ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 13, nº 3 (1970), p. 221-227

<http://www.numdam.org/item?id=AIHPA_1970__13_3_221_0>

Ann. Inst. Henri Poincaré,
Vol. XIII, n° 3, 1970, p. 221-227.

Section A :
Physique théorique.

Two simple applications of the deformation of Lie algebras

by

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RÉSUMÉ. — 1. Déformer une somme directe $\mathscr{P} \oplus \mathscr{S}$ où $\mathscr{P}$ est l'algèbre de Lie du groupe de Poincaré et $\mathscr{S}$ celle d'un groupe de symétrie interne semi-simple, pouvait sembler *a priori* un moyen d'obtenir des formules de masses en évitant le théorème d'O'Raighfairtaigh. On a cependant des résultats aussi négatifs.

2. Dans le cas d'une représentation de masse non nulle du groupe de Poincaré, le formalisme des déformations permet de montrer simplement l'incompatibilité entre les propriétés de covariance et de commutation pour un opérateur en position relativiste. On retrouve par ailleurs une forme covariante connue.

SUMMARY. — 1. In order to obtain noss formulas, we try to deform a direct sum $\mathscr{P} \oplus \mathscr{S}$, where $\mathscr{P}$ is the Lie algebra of the Poincaré group, and $\mathscr{S}$ the Lie algebra of a semi-simple internal symmetry group.: this process has the advantage of by passing the O'Raighfairtaigh theorem. However, it leads be the same kind of negative results.

2. Using the deformation formalism in looking for a covariant position operator, the incompatibility between covariance and commutation properties can be shown very simply. We recover on the other nand a well-known covariant expression.

In this paper ([1]), we study two simple applications of the deformation formalism of Lie algebras concerning:

1. the relation of the internal symetries and the Poincaré Group in a deformation;

2. the construction of a covariant position operator.

§ 0. Let V ba a n-dimensional vector space and e_i, $i = 1, \ldots, n$ a fixed basis of V. *A deformation of a Lie algebra* μ (defined by its structure constants $\mu(e_i, e_j) = \Sigma_k C_{ij}^k e_k$) is a Lie algebra on the same vector space, with a law depending of one (or more) parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3 \ldots$ such that:

$$\lim_{\varepsilon_i \to 0} C_{ij}^k(\varepsilon_1, \varepsilon_2 \ldots) = C_{ij}^k$$

The initial algebra μ is in a generalized sense, a contraction of the algebra μ_{ε_i} [L.N].

Thus, in order to find all the algebras which can be contracted into a given one, it is necessary to have a classification of all the possible deformations of one algebra.

The first results in that direction are due to Gerstenhaber [G]. In particular, he proves the formal rigidity of the semi-simple algebras: all the deformations by formal series of a semi-simple algebra are isomorphic (over $k((t))$) to the inial algebra. Then, Nijenhuis and Richardson using a more geometrical view-point have found analogous rigidity theorems (on the algebraic manifold of all the Lie algebras defined on V, the orbits of the semi-simple algebras (on C) are open in the topology of Zariski...) [N.R]. Various other aspects of the theory has been developped in R. Hermann's articles [H], most of them being connected with a differential geometry view-point. Likewise, the interested reader will find some results also in [L.N.].

Finally, we mention that the formal and geometrical point of view can be compared, due to a powerful theorem of Artin [A.] [L.N.]. It allows one to construct a deformation by a formal series, without caring about the question of convergence.

§ 1. **Deformation of the direct sum** P + S **where** S **is a semi-simple algebra**

Instead of trying to find a group containing the Poincaré group $\mathscr{P}$ and an internal group $\mathscr{S}$ (a process leading to some well-known difficulties

([1]) The mathematical properties was the main part of the talk presented at Stokholm (Symposium Franco-Suédois de physique théorique). But since they have already been published elsewhere (see references in § 0) in this version, we concentrate on some simple applications.

[O'. R.], [F. S.], [J.]), one can try to find a deformation of the direct product $\mathscr{P} \times \mathscr{S}$. However, this leads to the same kind of negative results.

Only the Lie algebra aspect is considered here. Some non trivial phenomena may appear at the level of groups, but we do not know anything about them.

Let us consider first the deformation of the Lie algebra $\mathscr{P} \times \mathscr{S}$ alone. One has the following result:

The only non trivial deformation of the direct sum $\mathscr{P} + \mathscr{S}$, where S is a semi-simple algebra, and P the Lie algebra of the Poincaré group, are the direct sums $O(4,1) + S$ *and* $O(3,2) + S$.

This theorem has been proved in Lyakhovsky [L.] and [L. N.] (by computing the suitable cohomology groups).

It can be also easily derived from a very useful recent theorem of Page and Richardson about the rigidity of the semi-simple subalgebras [Ri.].

Physically, one has to look at the deformation of the representation. Starting from a representation of $\mathscr{P}$ + S, « sum » of an irreducible representation of P and of S, it is not obvious that a mixing cannot be produced by a deformation in the algebra $O(3,2) + S$ (or $O(4,1) + S$). No general results can be obtained, except in the case of the « first order deformations » (depending linearly on the parameter). But in that case, the general form of these deformations can be written « around the semi-simple part » SL(2, C) + S; it is in obvious notations [1]

$$M_{\mu\nu}(\varepsilon) = M_{\mu\nu} \qquad S_{\alpha(\varepsilon)} = S_\alpha \qquad S_\alpha \in S \tag{1}$$
$$P_{\mu(\varepsilon)} = P_\mu + \frac{i\varepsilon}{2M_0}[\vec{N}^2 - \vec{M}^2, P_\mu]$$

Using the same interpretation as A. Böhm [B.], one obtains the « mass formula »:

$$P_\mu(\varepsilon)P^\mu(\varepsilon) = M^2(\varepsilon) = M_0^2 + \lambda 5(S + 1) \tag{2}$$

(λ depends on ε and M).

In other words, due to (1), any mixing is forbidden between internal and cinematical quantum numbers.

(1) If the group S contains SL(2, C), one can start from a semi-direct form equivalent to the direct sum, the « double structure » [F. N.], [Mi.]. This seems to open some possibilities since the same formulas (1) involves in that case the new generator $M'_{\mu\nu} = M_{\mu\nu} - S_{\mu\nu}$. The masse formula (2) involves then an « orbital » spin S'. But, for instance with Fronsdal's interpretation [F.], S' has to be zero!

§ 2. **Remarks on the position operator**

Recently, the discussion about a relativistic position operator has led to some very interesting and rigourous discussions [R.], [Be.]. Here, we will only point out some simple consequence of postulates which arise naturally in the context of deformation formalism.

Let us try and construct a four-vector V_μ representing a covariant position operator (*). We will only require that the 3-vector part $\vec{V} \to x$ in the non relativistic limit.

Let us postulate that V_μ admits the following development:

$$\begin{aligned} V_i &= x_i + \frac{1}{C^2} x_i^1 + \frac{1}{C^4} x_i^2 + \dots \\ V_0 &= A + \frac{1}{C^2} A^1 + \frac{1}{C^4} A^2 + \dots \end{aligned} \tag{3}$$

The generators of the Lorentz algebra ($\underline{J}_i$, $\underline{K}_i$) can be expressed as deformations of those of the homogeneous Galile algebra (J_i, K_i). As a result of the Richardson's theorem $\underline{J}_i = J_i$, and

$$\underline{K}_i = K_i + \frac{1}{C^2} K_i^1 + \frac{1}{C^4} K_i^2 + \dots \tag{4}$$

In the following, we consider only the generators $\underline{K}_i$. The four-vector V_μ must obey the following commutation relations:

$$\begin{aligned} [\underline{K}_i, V_j] &= i\delta_{ij}\frac{V_0}{C^2} \\ [\underline{K}_i, V_0] &= iV_i \end{aligned} \tag{5}$$

At the limit $c \to \infty$, one obtains:

$$\begin{aligned} [K_i, X_j] &= 0 \\ [K_i, A] &= iX_i \end{aligned} \tag{6}$$

Considering the case of spin zero, and mass different from zero, we can use the representation $K_i = Mx_i$ (LL]. According to (6), the operator A must be of the form:

$$A = \frac{1}{2M}(\vec{x} \cdot \vec{P} + \vec{P} \cdot \vec{x}) + \varphi(x) \tag{7}$$

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(1) In the case $M \neq 0$, M being the mass of the representation of $\mathscr{P}$ considered.

Let us examine the first order term, that is the term $1/c^2$ in (5). One gets

$$[K_i, x_j^1] + [K_i^1, x_j] = i\delta_{ij}A$$
$$[K_i, A^1] + [K_i^1, A] = ix_i^1$$

This can be easily solved, using the known expression of

$$K_i^1 = \frac{1}{2M} P^2 x_i.$$

One obtains, writing $\{A, B\}$ for $AB + BA$:

$$\begin{aligned} x_i'^1 &= x_i^1 + \frac{1}{2M}\{\varphi(x), P_i\} \\ A'^1 &= A^1 + \frac{1}{4M^2}\{\varphi(x), P^2\} + C(x) \end{aligned} \tag{8}$$

where:

$$\begin{aligned} x_i^1 &= -\frac{i}{2M^2}P_i + \frac{1}{4M^2}\{\vec{P}\cdot\vec{x} + \vec{x}\cdot\vec{P}, P_i\} \\ A^1 &= \frac{1}{8M^3}\{\vec{P}^2, \vec{P}\cdot\vec{x} + \vec{x}\cdot\vec{P}\} - \frac{i}{2M^3}\vec{P}^2 \end{aligned} \tag{9}$$

correspond to the case $\varphi = 0$.

We now turn our attention to commutation properties of the components $V\mu$. To the first order, one has for $[V_i, V_j]$:

$$[x_i, x_j'^1] + [x_i'^1, x_j] = \frac{i}{M^2}(P_j x_i - P_i x_j) \tag{10}$$

Therefore, already from the first order result, the incompatibility between the commutation and covariance properties is manifest.

In the case $\varphi = 0$, it is possible to obtain a general solution for the equations (5). Order by order, one gets finally:

$$\vec{V} = \vec{x} + \frac{1}{2M^2c^2}\vec{P}\cdot(\vec{P}\cdot\vec{x} + \vec{x}\cdot\vec{P})$$

$$V_0 = \frac{1}{2M^2c^2}P_0(\vec{P}\cdot\vec{x} + \vec{x}\cdot\vec{P})$$

where

$$P_0 = \sqrt{\vec{P}^2c^2 + M^2c^4}$$

This can be expressed in term of the generators J_i and $\underline{K}_i$ of SL(2, C), leading to a formula valid also for the case of a non-zero spin

$$V_i = \frac{1}{2M^2c^2}(P_0\underline{K}_i + \underline{K}_iP_0 + \varepsilon_{ijk}(P_jJ_k + J_kP_j)$$

$$V_0 = \frac{1}{2M^2}(\vec{P}\cdot\vec{\underline{K}} + \vec{\underline{K}}\cdot\vec{P})$$

One can verify directly the hermiticity of this solution with respect to the relativistic scalar product.

Their commutation relations are easily seen to be:

$$[V_\mu, V_\nu] = \frac{1}{M^2c^2}iM_{\mu\nu}$$

(the operators V_μ generate with $M_{\mu\nu}$ a De Sitter algebra. See also section 4 of ([C.LN.S])).

Finally, one should mention that these operators are not new. They have already been discussed in a quite different context; see for example [C.].

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Manuscrit reçu le 19 février 1970.