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## Quantized charge in terms of pure geometry

by

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ABSTRACT. — Misner and Wheeler [1] have shown that unquantized charge may be regarded as a manifestation of a suitably chosen geometry. It will be proved that, under appropriate restrictions, quantized charge may be considered as a manifestation of some curved space-time geometry as well.

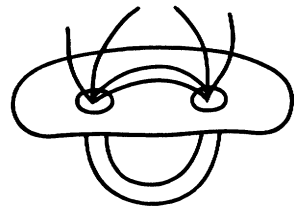
### I. INTRODUCTION

In their fundamental paper, « Classical physics and geometry », Misner and Wheeler [1] regarded classical physics as comprising gravitation, electromagnetism, *unquantized* charges and masses; all four concepts described in terms of empty curved space. In particular, unquantized charge appears in such a geometrical model as being a manifestation of lines of force trapped in a multiply-connected topology, as shown in the adjacent symbolic representation. This unquantized charge is described in terms of the source-free Maxwell equations

$$(1) \quad \operatorname{div} \vec{E} = 0$$

and

$$(2) \quad \operatorname{curl} \vec{H} - \frac{1}{c} \dot{\vec{E}} = 0.$$



One is therefore led to assign a new interpretation to charge in terms of electromagnetic fields that are subject to source-free Maxwell equations (1) and (2). This can be naturally done by considering the net flux of lines of force, as shown in the figure, through the « handle » of some suitable multiply-connected topology. I. e., it can be proved [1] (refer to section II) that this flux is conserved, thus justifying its identification with charge. Consequently, the two holes of the figure exhibit equal and opposite charges. One therefore obtains the familiar pattern of an electric dipole, which is consistent with the equation  $\text{div } \vec{E} = 0$ . This provides an *a posteriori* justification that the divergence-free field equations (1) and (2) permit the existence of electric charge within a suitable geometric framework.

The purpose of this paper is to provide a description of quantized charge in terms of pure geometry, aiming thus ultimately at a description of quantum theory of electricity and general quantum field theory in terms of fields which may be derived from geometry and not added to it.

## II. STATEMENT OF THE PROBLEM

In order to define unquantized charge as a manifestation of geometry, one may proceed, according to [1], as follows: Let  $M$  be some suitable  $C^k$ -manifold as specified below and let  $F^p = F^p(M)$  denote the real vector space of all exterior  $p$ -forms on  $M$ . Then we introduce the operator

$$(3) \quad d: F^p \rightarrow F^{p+1}$$

( $d$  denotes the exterior derivative,  $p = 0, 1, 2, \dots$ ).

The kernel of  $d$ ,  $\text{Ker } d$ , in each dimension is the space

$$(4) \quad \mathring{F}^p = \{ \omega \in F^p \mid d\omega = 0 \} = \text{Ker } d$$

of closed forms. The space

$$(5) \quad dF^{p-1} = \{ \omega \mid \omega = d\alpha \} = \text{Im } d,$$

which constitutes the image of  $d$ , is the space of exact forms. Clearly, according to Poincaré's Lemma  $d(d\omega) = 0, \forall \omega$ , one has:

$$(6) \quad dF^{p-1} \subseteq \mathring{F}^p.$$

The quotient space

$$(7) \quad R^p(M) := H^p = \mathring{F}^p/dF^{p-1}$$

denotes the  $p$ th de Rham cohomology group of  $M$ . Homology is introduced in the same way: let

$$(8) \quad C_p(M) = \left\{ c \mid c = \sum_i \lambda_i \sigma_i^p; \lambda_i \in \mathbb{R} \right\}$$

where

$$(9) \quad \sigma_i^p = \{ \phi : s^p \rightarrow M \}$$

denotes any singular  $p$ -simplex in  $M$  ( $s^p \subset \mathbb{R}^n$  stands for the Euclidean standard  $p$ -simplex [2]).

Upon introduction of the notion of boundary operator  $\partial$  which acts linearly on the chains  $c = \sum_i \lambda_i \sigma_i^p$ , that is

$$(10) \quad \begin{cases} \partial : C_p(M) \rightarrow C_{p-1}(M) & (p = 0, 1, 2, \dots) \\ \partial c = \sum_i \lambda_i \partial \sigma_i^p \end{cases}$$

one has:

$$(11) \quad \mathring{C}_p(M) = \{ c \in C_p(M) : \partial c = 0 \} = \text{Ker } \partial$$

constituting the vector space of  $p$ -cycles (closed chains). I. e. a cycle is a chain whose boundary vanishes. In particular, a bounding cycle or boundary  $b$  is a chain which is the boundary of a chain of one higher dimension,  $b = \partial c$ , that is

$$(12) \quad \partial C_{p+1}(M) = \{ c^p \mid c^p = b = \partial c^{p+1} \}$$

represents the vector space of bounding cycles. Since for each  $p$ -chain  $c$ ,  $\partial(\partial c) = 0$ , each boundary is a cycle, i. e.

$$\partial b = \partial(\partial c) = 0$$

and therefore

$$(13) \quad \partial C_{p+1} \subseteq \mathring{C}_p$$

The quotient space of homology classes is then given by

$$(14) \quad H_p(M) = \mathring{C}^p / \partial C_{p+1}$$

Now define a real bilinear mapping as follows:

$$(15) \quad \left\{ \begin{array}{l} (c, \omega) \rightarrow \int_c \omega \\ H_p \times H^p \xrightarrow{\text{into}} \mathbb{R} \end{array} \right.$$

For some fixed  $\omega$  this entails the existence of a homomorphism

$$(16) \quad h: H_p(M) \rightarrow \mathbb{R}$$

which is referred to as the periods of the closed form  $\omega$ . Otherwise stated: to each  $p$ -cycle  $c$  on  $M$  corresponds a period

$$(17) \quad \int_c \omega$$

of  $\omega$ .

Since unquantized charge is described in terms of source-free Maxwell equations, the two 2-forms

$$(18) \quad \left\{ \begin{array}{l} \omega = F_{\mu\nu} dx^\mu dx^\nu \\ \omega^* = *F_{\mu\nu} dx^\mu dx^\nu \end{array} \right.$$

( $*F^{\mu\nu} = \frac{1}{2!} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$  represents the dual Maxwell tensor) are closed, i. e. in charge-free space the Maxwell equations take the form

$$(19) \quad \left\{ \begin{array}{l} d\omega = 0 \\ d\omega^* = 0. \end{array} \right.$$

Consequently charge may be defined by means of the following periods:

$$(20) \quad \left\{ \begin{array}{l} \int_{c_2} \omega = 4\pi e = 4\pi \circ (\text{electric charge of } \{c_2\}) \\ \int_{c_2} \omega^* = 4\pi g^* = 4\pi \circ (\text{magnetic charge of } \{c_2\}), \end{array} \right.$$

that is,  $e$  and  $g^*$  represent the electric and magnetic charge of  $c_2$  respectively. The definition of charge according to (20) receives its *a posteriori* justification by the fact that it generalizes Gauss's law (refer to remark 7) and by showing that charge defined in this way represents a constant of motion, as displayed by the following

**THEOREM** (Misner-Wheeler [1]). — Unquantized charge regarded as lines of force trapped in a multiply-connected topology stays constant with time.

*Proof.* — This is readily verified by using the following property of periods: periods take the same value on homologous cycles. Indeed:

$$\int_{c'} \omega = \int_{c'' + \partial c} \omega = \int_{c''} \omega + \int_{\partial c} \omega \quad \text{by linearity of the integral}$$

$$\int_{\partial c} \omega = \int_c d\omega = 0 \quad \text{by Stoke's Theorem}$$

where  $c' - c'' = \partial c \Leftrightarrow c' \sim c''$  (which means that  $c'$  and  $c'' \in \dot{C}_p(M)$  are homologous).

From this property we infer:

$$\int_{c'_i} \omega - \int_{c''_i} \omega = 4\pi e'_i - 4\pi e''_i = 0$$

which yields the required conservation law, i. e. this constant of motion represents the unquantized charge.

*Remark 1.* — There is a possibility of associating unquantized charge with a topology as represented in our figure, since the holes are connected by a handle. This ensures that these cycles are homologous and that the above-mentioned theorem applies.

*Remark 2.* — Misner and Wheeler have classified spaces permitting charge. Such spaces may be represented by differentiable manifolds such as

(21)  $\mathbb{R} \times W_k$  (where  $W_k$  denotes a  $k$ -pierced sphere, i. e. spheres which are obtained by drilling  $k$  non-intersecting holes)

or

(22)  $\mathbb{R} \times T^3$  ( $T^3$  represents a 3-torus).

It turns out that the class of spaces permitting charge must display the property that its second Betti-number be  $\beta_2 \geq 1$ .

*Remark 3.* — The lines of force of the flux given by (20), which defines unquantized charge, may be continuously shrunk to extinction in the case where the underlying topology is simply connected. Indeed, by Stoke's Theorem, one has:

$$\int_{c = \partial c'} \omega = \int_{c'} d\omega = 0$$

for Maxwell's equations in charge-free space. Thus source-free Maxwell equations do not yield any definition of charge in terms of a simply-connected topology.

The problem which now arises is: which geometry has to be associated with the concept of quantized charge? In order to answer this problem, we must first analyze some properties of chains.

Chains are in duality with exterior forms, which means that to each property of exterior forms there corresponds a dual property of chains (refer to [3]). To the notion of an integral on a manifold there corresponds the dual notion of the Krobecker-index [2], [3], [4]. This index provides a relationship by means of which point charges may be associated with some appropriate topology. This will be explained in our subsequent discussion.

Consider 0-chains, which are by definition linear combinations of a finite number of points, i. e.

$$(23) \quad c^0 = \sum_k \lambda_k P_k \in C_0(M) \quad \lambda_k \in \mathbb{R}$$

Then the Kronecker-index is defined as being a linear functional on  $C_0(M)$ :

$$(24) \quad \begin{aligned} I: C_0(M) &\rightarrow \mathbb{R} \\ I(c^0) &= \sum_k \lambda_k \end{aligned}$$

and has all properties of an integral [3]. Then the following Theorem holds:

**THEOREM OF POINCARÉ** ([2], [3], [4]). — The necessary and sufficient condition for a  $p$ -cycle to be homologous to zero is given by

$$(25) \quad I(c^p \cdot c^{n-p}) = 0 \quad \forall c^{n-p} \in \dot{C}_{n-p}(M).$$

This constitutes the counterpart of de Rham's first theorem for closed forms.

*Remark 4.* — The composition law  $c^p \cdot c^{n-p}$  in (25) represents the set-theoretical intersection and the orientation of the chains  $c^p$  and  $c^{n-p}$ . Concerning the dimension of this intersection, the following rule holds [3]: given any  $n$ -dimensional manifold  $M^n$ , two chains of dimension  $p_1$  and  $p_2$

of this manifold intersect along a chain of dimension  $(p_1 + p_2 - n)$ .  $p_1 + p_2 < n$  yields no intersection at all.

Poincaré's Theorem entails [3] that one can associate with every closed  $p$ -form  $\omega \in \mathring{F}^p$  an  $(n - p)$ -cycle  $c^{n-p}$  according to the following relationship:

$$(26) \quad \int_{c^p} \omega = I(c^p \cdot c^{n-p}).$$

If one considers in particular a fundamental system of  $p$ -cycles

$$c_1, c_2, \dots, c_{\beta_p} \quad \beta_p \text{ } p\text{th Betti number,}$$

then (26) becomes in terms of the  $\beta_p$  fundamental cycles:

$$(27) \quad \int_{c_i} \omega = I(c \cdot c_i) \quad i = 1, \dots, \beta_p.$$

### III. QUANTIZED CHARGE AND GEOMETRY

Our next task is to define a space-time geometry  $M^4$  such that quantized charge can be regarded as a manifestation of this geometry.

Let  $M^4$  be a differentiable manifold of dimension 4 which represents the space-time continuum.  $M^4$  is supposed to be endowed with a pseudo-Riemannian structure, the metric of which is of the hyperbolic type. Special relativity is taken into account by means of the principal fibre bundle  $E(M^4)$  over the base space  $M^4$ .  $E(M^4)$  is defined as follows:

$$(28) \quad E(M^4) = \{ (x, \rho_x) \mid x \in M^4 \}$$

With respect to the set of orthonormal frames  $\rho_x = \{ e_0, e_k \} \in E(M^4)$ ,  $k = 1, 2, 3$ , the metric can be written on an open neighbourhood of  $M^4$

$$(29) \quad \begin{aligned} ds^2 &= g_{\mu\nu} \theta^\mu \theta^\nu = (\theta^0)^2 - \sum_k (\theta^k)^2, \\ g_{\mu\nu} &= e_\mu \cdot e_\nu = 0 \quad \text{for } \mu \neq \nu \\ g_{00} &= 1; \quad g_{kk} = -1 \end{aligned}$$

where the  $\theta^k$  denote Pfaffians. The structural group of  $E(M^4)$  is the complete Lorentz group  $L(4)$ .



The underlying topology of this space-time must display appropriate properties in order to exhibit quantized charge. This may be formulated by means of the following.

*Assumption.* — The space-time topology  $M^4$  has a structure such that point charges  $e_1, e_2, \dots, e_n$  distributed over the points  $\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_n \in M^4$  and satisfying

$$(23') \quad \tilde{c}^0 = 4\pi \sum_k e_k \tilde{P}_k \in C_0(M^4)$$

appear as a manifestation of this geometry.

The topological properties which  $M^4$  must have in order to be consistent with this assumption, follow from our subsequent Lemmata.

LEMMA 1. — Let  $E$  be a space permitting unquantized charge, then there exists a  $C^k$ -diffeomorphism ( $k \geq 1$ ).

$$(30) \quad \phi : E \rightarrow M^4$$

such that the mapping

$$(31) \quad \phi_* : \dot{C}_2(E) \rightarrow \dot{C}_2(M^4)$$

induced by  $\phi$ , satisfies the following condition:

$$(32) \quad \int_{\phi_* c_i} \tilde{\omega} = 4\pi e_i$$

where  $c_j \in \dot{C}_2(E)$  are fundamental cycles ( $j = 1, \dots, \beta_2$ ) and  $e_i$  ( $i = 1, \dots, n, n < \beta_2$ ) denote point charges located in  $\tilde{P}_i \in M^4$ .

*Remark 5.* — According to (31), the modulhomomorphism  $\phi_*$  maps cycles into cycles. Indeed, by virtue of the following general commutative diagram (33), we have:

$$(33) \quad \begin{array}{ccc} C_p(E) & \xrightarrow{\phi_*} & C_p(M^4) \\ \partial \downarrow & & \downarrow \partial \\ C_{p-1}(E) & \xrightarrow{\phi_*} & C_{p-1}(M^4) \end{array}$$

i. e.  $\forall c_i \in \dot{C}_2 : \partial(\phi_* c_i) = \phi_*(\partial c_i) = \phi_*(0) = 0 \Rightarrow \phi_* c_i \in \dot{C}_2(M^4)$ .

*Proof of Lemma 1.* — Consider the following modulhomomorphism  $\phi^*$

$$(34) \quad F^p(E) \xleftarrow{\phi^*} F^p(M^4)$$

which is naturally induced by the mapping (30). The adjacent commutative diagram (35) expresses explicitly relation (34):

$$(35) \quad \begin{array}{ccc} F^2(E) & \xleftarrow{\phi^*} & F^2(M^4) \\ \omega = \phi^*\tilde{\omega} & \xleftarrow{\sim} & \tilde{\omega} \\ d\downarrow & & \downarrow d \\ F^3(E) & \xleftarrow{\phi^*} & F^3(M^4) \\ d\omega = 0 & \xleftarrow{\sim} & d\tilde{\omega} = 4\pi\gamma \end{array}$$

where  $\tilde{\omega}$  and  $\phi^*\tilde{\omega}$  represent the electromagnetic field tensor  $F^{\mu\nu}(x^\mu)$  in different coordinate frames. The 3-form  $\gamma$  stands for the charge density

$$(36) \quad \gamma = \frac{1}{c}(i_1 dx^2 \wedge dx^3 + i_2 dx^3 \wedge dx^1 + i_3 dx^1 \wedge dx^2) \wedge dx^0 - \rho dx^1 \wedge dx^2 \wedge dx^3.$$

Thus diagram (35) is consistent with our aforementioned assumption about the structure of the space-time topology  $M^4$ . That is, (35) accounts for the conventional as well as for the source-free Maxwell equations, the latter being associated with spaces permitting charge, i. e.

$$(37) \quad \begin{cases} d\tilde{\omega} = 4\pi\gamma \Leftrightarrow F^{\mu\nu}_{, \nu} = 4\pi j^\mu & j^\mu = \left(\rho, \frac{1}{c}\tilde{i}\right) \\ d\omega = 0 \Leftrightarrow F^{\mu\nu}_{, \nu} = 0. \end{cases}$$

On account of the definition of an integral on a manifold [2], one obtains by virtue of diagram (35):

$$(38) \quad \int_{c_i} \omega = \int_{c_i} \phi^*\tilde{\omega} \stackrel{\text{def}}{=} \int_{\phi_*c_i} \tilde{\omega}.$$

The condition (38) is necessary for a distribution of charges according to assumption (23'). Indeed, let

$$\tilde{c}^0 = 4\pi \sum_i e_i \tilde{P}_i \in C_0(M^4).$$

There always exists a linear Kronecker functional  $\tilde{I}$ , such that

$$(39) \quad \begin{cases} \tilde{I}: C_0(M^4) \rightarrow \mathbb{R} \\ \tilde{c}^{n-p} \cdot \tilde{c}'_i{}^p \rightsquigarrow 4\pi e_i, \end{cases}$$

i. e. there exists a  $\tilde{c}^{n-p}$  such that, for every  $\tilde{P}_i \in M^4$  which carries some point charge  $e_i$ , one has

$$\tilde{P}_i \in \tilde{c}^{n-p} \in \dot{C}(M^4).$$

Furthermore,  $\tilde{P}_i \in \phi_* c_i \cap \tilde{c}'_i{}^p$  where  $\tilde{c}'_i{}^p$  denotes a cycle with the properties

$$(40) \quad \tilde{c}^{n-p} \cdot \tilde{c}'_i{}^p = \tilde{c}_i^0 \quad \text{where} \quad \sum_i \tilde{c}_i^0 = \tilde{c}^0$$

and

$$(41) \quad \int_{\phi_* c_i} \tilde{\omega} = \tilde{I}(\tilde{c}^{n-p} \cdot \tilde{c}'_i{}^p),$$

in particular:  $n = 4$ ,  $p = 2$ ,  $n - p = 2$ .

By virtue of (39) and (41) one obtains relationship (32). This entails that  $\int_{\phi_* c_i} \tilde{\omega}$  constitutes a generalization of Gauss's law  $\iint_{c^2} \tilde{E} d\vec{f} = 4\pi e$  ( $e$  denotes any point charge,  $c^2$  is a closed surface) within a suitable space-time geometry  $M^4$  (refer to [1] and our subsequent remark 7). The aforementioned formulae from (39) to (41) inclusive relate physics to geometry.

Thus we have proved the implication (23')  $\Rightarrow$  (32). The converse is also true, as can be checked easily.

*Remark 6.* — The demonstration of Lemma 1 is based upon the equivalence between the assumption (23') and relationship (32).

*Remark 7.* — Consider the affine tangent space  $T_x$  according to (28) which admits a structure of Minkowski space, and let  $x^0, x^1, \dots, x^3$  be the time and space coordinates of  $T_x$ . Then we set

$$(42) \quad \tilde{\omega} = (H_1 dx^1 + H_2 dx^2 + H_3 dx^3) dx^0 \\ + (E_1 dx^2 dx^3 + E_2 dx^3 dx^1 + E_3 dx^1 dx^2),$$

where  $E_i$  and  $H_i$  denote the 6 components of the electromagnetic field tensor  $F^{\mu\nu}(x^\mu)$ . Performing the integration of (32) over the 2-cycle  $\phi_* c_i \in \dot{C}_2(M^4)$  in a hyperplan  $x^0 = \text{constant}$ , this yields:

$$4\pi e = \int_{\phi_* c} \tilde{\omega} = \int E_k dx^i dx^j.$$

*Remark 8.* — The flux integral (32) is induced by the modulhomomorphism

$$\phi_* : \dot{C}_2(E) \rightarrow \dot{C}_2(M^4)$$

as well as by (20) and (38). Thus, quantized charge, which is associated with the topology of  $M^4$ , is determined by unquantized charge and its corresponding topology. This yields that the geometry which is associated with unquantized charge must determine partly the properties of the geometry of  $M^4$  which gives rise to point charge (refer to the proof of Lemma 3).

*Remark 9.* — The conjunction of equations (39) and (41) is different from statement (27), since  $\tilde{\omega}$  is not closed. Nevertheless, one can find a 2-cycle  $\tilde{c}_i = \phi_{*}c_i$  for which equation (41).becomes meaningful. Furthermore, it should be noted that  $\int_{\phi_{*}c_i} \tilde{\omega}$  does not represent any de Rham period, since  $\tilde{\omega}$  is no closed differential form. This integral therefore cannot represent any flux for unquantized charge of the kind displayed in [I].

LEMMA 2. — A necessary condition for the occurrence of quantized charge associated with the topology of  $M^4$  is given by

$$(43) \quad H_{\gamma}^3(M^4) = 0,$$

where  $\gamma$  stands for the charge density (36).

*Proof.* — Let  $\gamma \in F^3(M^4)$  be the 3-form given by (36). This form must satisfy the continuity equation  $\text{div } \vec{i} + \frac{\partial \rho}{\partial t} = 0$ , i. e.  $d\gamma = 0 \in F^4(M^4)$ , which means that  $\gamma$  is closed, i. e.  $\gamma \in \mathring{F}^3(M^4)$ .  $H_{\gamma}^3(M^4) = 0$  means that  $\forall \gamma : \gamma$  is an exact form, that is  $\exists \beta \in F^2(M^4)$  such that  $\gamma = d\beta$ .

Our assumption (23') about the charge distribution associated with the geometry  $M^4$  and the diagram (35) ensures the existence of such a  $\beta$ , since, by virtue of the Maxwell equations (37),  $d\tilde{\omega} = 4\pi\gamma$ , we have  $\beta = \frac{1}{4\pi} \tilde{\omega}$ .

A further property to which  $M^4$  is subject, in order that  $M^4$  may be associated with point charges, is given by the following

LEMMA 3. — A sufficient condition for the geometry  $M^4$  to give rise to quantized charges is that

$$(44) \quad \Pi_2(M^4) = 0$$

holds.

$\Pi_2(M^4)$  constitutes the second homotopy group of  $M^4$ . This group, or more generally the  $k$ -th homotopy group of  $M^4$ , is defined by

$$(45) \quad \Pi_k(M^4) = \{ [f] : S^k \rightarrow M^4 \}.$$

The elements of this group are homotopy classes of maps of the  $k$ -th sphere  $S^k$  into  $M^4$  (see reference [5]). In particular, [0] is represented by the constant map  $f_0(S^k) = \tilde{P} \in M^4$  ( $\tilde{P}$  denotes any point of  $M^4$ ). That is,  $\forall g \in [0] : g(S^k) \simeq \tilde{P}$  ( $\simeq$  means « homotopic to »).

*Proof of Lemma 3.* —  $\Pi_2(M^4) = 0$  means obviously that each  $f_j \in [f] \in \Pi_2(M^4)$  is homotopic to a constant map. Let  $\{c_i : i=1, \dots, \beta_2\}$  be  $\beta_2$  fundamental 2-cycles of  $\mathring{C}_2(E)$ . Some of these constitute cycles in the sense of equation (20). According to [I] and remark 2, these 2-cycles are homeomorphic to 2-spheres, that is, one has (by identifying these homeomorphic spaces)

$$(46) \quad S^2 = c_i^2 \in \mathring{C}_2(E) \quad i = 1, \dots, n; \quad n < \beta_2$$

and therefore there exists some  $f \in [f] \in \Pi_2(M^4)$  with the property

$$(47) \quad f(S^2) = \phi_* c_i^2 = \tilde{c}_i^2 \in \mathring{C}_2(M^4).$$

The condition (44) then amounts to saying that  $\tilde{c}_i^2$  is continuously deformable to the point  $\tilde{p}_i \in M^4$ , that is:

$$(48) \quad \tilde{c}_i^2 = \{ \phi : s^k \subset \mathbb{R}^n \rightarrow M^4 \} \simeq f_0 \quad (\text{refer to (9)})$$

where

$$(49) \quad c_i \in \mathring{C}_2(E) \xrightarrow{\phi_*} \phi_* c_i = \tilde{c}_i \in \mathring{C}_2(M^4).$$

Therefore, any 2-cycle  $c_i^2 \in \mathring{C}_2(E)$ , which accounts for unquantized charge and which satisfies the side condition (39), is mapped into a 2-cycle  $\tilde{c}_i^2 \in \mathring{C}_2(M^4)$  which may be continuously deformed to a point. This point carries the quantized charge in question.

*Remark 10.* — It is seen in Lemma 3 that a comparison between the geometries of the quantized and the unquantized charges, i. e.  $E$  and  $M^4$  is indispensable in order to obtain the property (44) which must be imposed on the space-time  $M^4$ . According to (30), i. e. (33), homologous 2-cycles  $\in \mathring{C}_2(E)$ , which represent unquantized charge, are mapped into *one* 2-cycle  $\in \mathring{C}_2(M^4)$ , which represents the corresponding quantized charge. This corresponds to the passage from equation  $\text{div } \vec{E} = 0$  to equation  $\text{div } \vec{E} = 4\pi\rho$ .

*Remark 11.* — The condition  $\Pi_2(M^4) = 0$  obviously implies the relationship  $\int_{\phi_* c_i} \tilde{\omega} = 4\pi e_i$  to hold (by means of the generalised Gauss's law). But, by virtue of remark 6, this amounts to saying that (44) entails the assumption (23'). Therefore, our assumption about the distribution of point charges receives its *a posteriori* justification through (44). Furthermore, the condition is not necessary for the existence of quantized charge.

Indeed there exist linearly independent cycles  $c_i^2 \in \mathring{C}_2(E)$  other than the cycles with property (20), to which correspond 2-cycles  $\phi_* c_i^2 = f(c_i^2) \in \mathring{C}_2(M^4)$ , which need not be homotopic to a point. This entails the existence of maps  $f \in \Pi_2(M^4)$  such that  $f \notin [0]$ .

*Remark 12.* — In the sequel, we make use of the following definition: the space  $M^4$  is said to be  $k$ -connected for  $k \geq 0$  if and only if it is path-connected (= 0-connected) and the  $k$ -th homotopy group  $\Pi_k(M^4)$  is trivial. This definition of  $k$ -connectedness and Lemma 3 yield that, if  $M_4$  is 2-connected, this space admits quantized charges. In this case, the 2-connectedness of  $M^4$  and the existence of an isomorphism between  $\Pi_1(M^4)$  and  $H_1(M^4)$  [6], entail the condition

$$(50) \quad H^1(M^4) = 0$$

which yields by de Rham's first theorem that each closed one-form on  $M^4$  is exact. Moreover, 2-connectedness of  $M^4$  would entail the relationship (51) to hold:

$$(51) \quad \Pi_3(M^4) \neq 0.$$

Thus, we infer by Hurewicz's Theorem [6]:  $H_3(M^4) \neq 0$  or equivalently  $H^3(M^4) \neq 0$ .

Lemma 3 is consistent with Lemma 2, since  $H_7^3(M^4) \subseteq H^3(M^4)$ .

Thus we may summarise our results as follows:

**THEOREM.** — Let the space-time geometry  $M^4$  be associated by means of a  $C^k$ -diffeomorphism ( $k \geq 1$ )  $\phi : E \rightarrow M^4$  with the topological space  $E$  permitting unquantized charges. Conditions for the space  $M^4$  to exhibit quantized charges as a manifestation of its topology are given by the necessary conditions

$$a) \quad \int_{\phi_* c_i} \tilde{\omega} = 4\pi e_i$$

and

$$b) \quad H_7^3(M^4) = 0$$

and the sufficient condition  $\Pi_2(M^4) = 0$ .

*Remark 13.* — Magnetic monopoles do not exist in a fully classical geometrical theory, as has been pointed out by Misner and Wheeler [1]. Indeed, if the electromagnetic field is derived from a vector potential

$\alpha = \sum_{\mu} A_{\mu} dx^{\mu} \in F^1$ , then there is a zero net flux through every surface  $c^2$  that is closed, and, according to (20), we have:

$$\int_{c^2} \omega = \int_{c^2} d\alpha = \int_{\partial c^2} \alpha = 0.$$

That is: the existence of a vector potential implies that there is no magnetic charge.

The picture of quantized charge which is associated with space-time, as described in this paper, should be referred to as a semi-classical picture. Indeed, all charges in such a theory must be integral multiples of a unit charge  $e$ . However, the reason for this way possibly only be explained within a framework of magnetic monopoles. In such a theory all charges would be integral multiples of  $e$  connected with the pole strength  $g$  of magnetic poles by the formula  $e \cdot g = n \cdot h$ ,  $n = 1, 2, \dots$ ,  $h =$  Planck constant, as exhibited by Dirac [7]. A geometrical theory with monopoles therefore provides a full quantization of electricity. It should be noted [8] that such a geometrical picture would require that the conventional Maxwell equations

$$(52) \quad d\omega = 4\pi\gamma \quad \text{and} \quad (53) \quad d\omega^* = 0,$$

which display a lack of symmetry, be remedied through the replacement of (53) by (53'):  $d\omega^* = 4\pi^*\gamma$  where  $^*\gamma$  represents a conserved magnetic current.

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