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Normal states and representations of the canonical commutation relations

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ABSTRACT. — A characterization of all representations of the canonical commutation relations which are quasi-equivalent to the Fock representation is given and a characterization of all finite density states is subsequently derived.

RÉSUMÉ. — Nous donnons une caractérisation de toutes les représentations des relations de commutation canoniques qui sont quasi-équivalentes à la représentation de Fock. Il s'ensuit un critère pour que la densité associée à un état soit finie.

I. INTRODUCTION

A criterion is derived which is necessary and sufficient to ensure that a representation of the C*-algebra associated with the canonical commutation relations is quasi-equivalent to the Fock representation. Previous authors [*1*] [*2*] [*3*] [*4*] have considered this problem and established criteria involving two conditions, firstly the regularity of the representation, and secondly the existence of a « number » operator; the various authors differ in their formulation of the second point. We will introduce a new formulation which has the advantages that it is directly expressed in terms of elements of the algebra, is applicable to all representations, and ensures regularity. We apply this new formulation to the characterization of states of finite density.

II. GENERAL FORMULATION

Let $\mathscr{L}$ be a real separable pre-Hilbert space. One constructs in a standard manner a complex Hilbert space $\mathscr{H}$, Fock space, and a map $f \to \{ U(f), V(f) \}$ from $\mathscr{L}$ to unitary operators $U(f)$, $V(f)$ on $\mathscr{H}$ such that for every $f, g \in \mathscr{L}$

a) $$U(f)U(g) = U(g)U(f), \qquad V(f)V(g) = V(g)V(f)$$

and

$$U(f)V(g) = V(g)U(f)e^{-i(f,g)}$$

where (f, g) is the natural scalar product on $\mathscr{L}$.

b) $t \in R \to U(tf)$ and $t \in R \to V(tg)$ are weakly (strongly) continuous in t at the origin.

We define the C*-algebra $\mathscr{A}$ associated with the Fock representation of the canonical commutation relations to be the uniform closure of the algebra generated on $\mathscr{H}$ by $\{ U(f), V(g);\ f, g \in \mathscr{L} \}$.

A representation π of $\mathscr{A}$ is defined to be regular if $t \in R \to \pi(U(tf))$ and $t \in R \to \pi(V(tg))$ are weakly (strongly) continuous in t at the origin for all $f, g \in \mathscr{L}$. A cyclic representation π is regular if and only if $t \in R \to (\Omega, \pi(U(tf))\Omega)$ and $t \in R \to (\Omega, \pi(V(tg))\Omega)$ are continuous at the origin for all $f, g \in \mathscr{L}$ where Ω is a cyclic vector. Similarly a state ω over $\mathscr{A}$ is regular if the analogous continuity properties hold for $t \in R \to \omega(U(tf))$ and $t \in R \to \omega(V(tg))$. If π_ω is the cyclic representation associated with a state ω then π_ω is regular if and only if ω is regular.

If π is a regular representation of $\mathscr{A}$ then Stone's theorem guarantees the existence of self-adjoint operators $\Phi_\pi(f)$ and $\Pi_\pi(f)$ on $\mathscr{H}_\pi$ such that

$$\pi(U(f)) = \exp \{ i\Phi_\pi(f) \}, \qquad \pi(V(f)) = \exp \{ i\Pi_\pi(g) \}$$

for all $f, g \in \mathscr{L}$. These operators are unbounded with dense domains $D(\Phi_\pi(f))$ and $D(\Pi_\pi(f))$ respectively. Using the commutation relations between $U(f)$ and $V(g)$ and the definitions

$$\lim_{t\to 0} \left\| \left[\frac{\pi(U(tf)) - 1}{t} - i\Phi_\pi(f) \right] \psi \right\| = 0 \qquad \psi \in D(\Phi_\pi(f))$$

$$\lim_{t\to 0} \left\| \left[\frac{\pi(V(tf)) - 1}{t} - i\Pi_\pi(f) \right] \psi \right\| = 0 \qquad \psi \in D(\Pi_\pi(f))$$

one checks that the domain $D_{\pi,f} = D(\Phi_\pi(f)) \cap D(\Pi_\pi(f))$ is also dense. Further the operator $a_\pi(f)$ defined by $D(a_\pi(f)) = D_{\pi,f}$ and

$$a_\pi(f)\psi = \Phi_\pi(f)\psi + i\Pi_\pi(f)\psi \qquad \psi \in D_{\pi,f}$$

is closed and has the property that

$$\| a_\pi(f)\psi \|^2 = \| \Phi_\pi(f)\psi \|^2 + \| \Pi_\pi(f)\psi \|^2 - \| \psi \|^2 \qquad \psi \in D_{\pi,f}$$

Thus introducing the self-adjoint operator $N_\pi(f) = a_\pi(f)^* a_\pi(f)$ one deduces that $D_{\pi,f} = D(\sqrt{N_\pi(f)})$ and

$$\| \sqrt{N_\pi(f)}\psi \|^2 = \| \Phi_\pi(f)\psi \|^2 + \| \Pi_\pi(f)\psi \|^2 - \| \psi \|^2 \qquad \psi \in D_{\pi,f}.$$

More generally if M is a finite dimensional subspace of $\mathscr{L}$ then one finds that the domain

$$D_{\pi,M} = \bigcap_{f \in M} D_{\pi,f}$$

is dense. Thus if $\mathscr{I} = \{ f \}$ is an orthonormal basis of M one can define the operators $a_\pi(f)$, $f \in \mathscr{I}$ on the common domain $D_{\pi,M}$. It follows immediately that the quadratic form

$$\psi \in D_{\pi,M} \to \sum_{f \in \mathscr{I}} \| a_\pi(f)\psi \|^2$$

is non-negative, closed, independent of the basis $\mathscr{I}$ and uniquely determines a self-adjoint operator $N_\pi(M)$ on $\mathscr{H}_\pi$ such that $D(\sqrt{N_\pi(M)}) = D_{\pi,M}$ and

$$\| \sqrt{N_\pi(M)}\psi \|^2 = \sum_{f \in \mathscr{I}} \| a_\pi(f)\psi \|^2 = \sum_{f \in \mathscr{I}} \{ \| \Phi_\pi(f)\psi \|^2 + \| \Pi_\pi(f)\psi \|^2 - \| \psi \|^2 \}$$

for each $\psi \in D_{\pi,M}$ [For the latter point see, for example, [5] Chapter VI].

III. A BASIC LEMMA

LEMMA 1. — *Let $\mathscr{H}$ be a Hilbert space and $t \in \mathbf{R} \to U(t)$ a one parameter group of unitary operators with the property that $t \in \mathbf{R} \to (\varphi, U(t)\psi)$ is continuous at the origin for all $\varphi, \psi \in \mathscr{H}$. Let A be the self-adjoint operator with domain D(A) which is such that $U(t) = \exp\{ iAt \}$. The following conditions are equivalent:*

1. $\psi \in D(A)$.
2. *There exists $C_\psi > 0$ independent of t such that*

$$\left\| \left[\frac{U(t) - 1}{t} \right] \psi \right\| \leqslant C_\psi$$

for all t. If these conditions are fulfilled then,

$$\| A\psi \| = \lim_{t\to 0} \left\| \left[\frac{U(t)-1}{t}\right]\psi \right\| = \sup_t \left\| \left[\frac{U(t)-1}{t}\right]\psi \right\|$$

Proof. — Introduce the spectral representation of U by

$$U(t) = \int dE(\lambda)e^{i\lambda t}$$

then for $\psi \in D(A)$ one has

$$\| A\psi \|^2 = \int d(\psi, E(\lambda)\psi)\lambda^2 < +\infty$$

But

$$\left\| \left[\frac{U(t)-1}{t}\right]\psi \right\|^2 = \int d(\psi, E(\lambda)\psi)\left|\frac{e^{i\lambda t}-1}{t}\right|^2$$
$$\leqslant \int d(\psi, E(\lambda)\psi)\lambda^2 = \| A\psi \|^2$$

and hence condition 1 implies condition 2. We prove the converse by showing that if 1 is false, then 2 is false. Now if $\psi \notin D(A)$, then for each $N > 0$ there is a λ_N such that

$$\int_{|\lambda|\leqslant\lambda_N} d(\psi, E(\lambda)\psi)\lambda^2 > 2N \| \psi \|^2$$

But one also has that

$$\left\| \left[\frac{U(t)-1}{t}\right]\psi \right\|^2 \geqslant \int_{|\lambda|\leqslant\lambda_N} d(\psi, E(\lambda)\psi)\lambda^2 \left|\frac{\sin \lambda t/2}{\lambda t/2}\right|^2$$

Now if we choose t_N such that

$$|\sin \lambda_N t_N/2|^2 = \frac{1}{2}|\lambda_N t_N/2|^2$$

it immediately follows that for $|t| \leqslant |t_N|$ and $|\lambda| \leqslant \lambda_N$ one has

$$|\sin \lambda t/2|^2 \geqslant \frac{1}{2}|\lambda t/2|^2$$

Thus

$$\left\| \left[\frac{U(t)-1}{t}\right]\psi \right\|^2 > N \| \psi \|^2 \qquad |t| \leqslant |t_N|$$

and condition 2 is clearly false.

Finally if $\psi \in D(A)$ then the limit given in the lemma exists and gives $\| A\psi \|$ by Stone's theorem but we further have

$$\begin{aligned}
\sup_t \left\| \left[\frac{U(t)-1}{t}\right]\psi \right\|^2 &\geqslant \lim_{t\to 0} \left\| \left[\frac{U(t)-1}{t}\right]\psi \right\|^2 \\
&= \int d(\psi, E(\lambda)\psi)\lambda^2 \\
&= \int d(\psi, E(\lambda)\psi) \sup_t \left| \frac{e^{i\lambda t}-1}{t} \right|^2 \\
&\geqslant \sup_t \int d(\psi, E(\lambda)\psi) \left| \frac{e^{i\lambda t}-1}{t} \right|^2 \\
&= \sup_t \left\| \left[\frac{U(t)-1}{t}\right]\psi \right\|^2
\end{aligned}$$

and the proof of the Lemma is complete.

Remark. — If we allow the value $+\infty$ we can assert that the limit

$$\lim_{t\to 0} \left\| \left[\frac{U(t)-1}{t}\right]\psi \right\|^2$$

exists for all $\psi \in \mathscr{H}$ because if $\psi \notin D(A)$ we have shown that

$$+\infty \geqslant \limsup_{t\to 0} \left\| \left[\frac{U(t)-1}{t}\right]\psi \right\|^2 \geqslant \liminf_{t\to 0} \left\| \left[\frac{U(t)-1}{t}\right]\psi \right\|^2 = +\infty$$

IV. MAIN THEOREMS

Using the foregoing lemma and material from [*2*] [*4*] we can now derive the following characterization of representations of $\mathscr{A}$ which are quasi-equivalent to the Fock representation, i. e. are unitarily equivalent to direct sums (not necessarily denumerable) of copies of the Fock representation.

THEOREM 1. — *Let π be a representation of $\mathscr{A}$ on $\mathscr{H}_\pi$ and let $\mathscr{I}$ be a finite family of orthonormal functions $f \in \mathscr{L}$. The following conditions are equivalent*:

1. *π is quasi-equivalent to the Fock representation.*
2. *The linear subspace of vectors $\psi \in \mathscr{H}_\pi$ with the property*

$$\sup_{\mathscr{I},t} \sum_{f\in\mathscr{I}} \left\{ \left\| \left[\frac{\pi(U(tf))-1}{t}\right]\psi \right\|^2 + \left\| \left[\frac{\pi(V(tf))-1}{t}\right]\Psi \right\|^2 - \| \psi \| \right\} < +\infty$$

is dense in $\mathscr{H}_\pi$.

and if π is cyclic then these conditions are equivalent to

3. *There exists a vector $\Omega \in \mathscr{H}_\pi$ cyclic for π and such that*

$$\sup_{\mathscr{I},t} \sum_{f \in \mathscr{I}} \left\{ \left\| \left[\frac{\pi(\mathrm{U}(tf)) - 1}{t} \right] \Omega \right\|^2 + \left\| \left[\frac{\pi(\mathrm{U}(tf)) - 1}{t} \right] \Omega \right\|^2 - \| \Omega \|^2 \right\} < +\infty .$$

Proof. — Consider condition 3. Clearly for this to be satisfied π must be regular but then we can use lemma 1 to rewrite the condition as:

$$\sup_{\mathscr{I}} \sum_{f \in \mathscr{I}} \{ \| \Phi_\pi(f)\Omega \|^2 + \| \Pi_\pi(f)\Omega \|^2 - \| \Omega \|^2 \} = \lim_{\mathrm{M} \to \mathscr{I}} \| \sqrt{\mathrm{N}_\pi(\mathrm{M})}\Omega \|^2 < +\infty .$$

Similarly condition 2 states that π is regular and there exists a dense set of vectors $\psi \in \mathscr{H}_\pi$ such that

$$\lim_{\mathrm{M} \to \mathscr{L}} \| \sqrt{\mathrm{N}_\pi(\mathrm{M})}\psi \|^2 < +\infty$$

In these latter forms the implications $1 \Rightarrow 2$ and $1 \Rightarrow 3$ are well known. Further for $f, g \in \mathscr{L}$ and $\Omega \in \mathrm{D}(\sqrt{\mathrm{N}_\pi(\mathrm{M})})$ one can use the commutation relations to calculate that

$$\| \sqrt{\mathrm{N}_\pi(\mathrm{M})}\pi(\mathrm{U}(f)\mathrm{V}(g))\Omega \|^2 \leqslant 2 \| \sqrt{\mathrm{N}_\pi(\mathrm{M})}\Omega \|^2 + | f |^2 + | g |^2$$

and the parallelogram inequality to deduce that

$$\| \sqrt{\mathrm{N}_\pi(\mathrm{M})}(\psi_1 + \psi_2) \|^2 \leqslant 2 \| \sqrt{\mathrm{N}_\pi(\mathrm{M})}\psi_1 \|^2 + 2 \| \sqrt{\mathrm{N}_\pi(\mathrm{M})}\psi_2 \|^2$$

for $\psi_1, \psi_2 \in \mathrm{D}(\sqrt{\mathrm{N}_\pi(\mathrm{M})})$. Thus $3 \Rightarrow 2$.

The remaining implication $2 \Rightarrow 1$ is essentially demonstrated in [*2*] [*4*] and so we will omit the proof. One can for example extract the desired result from Theorem 4 of [*4*] (cf. also remark 2 after this theorem). Note that the C*-algebra used in [*4*] differs from ours but it follows from the Stone-Von Neumann theorem that each regular representation of $\mathscr{A}$ has a unique regular extension to the algebra of [*4*] and thus by extension, application of the result of [*4*], and restriction, one obtains the result.

Remark. — If π is a representation of $\mathscr{A}$ on $\mathscr{H}_\pi$ we can introduce the form:

$$n_\pi(\psi) = \sup_{\mathscr{I},t} \sum_{f \in \mathscr{I}} \left\{ \left\| \left[\frac{\pi(\mathrm{U}(tf)) - 1}{t} \right] \psi \right\|^2 + \left\| \left[\frac{\pi(\mathrm{V}(tf)) - 1}{t} \right] \psi \right\|^2 - \| \psi \|^2 \right\}$$

as a functional over the domain $\mathrm{D}(n_\pi) \subset \mathscr{H}_\pi$ which is defined as the vectors for which $n_\pi(\psi) < +\infty$. Arguments similar to the above show that n_π is a closed non-negative quadratic form but of course it is not necessarily densely defined, e. g. if π is not regular $D(n_\pi)$ could be empty. However the closure $\overline{D(n_\pi)}$ of $D(n_\pi)$ in the stromg topology of $\mathcal{H}_\pi$ is invariant under the action of π and hence one deduces that the restriction of π to $\overline{D(n_\pi)}$ is quasi-equivalent to the Fock representation whilst the restriction of π to the complement of $\overline{D(n_\pi)}$ in $\mathcal{H}_\pi$ is disjoint with the Fock representation. Further one deduces that the form n_π uniquely determines a self-adjoint operator N_π, the number operator, which is densely defined on the closed subspace $\overline{D(n_\pi)}$, has the property $D(n_\pi) = D(\sqrt{N_\pi})$, and is such that

$$\begin{aligned} n_\pi(\psi) &= \| \sqrt{N_\pi}\psi \|^2 \\ &= \lim_{M\to\mathcal{L}} \lim_{t\to 0} \sum_{f\in\mathcal{I}_M} \left\{ \left\| \left[\frac{\pi(U(tf)) - 1}{t}\right]\psi \right\|^2 + \left\| \left[\frac{\pi(V(tf)) - 1}{t}\right]\psi \right\|^2 - \| \psi \|^2 \right\} \end{aligned}$$

for $\psi \in D(n_\pi)$ where $\mathcal{I}_M$ is an orthonormal basis of the subspace M of $\mathcal{L}$; the first limit can be taken over any increasing sequence of finite dimensional subspaces M of $\mathcal{L}$ or over the net of all finite-dimensional subspaces.

An important set of representations satisfying the conditions of Theorem 1 are those given by the GNS construction from normal states. A normal state over $\mathcal{A}$ is a state ω_ρ determined by a density matrix (a non-negative trace-class operator with trace-norm unity) ρ on the Fock space $\mathcal{H}$ in the form

$$\omega_\rho(A) = \mathrm{Tr}_{\mathcal{H}}(\rho A) \qquad A \in \mathcal{A}$$

The representation π_ρ associated with ω_ρ is a denumerable sum of copies of the Fock representation. In physical applications it is often useful to have a characterization of the normal states ω_ρ for which the associated cyclic vector Ω_ρ is in the domain of the corresponding number operator N_{π_ρ} or $N_{\pi_\rho}^{\frac{1}{2}}$. The following Theorem is the direct analogue of Theorem 1 of [6] for the anti-commutation relations.

THEOREM 2. — *Define n as a function from the state space* $E_{\mathcal{A}}$ *of* $\mathcal{A}$ *to* $[0, +\infty]$ *by*

$$\omega \in E_{\mathcal{A}} \to n(\omega) = \sup_{\mathcal{I},t} \sum_{f\in\mathcal{I}} \omega\left(\left|\frac{U(tf) - 1}{t}\right|^2 + \left|\frac{V(tf) - 1}{t}\right|^2 - 1\right)$$

where $\mathcal{I}$ *is a finite family of orthonormal functions* $f \in \mathcal{L}$.

It follows that n is affine and lower semi-continuous in the weak-topology of* $\mathcal{A}$. *The following conditions are equivalent.*

1. $n(\omega) < +\infty$.
2. *ω is normal and the associated density matrix ρ_ω has the property*

$$\mathrm{Tr}_{\mathscr{H}}\,(\rho_\omega^{\frac{1}{2}} N \rho_\omega^{\frac{1}{2}}) = \mathrm{Tr}_{\mathscr{H}}\,(N^{\frac{1}{2}} \rho_\omega N^{\frac{1}{2}}) < +\infty$$

where N *is the number operator on Fock space.*

If these latter conditions are satisfied then

$$n(\omega) = \mathrm{Tr}_{\mathscr{H}}\,(N^{\frac{1}{2}} \rho_\omega N^{\frac{1}{2}})$$

Proof. — n is lower semi-continuous because it is the upper envelope of a family of continuous functions. It is clearly affine on the subset $\{ \omega \in E_{\mathscr{A}}\,;\, n(\omega) = +\infty \}$ but it is also affine on the complementary subset due to the last characterization of the number operator given in the above remark. The rest of the theorem follows by the arguments of [*6*] through use of Theorem 1 above, the associated characterization of the number operator, and the results of [*2*] [*4*] for normal states.

COROLLARY 1. — *Let* D_m, $m \geqslant 0$ *denote the subset of normal states* ω_ρ *over* $\mathscr{A}$ *determined by density matrices* ρ *on* $\mathscr{H}$ *with the property that*

$$\mathrm{Tr}\,(N^{\frac{1}{2}} \rho N^{\frac{1}{2}}) \leqslant m.$$

It follows that D_m *is closed in the weak* topology of* $\mathscr{A}$

This conclusion is a direct consequence of the lower semi-continuity of n and its characterization given in Theorem 2.

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