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Fields on a homogeneous space of the Poincaré group

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Section A :
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Fields on a homogeneous space of the Poincaré group

by

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ABSTRACT — Fields describing one particle or an infinite number of particles are defined on an eight-dimensional homogeneous space of the Poincaré group. The relativistic invariance and causality properties are discussed in particular in connection with a local coupling of the fields.

1. INTRODUCTION

The failure of the meson theories to cope with the problems of strong interactions has caused a wide spread resignation as to the role of field theory in hadron physics [*1*]. Instead one has tried to save only certain properties inherent in a field theory approach. The most intensively studied such property is the analyticity in the variables of the scattering amplitudes. We have seen an enormous activity on this subject starting with dispersion relations [*2*], hope and dispair in the Regge pole theory [*3*] and now lately the group theoretical approach to partial wave expansion in a crossed channel [*4*]. The failure of field theory has been blamed on the large coupling constant which would make a perturbation expansion completely unreliable. We know well our unability to do more than calculating the first few terms in this expansion and the success of quantum electrodynamics is altogether based on a perturbation scheme.

What is it that goes wrong when one tries to fit the experimental data with the first few terms in say the pseudo-scalar meson theory. In short both energy dependences and momentum transfer distributions. The higher the energy the more drastic become the discrepancies. While the experiments show rapidly decreasing differential cross sections of the form $\sim e^{At}$ in two-body reactions, where t is the momentum transfer squared and A a constant, the first Born approximation gives a more or less flat distribution [*5*]. Several attempts have been made to remedy the diseases, but they are all of a phenomenological nature.

Turning to more mathematical questions it is known that all the ordinary meson field theories contain divergencies. Some are renormalizable others not. Also here one has tried to circumvent the difficulties by studying only general principles starting from axiomatic field theory or vacuum expectation values without any prescribed dynamics [*6*, *7*]. Although partial results for the analyticity properties have been obtained it is clear that one is far from a theory which makes real contact with experiments.

Despite all the troubles a field theory has in its wake the following properties making it attractive

1° It will automatically imply the analyticity properties of the scattering amplitudes such as poles and thresholds for different channels and crossing symmetry.

2° It will automatically imply unitarity of the S-matrix.

3° It will effectively limit the possible choices of dynamics.

4° Provided the perturbation expansion converges rapidly it will lead to a practical theory.

In several papers [*8*, *9*, *10*] we have studied the possibility of extending the Minkowski space with an internal spin space. The extended space is of dimension 8. In a subsequent paper [*11*] a field theory on this space was proposed. Through its lowest order graphs it seemed to be able to explain the peripheral nature of hadron collisions. Already in 1964 Lurçat [*12*] proposed a similar extension, namely to consider fields on the Poincaré group manifold. Both our 8-dimensional and this 10-dimensional space are examples of homogeneous spaces of the Poincaré group. One may therefore ask oneself if it would not be wise first to classify the possible homogeneous spaces and then select the best one. This was done in ref. [*13*]. Our 8-dimensional space is in fact most « economical » in the sense that it is of lowest dimension satisfying certain requirements. In section 2 we give a short description of the class of homogeneous spaces.

Section 3 contains the definition of one-particle states and free fields. The connection between spin and statistics is discussed. In section 4 we try to introduce interactions.

2. CLASSIFICATION OF HOMOGENEOUS SPACES

A homogeneous space, H, of a group G has the following properties.

a) It is a topological space on which the group G acts continuously i. e. let y be a point in H, then gy ($g \in G$) is defined and is again a point in H.

b) This action is transitive i. e. given any two points y_1, and y_2 in the space then it is always possible to find a group element $g \in G$ such that

$$y_2 = gy_1 \tag{1}$$

Denote by S_0 the maximal subgroup of G which leaves the point y_0 invariant. S_0 is called the stabilizer of y_0. The stabilizer S of another point y is conjugate to S_0 in G and one can establish a mapping between the points of H and the points in the coset space G/S_0. Therefore the enumeration of the different H of the Poincaré group P amounts to an enumeration of the subgroups of P up to conjugation.

We shall now make an important restriction on the class of homogeneous spaces we are going to consider. We require the H always contains the Minkowski space M which means that four parameters of H can be denoted by $x(x^\mu)$. P must also act on x in the usual way. Now this means that the stabilizer must be a subgroup of the Lorentz group L.

The restriction to homogeneous spaces containing the Minkowski space is done for physical reasons. We think that it would be difficult to make an interpretation if the Minkowski space is not present. In this way we are however led to the starting point of Finkelstein [*14*]. We can thus use his classification of homogeneous spaces. Notice, however, that he considers only stabilizers which are generated from the Lie algebra. Spaces having for instance discrete stabilizers are missing.

Consider now the action of P on the homogeneous space H = P/S. If we parametrize P in the form

$$g = g_x g_\Sigma \tag{2}$$

where g_x is a translation x and g_Σ is a homogeneous Lorentz transformation Σ, then the points of H are parametrized by x and Σ modulo an element of S. The action of P is given by left multiplication

$$g_a g_\Lambda g_x g_\Sigma = g_{a+\Lambda x} g_{\Lambda\Sigma} \text{ mod } S. \tag{3}$$

Denoting a point in L/S by z we therefore have

$$(x, z) \xrightarrow{(a,\Lambda)} (a + \Lambda x, \Lambda z) \tag{4}$$

and the action splits into an orbital part on x and an « internal » part on z. For the infinitesimal generators of P this means that they can be written

$$\begin{aligned} P_\mu &= i\frac{\partial}{\partial x^\mu} \\ M_{\mu\nu} &= ix_\mu\frac{\partial}{\partial x^\nu} - ix_\nu\frac{\partial}{\partial x^\mu} + S_{\mu\nu} \end{aligned} \tag{5}$$

where $S_{\mu\nu}$ are differential operators only in the variable z. The explicit expression for $S_{\mu\nu}$ of course depends on the homogeneous space at hand.

A suitable parametrization of the Lorentz group or its covering group $\overline{L}$ is given by

$$\Lambda = e^{-i\varphi L_{12}}e^{-i\theta L_{31}}e^{-i\psi L_{12}}e^{isL_{03}}e^{-it[L_{01}+L_{31}]}e^{-iu[L_{02}-L_{23}]} \tag{6}$$

where

$$\begin{aligned} L_{ij} &= \frac{1}{2}\varepsilon_{ijk}\sigma_k \\ L_{0k} &= \frac{i}{2}\sigma_k \end{aligned} \tag{7}$$

and σ_1, σ_2, σ_3 are the Pauli matrices. If $\Lambda \in \overline{L}$ the ranges of angles are

$$\begin{aligned} 0 \leqslant \varphi + \psi \leqslant 4\pi \\ -2\pi \leqslant \varphi - \psi \leqslant 2\pi \end{aligned} \tag{8}$$

while for $\Lambda \in L$ φ and ψ belong to the interval $[0, 2\pi]$. The ranges of the other parameters are in both cases

$$\begin{aligned} 0 \leqslant \theta \leqslant \pi \\ -\infty < s, t, u < \infty \end{aligned} \tag{9}$$

The homogeneous space with a stabilizer generated by $(L_{02} - L_{23})$ is of course parametrized by $(\varphi, \theta, \psi, s, t)$. Similarly for other stabilizers one has to write Λ as a product with an element of S to the right. Then those parameters which are outside S furnish a parametrization of G/S. In table I the different spaces are characterized. Besides the parameters of equation 6 the following ones also appear

$$\begin{aligned} \tilde{t} &= e^s[t \cos\psi - u \sin\psi] \\ \tilde{u} &= e^s[u \cos\psi + t \sin\psi] \\ \tilde{\psi} &= \psi - s \cot f/2 \quad (0 < f \leqslant \pi) \end{aligned} \tag{10}$$

Sometimes a four-vector is used. Table I also indicates when the homogeneous space can carry a half-integral spin wave function. This is so when the stabilizer is a subgroup of $\bar{L}$ that does not contain any compact subgroup. The invariant measure (unique up to constant) is indicated in case it exists. For further details we refer the reader to reference 13.

If one wants to construct wave functions or fields depending on the variables (x, z) it is natural to consider only scalar valued functions since the spin degree of freedom can be carried by the z-variables. Then the number of parameters of H is related to the number of wave equations and the type of particle [*15*]. Call the dimension of the space d and the number of wave equations e. Then $d - e = 4$ for a massive particle with spin and $d - e = 3$ for a massive particle without spin and a massless particle. Looking at the table we see that the case [*4*] requires a minimum number of wave equations if we insist on the existence of an invariant measure and half-integral spin representations. This space is the one we are going to consider in the sequel.

3. DEFINITION OF ONE-PARTICLE STATES AND FREE FIELDS ON THE SPACE [*4*]

The space on which wave functions and fields will be defined is called the carrier space. Our space [*4*] in table I is the topological product of the Minkowski space and a 4-dimensional internal space. The latter is homeomorphic to the topological product of the group space of SU(2) and a straight line. We shall denote a point in the carrier space by $(x^\mu, \varphi, \theta, \psi, s)$. Here x^μ is a point in space-time, (φ, θ, ψ) are Euler angles in SU(2) and s is the coordinate on the line. Another way of characterizing this 8-dimensional space is to define it operationally using the classical electromagnetic field (radar pulses). For further elaborations on these points we refer the reader to paper I and II.

We shall now define one-particle states with the helicity convention for the spin projections, limiting ourselves throughout this paper to positive mass (m) particles. As a consequence of the large dimension of the carrier space not only the spin degree of freedom but also two gauge transformations can be realized on it. These are rotations in the angle ψ and translations in the variable s. According to paper III equation (23) the wave function for a particle at rest is

$$|\vec{p} = 0\ \lambda\rangle = \frac{1}{\sqrt{2(2\pi)^3}} e^{imx^0} e^{-(\alpha + i\beta)s} \mathrm{D}^j_{\lambda n}(\varphi\theta\varphi). \tag{11}$$

$D^j_{\lambda n}$ is the Wigner D-function, j is the spin and λ is the helicity. The constants n and $(\alpha + i\beta)$ which are the conserved quantum numbers resulting from the gauge transformations mentioned above will be interpreted later on. Here we just notice that a particle is characterized not merely by mass and spin but also by these two additional quantum numbers. The real α and β are connected to the ν in paper III through

$$\begin{aligned} \operatorname{Im} \nu &= \alpha - 1 \\ \operatorname{Re} \nu &= -\beta \end{aligned} \tag{12}$$

One arrives at an arbitrary helicity state by making in succession an acceleration ε along the z-axis and a rotation v around the y-axis and a rotation Φ around the z-axis. All Poincaré transformations are coordinate transformations on the carrier space, i. e., substitutions on the arguments $(x^\mu \varphi\theta\psi s)$ (see equation 3). In this case one obtains with the help of equations (11, 15) of III the following formulae

$$a \to a'$$
$$= \frac{e^{-\varepsilon/2}\left[\cos\frac{v}{2}e^{i\Phi/2}a - \sin\frac{v}{2}e^{-i\Phi/2}b^*\right]}{[\cosh\varepsilon - \sinh\varepsilon(\sin\theta\cos\varphi\sin v\cos\Phi + \sin\theta\sin\varphi\sin v\sin\Phi + \cos\theta\cos v)]^{\frac{1}{2}}}$$
$$b \to b'$$
$$= \frac{e^{\varepsilon/2}\left[\cos\frac{v}{2}e^{i\Phi/2}b + \sin\frac{v}{2}e^{-i\Phi/2}a^*\right]}{[\cosh\varepsilon - \sinh\varepsilon(\sin\theta\cos\varphi\sin v\cos\Phi + \sin\theta\sin\varphi\sin v\sin\Phi + \cos\theta\cos v)]^{\frac{1}{2}}} \tag{13}$$
$$e^s \to e^{s'}$$
$$= e^s[\cosh\varepsilon - \sinh\varepsilon(\sin\theta\cos\varphi\sin v\cos\Phi + \sin\theta\sin\varphi\sin v\sin\Phi + \cos\theta\cos v)]$$
$$x^0 \to \cosh x^0 - \sinh\varepsilon\sin v\cos\varphi x^1 - \sinh\varepsilon\sin v\sin\varphi x^2 - \sinh\varepsilon\cos v x^3$$

where

$$a = \cos\frac{\theta}{2}\exp -\frac{i}{2}(\varphi + \psi)$$
$$b = -\sin\frac{\theta}{2}\exp -\frac{i}{2}(\varphi - \psi)$$

and tgh ε is the velocity of the acceleration. Define now the momentum through

$$p^\mu = m(\cosh\varepsilon, \sinh\varepsilon\sin v\cos\Phi, \sinh\varepsilon\sin v\sin\Phi, \sinh\varepsilon\cos v) \tag{15}$$

a lightlike vector

$$k^\mu = e^s[1, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta] \tag{16}$$

and a function

$$S^{j}_{\lambda n}(p\varphi\theta\psi) = D^{j}_{\lambda n}(\varphi', \theta', \psi') \tag{17}$$

where φ', θ' and ψ' are obtained from equations 13. Then the wave function for a particle with momentum $\vec{P}$ and helicity λ is given by

$$|\,\vec{p}\lambda\,\rangle = \frac{1}{\sqrt{2(2\pi)^3}} \frac{e^{ipx}S^{j}_{\lambda n}(p\varphi\theta\psi)}{\left(\dfrac{pk}{m}\right)^{\alpha+i\beta}} \tag{18}$$

In the appendix some properties of the $S^{j}_{\lambda n}$ function are resumed. One sees from equation (A5) that the wave function (18) contains a denominator

$$\left(\frac{p\mathscr{H}}{m}\right)^{\alpha+i\beta+j}$$

where

$$\mathscr{H} = e^{-s}k \tag{19}$$

The « conventional » one-particle realizations can be obtained from equation (18) by putting $\alpha + i\beta = -j$. Then one can forget about the internal space as we have shown in paper III. Of course we shall keep α and β free and we believe that these numbers characterize in some way the internal structure of the particle. Thus while the electron is well described by $\alpha + i\beta = -\frac{1}{2}$, a hadron may require other values of these parameters.

The wave function for the anti-particle is defined to be

$$|\,\vec{p}\lambda\,\rangle_A = \frac{1}{\sqrt{2(2\pi)^3}} \frac{e^{ipx}S^{j}_{\lambda -n}(p\varphi\theta\psi)}{\left(\dfrac{pk}{m}\right)^{\alpha-i\beta}} \tag{20}$$

This choice is dictated first of all by the requirement that the complex conjugate of equation 18 should give the anti-particle state with the momentum equal to $(-p)$. Then we can expect to get the correct crossing properties of the scattering matrix elements. Comparing with equations 43 and 45 of III we see that the total inversion $\Sigma\tau$ introduced there should be identified with the ordinary CPT transformation. Thus n and β turn into $-n$ and $-\beta$ for the anti-particle. We shall assume in this paper that n is half the baryon number, and that Σ is the ordinary space inversion multiplied by the baryon-antibaryon conjugation.

Introducing momentum space wave functions through

$$f(x^\mu \varphi\theta\psi s) = \int \frac{d^3q}{q_0} e^{iqx} \hat{f}(\vec{q}\varphi\theta\psi s) \tag{21}$$

$$q_0 = \sqrt{m^2 + \vec{q}^2}$$

one sees that the helicity states (18) and (20) in the momentum realization are

$$\begin{aligned} \widehat{|\vec{p}\lambda\rangle} &= \frac{1}{\sqrt{2(2\pi)^3}} p_0 \delta(\vec{p} - \vec{q}) \frac{S^j_{\lambda n}(p\varphi\theta\psi)}{\left(\frac{pk}{m}\right)^{\alpha + i\beta}} \\ \widehat{|\vec{p}\lambda\rangle}_A &= \frac{1}{\sqrt{2(2\pi)^3}} p_0 \delta(\vec{p} - \vec{q}) \frac{S^j_{\lambda - n}(p\varphi\theta\psi)}{\left(\frac{pk}{m}\right)^{\alpha - i\beta}} \end{aligned} \tag{22}$$

In paper III we also introduced a scalar product thus defining a Hilbert space in which P is represented unitarily. It is

$$(f_1, f_2) = \int\int \frac{d^3q \sin\theta d\theta d\varphi}{q^0 \left(\frac{qk}{m}\right)^{-2\alpha} \left(\frac{q\mathscr{H}}{m}\right)^2} \hat{f}_1^* \hat{f}_2 \tag{23}$$

Then the one-particle states are normalized according to

$$\langle \vec{p}\lambda | \vec{p}'\lambda' \rangle = \frac{1}{(2\pi)^2} p_0 \delta(\vec{p} - \vec{p}') \frac{\delta_{\lambda\lambda'}}{(2j + 1)} \tag{24}$$

For the sake of clarity we want to stress the difference between the variables $(\vec{p}, \lambda, m, j, n, \alpha + i\beta)$ which are labels or quantum numbers of the states, and $(x^\mu, \varphi, \theta, \psi, s)$. The states are functions of the latter variables in an explicit realization on the carrier space. Thus the spin quantum numbers are *not* made continuous as in the Regge theory. On the other hand the continuous angles (φ, θ, ψ) are not eigenvalues of any operators. However, this is no new situation since it is well known that the Minkowski coordinates x^μ *cannot be considered as operators or eigenvalues thereof.*

Once the wave functions for particles and anti-particles are defined one can introduce free fields in the conventional way [*16*]. In our case the wave functions depend on 8 variables and therefore the fields will also depend on these 8 variables. In the conventional field theory the fields depend only on the 4 Minkowski coordinates. In order to represent the spin degree of freedom one has there to consider many-component fields. This will not be necessary here since the carrier space is large enough to carry the spin degree of freedom (and two gauge transformations in addition).

Introduce now the annihilation operators $b(\vec{p}\lambda)$ and $c(\vec{p}\lambda)$ for particles and anti-particles. Their Hermitean conjugates $b^*(\vec{p}\lambda)$ and $c^*(\vec{p}\lambda)$ are the creation operators for the particles and anti-particles. The non zero commutation relations are

$$\begin{aligned}[b(\vec{p}\lambda), b^*(\vec{p}', \lambda')]_\mp &= p_0\delta(\vec{p} - \vec{p}')\delta_{\lambda\lambda'} \\ [c(\vec{p}\lambda), c^*(\vec{p}', \lambda')]_\mp &= p_0\delta(\vec{p} - \vec{p}')\delta_{\lambda\lambda'}\end{aligned} \tag{25}$$

where the $(-)$ sign is chosen for bosons and the $(+)$ sign for fermions. Define the field

$$\Psi(x, z) = \frac{1}{\sqrt{2(2\pi)^3}} \sum_\lambda \int_{p_0=\sqrt{m^2+\vec{p}^2}} \frac{d^3p}{p_0} \left\{ \frac{e^{ipx} S^j_{\lambda n}(p\varphi\theta\psi)}{\left(\frac{pk}{m}\right)^{\alpha+i\beta}} b^*(\vec{p}\lambda) + \frac{e^{-ipx} S^{j*}_{\lambda-n}(p\varphi\theta\psi)}{\left(\frac{pk}{m}\right)^{\alpha+i\beta}} c(\vec{p}\lambda) \right\} \tag{26}$$

where $z \equiv (\varphi\theta\psi s)$. Then its Hermitean conjugate is

$$\Psi^*(x, z) = \frac{1}{\sqrt{2(2\pi)^3}} \sum_\lambda \int \frac{d^3p}{p_0} \left\{ \frac{e^{-ipx} S^{j*}_{\lambda n}(p\varphi\theta\psi)}{\left(\frac{pk}{m}\right)^{\alpha-i\beta}} b(\vec{p}\lambda) + \frac{e^{ipx} S^j_{\lambda-n}(p\varphi\theta\psi)}{\left(\frac{pk}{m}\right)^{\alpha-i\beta}} c^*(p\lambda) \right\} \tag{27}$$

By means of equation (25) we can now calculate the commutator

$$[\Psi(x_1 z_1), \Psi^*(x_2 z_2)]_\mp \equiv i\Delta_\mp(x_1 - x_2, z_1, z_2) \tag{28}$$

One finds

$$\Delta_\mp(x_1, z_1, z_2) = \frac{-i}{2(2\pi)^3} \int_{p_0=\sqrt{m^2+\vec{p}^2}} \frac{d^3p}{p_0} (e^{-ipx} \mp e^{ipx}) \cdot \sum_\lambda \frac{S^j_{\lambda n}(p\varphi_1\theta_1\psi_1) S^{j*}_{\lambda n}(p\varphi_2\theta_2\psi_2)}{\left(\frac{pk_1}{m}\right)^{\alpha+i\beta} \left(\frac{pk_2}{m}\right)^{\alpha-i\beta}} \tag{29}$$

In order that the theory satisfies micro-causality $\Delta_\mp$ must be zero when x is space-like. For ordinary fields one then concludes that one must choose the $(-)$ sign for integer spins and the $(+)$ sign for half-integer spins [16]. The proof uses the fact that the integrand contains an even polynomial in p in the first case and an odd one in the second case. Here we do not have polynomials in p but more general functions. Therefore $\Delta_{\mp}$ will not be causal except for special values of α and β. Consequently we cannot derive the spin-statistics theorem. However, if we choose $\alpha = -j$ and $\beta = 0$ the theorem will follow. In a more general case when α and β do not fulfil these equalities G. Fuchs has shown [17] that commutativity for space-like $x_1 - x_2$ can be saved for certain relative values of z_1 and z_2 provided one chooses the usual connection between spin and statistics. In this sense the spin-statistics theorem can again be proved from a weakened causality axiom. In fact this is the only point which has to be relaxed in the Wightman axiom system.

The next important quantity is the vacuum expectation value of time ordered products of fields. Using equations (25, 26, 27) we easily derive

$$\langle 0 \,|\, \mathrm{T}(\Psi(x_1 z_1)\Psi^*(x_2 z_2)) \,|\, 0 \rangle = \frac{1}{2}\Delta_{\mathrm{F}}(x_1 - x_2, z_1, z_2) \tag{30}$$

where

$$\begin{aligned} \frac{1}{2}\Delta_{\mathrm{F}}(x_1 - x_2, z_1, z_2) &= \langle 0 \,|\, \Psi(x_1 z_1)\Psi^*(x_2 z_2) \,|\, 0 \rangle \\ &= \frac{1}{2(2\pi)^3} \int_{p_0=\sqrt{m^2+\vec{p}^2}} \frac{d^3p}{p_0} e^{-ip(x_1-x_2)} \cdot \sum_{\lambda} \frac{S^j_{\lambda n}(p\varphi_1\theta_1\psi_1) S^{j*}_{\lambda n}(p\varphi_2\theta_2\psi_2)}{\left(\frac{pk_1}{m}\right)^{\alpha+i\beta}\left(\frac{pk_2}{m}\right)^{\alpha-i\beta}} \\ &\qquad \text{for } x_{10} > x_{20} \\ \frac{1}{2}\Delta_{\mathrm{F}}(x_1 - x_2, z_1, z_2) &= \mp \langle 0 \,|\, \Psi^*(x_2 z_2)\Psi(x_1 z_1) \,|\, 0 \rangle \\ &= \mp \frac{1}{2(2\pi)^3} \int_{p_0=\sqrt{m^2+\vec{p}^2}} \frac{d^3p}{p_0} e^{ip(x_1-x_2)} \cdot \sum_{\lambda} \frac{S^j_{\lambda n}(p\varphi_1\theta_1\psi_1) S^{j*}_{\lambda n}(p\varphi_2\theta_2\psi_2)}{\left(\frac{pk_1}{m}\right)^{\alpha+i\beta}\left(\frac{pk_2}{m}\right)^{\alpha-i\beta}} \\ &\qquad \text{for } x_{10} < x_{20} \end{aligned} \tag{31}$$

One can show that the following expression for Δ_{F} holds

$$\begin{aligned} \Delta_{\mathrm{F}}(x, z_1, z_2) = \frac{i}{(2\pi)^4} \int_{p_0=\sqrt{m^2+\vec{p}^2}} \frac{d^3p}{p_0} \int_{-\infty}^{\infty} dq_0 e^{-iq_0x_0} e^{i\vec{p}\vec{x}} \\ \left\{ \sum_{\lambda} \frac{S^j_{\lambda n}(p\varphi_1\theta_1\psi_1) S^{j*}_{\lambda n}(p\varphi_2\theta_2\psi_2)}{\left(\frac{pk_1}{m}\right)^{\alpha+i\beta}\left(\frac{pk_2}{m}\right)^{\alpha-i\beta}(q_0 - p_0 + i\varepsilon)} \right. \\ \left. \mp \sum_{\lambda} \frac{S^j_{\lambda n}(\tilde{p}\varphi_1\theta_1\psi_1) S^{j*}_{\lambda n}(\tilde{p}\varphi_2\theta_2\psi_2)}{\left(\frac{\tilde{p}k_1}{m}\right)^{\alpha+i\beta}\left(\frac{\tilde{p}k_2}{m}\right)^{\alpha-i\beta}(q_0 + p_0 - i\varepsilon)} \right\} \end{aligned} \tag{32}$$

where $\tilde{p} = (p^0, -\vec{p})$ and the integration in q_0 is along the curve in figure 1.

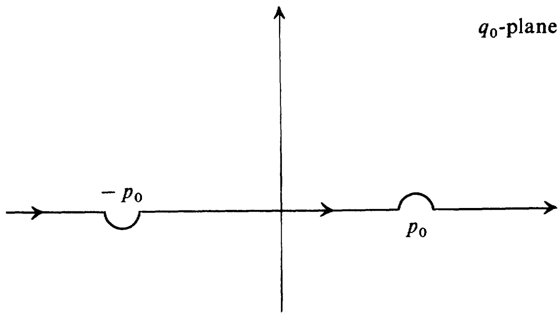

FIG. 1. — Integration contour for the function Δ_F.

We remark that Δ_F is *in general not* a relativistically invariant function. One source of non-invariance is the appearance of the vectors p and $\tilde{p}$. When a Lorentz transformation Λ acts on the arguments x, z_1 and z_2 then $(x_0, \vec{x})$, k_1 and k_2 are transformed like four-vectors. This transformation can be thrown over on the integration variables (q_0, p) which are transformed into $(q_0', \vec{p}')$. However, this $\vec{p}'$ will not be the same as that which appears in the scalar product (pk_1) or (pk_2) and therefore we cannot rewrite the integral in the form (32) exept when $\alpha + i\beta = -j$. Such difficulties are expected since we do not have causality in general and therefore the time-ordering is not relativistically invariant either. The question is whether non-causality (which may be physically relevant) can be combined with relativistic invariance for interacting fields.

4. INTERACTING FIELDS

In conventional field theory the S-operator can be expressed through the Dyson formula [*1*]

$$S = T \exp\left(- ig \int d^4\chi \mathscr{H}(x) \right) \tag{33}$$

where T is the time ordering operator used in equation (30). The Hamiltonian density $\mathscr{H}(x)$ has in general been assumed to have the form of an invariant local product of fields. The locality has no firm physical basis but it has worked well in quantum electrodynamics. Furthermore non-local theories have turned out to be very difficult to define and handle.

Now that we have at our disposal fields defined on a manifold larger than the Minkowski space we could try a formula similar to equation (33). The first important fact is that there exists a measure on the carrier space which is invariant under the Poincaré group. According to table I it is

$$d\mu = d^4\chi dz \tag{34}$$

where

$$dz = e^{2s}ds \sin\theta d\theta d\varphi d\psi. \tag{35}$$

In reference [11] we therefore proposed the following generalization of equation (33)

$$S = T \exp\left(-ig\int\int d^4\chi dz\mathscr{H}(x, z)\right) \tag{36}$$

where

$$\mathscr{H}(x, z) = \prod_{i=1}^{l} \Psi_i(x, z) \tag{37}$$

and Ψ_i are fields corresponding to the interacting elementary particles. We showed that such a formula leads to scattering amplitudes which have built-in form factors. More precisely expanding equation (36) up to second order we got for the process

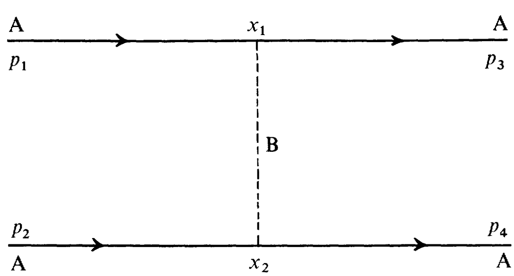

FIG. 2. — OPE graph for (AA)-scattering.

$$S_{fi} = \delta_{fi} - \frac{iG^2}{16\pi^2}\delta^4(p_1 + p_2 - p_3 - p_4)\frac{F^2(t)}{t - \mu^2} \tag{38}$$

where

$$F(t) = \int\int \frac{\sin\theta d\theta d\varphi}{\left(\dfrac{p_3\mathscr{H}}{m}\right)^{1+i\beta}\left(\dfrac{p_1\mathscr{H}}{m}\right)^{1-i\beta}} \tag{39}$$

The quantum numbers of the particle A are mass m, spin zero, $\alpha = 1$, β, and for the particle B mass μ, spin zero, $\alpha = 0$, $\beta = 0$, $p_1 \ldots p_4$ are the particle momenta and $\mathscr{H} = e^{-s}k$. The coupling constant G differs from g of equation (36) by an infinite renormalization constant. The function F(t) is a strongly decreasing function of $t = (p_1 - p_3)^2$ for small values of t. The slope depends on β. Now this result is encouraging since it is a hint that the peripheral nature of high energy hadron scattering might have its explanation in terms of a field theory on our homogeneous space. However, although equation (38) is relativistically invariant the whole S-matrix will not be so. This follows from the conditions

$$\sum_{i=1}^{l} \alpha_i = 2$$
$$\sum_{i=1}^{l} \beta_i = 0 \qquad (40)$$
$$\sum_{i=1}^{l} n_i = 0$$

which are consequences of the assumption of local interaction (see ref. *11*). The α_i are thus not all free and the propagator for the A-particle will not be invariant (see section 4).

Since the non-invariance of the S-matrix is due to the non-locality of the field *and* the T-ordering the invariance may be restored either by making the fields causal or by omitting the T-ordering. The latter remedy [*18*] would normally require quadri-linear couplings since the propagators then have the momenta on the mass shells. Vertices involving only three lines can have all momenta on the mass shells only for special values of the masses and never when the same particle is absorbed and emitted.

The free fields can be made causal, however, only at the expense of giving up the irreducibility of the fields under the Poincaré group. Define annihilation operators $b(\vec{p}\lambda jn\beta)$ and creation operators b^* for each spin j and quantum numbers n and β with the commutation relations

$$[b(\vec{p}j\lambda n\beta), b^*(\vec{p}'j'\lambda'n'\beta']_{\mp} = p_0\delta(\vec{p} - \vec{p}')\frac{\delta_{jj'}}{2j+1}\delta_{\lambda\lambda'}\delta_{nn'}\delta(\beta - \beta') \qquad (41)$$

A similar relation holds for the operator c and c^* of the anti-particles. Define furthermore the big field

$$\Psi_B(x, z) = \frac{1}{\sqrt{2(2\pi)^3}} \sum_{j\lambda n} \int d\beta \int \frac{d^3p}{p_0} (2j+1) \left\{ \frac{e^{ipx} S^j_{\lambda n}(p\varphi\theta\psi)}{\left(\frac{pk}{m}\right)^{\alpha+i\beta}} b^*(\vec{p}j\lambda n\beta) + \frac{e^{-ipx} S^{j*}_{\lambda-n}(p\varphi\theta\psi)}{\left(\frac{pk}{m}\right)^{\alpha+i\beta}} c(\vec{p}j\lambda n\beta) \right\} \quad (42)$$

This field thus is highly reducible under the Poincaré group containing an infinite number of particles with different spins but the same masses [19]. We shall, however, distinguish between half-integer spin field and integer spin field by letting j take the values $\frac{1}{2}, \frac{3}{2}, \ldots$ or $0, 1, 2, \ldots$ respectively. The causal commutator for such a field

$$\Delta^B_\mp(x_1 - x_2, z_1, z_2) \equiv -i[\Psi_B(x_1z_1), \Psi^*_B(x_2z_2)]_\mp \quad (43)$$

becomes

$$\Delta^B_\mp(x, z_1, z_2) = -im^{2\alpha} \int \frac{d^3p}{p_0} [e^{-ipx} \mp e^{ipx}] (pk_1)^{2(1-\alpha)} e^{-2s_1} \delta(s_1 - s_2)\delta(\varphi_1 - \varphi_2)\delta(\cos\theta_1 - \cos\theta_2)\delta(\psi_1 - \psi_2),$$
$$\varphi_1, \varphi_2, \psi_1, \psi_2 \in [0, 2\pi] \quad (44)$$

Now just as in reference 16 $\Delta_\mp$ will be zero for x space-like if the exponent $2(1-\alpha)$ is an even positive integer for bosons and an odd positive integer for fermions. This means that α must be half integer not greater than $\frac{1}{2}$ for fermions and integer not greater than 1 for bosons in order to have microcausality. Let us now see what happens to the propagator function

$$\Delta^B_F(x_1 - x_2, z_1, z_2) = 2\langle 0 | T\{\Psi_B(x_1z_1)\Psi^*_B(x_2z_2)\} | 0 \rangle \quad (45)$$

From equation 32 we get summing over β, n and j (the same expression results both in the integral spin and in the half integral spin case if we restrict the angles φ and ψ to the interval $[0, 2\pi]$)

$$\Delta_F^B(x, z_1, z_2) = i\frac{m^{2\alpha}e^{-2s_1}}{\pi}\int\frac{d^3p}{p_0}\int_{-\infty}^{\infty} dq_0$$
$$e^{-iq_0x_0}e^{i\bar{p}\bar{x}}\left[\frac{(pk_1)^{2(1-\alpha)}}{q_0 - p_0 + i\varepsilon} \mp \frac{(\tilde{p}k_1)^{2(1-\alpha)}}{q_0 + p_0 - i\varepsilon}\right]$$
$$\delta(s_1 - s_2)\delta(\varphi_1 - \varphi_2)\delta(\cos\theta_1 - \cos\theta_2)\delta(\psi_1 - \psi_2) \quad (46)$$

In order that Δ_F^B be an invariant function the bracket

$$\frac{(pk_1)^{2(1-\alpha)}}{p_0(q_0 - p_0 + i\varepsilon)} \mp \frac{(\tilde{p}k_1)^{2(1-\alpha)}}{p_0(q_0 + p_0 - i\varepsilon)} \quad (47)$$

must be an invariant function since $e^{-iq_0x_0}e^{i\bar{p}\bar{x}}$ and the distribution on z are invariant. Now this is possible only if $\alpha = 1$ and one chooses the upper sign or if $\alpha = \frac{1}{2}$ and one chooses the lower sign. We therefore find the interesting relations

$$\alpha = 1 \quad \text{for boson field}$$
$$\alpha = \frac{1}{2} \quad \text{for fermion field.}$$

The requirement that the propagator function be relativistically invariant is much more restrictive than the requirement that the fields be causal. To sum up: infinite component fermion fields on the homogeneous space can be constructed according to equation (42) which satisfy microcausality if α is half integer $\leqslant \frac{1}{2}$. If furthermore $\alpha = \frac{1}{2}$ the propagator function will be relativistically invariant. Infinite component boson fields can be similarly constructed satisfying microcausality if α is integer $\leqslant 1$. If furthermore $\alpha = 1$ the propagator-function will be relativistically invariant. The spin-statistics theorem does not follow since α is not related to the integer or half-integer character of j.

This enables us to construct a relativistically invariant interaction field theory between the big fields. The first of equations 40 must hold again if we assume local point-wise coupling. Therefore only two types of coupling is possible: tri-linear with two fermion fields and one boson field or quadri-linear with four fermion fields. What we have constructed is an infinite component field theory with a quite particular realization and with an interaction which is neatly confined once one has accepted a local coupling. Since the big field contains many spins it must contain many particles. However, in the free field they all have the same mass [*20*]. Of course the masses might be renormalized due to the interaction so that a mass spectrum could occur. If we take the half-integer spin field we would of course like to identify the spin $\frac{1}{2}$ component with the nucleon. But what β and n should we take? n can only take two values in that case but β is continuous. The calculations in references [*10, 11*] suggest that β is a structure parameter of the hadron. Therefore a certain value which has to be tried out by comparison with experiments may have to be chosen. But it might also be that the proton field is a linear combination with differnet β-values and a weight function $F(\beta)$.

To see what sort of expressions one will get we calculate the term in the scattering amplitude corresponding to the diagram

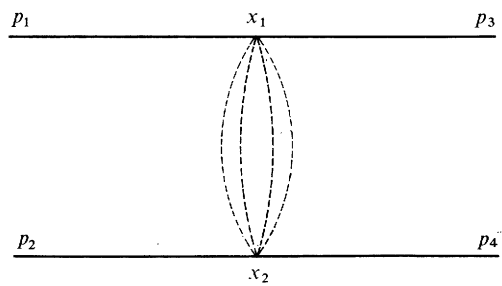

FIG. 3. — One-field exchange diagram for elastic scattering.

The exchanged field is the boson field with $\alpha = 1$ and mass μ. The scattered particles have momenta p_i and quantum numbers $(m, j, \lambda_i, n, \beta)$. We therefore get using similar Feynman rules as those of reference [*11*]

$$S_{fi} = \delta_{fi} - \frac{g^2}{4(2\pi)^6} \int\int\int\int d^4x_1 d^4x_2 dz_1 dz_2 \cdot \frac{1}{2} \Delta_F^B(x_1 - x_2, z_1, z_2)$$

$$\cdot \frac{e^{ip_3x_1} S^j_{\lambda_3 n}(p_3\varphi_1\theta_1\psi_1) e^{-ip_1x_1} S^{j*}_{\lambda_1 n}(p_1\varphi_1\theta_1\psi_1)}{\left(\frac{p_3k_1}{m}\right)^{\frac{1}{2}+i\beta} \left(\frac{p_1k_1}{m}\right)^{\frac{1}{2}-i\beta}}$$

$$\frac{e^{ip_4x_2} S^j_{\lambda_4 n}(p_4\varphi_2\theta_2\psi_2) e^{-ip_2x_2} S^{j*}_{\lambda_2 n}(p_2\varphi_2\theta_2\psi_2)}{\left(\frac{p_4k_2}{m}\right)^{\frac{1}{2}+i\beta} \left(\frac{p_2k_2}{m}\right)^{\frac{1}{2}-i\beta}}$$

Introducing the expression (46) for Δ_F^B we get

$$S_{fi} = \delta_{fi} - i(g\mu)^2(2\pi)^3\delta(p_i - p_f)\delta(0)$$

$$\frac{1}{t-\mu^2}\int\int \sin\theta d\theta d\varphi \frac{S^j_{\lambda_3 n}(p_3\varphi\theta\psi)S^{j*}_{\lambda_1 n}(p_1\varphi\theta\psi)}{\left(\frac{p_3\mathscr{H}}{m}\right)^{\frac{1}{2}+i\beta}\left(\frac{p_1\mathscr{H}}{m}\right)^{\frac{1}{2}-i\beta}}\frac{S^j_{\lambda_4 n}(p_4\varphi\theta\psi)S^{j*}_{\lambda_2 n}(p_2\varphi\theta\psi)}{\left(\frac{p_4\mathscr{H}}{m}\right)^{\frac{1}{2}+i\beta}\left(\frac{p_2\mathscr{H}}{m}\right)^{\frac{1}{2}-i\beta}}$$

where $t = (p_1 - p_3)^2$ and $\delta(0)$ means conservation of β.

The second term is the scattering amplitude to this order of approximation of the perturbation expansion. It is a special case of the expression proposed in paper IV. We can expect a sharp drop of this amplitude as t becomes negative. We will also have the somewhat strange crossing properties discussed in paper IV. This is to say the amplitude will at the same time describe the three crossed channels but the path of analytical continuation cannot be kept in the finite complex $s - t$. plane. In order to reach the physical sheet for a crossed channel the path of continuation has to pass a branch point at infinity. These analytic properties are a consequence of the denominators $(pk)^{\alpha+i\beta}$ in the wave functions. Thus the situation is the same whether we have an infinite component or a one-component field theory on the homogeneous space. Notice that the normal analyticity requires both finite number of components and microcausality.

APPENDIX

The Wigner D-functions are defined through

$$D^j_{\lambda n}(\varphi\theta\psi) = \sum_\chi (-1)^\chi \frac{\sqrt{(j+n)!(j-n)!(j+\lambda)!(j-\lambda)!}}{(j-\lambda-\chi)!(j+n-\chi)!\chi!(\chi+\lambda-n)!} a^{j-\lambda-\chi}(a^*)^{j+n-\chi}b^\chi(b^*)^{\chi+\lambda-n} \quad \text{(A1)}$$

where

$$a = \cos\frac{\theta}{2}\exp -\frac{i}{2}(\varphi+\psi)$$

$$b = -\sin\frac{\theta}{2}\exp -\frac{i}{2}(\varphi-\psi)$$

$$2j = 0, 1, 2, \ldots, \quad 0 \leqslant \varphi + \psi \leqslant 4\pi$$
$$-j \leqslant \lambda, n \leqslant j, \quad -2\pi \leqslant \varphi - \psi \leqslant 2\pi \quad \text{(A2)}$$

and have the following properties

$$D^{j*}_{\lambda n}(\varphi\theta\psi) = (-1)^{n-\lambda}D^j_{-\lambda-n}(\varphi\theta\psi) \quad \text{(A3)}$$

$$\int_0^\pi\int_0^{4\pi}\int_{-2\pi}^{2\pi} \sin\theta d\theta d(\varphi+\psi)d(\varphi-\psi)D^{j*}_{\lambda n}D^{j'}_{\lambda' n'} = \frac{32\pi^2}{2j+1}\delta_{jj'}\delta_{\lambda\lambda'}\delta_{nn'} \quad \text{(A4)}$$

Making the substitutions (13) we get the following expression for the function $S^j_{\lambda n}$ defined in equation (17)

$$S^j_{\lambda n}(p\varphi 0\psi) = \sum_{\chi} (-1)^{\chi} \frac{\sqrt{(j+n)!(j-n)!(j+\lambda)!(j-\lambda)!}}{(j-\lambda-\chi)!(j+n-\chi)!\chi!(\chi+\lambda-n)!}$$

$$\frac{\left[\frac{p_0 + |\vec{p}|}{m}\right]^{(2\chi+\lambda-n-j)}}{\left[\frac{p\mathscr{H}}{m}\right]^{j}} \left[\cos\frac{v}{2} e^{i\Phi/2} a - \sin\frac{v}{2} e^{-i\Phi/2} b^*\right]^{j-\lambda-\chi}$$

$$\left[\cos\frac{v}{2} e^{-i\Phi/2} a^* - \sin\frac{v}{2} e^{i\Phi/2} b\right]^{j+n-\chi}$$

$$\left[\cos\frac{v}{2} e^{i\Phi/2} b + \sin\frac{v}{2} e^{-i\Phi/2} a^*\right]^{\chi}$$

$$\left[\cos\frac{v}{2} e^{-i\Phi/2} b^* + \sin\frac{v}{2} e^{i\Phi\cdot 2} a\right]^{\chi+\lambda-n} \tag{A5}$$

where p is the four momentum of the particle, m the mass and

$$\vec{p} = |\vec{p}|\,(\sin v \cos\Phi, \sin v \sin\Phi, \cos v) \tag{A6}$$

Table I

Homogeneous spaces of P and $\overline{P}$ belonging to continuous stabilizers of L and $\overline{L}$. The ranges of the parameters are $-\infty < x^\mu, s, t, \tilde{u}, \tilde{t}, u < \infty$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi + \psi \leqslant 4\pi$, $-2\pi \leqslant \varphi - \psi \leqslant 2\pi$ for $\overline{P}$ and $0 \leqslant \varphi$, $\psi \leqslant 2\pi$ for P.

Stabilizer generated by	Dimension	Parameters	Half integral spin	Invariant measure	Notation of Finkelstein
0	10	$x^\mu, \varphi, \theta, \psi, s, t, u$	Yes	$d^4xe^{2s}dsdtdud\varphi d\cos\theta d\psi$	[6]
$L_{02}-L_{23}$	9	$x^\mu, \varphi, \theta, \psi, s, t$	Yes	$d^4xe^{2s}dsdtd\varphi d\cos\theta d\psi$	[5]
L_{12}	9	$x^\mu, \varphi, \theta, s, \tilde{t}, \tilde{u}$	No	$d^4xdsd\tilde{t}d\tilde{u}d\varphi d\cos\theta$	[5_0]
$\cos f/2L_{12}+\sin f/2L_{03}$ $0<f\leqslant\pi$	9	$x^\mu, \varphi, \theta, \tilde{\psi}, \tilde{t}, \tilde{u}$	Yes	$d^4xd\tilde{t}d\tilde{u}d\varphi d\cos\theta d\tilde{\psi}$	[5_f]
$L_{02}-L_{23}$ $L_{01}+L_{31}$	8	$x^\mu, \varphi, \theta, \psi, s$	Yes	$d^4xe^{2s}dsd\varphi d\cos\theta d\psi$	[4]
$L_{02}-L_{23}, L_{03}$	8	$x^\mu, \varphi, \theta, \psi, \tilde{t}$	Yes	No	[4′]
L_{12}, L_{03}	8	$x^\mu, \varphi, \theta, \tilde{t}, \tilde{u}$	No	$d^4xd\tilde{t}d\tilde{u}d\varphi d\cos\theta$	[4″]
L_{12}, L_{23}, L_{31}	7	$x^\mu, q^\mu, q^\mu q_\mu=1$	No	$d^4x\dfrac{d^3q}{q_0}$	[3]
L_{12}, L_{01}, L_{02}	7	$x^\mu, q^\mu, q^\mu q_\mu=-1$	No	$d^4x\dfrac{dq^0dq^1dq^2}{q^3}$	[3]
$L_{12}, L_{02}-L_{23}$ $L_{01}+L_{31}$	7	$x^\mu, \varphi, \theta, s$	No	$d^4xe^{2s}dsd\varphi d\cos\theta$	[3_0]
$\cos f/2L_{12}+\sin f/2L_{03}$ $L_{02}-L_{23}$ $L_{01}+L_{31}$ $0<f\leqslant\pi$	7	$x^\mu, \varphi, \theta, \tilde{\psi}$	Yes	No	[3_f]
L_{12}, L_{03} $L_{02}-L_{23}$ $L_{01}+L_{31}$	6	x^μ, φ, θ	No	No	[2]
$L_{\mu\nu}$	4	x^μ	No	d^4x	[0]

REFERENCES

[*1*] See S. S. SCHWEBER, *Relativistic quantum field theory*. Row, Peterson, 1961.
[*2*] See H. LEHMANN, *Suppl. Nuovo Cimento*, **14**, 1959, 153.
[*3*] See L. BERTOCCHI and E. FERRARI, in *High energy physics*, vol. II. Edited by E. H. S. Burhop, 1967.
[*4*] M. TOLLER, *Nuovo Cimento*, **54**A, 1968, 295.
E. T. HADJIOANNOU, *Nuovo Cimento*, **44**, 1966, 185.
G. DOMOKOS, *Phys. Lett.*, **24**B, 1967, 293.
[*5*] E. FERRARI and F. SELLERI, *Nuovo Cimento*, **27**, 1962, 1455. Suppl. *Nuovo Cimento*, **24**, 1962, 453.
[*6*] A. S. WIGHTMAN, *Phys. Rev.*, **101**, 1956, 860.
[*7*] H. LEHMAN, K. SYMANZIK and W. ZIMMERMAN, *Nuovo Cimento*, **1**, 1955, 1425, **6**, 1957, 319.
[*8*] A. KIHLBERG, *Arkiv Fysik*, **28**, 1964, 121. Hereafter referred to as paper I. *Arkiv. Fysik*, **32**, 1966, 263 (paper II).
[*9*] A. KIHLBERG, *Nuovo Cimento*, **53**, 1968, 592 (paper III).
[*10*] A. KIHLBERG, *Arkiv Fysik*, **39**, 1969, 77 (paper IV).
[*11*] A. KIHLBERG, *On a new field theory*. Report 68-9 Institute of Theoretical Physics, Göteborg, 1968 (A wrong conjecture about the microcausality was made in this paper. Therefore the S-matrix defined does not always lead to relativistically invariant amplitudes).
[*12*] F. LURÇAT, *Physics*, **1**, 1964, 95.
[*13*] H. BACRY and A. KIHLBERG, *J. Math. Phys.*, **10**, 1969, 2132.
[*14*] D. FINKELSTEIN, *Phys. Rev.*, **100**, 1955, 942.
[*15*] H. BACRY, *Commun. Math. Phys.*, **5**, 1967, 97.
[*16*] S. WEINBERG, *Phys. Rev.*, **133**, 1964, B1318.
[*17*] G. FUCHS, *Field theories on homogeneous spaces of the Poincaré group*. Preprint École Polytechnique, 1968.
[*18*] J. NILSSON and H. V. R. PIETSCHMAN, *Phys. Rev.*, **142**, 1966, 1173.
L. REDEI, private communication.
[*19*] A list of references on infinite component field theories is given in ref. 13.
[*20*] Concerning the possibility of having a mass splitting in the free fields, see I. T. GRODSKY and R. F. STREATER, *Phys. Rev. Lett.*, **20**, 1968, 695.

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