)-(*b*) correspond to the characterization of a flat connection, i. e.

$$\nabla_{\frac{\partial}{\partial x^{\mu}}} . \psi_{\alpha} = \frac{\partial \psi_{\alpha}}{\partial x^{\mu}}.$$

Conversely, starting the other way round by introducing first the restricted holonomy group of the connection of P (M^n) (refer to our subsequent remark 13), one can always determine the homologous field component of some Yang-Mills field in terms of the base space M^n of P (M^n). In fact, the Poincaré group of M^n vanishes, i. e.

(32) $$\Pi_1\,(M^n) = 0 \qquad (M^n : \text{arcwise connected})$$

since the restricted holonomy group corresponds to all loops that are homotopic to zero. Hence

(33) $$\gamma = c^1 \in \overset{\circ}{C}_1\,(M^n) \qquad \text{(vector space of 1-cycles).}$$

Since the connection form itself may be associated with some Yang-Mills field, the one-cycle (33) may be interpreted as being the homologous field component of this Yang-Mills field.

REMARK 13. — Within the context of the Ambrose-Singer-Theorem one is compelled to confine oneself to the restricted holonomy group, since only in this case can one obtain a description of the holonomy Lie algebra in terms of the curvature form.

Suppose in particular Φ_x to be a one parameter Abelian group. This entails the vanishing of the structure constants (28), $c^{\alpha}_{\rho\sigma}$. That is, (15) reduces to

(34) $$F_{\mu\nu}\,(x) = \frac{\partial B_{\mu}}{\partial x^{\nu}} - \frac{\partial B_{\nu}}{\partial x^{\mu}}.$$

This constitutes the electromagnetic field-tensor, due to some charge distribution ρ. If (34) is regarded as being derived from the geometry M^4 (curved space-time manifold), this field is characterized by property (35) of M^4 as given below and which actually constitutes a necessary condition for the occurrence of charge associated with the topology of M^4 [9]:

(35) $$H^3_{\gamma}\,(M^4) = 0$$

[$H^3_\gamma(M^4)$ stands for the third de Rham group " restricted " to γ, i. e. $H^3_\gamma(M^4) \subset H^3(M^4)$], where

$$(36) \quad \gamma = \frac{1}{c}(i_1\, dx^2\, dx^3 + i_2\, dx^3\, dx^2 + i_3\, dx^1\, dx^2)\, dx^0 + \rho\, dx^1\, dx^2\, dx^3 \in F^3(M^4).$$

Otherwise stated : If the holonomy group is subject to some conditions of the aforementioned kind, which amounts to imposing on the geometry the conditions (30) and (35), respectively, one reduces the Yang-Mills field to a new type of field, the electromagnetic field $(\overset{2}{\omega}, c_2)$ [formula (34)].

Conclusion. — Our Theorem is to be understood as a contribution to a rigourous geometric description of strongly interacting fields along the same lines as developped by Sakurai [10]. In this theory

$$(37) \quad \begin{cases} L_B = b\, B^{(B)}_\mu . j^\mu_{(B)} & [b\ (+1) \text{ is the baryon-charge for the baryon fields } (N, \Lambda, \Sigma, \Xi)], \\ L_Y = y\, B^{(Y)}_\mu . j^\mu_{(Y)} & (y : \text{Hypercharge}), \\ L_I = i\, B^{(I)}_\mu . j^\mu_{(I)} & (i : \text{Isospin}; B_\mu : \text{Yang-Mills-Field}) \end{cases}$$

are the fundamental interaction Lagrangians of strong interactions. The corresponding isospin-, hypercharge- and baryonic currents are 3-forms $\in F^3(M^n)$ according to formula (36). A corresponding geometric description can be given in terms of jet bundles, i. e. the Lagrangian (37) will be represented in such a framework by a real-valued function [7] :

$$(38) \qquad L : \quad J^1(P(M^n)) \to \mathbf{R},$$

where $J^1(P(M^n))$ is the jet bundle of first order associated with $P(M^n)$.

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