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Section A :
Physique théorique.

Spinorial solution associated to a radially symmetric radiation field in general relativity

by

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ABSTRACT. — This work concerns in the application of the spinorial formalism to the problem of gravitational radiation in empty space-time. The purpose of using this formalism is to find a set of spinorial quantities satisfying the Trautman boundary conditions, or alternatively, describing a radiation field. Such quantities may be determined from a tetrad of null vectors. The problem presently considered is that of an axially symmetric distribution, as the source of gravitational radiation, as was first considered by Bondi.

The solutions of the spinorial equations for gravitation are found, and it is shown that they imply in the tensorial solution found by Bondi. Consequently the Trautman boundary condition is satisfied by the spinor variables. Finally, the radiation conditions for the spinorial curvature are determined, using a similarity with the same conditions for the Riemann tensor.

1. DETERMINATION OF THE RADIATION METRIC FROM THE TETRAD FIELD

We deal with a four-dimensional Riemannian space with signature-2, a tetrad system of vectors is introduced into this space, with the following specification, K_μ and n_μ are real null vectors, and m_μ along with its complex conjugate $\overline{m}_\mu$ are complex null vectors. The vector n_μ can

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be defined from a pair of real, orthogonal unit space-like vectors ξ_μ and η_μ by

$$m_\mu = \frac{1}{\sqrt{2}}(\xi_\mu - i\,\eta_\mu).$$

The orthogonality properties of these vectors are :

$$(1)\qquad \begin{cases} K_\mu K^\mu = m_\mu m^\mu = \overline{m}_\mu \overline{m}^\mu = n_\mu n^\mu = 0, \\ K_\mu n^\mu = -\, m_\mu \overline{m}^\mu = 1, \\ K_\mu m^\mu = K_\mu \overline{m}^\mu = n_\mu m^\mu = n_\mu \overline{m}^\mu = 0. \end{cases}$$

It is convenient to introduce the tetrad notation,

$$h_{\mu(\alpha)} = (K_\mu, n_\mu, m_\mu, \overline{m}_\mu), \qquad \alpha, \mu = 0, 1, 2, 3.$$

The tetrad indices are raised (or lowered by the flat-space metric $\tilde{\eta}^{(\alpha)(\beta)}$ (or $\tilde{\eta}_{(\alpha)(\beta)}$), where :

$$(2)\qquad \tilde{\eta}_{(\alpha)(\beta)} = \tilde{\eta}^{(\alpha)(\beta)} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \mathbf{0} \\ & \mathbf{0} & 0 & -1 \\ & & -1 & 0 \end{pmatrix}.$$

The following relation is easily seen to be true,

$$(3)\qquad g_{\mu\nu} = h_{\mu(\alpha)}\, h_{\nu(\beta)}\, \tilde{\eta}^{(\alpha)(\beta)} = K_\mu n_\nu + n_\mu K_\nu - m_\mu \overline{m}_\nu - \overline{m}_\mu m_\nu.$$

However, the tetrads vectors of (3) are not the unique possible solutions for $g_{\mu\nu}(x^\alpha)$. Indeed, the whole class of tetrad fields defined by

$$\hat{h}_{\mu(\alpha)}(x) = h_{\mu(\gamma)}(x)\, M^{(\gamma)}{}_{(\alpha)}$$

with M an arbitrary point-dependent matrix satisfying

$$M^T\, \tilde{\eta}\, M = \tilde{\eta}$$

will also be solutions for the same metric $g_{\mu\nu}(x^\alpha)$.

To define the tetrad on each point P of space-tine is equivalent to define 16 real functions, with which we are able to construct the metric tensor, $g_{\mu\nu}(x^\alpha)$, according to equation (3). The $h_{\mu(\alpha)}$ have 16 independent components but $g_{\mu\nu}$ has only 10. We are then left with 6 extra degrees of freedom, which can be related to 6 continuous transformation parameters. There parameters allow us to define different tetrads in P, obtained one from the other by « internal local rotations ». The number of independent functions in the tetrad may be further reduced by using the symmetry properties of a given problem; in this work, as will be seen later, $h_{\mu(\alpha)}$ will depend on four functions, corresponding to the particular orientations of the tetrad axes consistent with the chosen symmetry.

2. TETRAD FIELDS ON A NULL HYPERSURFACE

It is always possible to introduce into the Riemannian manifold a family of hypersurfaces defined by $\omega(x) = 0$. If $\omega_{,\mu}$, the normal on each point, is a null vector, the hypersurface is said to be null; therefore, we chose,

$$g^{\mu\nu}\,\omega_{,\mu}\,\omega_{,\nu} = 0.$$

A null hypersurface defines a congruence of null geodesics.

Bondi *et al.* [1] were the first to obtain solutions of the Einstein's equations by applying the Trautman boundary conditions in the case of a limited radiating source. The initial conditions were given on a null hypersurface, and not on a space-like one. In this case, in order to find solutions like the solutions of Bondi it is first of all necessary to construct a family of null hypersurfaces.

Robinson and Trautman [2] have shown that if one chooses as coordinates $\omega(x) = x^0$, and an affine parameter along the geodesics $r = x^1$, and two coordinates x^2, x^3 that label the geodesics on each surface $\omega(x) =$ constant, the metric takes the form

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & g^{11} & g^{12} & g^{13} \\ 0 & g^{21} & g^{22} & g^{23} \\ 0 & g^{31} & g^{32} & g^{33} \end{pmatrix},$$

for (*)

$$K^{\mu} = (0, 1, 0, 0), \qquad n^{\mu} = (1, \alpha, \chi^3, \chi^4),$$
$$m^{\mu} = (0, \omega, \beta^3, \beta^4), \qquad \overline{m}^{\mu} = (0, \overline{\omega}, \overline{\beta}^3, \overline{\beta}^4).$$

3. FUNDAMENTAL SPINORS FOR THE AXIALLY SYMMETRIC RADIATION FIELD

Riemannian geometry may be described [3] by means of spinorial objects $\sigma_\mu(x)$ and $\tau_\mu(x)$, introduced in the 4-dimensional manifold through the relations

$$\sigma_\mu(x^\alpha) = h_\mu{}^{(\alpha)}(x)\,\mathring{\sigma}_\alpha, \qquad \tau_\mu(x^\alpha) = \varepsilon\,.\,\overset{*}{\sigma}_\mu(x^\alpha)\,.\,\varepsilon,$$
$$\mathring{\sigma}_\alpha = (I_{2\ 2}, \mathring{\sigma}_K \text{ being the Pauli matrices}).$$

(*) $m^{\mu} = \omega\delta^{\mu}_{2} + \beta^k\,\delta^{\mu}_{k}$, $n^{\mu} = \delta^{\mu}_{1} + \alpha\delta^{\mu}_{2} + X^k\,\delta^{\mu}_{k}$ for $k = 2, 3$; and ω, β^r, α and X^k arbitrary functions of the coordinates.

They are introduced as the fundamental objetcts with which we describe the gravitational field, instead of the $g_{\mu\nu}(x)$. Besides, in our problem we consider axial and reflection symmetries in the 4-space.

A coordinate system is chosen such that $x^0 = \omega(x) = u$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, where γ is the « luminosity distance », φ the lattitude and θ the longitude; for the angle φ the axial symmetry is straight foward defined in an uniform way. In the radiation zone, we are supposed to cheek if $\sigma_\mu(x)$ obey the Trautman's conditions. As those conditions reduce to simpler form in the cartesian coordinate system, it is advisable that σ_μ be asymptotically written in such a system.

In the coordinates system chosen, the $h_\mu{}^{(\alpha)}$ are functions of u, γ, and θ only and the $g_{\mu\nu}(x)$ are obtained from

$$(4) \qquad g_{\mu\nu}(x) = h_{\mu(\alpha)}(x)\, h_{\nu(\beta)}(x)\, \tilde{\eta}^{(\alpha)(\beta)}.$$

It has already been shown by Bondi that the $g_{\mu\nu}$ which satisfy the properties of axial and reflection symmetries depend on four functions U, β, V, γ, which are themselves functions of u, r and θ.

One of the possible choices for $h_\mu{}^{(\alpha)}$ which results in $g_{\mu\nu}$ of this form is

$$(5) \qquad h_\mu{}^{(\alpha)} = \begin{pmatrix} a & b & b & 0 \\ c & c & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & e \end{pmatrix}.$$

In this case, we will have

$$(6) \quad \begin{cases} \sigma_0 = a\,\mathring{\sigma}_0 + b\,(\mathring{\sigma}_1 + \mathring{\sigma}_2) & \text{and} \quad \tau_0 = -\,a\,\mathring{\sigma}_0 + b\,(\mathring{\sigma}_1 + \mathring{\sigma}_2), \\ \sigma_1 = c\,(\mathring{\sigma}_0 + \mathring{\sigma}_1) & \text{»} \quad \tau_1 = c\,(-\,\mathring{\sigma}_0 + \mathring{\sigma}_1), \\ \sigma_2 = d\,\mathring{\sigma}_2 & \text{»} \quad \tau_2 = d\,\mathring{\sigma}_2, \\ \sigma_3 = e\,\mathring{\sigma}_3 & \text{»} \quad \tau_3 = e\,\mathring{\sigma}_3. \end{cases}$$

From equation (4), we obtain the relation among U, V, β, γ, and a, b, c, d, e. It is important to recall that the $h_\mu{}^{(\alpha)}$ are not uniquelly determined.

From equation (6), using the metric $g^{\mu\nu}$, we obtain the contravariant components and as,

$$\begin{cases} \sigma^0 = c\,e^{-2\beta}\,(\mathring{\sigma}_0 + \mathring{\sigma}_1), \\ \sigma^1 = (a\,e^{-2\beta} - \mathrm{V}\,e^{-2\beta}\,r^{-1}\,c)\,\mathring{\sigma}_0 + (b\,e^{-2\beta} - \mathrm{V}\,e^{-2\beta}\,r^{-1}\,c)\,\mathring{\sigma}_1, \\ \sigma^2 = \mathrm{U}\,c\,e^{-2\beta}\,(\mathring{\sigma}_0 + \mathring{\sigma}_1) - e^{-2\gamma}\,r^{-2}\,d\,\mathring{\sigma}_2, \\ \sigma^3 = \dfrac{-\,e^{2\gamma}\,r^{-2}}{\sin^2\theta}\,\mathring{\sigma}_3 \end{cases}$$

and

$$\begin{cases} \tau^0 = c\, e^{-2\beta} (-\overset{\circ}{\sigma}_0 + \overset{\circ}{\sigma}_1), \\ \tau^1 = -(a\, e^{-2\beta} - \mathrm{V}\, e^{-2\beta}\, r^{-1}\, c)\, \overset{\circ}{\sigma}_0 + (b\, e^{-2\beta} - \mathrm{V}\, e^{-2\beta}\, r^{-1}\, c)\, \overset{\circ}{\sigma}_1, \\ \tau^2 = \mathrm{U}\, c\, e^{-2\beta} (-\overset{\circ}{\sigma}_0 + \overset{\circ}{\sigma}_1) - e^{-2\gamma}\, r^{-2}\, d\, \overset{\circ}{\sigma}_2, \\ \tau^3 = -\dfrac{e^{2\gamma}\, r^{-2}}{\sin^2\theta}\, \overset{\circ}{\sigma}_3. \end{cases}$$

Solutions of the Sachs equations. — Starting from a variational principle, Sachs [3] obtained the field equations, with σ_μ and τ_μ as the dynamical objects. These equations are

$$\mathrm{P}^{+}_{\mu\alpha}\, \sigma^\alpha + \sigma^\alpha\, \mathrm{P}_{\mu\alpha} = 0; \tag{7}$$

$$\mathrm{P}_{\mu\alpha} = \Omega_{\mu,\alpha} - \Omega_{\alpha,\mu} + [\Omega_\mu, \Omega_\alpha]; \tag{8}$$

where

$$\Omega_\mu = -\frac{1}{4}\left[\sigma^\lambda_{,\mu}, \tau_\lambda + \left\{ {\lambda \atop \mu\nu} \right\} \sigma^\nu\, \tau_\lambda\right], \tag{9}$$

and $\left\{ {\lambda \atop \mu\alpha} \right\}$ are the Christoffel symbols. After calculating the affine connection coefficients and the components of the spin curvature, it is possible to proceed with the resolution of (7). These coefficients are :

$$\Omega_0 = -\frac{1}{4}[\mathrm{A}^{(0)}\, \overset{\circ}{\sigma}_0 + \mathrm{B}^{(0)}\, \overset{\circ}{\sigma}_1 + \mathrm{D}^{(0)}\, \overset{\circ}{\sigma}_2 + i\, \mathrm{E}^{(0)}\, \overset{\circ}{\sigma}_3],$$

$$\Omega_1 = -\frac{1}{4}[\mathrm{A}^{(1)}\, \overset{\circ}{\sigma}_0 + \mathrm{B}^{(1)}\, \overset{\circ}{\sigma}_1 + \mathrm{D}^{(1)}\, \overset{\circ}{\sigma}_2 + i\, \mathrm{E}^{(1)}\, \overset{\circ}{\sigma}_3],$$

$$\Omega_2 = -\frac{1}{4}[\mathrm{A}^{(2)}\, \overset{\circ}{\sigma}_0 + \mathrm{B}^{(2)}\, \overset{\circ}{\sigma}_1 + \mathrm{D}^{(2)}\, \overset{\circ}{\sigma}_2 + i\, \mathrm{E}^{(2)}\, \overset{\circ}{\sigma}_3],$$

$$\Omega_3 = -\frac{1}{4}[i\, \mathrm{B}^{(3)}\, \overset{\circ}{\sigma}_1 + i\, \mathrm{D}^{(3)}\, \overset{\circ}{\sigma}_2 + \mathrm{E}^{(3)}\, \overset{\circ}{\sigma}_3];$$

The components of the spin curvature, $\mathrm{P}_{\mu\alpha}$, are calculated from (8); using (9), it results that :

$$\begin{aligned} \mathrm{P}_{10} = \frac{1}{4}\Big[& (\mathrm{A}^{(0)}_{,1} - \mathrm{A}^{(1)}_{,0})\, \overset{\circ}{\sigma}_0 + \left(\mathrm{B}^{(0)}_{,1} - \mathrm{B}^{(1)}_{,0} + \frac{1}{2}\mathrm{E}^{(1)}\,\mathrm{D}^{(0)} - \frac{1}{2}\mathrm{E}^{(0)}\,\mathrm{D}^{(1)}\right) \overset{\circ}{\sigma}_1 \\ & + \left(\mathrm{D}^{(0)}_{,1} - \mathrm{D}^{(1)}_{,0} + \frac{1}{2}\mathrm{E}^{(0)}\,\mathrm{B}^{(1)} - \frac{1}{2}\mathrm{B}^{(0)}\,\mathrm{E}^{(1)}\right) \overset{\circ}{\sigma}_2 \\ & + i\left(\mathrm{E}^{(1)}_{,0} - \mathrm{E}^{(0)}_{,1} + \frac{1}{2}\mathrm{D}^{(1)}\,\mathrm{B}^{(0)} - \frac{1}{2}\mathrm{B}^{(1)}\,\mathrm{D}^{(0)}\right) \overset{\circ}{\sigma}_3 \Big], \end{aligned}$$

$$P_{20} = \frac{1}{4}\Big[(A^{(0)}_{,2} - A^{(2)}_{,0})\,\mathring{\sigma}_0 + \Big(B^{(0)}_{,2} - B^{(2)}_{,0} + \frac{1}{2}E^{(2)}D^{(0)} - \frac{1}{2}E^{(0)}D^{(2)}\Big)\mathring{\sigma}_1$$
$$+ \Big(D^{(0)}_{,2} - D^{(2)}_{,0} + \frac{1}{2}E^{(0)}B^{(2)} - \frac{1}{2}B^{(0)}E^{(2)})\,\mathring{\sigma}_2$$
$$+ i\Big(E^{(2)}_{,0} - E^{(0)}_{,2} + \frac{1}{2}B^{(0)}D^{(2)} - \frac{1}{2}B^{(2)}D^{(0)}\Big)\mathring{\sigma}_3\Big],$$

$$P_{30} = \frac{1}{4}\Big[i\Big(B^{(3)}_{,0} + \frac{1}{2}D^{(3)}E^{(0)} + \frac{1}{2}D^{(0)}E^{(3)}\Big)\mathring{\sigma}_1$$
$$+ i\Big(D^{(3)}_{,0} - \frac{1}{2}E^{(0)}B^{(3)} - \frac{1}{2}B^{(0)}E^{(3)}\Big)\mathring{\sigma}_2$$
$$+\Big(- E^{(3)}_{,0} + \frac{1}{2}B^{(0)}D^{(3)} - \frac{1}{2}D^{(0)}B^{(3)}\Big)\mathring{\sigma}^3\Big],$$

$$P_{31} = \frac{1}{4}\Big[i\Big(B^{(3)}_{,2} + \frac{1}{2}E^{(1)}D^{(3)} + \frac{1}{2}E^{(3)}D^{(1)}\Big)\mathring{\sigma}_1$$
$$+ i\Big(D^{(3)}_{,1} - \frac{1}{2}B^{(3)}E^{(1)} - \frac{1}{2}E^{(3)}D^{(1)}\Big)\mathring{\sigma}_2$$
$$+\Big(- E^{(3)}_{,1} + \frac{1}{2}D^{(3)}B^{(1)} - \frac{1}{2}D^{(1)}B^{(3)}\Big)\mathring{\sigma}_3\Big],$$

$$P_{32} = \frac{1}{4}\Big[i\Big(B^{(3)}_{,2} + \frac{1}{2}D^{(3)}E^{(2)} + \frac{1}{2}E^{(3)}D^{(2)}\Big)\mathring{\sigma}_1$$
$$+ i\Big(D^{(3)}_{,2} - \frac{1}{2}B^{(3)}E^{(2)} - \frac{1}{2}E^{(3)}B^{(2)}\Big)\mathring{\sigma}_2$$
$$+\Big(- E^{(3)}_{,2} + \frac{1}{2}D^{(3)}B^{(2)} - \frac{1}{2}B^{(3)}D^{(2)}\Big)\mathring{\sigma}_3\Big].$$

Substituting these expressions in equation (7), we can write the equation for $\mu = 0$, as

$$\mathcal{A}^{(0)}\,\mathring{\sigma}_0 + \mathcal{B}^{(0)}\,\mathring{\sigma} + \mathcal{C}^{(0)}\,\mathring{\sigma}_2 = 0; \tag{10}$$

since the four $\mathring{\sigma}_\mu$ are linearly independent, we get

$$\mathcal{A}^{(0)} = \mathcal{B}^{(0)} = \mathcal{C}^{(0)} = 0,$$

with similar results for $\mu = 1, 2, 3$.

We obtain ten equations; nevertheless, the theory of general relativity admits simultaneously the gauge group as symmetry and invariance groups, consequently, it will be described by fields equations not entirely independent (as consequence of the four Bianchi identities there will be only four independent equations), namely by the equations

$$\left(\beta_{,1} - \frac{1}{2} r \gamma_1^2\right) r^{-1} = 0,$$

$$[r^4 e^{2(\gamma-\beta)} U_{,1}] - 2r^2 [\beta_{,12} - \gamma_{,12} + 2\gamma_{,1}\gamma_{,2} - 2\beta_{,2} r^{-1} - 2\gamma_{,1} \operatorname{cotg}\theta] = 0,$$

$$2V_{,1} + \frac{1}{2} r^4 e^{2(\gamma-\beta)} U_{,1}^2 - r_2 U_{,12} - 4rU_{,2} - r^2 U_{,1} \operatorname{cotg}\theta - 4rU\operatorname{cotg}\theta$$
$$+ 2e^{2(\beta-\gamma)}[-1 - (3\gamma_{,2} - \beta_{,2})\operatorname{cotg}\theta$$
$$- \gamma_{,22} + \beta_{,22} + \beta_{,2}^2 + 2\gamma_{,2}(\gamma_{,2} - \beta_{,2})] = 0,$$

$$2r(r\gamma)_{,01} + (1 - r\gamma_{,1})V_{,1} - (r\gamma_{,11} + \gamma_{,1})V - r(1 - r\gamma_{,1})U_{,2}$$
$$- r^2(\operatorname{cotg}\theta - \gamma_{,2})U_{,1} + r(2r\gamma_{,12} + 2\gamma_{,2} + r\gamma_{,1}\operatorname{cotg}\theta - 3\operatorname{cotg}\theta)$$
$$+ e^{2(\beta-\gamma)}[-1 - (3\gamma_{,2} - 2\beta_{,2})\operatorname{cotg}\theta - \gamma_{,22} + 2\gamma_{,2}(\gamma_{,2} - \beta_{,2})]U = 0.$$

It should be noted that this set of equations which was obtained from a spinorial formalism, are the same as those obtained by Bondi from a tensorial formalism.

5. SPINORIAL FORM OF THE TRAUTMAN BOUNDARY CONDITIONS

Trautman assumed a coordinate system such that $g_{\mu\nu}(x)$ is written asymptotically in the form

(11) $$g_{\mu\nu} = \eta_{\mu\nu} + \lambda_{\mu\nu}(r^{-1}),$$

where $\eta_{\mu\nu} = \operatorname{diag}(+1, -1, -1, -1)$, where r is the " luminosity distance " along the radius. He further assumed that $g_{\mu\nu}$ obtained from equation (11), satisfies

(12) $$g_{\mu\nu,\rho} = \lambda_{\mu\nu,\rho} + O(r^{-2}) = i_{\mu\nu} K_\rho + O(r^{-2}),$$

where $i_{\mu\nu}$ obeys the condition

(13) $$\left(i_{\mu\nu} - \frac{1}{2} i \eta_{\mu\nu}\right) K^\nu = O(r^{-2}).$$

Presently, we wish to determine the spinorial form corresponding to the asymptotical conditions (12) and (13) of the radiation field.

Asymptotically, we admit a cartesian coordinates system such that σ_μ is written as

(14) $$\sigma_\mu = \mathring{\sigma}_\mu{}^c + \Gamma_\mu(r^{-1}),$$

where $\mathring{\sigma}_\mu{}^c$ are the cartesian components of σ_μ; for simplifying the notation, from now on, $\mathring{\sigma}_\mu{}^c$ will be written as $\mathring{\sigma}_\mu$.

Then, defining

$$\tilde{\Gamma}_\mu = \Gamma_\mu + \frac{1}{4}\mathring{\sigma}_\mu \operatorname{tr}(\mathring{\tau}_\alpha \Gamma^\alpha), \tag{15}$$

imposing, asymptotically, that

$$\tilde{\Gamma}_{\mu,\nu} = \tilde{\Gamma}_\mu K_\nu + O(r^{-2}), \tag{16}$$

and, using the relations (11), (12), (13) and (14), we obtain

$$\operatorname{tr}\left(\tilde{\Gamma}_\nu{}^{,\nu}\, \mathring{\tau}_\mu + \tilde{\Gamma}_\mu{}^{,\nu}\, \mathring{\tau}_\nu\right) = O(r^{-2}); \tag{17 a}$$

with the aid of (16) and (17-*a*) we write

$$\operatorname{tr}\left[\left(\tilde{\Gamma}_\nu\, \mathring{\tau}_\mu + \tilde{\Gamma}_\mu\, \mathring{\tau}_\nu\right) K^\nu\right] = O(r^{-2}); \tag{17}$$

which is the spinorial form of the Trautman conditions.

6. APPLICATION OF THE TRAUTMAN CONDITIONS

If the objects σ_μ and τ_μ, defined on section 3, are to correspond to a radiation field, they must verify the Trautman conditions.

Recording the approach on the last section, we must perform a coordinate transformation such that we have the cartesian components of the fundamental objects. The transformation from the system (u, r, θ, φ) to the system (t, x, y, z) is given by

$$\sigma_\mu(x) = A^\alpha{}_\mu\, \sigma_\alpha(x),$$

where $x' = (t, x, y, z)$; $x = (u, r, \theta, \varphi)$; $u = t - r$; $c = 1$ or by

$$\sigma_\mu(x') = (A^T)_\mu{}^\alpha\, \sigma_\alpha(x),$$

Explicitly

$$A = \begin{pmatrix} 1 & -\frac{x}{r} & -\frac{y}{r} & -\frac{z}{r} \\ 0 & \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ 0 & \frac{xz}{\rho\, r^2} & \frac{yz}{\rho\, r^2} & -\frac{\rho}{r^2} \\ 0 & -\frac{y}{\rho^2} & \frac{x}{\rho^2} & 0 \end{pmatrix}$$

with $\rho = (x^2 + y^2)^{\frac{1}{2}}$.

Consequently,

$$
(18)\quad \begin{cases}
\sigma_0(x') = \sigma_0(x), \\
\sigma_1(x') = -\dfrac{x}{r}\sigma_0(x) + \dfrac{x}{r}\sigma_1(x) + \dfrac{xz}{\rho\, r^2}\sigma_2(x) - \dfrac{y}{\rho^2}\sigma_3(x), \\
\sigma_2(x') = -\dfrac{y}{r}\sigma_0(x) + \dfrac{y}{r}\sigma_1(x) + \dfrac{yz}{\rho\, r^2}\sigma_2(x) + \dfrac{x}{\rho^2}\sigma_3(x), \\
\sigma_3(x') = -\dfrac{z}{r}\sigma_0(x) + \dfrac{z}{r}\sigma_1(x) - \dfrac{\rho}{r^2}\sigma_2(x).
\end{cases}
$$

Analogously, the matrices $\mathring{\sigma}_\mu(x)$ are related to $\mathring{\sigma}_\mu(x')$ through

$$
(19)\quad \begin{cases}
\mathring{\sigma}_0(x) = \mathring{\sigma}_0(x'), \\
\mathring{\sigma}_1(x) = \dfrac{x}{r}\mathring{\sigma}_1 + \dfrac{y}{r}\mathring{\sigma}_2 + \dfrac{z}{r}\mathring{\sigma}_3, \\
\mathring{\sigma}_2(x) = \dfrac{xz}{\rho}\mathring{\sigma}_1 + \dfrac{yz}{\rho}\mathring{\sigma}_2 - \rho\mathring{\sigma}_3, \\
\mathring{\sigma}_3(x) = -y\,\mathring{\sigma}_1 + x\,\mathring{\sigma}_2,
\end{cases}
$$

Substituting (19) into (18), [with and given by (3)-(6)], we find that

$$
(20)\quad \begin{cases}
\sigma_0(x') = \mathring{\sigma}_0 + \Gamma_0, \\
\sigma_1(x') = \mathring{\sigma}_1 + \Gamma_1, \\
\sigma_2(x') = \mathring{\sigma}_2 + \Gamma_2, \\
\sigma_3(x') = \mathring{\sigma}_3 + \Gamma_3,
\end{cases}
$$

where

$$
(21)\quad \begin{cases}
\Gamma_0 = \dfrac{a}{r}\mathring{\sigma}_0 + \dfrac{xz}{\rho\, r^2}b\,\mathring{\sigma}_1 - \dfrac{yz}{\rho\, r^2}b\,\mathring{\sigma}_2 - \dfrac{\rho}{r^2}b\,\mathring{\sigma}_3, \\
\Gamma_1 = \left(-\dfrac{x}{r}\dfrac{a}{r} + \dfrac{x}{r}\dfrac{c}{r}\right)\mathring{\sigma}_0 + \left(-\dfrac{x^2 z}{\rho\, r^3}b + \dfrac{x^2}{r^2}\dfrac{c}{r} + \dfrac{x^2 z^2}{\rho^2 r^3}d + \dfrac{y^2}{\rho^2}\dfrac{e}{r}\right)\mathring{\sigma}_1 \\
\qquad + \left(-\dfrac{xyz}{\rho\, r^3}b + \dfrac{xy}{r^3}c + \dfrac{xyz^2}{\rho^2 r^3}d - \dfrac{xy}{\rho^2 r}e\right)\mathring{\sigma}_2 \\
\qquad + \left(-\dfrac{xz}{r^3}b + \dfrac{x\rho}{r^3}b + \dfrac{xz}{r^3}c - \dfrac{xz}{r^3}d\right)\mathring{\sigma}_3, \\
\Gamma_2 = \left(-\dfrac{y}{r}\dfrac{a}{r} + \dfrac{y}{r}\dfrac{c}{r}\right)\mathring{\sigma}_0 + \left(-\dfrac{yxz}{\rho\, r^3}b + \dfrac{xy}{r^3}c + \dfrac{xyz^2}{\rho^2 r^3}d - \dfrac{xy}{\rho^2 r}c\right)\mathring{\sigma}_2 \\
\qquad + \left(-\dfrac{y^2 z}{\rho\, r^3}b + \dfrac{y^3}{r_0}c + \dfrac{y^2 z^2}{\rho\, r^3}d + \dfrac{x^2}{\rho^2}\dfrac{e}{r}\right)\mathring{\sigma}_2 \\
\qquad + \left(\dfrac{y\rho}{r^3}b + \dfrac{yz}{r^3}c - \dfrac{yz}{r^3}d\right)\mathring{\sigma}_3, \\
\Gamma_3 = \left(-\dfrac{z}{r}\dfrac{a}{r} + \dfrac{z}{r}\dfrac{c}{r}\right)\mathring{\sigma}_2 + \left(-\dfrac{xz}{\rho\, r^3}b + \dfrac{xz}{r^3}c - \dfrac{xz}{r^3}d\right)\mathring{\sigma}_1 \\
\qquad + \left(-\dfrac{yz^2}{\rho\, r^3}d + \dfrac{yz}{r^3}c - \dfrac{yz}{r^3}d\right)\mathring{\sigma}_2 + \left(\dfrac{z\rho}{r^3}b + \dfrac{z^2}{r^3}c + \dfrac{\rho}{r^3}d\right)\mathring{\sigma}_3.
\end{cases}
$$

It is now possible to obtain an asymptotical expansion under the desired form. Although the Γ_μ in the above expression are not yet of the order r^{-1}, it is possible to overcome this difficulty, by imposing that they are of this order. This imposition will create restrictions on the functions a, b, c, d and e.

If the asymptotic form (20) is obtained, the verification of the boundary conditions turn into a matter of pure calculation.

We know that if $R_{\mu\nu\rho\sigma}$, is Petrov type N, and satisfies asymptotically an equation of the form

$$K^\mu R_{\mu\nu\rho\sigma} = O(r^{-2}), \tag{22}$$

with K^μ light-like, then $g_{\mu\nu}$ describes a radiation field. Besides, the Riemann tensor constructed with the $g_{\mu\nu}$ assumed by Trautman, satisfies the equation (22) and consequently, both methods are equivalent for verifying the radiation character of a given field.

We will look for an analogous equation in the spinorial formalism, that is, verified by the spinorial curvature constructed with the σ_μ which obey the Trautman's condition.

It follows from the Bianchi identities that the curvature spinor, χ_{ABCD}, must satisfies the equation [4] :

$$\Delta^{A\dot{E}} \chi_{ABCD} = 0, \tag{23}$$

where

$$\Delta^{A\dot{E}} \equiv \sigma^{\mu A\dot{E}} \partial_{;\mu};$$

in empty space, χ_{ABCD} is denoted as ψ_{ABCD}, and (23) becomes

$$\Delta^{A\dot{E}} \psi_{ABCD} = 0. \tag{24}$$

It has already been shown [5] that the spinor curvature and the spin curvature, $P_{\mu\alpha CD}$, are related through

$$\chi_{ABCD} = -2\, \sigma^{\mu \dot{F}}{}_{A}\, \sigma^{\alpha}{}_{B\dot{F}}\, P_{\mu\alpha CD}; \tag{25}$$

The value of ψ_{ABCD} calculated from equation (25) with the approximation (14) is

$$\left\{ \begin{aligned} \psi_{ABCD} &\simeq -\frac{1}{8} (\mathring{\sigma}^{\rho}\, \mathring{\tau}^{\sigma})_{BA}\, (\mathring{\tau}^{\Omega}\, \mathring{\sigma}^{\lambda})_{CD}\, \delta^{\nu\beta}{}_{\rho\sigma}\, \delta^{\mu\alpha}{}_{\lambda\Omega}\, \mathrm{tr}\, (\Gamma_\beta\, \mathring{\tau}_\mu + \mathring{\tau}_\beta\, \Gamma_\mu)_{,\nu\alpha}, \\ \partial^{A\dot{E}}\, \psi_{ABCD} &\simeq -\frac{1}{8} (\mathring{\sigma}^{\rho}\, \sigma^{\sigma})_{BA}\, (\mathring{\tau}^{\Omega}\, \mathring{\sigma}^{\lambda})_{CD}\, \delta^{\nu\beta}{}_{\rho\sigma}\, \delta^{\mu\alpha}{}_{\lambda\Omega}\, \mathrm{tr}\, (\Gamma_\beta\, \mathring{\tau}_\mu + \mathring{\tau}_\beta\, \Gamma_\mu)_{,\nu\alpha}. \end{aligned} \right. \tag{26}$$

In the approximation of equation (14) the derivative equals the coderivative; therefore, we have just shown that ψ_{ABCD}, constructed with

σ_μ suitable to Trautman's method, satifies the équation

$$\Delta^{A\dot{E}}\, \psi_{ABCD} \simeq \partial^{A\dot{E}}\, \psi_{ABCD} = O\,(r^{-2}) \tag{28}$$

which is analogous to equation (22); in order that (27) yields (28) it is necessary to impose that Γ is of order r^{-1} which again implies in restrictions on the functions a, b, c, d and e; a result to be expected. By the method of construction of equation (28), we can see that still here the two theories are equivalent.

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