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A spectrum generating nilpotent group for the relativistic free particle

by

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ABSTRACT. — We give an embedding of the full pseudoorthogonal group into the symplectic and antisymplectic matrices on the vector space of position, time, momentum and energy variables. By means of the relativistic mass operator we define a nilpotent Lie group which is to describe after unitary representation the physical concepts of mass, position, time, momentum, energy, space inversion and time reflection of a relativistic free particle. The relation to the canonical Weyl algebra of relativistic quantum mechanics is given.

RÉSUMÉ. — On donne une injection du groupe pseudo-orthogonal complet dans les matrices symplectiques et antisymplectiques d'un espace vectoriel des variables de position, temps, impulsion et énergie. Avec l'aide de l'opérateur de masse relativiste, on définit un groupe de Lie nilpotent qui, après la représentation unitaire, doit décrire les conceptions physiques de masse, position, temps, impulsion, énergie, réflexion de l'espace et du temps d'une particule relativiste libre. La relation avec l'algèbre canonique de Weyl de la mécanique quantique relativiste est décrite.

§ 1: INTRODUCTION

In § 2 we describe an embedding of the full n-dimensional pseudo-orthogonal group (which consists of four connectivity components) into the full symplectic matrix group in $2n$ dimensions (which consists of two connectivity components, the symplectic and the antisymplectic matrices). Among the various possibilities for this embedding into the symplectic only and into the full symplectic group we choose one which gives no inversion of the energy components. Momentum inversion is then found only in the two group components of space inversion and time reflection. If we take for Hamilton operator the mass operator of relativistic physics (i. e. the first Casimir operator of the Poincaré group) we get a nilpotent Lie group which is a special case of the solvable groups described in [I]. Inclusion of space inversion and time reflection (which was not done in [I]) then defines an enlarged nonconnected group of four connectivity components which we suggest to describe mass, position, time, momentum, energy space inversion and time reflection of a relativistic free particle after representation into the operators on a (necessarily infinite-dimensional) Hilbert space. This group gives no concept of angular momentum, orbital or spin, contrary to the Poincaré group which conversely gives no satisfactory concept of position. Perhaps this difficulty can be overcome by inclusion of the pseudoorthogonal group as was done for the nonrelativistic Galilei group in [I; § 10]. We easily get a matrix representation of that enlarged group (which containes the full Poincaré group as a subgroup) from (17) below. To describe the relativistic free particle satisfactoryly the nilpotent group in question must have (at least) two sorts of Hilbert space representations corresponding to zero mass and positive mass either. If there are additional representations with masses not yet experimentally observed (like imaginary masses), we will be confronted to the problem of « unphysical » representations as in the representation theory of the Poincaré group.

In § 4 we describe the embedding of the Lie algebra of the discussed group into the canonical Weyl algebra which is constructed over the Heisenberg subalgebra of position, time, momentum and energy variables. Since position and time, momentum and energy are treated on equal footing we will get time and energy operators after representation, which must lead to a satisfactory interpretation of time and energy for the relativistic free particle, as well as of position and momentum.

§ 2: EMBEDDING OF THE FULL PSEUDORTHOGONAL GROUPS INTO THE FULL SYMPLECTIC GROUPS

Let V be a finite-dimensional vector space, æ a bilinear form on V and Aut (V) the group of invertible endomorphisms on V. We define

$$(1)\quad \begin{aligned} &\mathrm{Aut}\,(V, æ) := \{\, G \in \mathrm{Aut}\,(V)/æ(Gæ, Gy) = æ(x, y) \text{ for all } x, y \in V\} \\ &\mathrm{der}\,(V, æ) := \{\, M \in \mathrm{End}\,(V)/æ(Mæ, y) + æ(x, My) = 0 \text{ for all } x, y \in V\}. \end{aligned}$$

We write the general element of the symplectic vector space E

$$x = \sum_{i=1}^{n_q} \xi^i q_i + \sum_{r=n_q+1}^{n} \xi^r t_r + \sum_{i=1}^{n_q} \xi_i p^i + \sum_{r=n_q+1}^{n} \xi_r h^r$$

$$E = E_q \oplus E_t \oplus E_p \oplus E_h$$

where $\dim (E_q) = \dim (E_p) = n_q$, $\dim (E_t) = \dim (E_h) = n_t$, $n_q + n_t = n$. We also write $E_q \oplus E_t =: E_1$, $E_p \oplus E_h =: E_2$ and $x = x_1 + x_2$ according to this decomposition. In the following $i, k, l, m = 1, \ldots, n_q$ and $r, s, t, u = n_q + 1, \ldots, n_q + n_t = n$. Hence we have $\dim (E) = 2n$. Let id_q, id_t, id_p and id_h be the identities in E_q, E_t, E_p and E_h respectively, and $\begin{pmatrix} \mathrm{id}_q & 0 \\ 0 & -\mathrm{id}_t \end{pmatrix} =: I_{qt}$. Then the $2n \times 2n$ matrix $\begin{pmatrix} I_{qt} & 0 \\ 0 & I_{ph} \end{pmatrix} =: -P$ is in $\mathrm{Spl}\,(2n, \mathbb{R}, E)$. We define $\tau(x, y) := \sigma(\mathfrak{J}x, -Py) =: \tau_1(x_1, y_1) + \tau_2(x_2, y_2)$, which induces on the vector spaces E, E_1, E_2 symmetric nondegenerate indefinite bilinear forms τ, τ_1, τ_2. If we denote by $\mathbb{C}_4$ the commutative group of the elements id_n, $-\mathrm{id}_n$, $-I_{qt}$, I_{qt} (the two latter are called *space inversion* and *time reflection* respectively) the full pseudoorthogonal group in E_1 is

$$\mathrm{Aut}\,(E_1, \tau_1) = SO^{\uparrow}_{+}(n_q, n_t\,; \mathbb{R}, E_1) \otimes \mathbb{C}_4 = SO^{\uparrow}_{+}(n_q, n_t\,; \mathbb{R}, E_1) \cup$$
$$-I_{qt}SO^{\uparrow}_{+}(n_q, n_t\,; \mathbb{R}, E_1) \cup I_{qt}SO^{\uparrow}_{+}(n_q, n_t\,; \mathbb{R}, E_1) \cup -\mathrm{id}_n SO^{\uparrow}_{+}(n_q, n_t\,; \mathbb{R}, E_1)$$

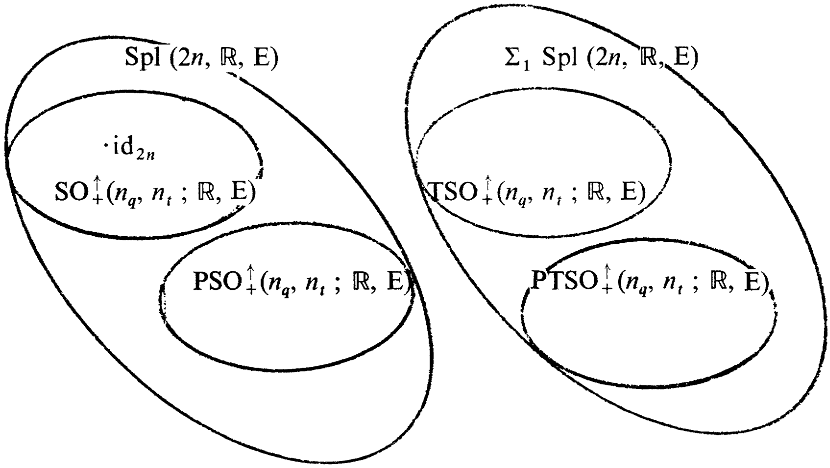
(where $SO^{\uparrow}_{+}(n_q, n_t; \mathbb{R}, E_1)$ is its identity component). We want to embed this group into the *full symplectic group* in two dimensions, which is $\mathrm{SpA}(2n, \mathbb{R}, E) := \mathrm{Spl}(2n, \mathbb{R}, E) \cup \Sigma_1\, \mathrm{Spl}(2n, \mathbb{R}, E)$. This embedding is given by

$$(2)\qquad G \mapsto \begin{pmatrix} G & 0 \\ 0 & (G^T)^{-1} \end{pmatrix} =: G \qquad G \in SO^{\uparrow}_{+}(n_q, n_t\,; \mathbb{R}, E_q)$$

for the identity component of $\mathrm{Aut}\,(E_1, \tau_1)$. We have $G \in \mathrm{Aut}\,(E_1, \tau_1)$ iff $G \in \mathrm{Aut}(E, \sigma) \cap \mathrm{Aut}(E, \tau)$. This $2n$-dimensional representation of $SO^{\uparrow}_{+}(n_q, n_t; \mathbb{R}, E_1)$ is denoted by $SO^{\uparrow}_{+}(n_q, n_t; \mathbb{R}, E)$ in the future. To continuate this representation to the full pseudoorthogonal group we consider among the various representations of $\mathbb{C}_4$ in Spl $(2n, \mathbb{R}, E)$ and SpA $(2n, \mathbb{R}, E)$ the « physical » one

$$(3)\qquad \begin{aligned} &\mathrm{id}_n \mapsto \mathrm{id}_{2n} && -I_{qt} \mapsto P \\ &I_{qt} \mapsto \begin{pmatrix} I_{qt} & 0 \\ 0 & -I_{ph} \end{pmatrix} =: T && -id \mapsto \begin{pmatrix} -\mathrm{id}_n & 0 \\ 0 & \mathrm{id}_n \end{pmatrix} = PT \end{aligned}$$

with id_{2n}, $P \in Spl\ (2n, \mathbb{R}, E)$ and T, $PT \in \Sigma_1\ Spl\ (2n, \mathbb{R}, E)$. We choose the terminus physical for this representation because it has the physical transformation properties of the *momentum* components x_p and no reflection of the *energy* components x_h of $x \in E$. If we want to embed projective Hilbert space representations of the full pseudoorthogonal group into those of the full symplectic group (for instance when we use the latter as a socalled noninvariance group of the relativistic free particle) we have to represent id_{2n} and P by unitary and T and PT by antiunitary operators [2; p. 7).



The two components of Aut (E_1, τ_1) in Spl $(2n, \mathbb{R}, E)$ may be called *causal* subgroup of Aut (E_1, τ_1) [2; p. 2]. The Lie algebra of $SO_+^{\uparrow}(n_q, n_t; \mathbb{R}, E_1)$ is der (E_1, τ_1). The condition for M to be in der (E_1, τ_1) is in matrix form $M^T I_{qt} + I_{qt} M = 0$. The embedding of der (E_1, τ_1) in spl $(2n, \mathbb{R}, E)$ follows then form [I; p. 77]

$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & I_{qt} M I_{qt} \end{pmatrix}. \tag{4}$$

§ 3: THE NILPOTENT GROUP OF THE RELATIVISTIC FREE PARTICLE

For the nilpotent $2n \times 2n$ matrix $\begin{pmatrix} 0 & -I_{ph} \\ 0 & 0 \end{pmatrix} =: F \in spl\ (2n, \mathbb{R}, E)$ the one-parameter subgroup $\exp(\alpha F) = id_n + \alpha F$ of Spl $(2n, \mathbb{R}, E)$ consists of unipotent matrices. This group is isomorphic as a Lie group to the straight line by $\exp(\mathbb{R}F) \mapsto \mathbb{R}F$, if we write the latter as the additive group $\mathbb{R}F$. For the relativistic free particle we consider the following nilpotent groups on the analytic manifolds $\exp(\mathbb{R}F) \times E \times \mathbb{R}$, $\exp(\mathbb{R}F) \times E \times e^{i\mathbb{R}}$, $\mathbb{R}F \times E \times \mathbb{R}$ with compositions

(5) $(e^{\alpha F}, x, \beta)(e^{\alpha F}, y, \beta') := (e^{(\alpha+\alpha')F}, x + e^{\alpha F}y, \sigma(x, e^{\alpha F}y) + \beta + \beta')$

(6) $(e^{\alpha F}, x, e^{i\beta})(e^{\alpha' F}, y, e^{i\beta'}) := \left(e^{(\alpha+\alpha')F}, x + e^{\alpha F}y, e^{i(\sigma(x, e^{\alpha F}y) + \beta + \beta')}\right)$

(7) $(\alpha F,\ x,\ \beta)(\alpha' F,\ x,\ \beta') := ((\alpha + \alpha')F,\ x + e^{\alpha F}y,\ \sigma(x,\ e^{\alpha F}y) + \beta + \beta')$

respectively. These $2n + 2$-dimensional groups are connected, noncompact, and locally isomorphic with Lie algebras

$$(8) \qquad [(\alpha F,\ x,\ \beta),\ (\alpha' F,\ y,\ \beta')]_- = (0,\ \alpha F y - \alpha' F x,\ \sigma(x,\ y))$$

on the vector spaces $\mathbb{R}F \oplus E \oplus \mathbb{R}$ (after a suitable change of basis). The groups (5) and (7) are simply connected and therefore global isomorphic. The group (6) is infinitely connected since the kernel of the covering homomorphism

$$(9) \qquad (e^{\alpha F},\ x,\ \beta) \mapsto (e^{\alpha F},\ x,\ e^{i\beta})$$

is $2\pi\mathbb{Z}$. (5) and (6) are central extensions of the Lie groups $e^{\mathbb{R}F} \otimes E$ cf. [*1*; theorem 12] by $\mathbb{R}$ and $e^{i\mathbb{R}}$ respectively. The proofs of these statements are given in [I; § 2, 3, 10]. For $S = e^{\alpha F}$ we get a faithful $2n + 2$-dimensional representation of (5) from [I; (42)]. We dont know similar representations for (6). We study the group (6) besides the group (5) since it is a central extension of $\exp(\mathbb{R}F) \otimes E$ by the torus, which is the standard connection we need for the ray representation theory of the latter [*3*].

A $2n + 2$-dimensional representation of the Lie algebras (8) is given in [I; (44)] for $V = \alpha F$. The proof of [*1*; § 10] for $n \geqslant 2$ is applicable to our special case with only minor modifications. Hence we get for the derivation Lie algebra of (8) the set of matrices

$$(10) \qquad \begin{pmatrix} v & 0 & 0 \\ -F\mathfrak{J}p & W + \frac{1}{2}v\Sigma_3 + \beta id_{2n} & 0 \\ \rho & p^T & \beta \end{pmatrix}$$

with $v, \rho, \beta \in \mathbb{R}$, $\Sigma_3 := \begin{pmatrix} id_n & 0 \\ 0 & -id_n \end{pmatrix}$ and $W = \begin{pmatrix} L & B \\ 0 & I_{qt}LI_{qt} \end{pmatrix}$,

B a symmetric $n \times n$ matrix, $L \in \operatorname{der}(E_1,\ \tau_1)$, i. e. $L^T I_{qt} + I_{qt}L = 0$. It is straightforward to show that because of [I; theorem 38] the automorphism group of (8) is given by the set of $2n + 2 \times 2n + 2$ matrices

$$
\text{(11)} \qquad \begin{pmatrix} \delta & 0 & 0 \\ -\frac{1}{\alpha}GF\mathfrak{J}b & G & 0 \\ \tau & b^T & \alpha \end{pmatrix} \qquad \begin{matrix} 0 \neq \alpha \in \mathbb{R} \\ 0 \neq \delta \in \mathbb{R} \\ \tau \in \mathbb{R} \end{matrix}
$$

where the $2n \times 2n$ matrix G has the form

$$
G = \sqrt{|\alpha|}\left\{\frac{1 + \text{sign}\,\alpha}{2}\begin{pmatrix} A & B \\ 0 & \frac{1}{\delta}I_{qt}AI_{qt} \end{pmatrix} + \frac{1 - \text{sign}\,\alpha}{2}\begin{pmatrix} C & D \\ 0 & \frac{1}{\delta}I_{qt}CI_{qt} \end{pmatrix}\right\}
$$

and the $n \times n$ matrices $B^TI_{qt}AI_{qt}$ and $D^TI_{qt}CI_{qt}$ are symmetric, where as A and C are subject to $A^TI_{qt}A = \delta I_{qt}$ and $C^TI_{qt}C = -\delta I_{qt}$. The adjoint representations of the groups (5) and (6), which are the same for both, are given by [I; (47)], those of the Lie algebras (8) are given by [I; (46)].

§ 4: CONNECTION WITH THE CANONICAL FORMALISM

We discuss the relation of the above groups and their Lie algebras to the canonical formalism, i. e. to the associative infinite-dimensional Weyl algebra of [I] which we get from the sub algebra heis($2n$) in (8). For Hamilton operator we take the *mass operator* of the relativistic free particle

$$
\text{(12)} \qquad H_F := \frac{1}{2}\left(\sum_{i=1}^{n_q} p^i p^i - \sum_{r=n_q+1}^{n} h^r h^r\right), \qquad R(H_F) = F
$$

(R being the isomorphism [I; (74)]). The sub algebra $\mathbb{R}F \dotplus\!\!> \text{heis}(2n)$ of the Lie algebra $\text{spl}(2n, \mathbb{R}) \dotplus\!\!> \text{heis}(2n)$, which is formed by $\Lambda W_2 \oplus E \oplus \mathbb{R}1$ in weyl(E, σ), then is isomorphic to the Lie algebra (8), and the Lie algebra of linear transformations on $E \subset \text{weyl}(E, \sigma)$ given by ad $(E \oplus \Lambda W_2)|_E$ is isomorphic to the adjoint Lie algebra of (8), describes by [I_I; (18), (46)]. The centralizer of H_F in $E \oplus \Lambda W_2$, which is supposed to be the smallest Lie algebra the universal enveloping algebra of which after embedding in weyl (E, σ) is the centralizer of H_F in weyl (E, σ), is given by the $\frac{1}{2}n(n-1)$ elements

$$M_{ik} := \Lambda(q_i p^k - q_k p^i),\ M_{ir} := \Lambda(q_i h^r + t_r p^i),\ M_{rs} := \Lambda(t_r h^s - t_s h^r) \tag{13}$$

which fulfill the well known commutation relations of the pseudoorthogonal Lie algebra, together with the n elements p^i and h^r. The R-image [I ; (74) of the Lie algebra of the M's is given by the matrices (4), the exponentials of those matrices by (2). One easily proves that

$$\mathrm{ad}\,(M)|_E \in \mathrm{der}\,(E, \sigma) \cap \mathrm{der}\,(E, \tau)$$

for M a linear combination of the M's above. For the realization of some outer automorphisms of $\mathbb{R}F \mathbin{+\!\!\!>} \mathrm{heis}(2n)$ in weyl(E, σ) we get the corresponding results to [I ; § 10].

§ 5: INCLUSION OF SPACE INVERSION AND TIME REFLECTION

Consider the one-parameter nonconnected subgroup

$$\begin{aligned} \mathbb{C}_4 \otimes e^{\mathbb{R}F} &= e^{\mathbb{R}F} \cup Pe^{\mathbb{R}F} \cup Te^{\mathbb{R}F} \cup PTe^{\mathbb{R}F} \\ &= (\mathrm{id}_{2n} + \mathbb{R}F) \cup (P + \mathbb{R}Q) \cup (T - \mathbb{R}Q) \cup (PT - \mathbb{R}F) \end{aligned} \tag{14}$$

of SpA($2n$, $\mathbb{R}$, E), where $Q := \begin{pmatrix} 0 & \mathrm{id}_n \\ 0 & 0 \end{pmatrix}$, which has four connectivity components and consists no longer of unipotent matrices only. Its composition on the manifold $\mathbb{C}_4 \times e^{\mathbb{R}F}$ is given by

$$(Z, e^{\alpha F})(Z', e^{\alpha' F}) := \left(ZZ', e^{((-1)^{\nu'}\alpha + \alpha')F}\right) \tag{15}$$

where ν depends on Z and is zero for $Z \in \mathrm{Spl}(2n, \mathbb{R}, E)$ and one for $Z \in \Sigma_1\,\mathrm{Spl}\,(2n, \mathbb{R}, E)$. We have $(Z, e^{\alpha F})^{-1} = \left(Z, e^{(-1)^{\nu+1}\alpha F}\right)$; for the commutant of two elements we get $(\mathrm{id}_{2n}, \exp\,((-1)^{\nu'})((-1)^{\nu} - 1)((-1)^{\nu}\alpha + \alpha')F)$, which shows that the second commutant on $\mathbb{C}_4 \otimes e^{\mathbb{R}F}$ is the identity element $(\mathrm{id}_{2n}, \mathrm{id}_{2n})$; hence the group is nilpotent.

On the nonconnected manifold $\mathbb{C}_4 \times e^{\mathbb{R}F} \times E \times \mathbb{R}$ of four connectivity components we define a group composition by

$$\begin{aligned} &(Z, e^{\alpha F}, x, \beta)(Z', e^{\alpha' F}, y, \beta') := \\ &\qquad \left(ZZ', e^{((-1)^{\nu'}\alpha + \alpha')F}, x + i^{\nu} Z e^{\alpha F} y, i^{\nu}\sigma\left(x, Z e^{\alpha F} y\right) + \beta + \beta'\right). \end{aligned} \tag{16}$$

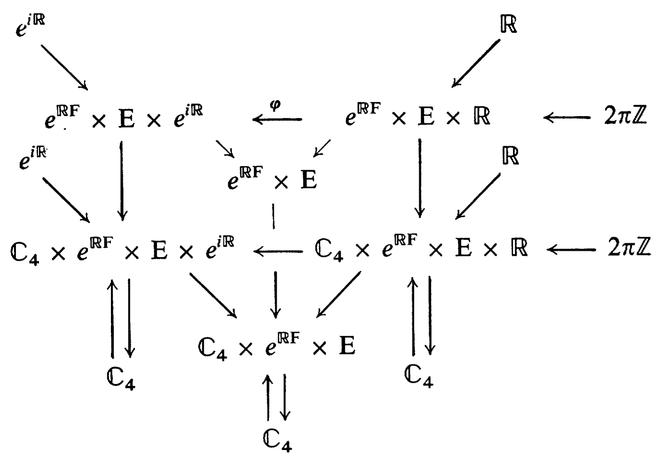
This group

$$\mathbb{C}_4 \otimes \left(e^{\mathbb{R}F} \otimes \text{Heis}(2n)\right) = \left(\mathbb{C}_4 \otimes e^{\mathbb{R}F}\right) \otimes (\text{Heis}(2n)) = e^{\mathbb{R}F} \otimes (\mathbb{C}_4 \otimes \text{Heis}(2n))$$

is again nilpotent. A faithful $2n + 2$-dimensional representation is given by the matrices

$$(17) \qquad (Z, e^{\alpha F}, x, \beta) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ x & i^\nu Z e^{\alpha F} & 0 \\ \beta & i^\nu x^T \mathfrak{J} Z e^{\alpha F} & 1 \end{pmatrix}$$

By substituting $\mathbb{R} \to e^{i\mathbb{R}}$ into the group (16) we get the corresponding results for the group (6), and by dropping that last factor we get the corresponding results for the *adjoint group* $\mathbb{C}_4 \otimes \left(e^{\mathbb{R}F} \otimes E\right)$. A faithful $2n + 2$-dimensional representation of this adjoint group is given by the matrices [7; (47) if we substitute $e^{\alpha F} \mapsto Ze^{\alpha F}$ and use $ZF = (-1)^\nu FZ$. We summarize the relation between the various groups of the relativistic case by the commutative diagramm of short exact sequences



(we dropped the trivial parts). φ is the covering homomorphism (9).

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