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Classification of the irreducible representations of the groups IO(n)

by

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ABSTRACT. — By IO(n) we denote the semidirect product $O(n) \boxed{S} T^n$, where O(n) is the n-dimensional orthogonal group, and T^n is the group of translations of the n-dimensional vector space of real n-tupels $(x_1, \ldots, x_n)$. We classify the irreducible representations of the group ISO(n), the identity component of the group IO(n), and determine the conditions under which an irreducible representation is unitary. These representations are extended in all possible ways to representations of the whole group IO(n). The general results are specialized to the cases $n = 2$, 3 and 4.

1. INTRODUCTION

Let R^n be the n-dimensional real vector space with elements $(x_1, \ldots, x_n)$, in which a positive definite quadratic form $x_1^2 + \ldots + x_n^2$ is given. By IO(n) we denote the group of linear transformations of this vector space which leave the distance $(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2$ between the vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ invariant. It is the semidirect product of the n-dimensional orthogonal group O(n) and the group T^n of pure translations, i. e., $IO(n) = O(n) \boxed{S} T^n$. The group IO($n$) consists of two disconnected pieces. The identity component is the semidirect product $ISO(n) = SO(n) \boxed{S} T^n$, where SO($n$) is the identity component of the group O(n), whose elements are the proper rotations. The other piece of IO(n), the coset with respect to the identity component, is the semidirect product of the improper rotations and the translations. The identity component ISO(n) is a normal subgroup of index two in the whole group IO(n). It is a noncompact, connected, non simply connected Lie group and has a basic set of $n(n+1)/2$ one parameter subgroups, $n(n-1)/2$ of which are for example the rotations in the coordinate planes. The remaining n ones are the pure translations along the coordinate axes. To each one parameter subgroup of the first type we assign a $(n+1) \times (n+1)$ matrix of the form

$$
g_{ij}(\alpha) = \begin{bmatrix}
1 & \cdots & & & & 0 \\
\vdots & \ddots & & & & \vdots \\
\vdots & & \cos\alpha & \cdots & \sin\alpha & & \vdots \\
\vdots & & \vdots & & \vdots & & \vdots \\
\vdots & & -\sin\alpha & & \cos\alpha & & \vdots \\
\vdots & & & & & \ddots & \vdots \\
\vdots & & & & & 1 & 0 \\
0 & \cdots & & & & 0 & 1
\end{bmatrix}
\begin{matrix} 1 \\ \\ i \\ \\ j \\ \\ n \\ n+1 \end{matrix}
$$
$$
\begin{matrix} 1 & \quad i & \quad j & \quad n & n+1 \end{matrix}
$$

and to the translations along the coordinate axes matrices of the type

$$
g_i(t) = \begin{bmatrix}
1 & \cdots & & & 0 \\
\vdots & \ddots & & & \vdots \\
\vdots & & \ddots & & t \\
\vdots & & & \ddots & \vdots \\
\vdots & & & 1 & 0 \\
0 & \cdots & & 0 & 1
\end{bmatrix}
\begin{matrix} 1 \\ \\ i \\ \\ n \\ n+1 \end{matrix}
$$
$$
\begin{matrix} 1 & \qquad & n & n+1 \end{matrix}
$$

The basis elements A_{ij} and T_i of the Lie algebra iso(n) are defined by

(1) $$A_{ij} = \frac{d}{d\alpha} g_{ij}(\alpha) \big|_{\alpha=0}$$

and

(2) $$T_i = \frac{d}{dt} g_i(t) \big|_{t=0}$$

respectively. A simple calculation shows that they obey the following commutation relations

(3) $$[A_{ij}, A_{kl}] = \delta_{jk}A_{il} + \delta_{il}A_{jk} - \delta_{ik}A_{jl} - \delta_{jl}A_{ik}$$

(4) $$[A_{ij}, T_k] = \delta_{jk}T_i - \delta_{ik}T_j$$

(5) $$[T_i, T_j] = 0$$

where all indices run from 1 to n. From relations (3), (4) and (5) it is easy to see that a representation of the Lie algebra iso(n) of the group ISO(n) is completely determined, if one knows the operators which correspond to the generators $A_{12}, A_{23} \ldots A_{n-1,n}$ and one of the translations, we take T_n, because the operators corresponding to the other generators A_{ij} and T_i can be expressed through them using (3), (4) and (5). In a unitary representation of the group the infinitesimal generators are represented by antihermitian operators, i. e., we have $D(A_{ij}) = -D^+(A_{ij})$ and $D(T_i) = -D^+(T_i)$.

The generators A_{ij} with $1 \leqq i < j \leqq n-1$ are a basis of the Lie algebra so(n) of the n-dimensional rotation group, the irreducible representations (IR's) of which have been determined by Gelfand and Zetlin [1]. Therefore the problem of determining the IR's of the Lie algebra iso(n) reduces to determining the operator which corresponds to the generator T_n. It turns out that this can be done by methods similar to those which were used in [1] to determine the IR's of the Lie algebras so(n). This is essentially a consequence of two facts: we reduce with respect to the maximal compact subalgebra, so that within an IR of iso(n) a state vector is completely labelled by discrete indices only. Further, in an IR of iso(n) an IR of so(n) occurs either with multiplicity one or not at all. From this it follows in addition that each IR of the Lie algebra iso(n) can be extended to an IR of the group ISO(n), because the representation space is a discrete sum of finite dimensional subspaces, i. e., the classification of the IR's of the groups ISO(n) is essentially reduced to determining the IR's of the Lie algebra iso(n). In section 2 we describe the IR's of the Lie algebras so(n). In section 3 we determine all IR's of the Lie algebras iso(n) and give the additional conditions, which must be fulfilled, so that the corresponding representation of the group is unitary. In section 4, the representations determined in section 3 are extended to the group IO(n). This is essentially based on a paper by A. H. Clifford [2] in which the connection between the representations of a group and those of a normal subgroup is derived. We need the general theorems of [2] only for the special case where the normal subgroup is of index 2, and therefore we describe this case in the appendix. In the last section we specialize the results of the preceding ones to the cases $n = 2$, 3 and 4.

2. THE IRREDUCIBLE REPRESENTATIONS OF THE LIE ALGEBRAS so(n)

We give in this section the results of [*1*] with some slight modifications of the notation which turn out to be convenient for the further calculations. The generators A_{ij} with $1 \leqq i < j \leqq n$ are defined by (1) and obey the commutation relations (3). There are some characteristic differences for $n = 2p$, even, or $n = 2p + 1$, odd. In either case an IR is determined by a set of p numbers m_{ij}, which are all integer or halfinteger at the same time. We denote a vector in a representation space by $| m_{ij} \rangle$, where m_{ij} is an abbreviation for a complete set of labels, which determine an IR and specify each vector within a representation space uniquely. For $n = 2p$ the complete scheme is

$$(6) \qquad | m_{ij} \rangle = \begin{bmatrix} m_{2p,1} & m_{2p,2} & \cdots & m_{2p,p-1} & m_{2p,p} \\ m_{2p-1,1} & m_{2p-1,2} & \cdots & m_{2p-1,p-1} & \\ m_{2p-2,1} & m_{2p-2,2} & \cdots & m_{2p-2,p-1} & \\ \vdots & \vdots & & & \\ m_{41} & m_{42} & & & \\ m_{31} & & & & \\ m_{21} & & & & \end{bmatrix}$$

and for $n = 2p + 1$

$$(7) \qquad | m_{ij} \rangle = \begin{bmatrix} m_{2p+1,1} & m_{2p+1,2} & \cdots & m_{2p+1,p-1} & m_{2p+1,p} \\ m_{2p,1} & m_{2p,2} & \cdots & m_{2p,p-1} & m_{2p,p} \\ m_{2p-1,1} & m_{2p-1,2} & \cdots & m_{2p-1,p-1} & \\ \vdots & \vdots & & & \\ m_{41} & m_{42} & & & \\ m_{31} & & & & \\ m_{21} & & & & \end{bmatrix}$$

The first lines in (6) and (7) determine an IR of so(n), the other labels specify a vector within a representation space. All m_{ij} are integer or halfinteger at the same time and obey the conditions

$$(8a) \qquad -m_{2k+1,1} \leqq m_{2k,1} \leqq m_{2k+1,1} \leqq \cdots \leqq m_{2k+1,k-1} \leqq m_{2k,k} \leqq m_{2k+1,k}$$

$$(8b) \qquad | m_{2k,1} | \leqq m_{2k-1,1} \leqq m_{2k,2} \leqq \cdots \leqq m_{2k,k-1} \leqq m_{2k-1,k-1} \leqq m_{2k,k}$$

The index k goes from 1 to $p - 1$ or p for n even or odd respectively. We denote by D(A) an operator in a representation space, where A is an element of the Lie algebra. If it is clear from the context we omit the complete specification of the representation. Only in cases where it might lead to some confusion do we use a more rigorous notation, for example $D^{(m_{2p,1},\ldots,m_{2p,p})}(A)$ instead of D(A) in the case of an IR of so($2p$). The operators $D(A_{i,i+1})$ for $1 \leqq i \leqq n-1$ are given by

$$D(A_{2k,2k+1})|l_{ij}\rangle = \sum_{j=1}^{k} A(l_{2k,j})|l_{2k,j}+1\rangle - \sum_{j=1}^{k} A(l_{2k,j}-1)|l_{2k,j}-1\rangle \tag{9}$$

$$D(A_{2k-1,2k})|l_{ij}\rangle = \sum_{j=1}^{k-1} B(l_{2k-1,j})|l_{2k-1,j}+1\rangle - \sum_{j=1}^{k-1} B(l_{2k-1,j}-1)|l_{2k-1,j}-1\rangle + iC_{2k}|l_{ij}\rangle \tag{10}$$

The matrix elements are

$$A(l_{2k,j}) = \frac{1}{2}\sqrt{\prod_{r=1}^{k-1}\left[\left(l_{2k-1,r}-\frac{1}{2}\right)^2-\left(l_{2k,j}+\frac{1}{2}\right)^2\right]} \sqrt{\frac{\prod_{r=1}^{k}\left[\left(l_{2k+1,r}-\frac{1}{2}\right)^2-\left(l_{2k,j}+\frac{1}{2}\right)^2\right]}{\prod_{\substack{r\neq j\\ r=1}}^{k-1}[l_{2k,r}^2-l_{2k,j}^2]\,[l_{2k,r}^2-(l_{2k,j}+1)^2]}} \tag{11}$$

$$B(l_{2k-1,j}) = \sqrt{\prod_{r=1}^{k-1}(l_{2k-2,r}^2-l_{2k-1,j}^2)} \sqrt{\frac{\prod_{r=1}^{k}(l_{2k,r}^2-l_{2k-1,j}^2)}{l_{2k-1,j}^2(4l_{2k-1,j}^2-1)\prod_{\substack{r\neq j\\ r=1}}^{k-1}(l_{2k-1,r}^2-l_{2k-1,j}^2)[(l_{2k-1,r}-1)^2-l_{2k-1,j}^2]}} \tag{12}$$

$$C_{2k} = \frac{\prod_{r=1}^{k-1} l_{2k-2,r} \prod_{r=1}^{k} l_{2k,r}}{\prod_{r=1}^{k-1} l_{2k-1,r}(l_{2k-1,r} - 1)} \tag{13}$$

The indices l_{ij} are connected with the m_{ij} in the following way

$$l_{2k+1,i} = m_{2k+1,i} + i \tag{14}$$

$$l_{2k,i} = m_{2k,i} + i - 1 \tag{15}$$

Let us describe now how these results are derived for so$(n + 1)$ if they are already known for so(n). According to what we said in the introduction, this problem reduces to determining the operator $D(A_{n,n+1})$. It is not difficult to show that the following commutation relations for $A_{n,n+1}$ can be derived from (3) and vice versa:

$$[A_{i,i+1}, A_{n,n+1}] = 0 \qquad \text{for} \qquad 1 \leqq i = n - 2 \tag{16}$$

$$[A_{n-1,n}, [A_{n,n+1}, A_{n-1,n}]] = A_{n,n+1} \tag{17}$$

$$[A_{n,n+1}, [A_{n,n+1}, A_{n-1,n}]] = - A_{n-1,n} \tag{18}$$

These equations mean that $A_{n,n+1}$ is a vector operator under transformations of SO(n), and therefore the action of $D(A_{n,n+1})$ can be written in the form (9) or (10) for n even or odd respectively. If this expression for $D(A_{n,n+1})$ is put into equation (17) one gets a set of recurrence relations for the matrix elements, a solution of which can be written in the form

$$A(l_{2p,j}) = \frac{1}{2}\sqrt{\prod_{r=1}^{p-1}\left[\left(l_{2p-1,r} - \frac{1}{2}\right)^2 - \left(l_{2p,j} + \frac{1}{2}\right)^2\right]}\cdot a(l_{2p,j}) \tag{19}$$

$$B(l_{2p+1,j}) = \sqrt{\prod_{r=1}^{p-1}(l_{2p,r}^2 - l_{2p+1,j}^2)}\cdot b(l_{2p+1,j}) \tag{20}$$

$$C_{2p+2} = \prod_{r=1}^{p-1} l_{2p,r}\cdot c_{2p+2} \tag{21}$$

The reduced matrix elements $a(l_{2p,j})$, $b(l_{2p+1,j})$ and c_{2p+2} depend on the indices of the uppermost line in the patterns (6) or (7) respectively only. The commutation relation (18) determines these reduced matrix elements with the result (11), (12) and (13). However, resulting from the commutation relations, the labels $l_{2p+1,i}$, $l_{2p+2,i}$ are not given by equations (14) and (15), but by the following ones:

$$l_{2p+1,i} = z_{2p+1,i} + i \tag{22}$$

$$l_{2p+2,i} = z_{2p+2,i} + i - 1 \tag{23}$$

The $z_{ij} = x_{ij} + iy_{ij}$ are complex numbers. The requirement of irreducibility restricts them to the discrete values m_{ij} with the range (8).

3. THE IRREDUCIBLE REPRESENTATIONS OF THE LIE ALGEBRAS iso(n)

In this section we determine an explicit expression for the operator $D(T_n)$ and give a complete classification of the IR's of the Lie algebras iso(n). The generator T_n is uniquely determined by the following commutation relations, which follow from (3), (4) and (5):

$$[A_{i,i+1}, T_n] = 0 \qquad \text{for} \qquad 1 \leqq i \leqq n-2 \tag{24}$$

$$[A_{n-1,n}, [T_n, A_{n-1,n}]] = T_n \tag{25}$$

$$[T_n, [T_n, A_{n-1,n}]] = 0 \tag{26}$$

Comparison of (24) and (25) with (16) and (17) shows that they differ only by the exchange of $A_{n,n+1}$ with T_n. That means, from (24) and (25) one obtains for the matrix elements the expressions (19), (20) and (21). The last commutation relation (26) differs from (18) by the right hand side, and one verifies by direct calculation that the matrix elements $A(l_{2p,j})$, $B(l_{2p-1,j})$ and C_{2p} are replaced by the following expressions, see also [3]:

$$A^0(l_{2p,j}) = \frac{1}{2} z_{2p+1,p} \sqrt{\prod_{r=1}^{p-1}\left[\left(l_{2p-1,r} - \frac{1}{2}\right)^2 - \left(l_{2p,j} + \frac{1}{2}\right)^2\right]} \times \sqrt{\frac{\prod_{r=1}^{p-1}\left[\left(l_{2p+1,r} - \frac{1}{2}\right)^2 - \left(l_{2p,j} + \frac{1}{2}\right)^2\right]}{\prod_{\substack{r=1\\ r\neq j}}^{p-1}(l_{2p,r}^2 - l_{2p,j}^2)[l_{2p,r}^2 - (l_{2p,j}+1)^2]}} \tag{27}$$

$$\text{(28)}\quad B^0(l_{2p+1,j}) = z_{2p+2,p+1}\sqrt{\prod_{r=1}^{p}(l_{2p,r}^2 - l_{2p+1,j}^2)}$$

$$\times\sqrt{\frac{\prod_{r=1}^{p}(l_{2p+2,r}^2 - l_{2p+1,j}^2)}{l_{2p+1,j}^2(4l_{2p+1,j}^2 - 1)\prod_{\substack{r=1\\ r\neq j}}^{p}(l_{2p+1,r}^2 - l_{2p+1,j}^2)[(l_{2p+1,r} - 1)^2 - l_{2p+1,j}^2]}}$$

$$\text{(29)}\qquad C_{2p+2}^0 = z_{2p+2,p+1}\frac{\prod_{r=1}^{p} l_{2p,r}\cdot l_{2p+2,r}}{\prod_{r=1}^{p} l_{2p+1,r}\cdot(l_{2p+1,r} - 1)}$$

The labels $l_{2p+1,i}$ $(1 \leqq i \leqq p - 1)$ in (27) and the labels $l_{2p+2,i}$ $(1 \leqq i \leqq p)$ in (28) and (29) are complex numbers, which are defined by (22) and (23). The $z_{ij} = x_{ij} + iy_{ij}$ have to be chosen in such a way that the so(n) labels have the correct range and that the representation of iso(n) is irreducible. We determine now the conditions for the constants z_{ij} which follow from these requirements. We treat the cases n even or odd separately and we begin with $n = 2p$, even.

The conditions for the so(n) labels are

$$\text{(30)}\qquad 0 \leqq |m_{2p,1}| \leqq m_{2p,2} \leqq \dots \leqq m_{2p,p}$$

This means that we must have $m_{2p,j}^{\max} = m_{2p,j+1}^{\min}$ for $1 \leqq j = p - 1$, and this requires

$$\text{(31)}\qquad A^0(m_{2p,j}^{\max}) = A^0(m_{2p,j+1}^{\min} - 1) = 0$$

These equations give $p - 1$ conditions for the p constants $z_{2p+1,j}$. We choose $1 \leqq j \leqq p - 1$ and get

$$\text{(32)}\quad \left(z_{2p+1,j} + j - \frac{1}{2}\right)^2 = \left(m_{2p,j}^{\max} + j - \frac{1}{2}\right)^2 = \left(m_{2p,j+1}^{\min} + j - \frac{1}{2}\right)^2$$

The solution of (32) is $z_{2p+1,j} = m_{2p+1,j} = m_{2p,j}^{\max} = m_{2p,j+1}^{\min}$ with the conditions

$$\text{(33)}\qquad |m_{2p,1}| \leqq m_{2p+1,1} \leqq m_{2p,2} \leqq \dots \leqq m_{2p+1,p-1} \leqq m_{2p,p}$$

The $m_{2p+1,j}$ are integer or halfinteger together with so(n) labels and have the range $m_{2p+1,j} = 0, \frac{1}{2}, 1, \ldots$ From (27) it is easy to see that that the representation is irreducible for arbitrary complex $z_{2p+1,p}$, because the zeros in the matrix elements, which are determined by (31), are the only ones. However, to avoid having the same representation occur more than once, we make the restriction $0 \leqq x_{2p+1,p}$. In a unitary representation of the group the operator $D(T_n)$ must be antihermitian. This requires that the matrix elements (27) are real. It is easy to see that the denominator in (27) is always positive, and the same is true for the expression

$$\prod_{r=1}^{p-1}\left[\left(l_{2p-1,r}-\frac{1}{2}\right)^2-\left(l_{2p,j}+\frac{1}{2}\right)^2\right]\left[\left(l_{2p+1,r}-\frac{1}{2}\right)^2-\left(l_{2p,j}+\frac{1}{2}\right)^2\right]$$

because $m_{2p\pm 1,j} \leqq m_{2p,j+1} \leqq m_{2p\pm 1,j+1}$. From this it follows that the requirement of antihermiticity for the operator $D(T_n)$ means that $y_{2p+1,p}=0$, i. e., $z_{2p+1,p}$ has to be real.

In the case $n = 2p + 1$ the condition for the so(n) labels is

$$0 \leqq m_{2p+1,1} \leqq m_{2p+1,2} \leqq \cdots \leqq m_{2p+1,p} \tag{34}$$

Thus we must have $m^{\min}_{2p+1,1} \leqq m_{2p+1,1}$ and $m^{\max}_{2p+1,j} = m^{\min}_{2p+1,j+1}$ for $1 \leqq j \leqq p-1$. This requires

$$B^0(m^{\max}_{2p+1,j}) = B^0(m^{\min}_{2p+1,j+1} - 1) = B^0(m^{\min}_{2p+1,1} - 1) = 0 \tag{35}$$

for $2 \leqq j \leqq p$. These equations give p conditions for the $p + 1$ labels $z_{2p+2,j}$. We choose $1 \leqq j \leqq p$ and get

$$z^2_{2p+2,1} = (m^{\min}_{2p+1,1})^2 \tag{36}$$

$$(z_{2p+2,j} + j - 1)^2 = (m^{\max}_{2p+1,j-1} + j - 1)^2 = (m^{\min}_{2p+1,j} + j - 1)^2 \tag{37}$$

The solution of these equations is

$$z_{2p+2,1} = m_{2p+2,1} = m^{\min}_{2p+1,1};$$
$$z_{2p+2,j} = m_{2p+2,j} = m^{\max}_{2p+1,j-1} = m^{\min}_{2p+1,j}$$

with the conditions

$$|m_{2p+2,1}| \leqq m_{2p+1,1} \leqq m_{2p+2,2} \leqq \cdots \leqq m_{2p+2,p} \leqq m_{2p+1,p} \tag{38}$$

The $m_{2p+2,j}$ are integer or halfinteger together with the so(n) labels;

$m_{2p+2,1}$ has the range $0, \pm\frac{1}{2}, \pm 1, \ldots$, the other labels $m_{2p+2,j}$ with $2 \leqq j \leqq p$ have the range $0, \frac{1}{2}, 1, \ldots$ The representation of iso(n) is irreducible for arbitrary complex $z_{2p+2,p+1}$. We restrict the real part again by $0 \leqq x_{2p+2,p+1}$, because we want a range for the parameters such that every irreducible representation occurs only once. By considerations similar to those in the case n even one finds that the operator $D(T_n)$ is antihermitian if $y_{2p+2,p+1} = 0$.

Let us summarize the results of this section in the following way: for $n = 2p$ the IR's of the Lie algebra iso(n) are determined by p numbers $m_{2p+1,1}, m_{2p+1,2}, \ldots, m_{2p+1,p-1}; z_{2p+1,p}$. The $m_{2p+1,i}$ for $1 \leqq i \leqq p-1$ are all integer or halfinteger together with the so(n) labels. They have the range $0, \frac{1}{2}, 1, \ldots$ and obey the condition

$$0 \leqq m_{2p+1,1} \leqq m_{2p+1,2} \leqq \cdots \leqq m_{2p+1,p-1} \tag{39}$$

The label $z_{2p+1,p} = x_{2p+1,p} + iy_{2p+1,p}$ is a complex number, its real part is restricted by $0 \leqq x_{2p+1,p}$. If in addition $y_{2p+1,p} = 0$ the operator $D(T_n)$ is antihermitian. The so(n) content is given by equation (33).

For $n = 2p + 1$ the IR's of iso(n) are determined by $p + 1$ numbers $m_{2p+2,1}, m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1}$. The $m_{2p+2,i}$ for $1 \leqq i \leqq p$ are integer or halfinteger together with the so(n) labels. The range of $m_{2p+1,1}$ is $0, \pm\frac{1}{2}, \pm 1, \ldots$, the $m_{2p+2,i}$ with $2 \leqq i \leqq p$ have the range $0, \frac{1}{2}, 1, \ldots$ They obey the condition

$$|m_{2p+2,1}| \leqq m_{2p+2,2} \leqq \cdots \leqq m_{2p+2,p} \tag{40}$$

The real part of $z_{2p+2,p+1} = x_{2p+2,p+1} + iy_{2p+2,p+1}$ is again restricted by $0 = x_{2p+2,p+1}$. If $y_{2p+2,p+1} = 0$ the operator $D(T_n)$ is antihermitian. The so(n) content is given by equation (38).

The IR's determined in this section are all pairwise inequivalent and exhaust all IR's of the Lie algebras iso(n).

4. EXTENSION TO THE GROUP IO(n)

As already mentioned in the introduction, the group IO(n) contains two disconnected pieces, the identity component ISO(n) and the coset with respect to it. Every element of the coset can be written uniquely as the product of an element of ISO(n) and a given representative of the coset. We take for this representative the element I which is defined through

$$\mathrm{I}x_i = \begin{cases} +x_i & \text{for} \quad 1 \leqq i \leqq n-1 \\ -x_i & \text{for} \quad i = n \end{cases} \tag{41}$$

and we assign to it a matrix of the form

$$\begin{bmatrix} +1 & & & & & & \\ & +1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & +1 & & \\ & & & & & & -1 & \\ & & & & & & & +1 \end{bmatrix} \begin{matrix} 1 \\ \\ \\ \\ \\ n-1 \\ n \\ n+1 \end{matrix} \tag{42}$$

$$1 \qquad\qquad n-1 \quad n \;\; n+1$$

It is easy to see that I obeys the following commutation relations

$$[\mathrm{I}, \mathrm{A}_{i,i+1}] = 0 \quad \text{for} \quad 1 \leqq i \leqq n-2 \tag{43a}$$

$$[\mathrm{I}, \mathrm{A}_{n-1,n}]_+ = 0 \tag{43b}$$

$$[\mathrm{I}, \mathrm{T}_i] = 0 \quad \text{for} \quad 1 \leqq i \leqq n-1 \tag{44a}$$

$$[\mathrm{I}, \mathrm{T}_n]_+ = 0 \tag{44b}$$

Here we used the notation $[\mathrm{A}, \mathrm{B}]_+ = \mathrm{AB} + \mathrm{BA}$. It turns out that the possible extensions to the group IO(n) are different for integer or half-integer values of the discrete labels which specify an IR of the identity component ISO(n). This is related to the fact that the n-dimensional rotation group is not simply connected so that in the halfinteger case the IR's of the Lie algebra iso(n) actually correspond to representations of a suitable covering group. Therefore we treat at first the integer case, where the results given in the appendix can be applied directly, and then we examine the changes which have to be made in the halfinteger case.

We begin with $n = 2p$. If the IR of ISO($2p$), which is specified by $m_{2p+1,1}, \ldots, m_{2p+1,p-1}$; $z_{2p+1,p}$ is selfconjugate in IO(n), there exists an operator C which obeys the following commutation relations

$$\mathrm{D}(\mathrm{I}^{-1}\mathrm{A}_{i,i+1}\mathrm{I}) = \mathrm{D}(\mathrm{A}_{i,i+1}) = \mathrm{C}^{-1}\mathrm{D}(\mathrm{A}_{i,i+1})\mathrm{C} \tag{45a}$$

(45*b*) $$D(I^{-1}A_{n-1,n}I) = -D(A_{n-1,n}) = C^{-1}D(A_{n-1,n})C$$

(45*c*) $$D(I^{-1}T_nI) = -D(T_n) = C^{-1}D(T_n)C$$

The index i in (45*a*) has the range $1 \leqq i \leqq 2p-2$. From these equations it follows that C has the form

(46) $$C\left| \begin{matrix} m_{2p+1,1}, & m_{2p+1,2}, & \ldots, & m_{2p+1,p-1}; & z_{2p+1,p} \\ m_{2p,1}, & m_{2p,2}, & \ldots, & m_{2p,p-1}, & m_{2p,p} \end{matrix} \right\rangle$$
$$= \sum_{(m'_{2p,1},\ldots,m'_{2p,p})} c(m_{2p-1,1},\ldots,m_{2p-1,p-1}; m_{2p,1},\ldots,m_{2p,p}; m'_{2p,1},\ldots,m'_{2p,p})$$
$$\left| \begin{matrix} m_{2p+1,1}, & m_{2p+1,2}, & \ldots, & m_{2p+1,p-1}; & z_{2p+1,p} \\ m'_{2p,1}, & m'_{2p,2}, & \ldots, & m'_{2p,p-1}, & m_{2p,p} \end{matrix} \right\rangle$$

Putting this expression into (45*b*) and (45*c*) one gets a set of conditions for the matrix elements at the right hand side of (46). The solution gives the following expression for C:

(47) $$C\left| \begin{matrix} m_{2p+1,1}, & m_{2p+1,2}, & \ldots, & m_{2p+1,p-1}; & z_{2p+1,p} \\ m_{2p,1}, & m_{2p,2}, & \ldots, & m_{2p,p-1}, & m_{2p,p} \end{matrix} \right\rangle$$
$$= \prod_{j=1}^{p-1}(-1)^{m_{2p-1,j}} \prod_{j=2}^{p}(-1)^{m_{2p,j}}$$
$$\left| \begin{matrix} m_{2p+1,1}, & m_{2p+1,2}, & \ldots, & m_{2p+1,p-1}; & z_{2p+1,p} \\ -m_{2p,1}, & m_{2p,2}, & \ldots, & m_{2p,p-1}, & m_{2p,p} \end{matrix} \right\rangle$$

There are no conditions for the labels $m_{2p+1,1}, \ldots, m_{2p+1,p-1}$ and $z_{2p+1,p}$, i. e., an arbitrary IR of ISO(2*p*) is selfconjugate in IO(2*p*). In the state vectors in equations (46) and (47) only the two uppermost lines of the Gelfand-Zetlin pattern are written down, the omitted labels are not changed in these equations. According to (A.3) the operator corresponding to I is given by

(48) $$D^{(m_{2p+1,1},\ldots,m_{2p+1,p-1};z_{2p+1,p})}(I) = \pm C$$

These are all possible extensions of an IR of ISO(2*p*) if the labels

$$m_{2p+1,1}, \ldots, m_{2p+1,p-1}$$

are integer.

If $n = 2p+1$ and an IR of ISO($2p+1$) is selfconjugate in IO($2p+1$), there must again exist an operator C which obeys the commutation relations (45). The range of the index i in (45*a*) is now $1 \leqq i \leqq 2p - 1$. From the commutation relations (45) it follows that C has the form

$$(49)\quad C\left| \begin{matrix} m_{2p+2,1}, m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1} \\ m_{2p+1,1}, m_{2p+1,2}, \ldots, m_{2p+1,p} \end{matrix} \right\rangle$$

$$= \sum_{(m'_{2p+1,1},\ldots,m'_{2p+1,p})} c(m_{2p,1}, \ldots, m_{2p,p}; m_{2p+1,1}, \ldots, m_{2p+1,p}; m'_{2p+1,1}, \ldots, m'_{2p+1,p}) \cdot \left| \begin{matrix} m_{2p+2,1}, m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1} \\ m'_{2p+1,1}, m'_{2p+1,2}, \ldots, m'_{2p+1,p} \end{matrix} \right\rangle$$

Again, putting this expression into (45*b*) and (45*c*) one gets a set of conditions for the matrix elements in (49). These conditions can only be fulfilled if $m_{2p+2,1} = 0$. That means, an IR of ISO($2p + 1$) is selfconjugate in IO($2p + 1$) only if $m_{2p+2,1} = 0$. In this case one gets for C

$$(50)\quad C\left| \begin{matrix} 0 \quad , m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1} \\ m_{2p+1,1}, m_{2p+1,2}, \ldots, m_{2p+1,p} \end{matrix} \right\rangle$$

$$= \prod_{j=1}^{p} (-1)^{m_{2p,j}} (-1)^{m_{2p+1,j}} \left| \begin{matrix} 0 \quad , m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1} \\ m_{2p+1,1}, m_{2p+1,2}, \ldots, m_{2p+1,p} \end{matrix} \right\rangle .$$

The operator representing I is given by

$$(51)\qquad D^{(0, m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1})}(I) = \pm C$$

If $m_{2p+2,1} \neq 0$, the representations of IO($2p + 1$) can be induced from those of ISO($2p + 1$) as described in the appendix. We take as representative of ISO($2p + 1$) the unit matrix E_{2p+1} and as representative of the coset the element I. Denoting the representation of IO($2p + 1$), which is induced by an irreducible representation

$$D(A) = D^{(m_{2p+2,1}, \ldots, m_{2p+2,p}; z_{2p+2,p+1})}(A),$$

by $\bar{D}(A)$, we get

$$(52)\quad \bar{D}(A_{i,i+1}) = \begin{bmatrix} D(A_{i,i+1}) & 0 \\ 0 & D(A_{i,i+1}) \end{bmatrix} \qquad 1 \leqq i \leqq 2p - 2$$

$$(53)\quad \bar{D}(A_{2p,2p+1}) = \begin{bmatrix} D(A_{2p,2p+1}) & 0 \\ 0 & -D(A_{2p,2p+1}) \end{bmatrix}$$

$$\bar{D}(T_{2p+1}) = \begin{bmatrix} D(T_{2p+1}) & 0 \\ 0 & -D(T_{2p+1}) \end{bmatrix} \tag{54}$$

$$\bar{D}(I) = \begin{bmatrix} 0 & D(E_{2p+1}) \\ D(E_{2p+1}) & 0 \end{bmatrix} \tag{55}$$

We go from $\bar{D}(A)$ to an equivalent representation

$$D^{(\pm m_{2p+2,1}, m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1})}(A)$$

with the transformation

$$D^{(\pm m_{2p+2,1}, m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1})}(A) = \begin{bmatrix} E & 0 \\ 0 & C \end{bmatrix}^{-1} \bar{D}(A) \begin{bmatrix} E & 0 \\ 0 & C \end{bmatrix} \tag{56}$$

where C is defined through

$$C \left| \begin{matrix} m_{2p+2,1}, m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1} \\ m_{2p+1,1}, m_{2p+1,2}, \ldots, m_{2p+2,p} \end{matrix} \right\rangle \tag{57}$$
$$= \prod_{j=1}^{p} (-1)^{m_{2p,j}} (-1)^{m_{2p+1,j}} \left| \begin{matrix} m_{2p+2,1}, m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1} \\ m_{2p+1,1}, m_{2p+1,2}, \ldots, m_{2p+1,p} \end{matrix} \right\rangle$$

The result has the following form

$$D^{(\pm m_{2p+2,1}, m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1})}(A_{i,i+1}) \tag{58}$$
$$= \begin{bmatrix} D^{(+ m_{2p+2,1}, m_{2p+2,2} \ldots m_{2p+2,p}; z_{2p+2,p+1})}(A_{i,i+1}) & 0 \\ 0 & D^{(- m_{2p+2,1}, m_{2p+2,2} \ldots m_{2p+2,p}; z_{2p+2,p+1})}(A_{i,i+1}) \end{bmatrix}$$

for $1 \leqq i \leqq 2p$,

$$D^{(\pm m_{2p+2,1}, m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1})}(T_{2p+1}) \tag{59}$$
$$= \begin{bmatrix} D^{(+ m_{2p+2,1}, m_{2p+2,2}, \ldots, m_{2p+2,p}, m_{2p+2,p+1})}(T_{2p+1}) & 0 \\ 0 & D^{(- m_{2p+2,1}, m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1})}(T_{2p+1}) \end{bmatrix}$$

and

$$D^{(\pm m_{2p+2,1}, m_{2p+2,2}, \ldots, m_{2p+2,p}; z_{2p+2,p+1})}(I) = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \tag{60}$$

Finally we have to discuss the case where the discrete labels, which specify an IR of ISO(n), are halfinteger, for this discussion see [*5*] [*6*] and [*7*].

The rotation group SO(n) has a twofold covering group CSO(n). The homomorphism from CSO(n) to SO(n) has the kernel $\pm e$, where e is the unit element of CSO(n). This covering group is uniquely determined. The group O(n) has two covering groups which we call $C_1O(n)$ and $C_2O(n)$. In one of them the elements corresponding to I, which is defined by (41), have the square $+ e$, in the other $- e$. Only the representations with integer values of the discrete labels are representations of the groups SO(n) and O(n) respectively. If the discrete labels have halfinteger values the representation specified by them actually determines a representation of one of the covering groups. According to the ambiguity in the choice of the covering group for O(n) there exist more possible extensions for halfinteger labels than for integer ones. The case of the groups ISO(n) and IO(n) is similar. ISO(n) has a single covering group CISO(n). IO(n) however has two covering groups which we call $C_1IO(n)$ and $C_2IO(n)$. In $C_1IO(n)$ the elements, which correspond to the element I of IO(n), have the square $+ e$, in $C_2IO(n)$ they have the square $- e$. It is easy to see that the whole discussion for $C_1IO(n)$ is completely analog the integer case. Therefore we are left with the halfinteger case for $C_2IO(n)$. For $n = 2p$ all IR's are again selfconjugate, and the discussion is similar to the integer case. Only the relation between $D(I^2)$ and C^2 is changed by the factor $- 1$. Therefore to the operator D(I) correspond now the matrices

$$(61) \qquad D^{(m_{2p+1,1},\ldots,m_{2p+1,p-1};z_{2p+1,p})}(I) = \pm iC$$

where C is given by (47).

For $n = 2p + 1$ and halfinteger values for the discrete labels there are no selfconjugate representations. The representations of $C_2O(n)$ have to be induced from those of the identity component. The formulas for the infinitesimal generators of the Lie algebra are the same as in the integer case, they are given by (52), (53) and (54). However, instead of (55) we get

$$(62) \qquad \bar{D}(I) = \begin{bmatrix} 0 & -D(e) \\ +D(e) & 0 \end{bmatrix}$$

We transform (52), (53) and (62) now with the matrix $\begin{bmatrix} E & 0 \\ 0 & -iC \end{bmatrix}$. For $A_{i,i+1}$ with $1 \leqq i \leqq 2p$ and T_{2p+1} this gives again (58) and (59) respectively. However, instead of (60) we get now

$$(63) \qquad D^{(\pm m_{2p+2,1},m_{2p+2,2},\ldots,m_{2p+2,p};z_{2p+2,p+1})}(I) = \begin{bmatrix} 0 & iC \\ iC & 0 \end{bmatrix}$$

where C is defined through (57).

At the end of this section let us again summarize the results, which we have found. If $n = 2p$ all IR's of ISO(n) are selfconjugate in IO(n). If the discrete labels $m_{2p+1,1}, \ldots, m_{2p+1,p-1}$ are integer, the two possible extensions are given by (47) and (48). If the discrete labels are halfinteger there is also the extension (61) possible which gives actually, a representation of the covering group $C_2IO(n)$.

If $n = 2p + 1$ and $m_{2p+2,1} = 0$, the IR's of ISO(n) are selfconjugate in IO(n) and the two possibilities for the operator D(I) are given by (50) and (51). If $m_{2p+2,1} \neq 0$ the IR's of ISO(n) are not selfconjugate. The operators corresponding to the generators $A_{i,i+1}$ for $1 \leqq i \leqq 2p$ and T_{2p+1} are always given by (58) and (59) respectively. In the integer case the only possible extension is given by (60). In the halfinteger case there exists in addition the extension by (63).

5. SOME SPECIAL CASES

In this section we specialize the results of the preceding ones to the cases $n = 2$, 3 and 4. We give explicit expressions for the operators $D(A_{i,i+1})$ for $1 \leqq i \leqq n - 1$ and $D(T_n)$. In the state vectors the labels which do not change are always omitted.

a) $n = 2$. In an IR a state is completely labelled by

(61) $$| m_{ij} \rangle = \begin{bmatrix} z_{31} \\ m_{21} \end{bmatrix}$$

From section 2 and 3 we get

(62) $$D^{(z_{31})}(A_{12}) | m_{21} \rangle = i m_{21} | m_{21} \rangle$$

(63) $$D^{(z_{31})}(T_2) \; | m_{21} \rangle = A^0(m_{21}) | m_{21} + 1 \rangle - A^0(m_{21} - 1) | m_{21} - 1 \rangle$$

There is only one matrix element

(64) $$A^0(m_{21}) = \frac{1}{2} z_{31}$$

The real part of z_{31} is restricted by $0 \leqq x_{31}$, If $y_{31} = 0$ the operator $D^{(z_{31})}(T_2)$ is antihermitian. The element I is defined by

(65) $$I x_i = \begin{cases} + x_1 \\ - x_2 \end{cases}$$

All IR's of ISO(2) are selfconjugate in IO(2). If m_{21} is integer the two extensions are

(66) $$D^{(z_{31})}(I) \mid z_{31}, m_{21} \rangle = \pm \mid z_{31}, -m_{21} \rangle$$

If m_{21} is halfinteger there exist in addition the two extensions

(67) $$D^{(z_{31})}(I) \mid z_{31}, m_{21} \rangle = \pm i \mid z_{31}, -m_{21} \rangle$$

b) $n = 3$. We describe at first the representations of the identity component ISO(3). In an IR of the Lie algebra iso(3) a state is completely labelled through

(68) $$\mid m_{ij} \rangle = \begin{bmatrix} m_{41} & z_{42} \\ m_{31} & \\ m_{21} & \end{bmatrix}$$

The operator $D^{(m_{41},z_{42})}(A_{12})$ is given by (62). The operators $D^{(m_{41},z_{42})}(A_{23})$ and $D^{(m_{41},z_{42})}(T_3)$ act in the following way on a state

(69) $$D^{(m_{41},z_{42})}(A_{23}) \mid m_{21} \rangle = A(m_{21}) \mid m_{21} + 1 \rangle - A(m_{21} - 1) \mid m_{21} - 1 \rangle$$

(70) $$D^{(m_{41},z_{42})}(T_3) \mid m_{31} \rangle = B^0(m_{31}) \mid m_{31} + 1 \rangle - B^0(m_{31} - 1) \mid m_{31} - 1 \rangle + iC_2^0 \mid m_{31} \rangle$$

The matrix elements are

(71) $$A(m_{21}) = \frac{1}{2}\sqrt{\left(m_{31} + \frac{1}{2}\right)^2 - \left(m_{21} + \frac{1}{2}\right)^2}$$

(72) $$B^0(m_{31}) = z_{42}\sqrt{m_{21}^2 - (m_{31} + 1)^2}\sqrt{\frac{m_{41}^2 - (m_{31} + 1)^2}{(m_{31} + 1)^2[4(m_{31} + 1)^2 - 1]}}$$

(73) $$C_2^0 = \frac{m_{21}m_{41}z_{42}}{(m_{31} + 1)m_{31}}$$

The real part of z_{42} is restricted by $0 \leqq x_{42}$. The operator $D^{(m_{41},z_{42})}(T_3)$ is antihermitian if $y_{42} = 0$. The group element I is defined through

(74) $$Ix_i = \begin{cases} + x_i & \text{for} \quad i = 1, 2 \\ - x_3 & \end{cases}$$

If $m_{41} = 0$ the representations of ISO(3) are selfconjugate in IO(3) and can be extended by the operator $D^{(0,z_{42})}(I)$, which is given by

(75) $$D^{(0,z_{42})}(I) \mid 0, z_{42}; m_{31}, m_{21} \rangle = \pm(-1)^{m_{21}}(-1)^{m_{31}} \mid 0, z_{42}; m_{31}, m_{21} \rangle$$

If $m_{41} \neq 0$ the IR's $D^{(\pm m_{41},z_{42})}(A)$ of IO(3) are induced from those of

ISO(3). From (58) and (59) we get for the generators $A_{i,i+1}$ with $i = 1, 2$ and T_3

$$D^{(\pm m_{41}, z_{42})}(A_{i,i+1}) = \begin{bmatrix} D^{(+m_{41}, z_{42})}(A_{i,i+1}) & 0 \\ 0 & D^{(-m_{41}, z_{42})}(A_{i,i+1}) \end{bmatrix} \tag{76}$$

$$D^{(\pm m_{41}, z_{42})}(T_3) = \begin{bmatrix} D^{(+m_{41}, z_{42})}(T_3) & 0 \\ 0 & D^{(-m_{41}, z_{42})}(I) \end{bmatrix} \tag{77}$$

If m_{41} is integer there is only one possible extension

$$D^{(\pm m_{41}, z_{42})}(I) = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \tag{78}$$

with

$$C \,|\, m_{41}, z_{42}\,; m_{31}, m_{21} \rangle = (-1)^{(m_{21}+m_{31})} \,|\, m_{41}, z_{42}\,; m_{31}, m_{21} \rangle \tag{79}$$

If m_{41} is halfinteger there exists in addition the possibility

$$D^{(\pm m_{41}, z_{42})}(I) = \begin{bmatrix} 0 & iC \\ iC & 0 \end{bmatrix} \tag{80}$$

with the same C.

c) $n = 4$. A complete labelling in an IR is

$$|\, m_{ij} \rangle = \begin{bmatrix} m_{51} & z_{42} \\ m_{41} & m_{42} \\ m_{31} & \\ m_{21} & \end{bmatrix} \tag{81}$$

The operators $D^{(m_{51}, z_{52})}(A_{12})$ and $D^{(m_{51}, z_{52})}(A_{23})$ are given by (62) and (69) respectively. $D^{(m_{51}, z_{52})}(A_{34})$ and $D^{(m_{51}, z_{52})}(T_4)$ act in the following way on a vector in a representation space:

$$\begin{aligned} &D^{(m_{51}, z_{52})}(A_{34}) \,|\, m_{31} \\ &\quad = B(m_{31}) \,|\, m_{31} + 1 \rangle - B(m_{31} - 1) \,|\, m_{31} - 1 \rangle + iC_2 \,|\, m_{31} \rangle \end{aligned} \tag{82}$$

$$\begin{aligned} &D^{(m_{51}, z_{52})}(A_{34}) \,|\, m_{41}, m_{42} \rangle \\ &\quad = A^0(m_{41}) \,|\, m_{41} + 1, m_{42} \rangle + A^0(m_{42}) \,|\, m_{41}, m_{42} + 1 \rangle \\ &\quad - A^0(m_{41} - 1) \,|\, m_{41} - 1, m_{42} \rangle - A^0(m_{42} - 1) \,|\, m_{41}, m_{42} - 1 \rangle \end{aligned} \tag{83}$$

The matrix element are

$$B(m_{31}) = \sqrt{m_{21} - (m_{31} + 1)^2} \tag{84}$$

$$\sqrt{\frac{[m_{41}^2 - (m_{31} + 1)^2][(m_{42} + 1)^2 - (m_{31} + 1)^2]}{(m_{31} + 1)^2[4(m_{31} + 1)^2 - 1]}}$$

(85) $$C_2 = \frac{m_{21}m_{41}(m_{42}+1)}{(m_{31}+1)m_{31}}$$

(86) $$A^0(m_{41}) = \frac{1}{2}z_{52}\sqrt{\left(m_{31}+\frac{1}{2}\right)^2 - \left(m_{41}+\frac{1}{2}\right)^2}$$
$$\sqrt{\frac{\left(m_{51}+\frac{1}{2}\right)^2 - \left(m_{41}+\frac{1}{2}\right)^2}{[(m_{42}+1)^2 - m_{41}^2][(m_{42}+1)^2 - (m_{41}+1)^2]}}$$

(87) $$A^0(m_{42}) = \frac{1}{2}z_{52}\sqrt{\left(m_{31}+\frac{1}{2}\right)^2 - \left(m_{42}+\frac{3}{2}\right)^2}$$
$$\sqrt{\frac{\left(m_{51}+\frac{1}{2}\right)^2 - \left(m_{42}+\frac{3}{2}\right)^2}{[m_{41}^2 - (m_{42}+1)^2][m_{41}^2 - (m_{42}+2)^2]}}$$

The real part x_{52} is again restricted by $0 \leqq x_{52}$, and $D^{(m_{51},z_{52})}(T_4)$ is antihermitian if $y_{52} = 0$. The element I is defined through

(88) $$Ix_i = \begin{cases} +\,x_i & \text{for} \quad 1 \leqq i \leqq 3 \\ -\,x_4 & \end{cases}$$

All IR's are selfconjugate in IO(4). If m_{51} is integer the only possible extensions are

(89) $$D^{(m_{51},z_{52})}(I)\,|\,m_{51}, z_{52}; m_{41}, m_{42}, m_{31}, m_{21}\rangle$$
$$= \pm(-1)^{(m_{31}+m_{42})}\,|\,m_{51}, z_{52}; -m_{41}, m_{42}, m_{31}, m_{21}\rangle$$

If m_{51} is halfinteger there exist in addition the possibilities

(90) $$D^{(m_{51},z_{52})}(I)\,|\,m_{51}, z_{52}; m_{41}, m_{42}, m_{31}, m_{21}\rangle$$
$$= \pm\, i(-1)^{(m_{31}+m_{42})}\,|\,m_{51}, z_{52}; -m_{41}, m_{42}, m_{31}, m_{21}\rangle$$

APPENDIX

In this appendix we describe the connection between the representations of a group and those of a normal subgroup of index 2. It is essentially based on a paper by A. H. Clifford [2]; see also [4] [5] and [6]. We give only the results we need. The interested reader can find the proofs and a more general treatment of the whole subject in these references.

Let G be a group, $H \subset G$ a normal subgroup of index 2. By h, h_i we always denote elements of H, by g, g_i elements of G which are not necessarily in H. However, let always $g_0 \notin H$, then $g_0 H$ is the coset with respect to H and we have $G = H + g_0 H$. If $D(h)$ is a representation of H, then also $D(g^{-1}hg) = D^*(h)$ with fixed $g \in G$ is a representation of H, because always $g^{-1}hg \in H$. The representation $D^*(h)$ is called the representation conjugate to $D(h)$. It may happen that the representation $D^*(h)$ is equivalent to $D(h)$ for a subset of G, in this case it is called selfconjugate in this subset. Trivially this is the case for $g \in H$, because then $D(g^{-1}hg) = D(g^{-1})D(h)D(g)$. However, in general the subset of G for which a given representation of H is selfconjugate, may be larger. It can be shown that this subset is always a subgroup of G, called the little group of the representation $D(h)$. If H is of index 2 in G, the little group of an arbitrary representation of H is either H itself or the whole group G.

Let $\bar{D}(g)$ be an irreducible representation of G. If G is restricted to H there are two possibilities which can occur. If $\bar{D}(g)$ remains irreducible, the representation $D(h)$, subduced by $\bar{D}(g)$, is selfconjugate in G. The other possibility is that $D(h)$ is reducible. In this case the little group of $D(h)$ is H itself. $\bar{D}(g)$ splits into the direct sum of two IR's $D_1(h)$ and $D_2(h)$ of H which are conjugate to each other.

We want to describe now how the IR's $D(h)$ of H, which are supposed to be known, can be extended to those of G. Such an extension is determined if we know the operator which corresponds to one representative g_0 of the coset. At first we consider the case where the representation $D(h)$ is selfconjugate in G. There exists an operator C with

$$D(g_0^{-1}hg_0) = C^{-1} \cdot D(h) \cdot C \tag{A.1}$$

for all $h \in H$, and consequently

$$D(g_0^{-2}hg_0^2) = D(g_0^{-2}) \cdot D(h) \cdot D(g_0^2) = C^{-2} \cdot D(h) \cdot C^2 \tag{A.2}$$

because $g_0^2 \in H$. It follows that $D(g_0^2) = C^2$, i. e., $D(g_0)$ is determined up to a sign and we have

$$D(g_0) = \pm C \tag{A.3}$$

The two possibilities of $D(h)$ corresponding to the different signs at the right hand side of (A.3) give two inequivalent representations of G.

The other possibility is that the little group of $D(h)$ is the group H itself. In this case the extension of $D(h)$ to an IR of G can be induced from $D(h)$. We take the unit element e and the element g_0 as representatives of H and the coset respectively. The representation $\bar{D}(g)$, induced by $D(h)$, is irreducible and given by

$$\bar{D}(g) = \begin{bmatrix} D(g) & D(gg_0) \\ D(g_0^{-1}g) & D(g_0^{-1}gg_0) \end{bmatrix} \tag{A.4}$$

where g is an arbitrary element of G. For $g \in H$ it follows from (A.4)

$$\bar{D}(g) = \begin{bmatrix} D(g) & 0 \\ 0 & D(g_0^{-1}gg_0) \end{bmatrix} \tag{A.5}$$

and for $g \notin \mathrm{H}$

$$\bar{\mathrm{D}}(g) = \begin{bmatrix} 0 & \mathrm{D}(gg_0) \\ \mathrm{D}(g_0^{-1}g) & 0 \end{bmatrix} \tag{A.6}$$

From (A.5) and (A.6) one sees that the representation space of $\bar{\mathrm{D}}(g)$ is a system of imprimitivity for G. For $g = g_0$ one gets from (A.6)

$$\bar{\mathrm{D}}(g_0) = \begin{bmatrix} 0 & \mathrm{D}(g_0^2) \\ \mathrm{D}(e) & 0 \end{bmatrix} \tag{A.7}$$

The extensions of the IR's of H to the whole group G, described in this appendix, exhaust all possibilities which lead to inequivalent representations of G.

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