Annales de l'I. H. P., section A

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Annales de l'I. H. P., section A, tome 13, nº 3 (1970), p. 215-220
<http://www.numdam.org/item?id=AIHPA_1970__13_3_215_0>

Ann. Inst. Henri Poincaré,
Vol. XIII, n° 3, 1970, p. 215-220.

Section A :
Physique théorique.

An exponentiation theorem for unbounded derivations

by

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ABSTRACT — We give a sufficient (and necessary) condition to define the exponential of unbounded derivations in C*-algebras.

1. DEFINITIONS

Let $\mathscr{A}$ be a Banach algebra, a derivation is a linear function D from a dense sub-algebra $\mathscr{A}^{(1)}$ of $\mathscr{A}$, into $\mathscr{A}$, such that

$$\forall x \in \mathscr{A}^{(1)} \quad \forall y \in \mathscr{A}^{(1)} \qquad D(xy) = D(x)y + xD(y) \tag{1.1}$$

For a *-Banach algebra $\mathscr{A}$, the derivation D is said to be hermitian if:

$$\forall x \in \mathscr{A}^{(1)} \qquad x^* \in \mathscr{A}^{(1)} \quad \text{and} \quad D(x^*) = (D(x))^*.$$

The set of the elements x in $\mathscr{A}$ such that the function

$$\zeta \to \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} D^n(x)$$

exists and is analytic in some neighbourhood of 0, is called « the set of the analytic elements » with respect to this derivation and is written $\mathscr{A}^{(a)}$.

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2. THEOREM

Let $\mathscr{A}$ be a C*-algebra, D an hermitian closed derivation of $\mathscr{A}$, such as $\mathscr{A}^{(a)}$ is dense in $\mathscr{A}$, then D induces a strongly continuous group $\{\alpha_t \mid t \in \mathbb{R}\}$ of automorphisms of $\mathscr{A}$.

Proof. — If $x \in \mathscr{A}^{(a)}$, $\exists t_x > 0$ such that $t \in \mathrm{R}$, $|t| \leqslant t_x$ we can define:

$$\alpha_t(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{D}^n(x)$$

which is absolutely convergent in $\mathscr{A}$.

$\alpha_t(x) \in \mathscr{A}^{(a)}$, since for $|t'| < t_x - |t|$ we shall show that:

$$\alpha_{t'}(\alpha_t(x)) = \alpha_{t+t'}(x). \tag{2.1}$$

We write

$$y = \alpha_t(x); \quad y_j = \sum_{n=0}^{j} \frac{\mathrm{D}^n(x)}{n!} t^n; \quad y_j \in \mathscr{A}^{(1)}.$$

$$\mathrm{D}(y_j - y_k) = \sum_{n=k+1}^{j} \frac{\mathrm{D}^{n+1}(x)}{n!} t^n$$

now $\sum_{n=0}^{\infty} \frac{\mathrm{D}^n(x)}{n!} t^n$ is analytic on $]-t_x, t_x[$, therefore ([*1*], 9.3.5) $(\alpha_t(x))'_t$ is absolutely and uniformly converging on the same interval

$$(\alpha_t(x))'_t = \sum_{n=1}^{\infty} \frac{\mathrm{D}^n(x)}{(n-1)!} t^{n-1} = \sum_{n=0}^{\infty} \frac{\mathrm{D}^{n+1}(x)}{n!} t^n$$

We write $z_j = \sum_{n=0}^{j} \frac{\mathrm{D}^{n+1}(x)}{n!} t^n$, then $(z_j)_j$ is a Cauchy sequence for $\|\cdot\|$ and:

$$z_j - z_k = \sum_{n=k+1}^{j} \frac{\mathrm{D}^{n+1}(x)}{n!} t^n = \mathrm{D}(y_j - y_k).$$

So $\mathrm{D}(y_j - y_k)$ converges to 0 as j and k go to infinity. Let

$$z = \lim_{j,\infty} \mathrm{D}(y_j).$$

Now, $y = \lim_{j,\infty} y_j$. As D is closed, $z = D(y)$

$$D(y_j) = \sum_{n=1}^{j} \frac{D^n(D(x))}{n!} t^n, \qquad \lim_{j,\infty} D(y_j) = \alpha_t(D(x)).$$

hence

$$D(\alpha_t(x)) = \alpha_t(D(x)) \tag{2.2}$$

and consequently

$$\alpha_{t'}(\alpha_t(x)) = \lim_{l,\infty} \sum_{k=0}^{l} \frac{D^k}{k!}\left(\sum_{n=0}^{\infty} \frac{D^n(x)}{n!} t^n\right) t'^k$$
$$= \lim_{l,\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{l} \frac{D^{k+n}(x)}{k!n!} t^n t'^k$$

which is absolutely converging as l goes to infinity, so we can rearrange the terms:

$$\sum_{m=0}^{\infty} \frac{D^m(x)}{m!}(t + t')^m = \alpha_{t+t'}(x).$$

Through elementary calculations, taking advantage of the absolutely convergence of the series and of the continuity of * one gets:

$$\alpha_t(\lambda x + \lambda y) = \lambda\alpha_t(x) + \mu\alpha_t(y) \tag{2.3}$$
$$\alpha_t(xy) = \alpha_t(x)\alpha_t(y) \tag{2.4}$$
$$\alpha_t(x^*) = (\alpha_t(x))^* \tag{2.5}$$

for $t \in R$ sufficiently small.

Moreover $\forall t \in R$, $\exists m \in N \mid, \ |t| < mt_x$; we write

$$\alpha_t(x) = \left[\frac{\alpha_t}{m}\right]^m (x)$$

α_t is now well defined for all $t \in R$ on $\mathscr{A}^{(a)}$ and fulfils (2.1) and (2.2) for every x in $\alpha^{(a)}$,

α_t is a *-algebra isomorphism applying $\mathscr{A}^{(a)}$ into $\mathscr{A}^{(a)}$ and $\forall x \in \mathscr{A}^{(a)}$, $t \to \alpha_t(x)$ is an analytic function. We shall extend α_t to $\mathscr{A}$. We can assume that $\mathscr{A}$ has a unit element, for, if not, we can define D on $\tilde{\mathscr{A}} = C \times \mathscr{A}$, the algebra obtained from $\mathscr{A}$ by adjunction of a unit element,

$$D(\lambda, x) = (0, D(x)).$$

Moreover, we can assume that $e \in \mathscr{A}^{(1)}$; because if not one settles: $D(e) = 0$.

Note that $\alpha_t(e) = e$ because $D(e) = 0$. If $y = \alpha_0(y)$ is invertible, there exists a neighbourhood of 0 such that $\alpha_t(y)$ is invertible. Now if $t \to \alpha_t(y)$ is analytic, then $t \to (\alpha_t(y))^{-1}$ is also analytic. We can put $\alpha_t(y^{-1}) = (\alpha_t(y))^{-1}$ so $y \in \mathscr{A}^{(a)} \Rightarrow y^{-1} \in \mathscr{A}^{(a)}$ for $x \in \mathscr{A}^{(a)}$; $\lambda \in \mathbb{C}$.

$x - \lambda e$ invertible $\Rightarrow \exists y$ and $(x - \lambda e)y = e$
$\Rightarrow \alpha_t$ is well defined on y and $[\alpha_t(x) - \lambda e]\alpha_t(y) = e$

therefore $(\alpha_t(x) - \lambda e)$ is invertible; hence $\text{Spec}'\, \alpha_t(x) \subset \text{Spec}'\, x$.

On the other hand, for an hermitian element y of $\mathscr{A}$:

$$\| y \| = \sup_{\zeta \in \text{Spec}' y} | \zeta |$$

([1], 15.4.14.1); hence:

$$\| \alpha_t(x) \|^2 = \| \alpha_t(x^*x) \| = \sup_{\zeta \in \text{Spec}' \alpha_t(x^*x)} | \zeta | \leqslant \sup_{\zeta \in \text{Spec}\, x^*x} | \zeta | = \| x^*x \| = \| x \|^2$$

and finally $\| \alpha_t(x^*x) \| = \| x \|$ on $\mathscr{A}^{(a)}$. We extend α_t to $\mathscr{A}$ (2.1) to (2.5) still hold $\forall x \in \mathscr{A}$, $\exists (y_n)n$, $y_n \in \mathscr{A}^{(a)}$ and $x = \lim_n y_n$. Therefore

$$\lim_n \| \alpha_t(x) - \alpha_t(y_n) \| = 0.$$

$t \to \alpha_t(x)$ is continuous as a uniform limit of continuous functions. So that the one-parameter unitary group $\{ \alpha_t \mid t \in \mathrm{R} \}$ is strongly continuous.

Comment. — We get an extension to C*-algebras of the work of E. Nelson on Hilbert spaces ([5]).

3. CONVERSE PROPOSITION

We give a new proof of the result of Kastler-Pool-Poulsen [4], which improves some one of I. Guelfand [3].

Let $\mathscr{E}$ be a Banach space, $\{ \alpha_t \}_{t \in \mathbb{R}}$ a strongly continuous one-parameter group of uniformly bounded linear operators, i. e.

$$\exists \mathrm{M} > 0 \, \forall t \in \mathbb{R} \qquad \| \alpha_t \| \leqslant \mathrm{M}$$

$\forall x \in \mathscr{E}$, $\forall \rho \in \mathscr{L}^1_{\mathbb{C}}(\mathbb{R})$; let $\alpha(\rho)x = \int_{-\infty}^{+\infty} \alpha_t(x)\rho(t)dt$, which exists in the Bochner's sense since $\| \alpha_t(x)\rho(t) \| \leqslant \mathrm{M} \| x \| \, | \rho(t) |$ and one has that:

$$t \to \| \alpha_t(x)\rho(t) \| \in \mathscr{L}^1_{\mathbb{C}}(\mathbb{R}).$$

PROPOSITION. — $\mathscr{E}^{(e)}$ $(= \{ x \in \mathscr{E} \mid t \in \mathbb{R} \to \alpha_t(x)$ is entire $\})$ is dense in $\mathscr{E}$.

Proof. — Let ρ be a function in $C^{\mathbb{R}}$ so that $\hat{\rho} \in \mathscr{D}$. Then $\rho \in \mathscr{S}$, and $\rho \in \mathscr{L}^1_{\mathbb{C}}(\mathbb{R})$. Moreover, suppose that $\int_{-\infty}^{+\infty} \rho(t)dt = 1$. We notice that $\forall \varepsilon > 0, \exists \eta > 0$ so that

$$\int_{-\infty}^{-\eta} |\rho(t)|\,dt \leqslant \varepsilon \quad \text{and} \quad \int_{\eta}^{\infty} |\rho(t)|\,dt \leqslant \varepsilon.$$

Now if $\rho_n^{(t)} = n\rho(nt)$, $\int_{-\infty}^{-\frac{\eta}{n}} |\rho_n(t)|\,dt \leqslant \varepsilon$ and $\int_{\frac{\eta}{n}}^{\infty} |\rho_n(t)|\,dt \leqslant \varepsilon$

$$\|\alpha(\rho_n)x - x\| = \left\| \int_{-\infty}^{+\infty} \rho_n(t)\alpha_t(x)dt - \int_{-\infty}^{+\infty} \rho_n(t)xdt \right\|$$
$$\leqslant \int_{-\infty}^{-\frac{\eta}{n}} |\rho_n(t)|\ \|\alpha_t(x) - x\|\,dt + \int_{-\frac{\eta}{n}}^{+\frac{\eta}{n}} \dots + \int_{\frac{\eta}{n}}^{\infty} \dots$$

Now, $\forall \varepsilon > 0, \exists n_0$ such that $\forall n \geq n_0$ and $|t| \leqslant \frac{\eta}{n} \Rightarrow \|\alpha_t(x) - x\| \leqslant \varepsilon$.

On the other hand $\|\alpha_t(x)\| \leqslant M\|x\|$, hence

$$\|\alpha(\rho_n)x - x\| \leqslant [2(M + 1)\|x\| + 1]\varepsilon \quad \text{and} \quad x = \lim_n \alpha(\rho_n)x.$$

We prove that $\alpha(\rho_n)x \in \mathscr{E}^{(e)}$ $\forall x \in \mathscr{E}$. Indeed:

$$\alpha_r(\alpha(\rho_n)x) = \alpha_r\left(\int_{-\infty}^{+\infty} \rho_n(t)\alpha_t(x)dt \right) = \int_{-\infty}^{+\infty} \rho_n(t)\alpha_{r+t}(x)dt$$
$$= \int_{-\infty}^{+\infty} \rho_n(t - r)\alpha_t(x)dt$$
$$= (\rho_n * h)(r)$$

where $h(r) = \alpha_r(x)$.

Now, h being continuous and bounded, $\rho_n * h \in \mathscr{S}$ and $\widehat{\rho_n * h} = \hat{\rho}_n\hat{h}$ is a distribution (cf. [*6*]) with compact support, hence due to the Paley-Wiener theorem $\rho_n * h$ is an entire function.

ACKNOWLEDGEMENTS

The author is very indebted to Professor D. Kastler for his encouragments and for his constant interest in this work. Thanks also are due to Dr. J. Manuceau, who suggested the problem and for his constant aid. M. A. Messager made us able to illuminate some delicate points. We are indebted to Mr. M. Sirugue et Mr. D. Testard for reading the manuscript.

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Manuscrit reçu le 4 mars 1970.