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Relativistic quantum mass distributions on velocity space

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ABSTRACT. — It is shown that by employing mass distributions which are allowed to become negative in some regions of a mixed fluid velocity space a very suitable classical basis for Schrödinger quantum theory can be generated. Further, it is shown that the negative mass regions arise naturally from relativistic considerations and thus the old problem of the negative regions of the quantum phase space distributions is resolved.

INTRODUCTION

In previous papers [*1*] [*2*] [*3*] [*4*] [*5*] [*6*], we have examined various aspects of the problem of putting conventional quantum mechanics on to a classical or classical-like basis. In reference [*2*], we discussed difficulties which occur in some works [*18*] [*19*] [*20*] of related motivation. This line of work has begun to reveal features of the quantum area which have previously gone unnoticed. Particularly important in this respect is the recognition of the existence of an underlying thermal equilibrium [*6*]. In this last mentioned reference, it was shown that conventional quantum theory, as expressed by the Schrödinger equation, could be regarded as being the consequence of a local thermal equilibrium between two subquantum fluids. However, there remained in that account one feature which did not seem to be of immediate classical clarity.

This feature is the fairly well known point that the phase space distributions which can be directly correlated with quantum states always have negative regions [7] [8] [9] [10]. It is this particular point which does require some closer examination because, for one thing, it appears to be crucial in making sense of the « thermal equilibrium ». In this paper, we shall look closer into the structure underlying the results of reference [6]. However, we shall make a self-contained, and somewhat different formulation in that, firstly, we shall find it more convenient to work with mass distributions on a velocity space rather than probability distributions on a phase space and, secondly, we wish to deal explicitly and immediately with the question of the negative regions of our distributions. The need for these changes arises from the relativistic approach which we shall now adopt and, indeed, which seems to resolve all the difficulties.

STRUCTURE AND MOTIVATION

We shall take the basic conceptions of classical relativistic non-quantum physics as our building material and make it our objective to build Schrödinger quantum mechanics from this basis. Our approach will be fundamentally statistical though here we shall work with mass distributions rather than probabilities. The usual direct substitution of operators for classical dynamical variables and the introduction of operands (wave functions) which have no direct intuitive significance, we wish to avoid. It can, of course, be argued that such efforts are unnecessary because the direct approach obviously works and leads to equations which can be solved to give correct physical information. However, the replacement of the classical momentum variable, p, for example by the operator $-i\hbar\frac{\partial}{\partial q}$ is undoubtedly a drastic philosophical step. If, on the other hand, we can show that a more continuous, intuitively acceptable, and more generally suggestive derivation of quantum mechanics can be made, then we will have established the possibility of a deeper understanding of the quantum area emerging. There is no doubt that a good analogy, when found, invariably suggests further lines of research and some times can give penetrating insight. This author suggests as, indeed, others [11] [12] have done before that the fluid character of the quantum process is even better than just a good analogy. The quantum process « is » a fluid interaction. If this contention is correct, and the evidence [6] [11] [12] in favour of it seems to be mounting, then the pursuit of this line will certainly lead to a deeper understanding of nature. Before proceeding to add to the evidence we would just like to mention one specific and important topic where a more classical like formulation of quantum theory could be particularly helpful. It seems that the technical problem of actually making use of the « Feynman integral » [4] is equivalent to the problem of reformulating quantum theory. This particular problem has, in fact, been this authors main motivation. Philosophical questions aside, let us now return to the technicalities.

SMALL VELOCITIES AND NEGATIVE MASS

Consider now the classical relativistic relation between energy, momentum and rest mass for a free particle

$$\left(\frac{E}{c}\right)^2 = P^2 + (m_0 c)^2. \tag{1}$$

Alternatively, we have the parametric forms and their approximations,

$$E = \pm \frac{m_0 c^2}{(1 - v^2/c^2)^{\frac{1}{2}}} \sim \pm\, m_0 c^2 \pm \frac{m_0 v^2}{2}, \tag{2}$$

and

$$P = \frac{m_0 v}{(1 - v^2/c^2)^{\frac{1}{2}}} \sim m_0 v, \tag{3}$$

the approximations holding when $v \ll c$.

Thus having decided how p is to depend on v, the two possibilities in equation (2) follow from (1). It is usual to regard the two possibilities in (2) as coming from the matter and anti-matter states according to the sign taken and with this we have no need to argue. However, the point we wish to make here is that in going from the relativistic classical forms to fluid like forms of dynamic there is no mathematical reason why, under some circumstances other than the high energy situation, the negative sign of mass should not be important. The physical reason for rejecting the negative sign for the classical non-quantum non-relativistic discrete particle situation is obvious, but for our purpose which is the transition from the relativistic domain to a statistical fluid situation equivalent to the Schrödinger equation there is no obvious reason for rejecting the negative sign. In the hydrodynamical situation negative mass contributions could well be masked by dominantly positive mass contributions but, if the negative mass is present, it will make a significant contribution to the form the equations will take. Thus in going from the relativistic region to the « hydrodynamical » quantum region the qualitatively new possibility of positive-negative mass mixtures emerges. Further, from this point of view, the idea of a structured « vacuum » becomes meaningful. Even the simplest undisturbed vacuum state of zero energy can be expressed as the superposition of a pair of equal and opposite sign mass distributions, $m_+(x)$ and $m_-(x) = - m_+(x)$, such that

$$m_+(x) + m_-(x) \equiv 0. \tag{4}$$

Local fluctuations about such a state are easily conceivable. Such fluctuations can still leave the total vacuum energy zero,

$$\int_{-\infty}^{+\infty} \mathrm{E}\ (\text{vacuum})\, dx \equiv 0 \tag{5}$$

where E (vacuum) is the local vacuum energy density for the fluctuating state. The preceding discussion now suggests that we can include some additional basic structural conceptions on which to build our statistical formulation. Let us assume then that at the atomic level the physical universe can be decomposed into two fundamental things. Let these two things be the « vacuum » for one, and the (bare) « particles » for the other. This, of course, is not such a revolutionary idea because vacuum effects have been extensively studied in quantum field theory [*13*] [*14*] [*15*] [*16*] and their importance in high energy physics would seem to be unquestionable. However, here we introduce the « vacuum » contribution in a novel way and we shall demonstrate that such ideas are important even for the low energy Schrödinger equation. The justification for our approach and particular collection of basic assumptions will lie in the insight we obtain over and above what is obtained by the orthodox approach. Having separated our universe into « vacuum » and « particles », we now assume, in line with the previous discussion, a second separation of both vacuum and particles into positive and negative mass contributions. Further we must be clear that the negative mass we have in mind here is really negative. We are not talking about holes in a continuum of negative energy states [*17*]. Thus we are not working with matter and anti-matter in quite the usual sense. If a numerical quantity of our positive mass coincides with an equal numerical quantity of our negative mass, then the total mass resulting will be zero and no energy will be released. Further, we see no need for the violation of any of the usual conservation laws in this scheme. Let us now give these ideas some mathematical form by showing that there is a method of deriving the Schrödinger equation from relativistic and statistical considerations which shows that negative mass contributions are an important part of its fundamental structure. In this derivation, we shall avoid the small velocity limit which is usually used when the operator

$$\tilde{a} = -\frac{1}{m_0} i\hbar \frac{\partial}{\partial x}$$

replaces the velocity in the approximation (2) to give the free Schrödinger Hamiltonian,

$$\mathcal{H} = -\frac{\hbar^2}{2m_0}\frac{\partial^2}{\partial x^2}. \tag{6}$$

Thus we shall show that the selection of positive mass only which is apparently made by the usual procedure (taking the plus sign in (2)) does not in fact occur. That positive mass appears to be separated out cleanly by the usual argument is a misleading consequence of the replacement of a function by an operator and, indeed, of the very use of the approximation (2).

Vacuum and negative energy effects have not previously convincingly [*18*] [*19*] [*20*] [*21*] been demonstrated to be an essential part of the structure of Schrödinger quantum mechanics.

THE RELATIVISTIC BASE

What normally has been regarded as being a particle, we are now regarding as being decomposable into two distinguishable parts. A convenient terminology for these parts is « the solute » for one and « the solvent » for the other. The solute is what in high energy physics might be called the bare particle and the solvent (or vacuum) is that part which is composed of the vacuum's recognition of the very existence of the bare particle. The form this recognition will take depends partly on the state of motion of the bare particle. We shall now employ subscripts « 1 » and « 2 » to denote the solute and solvent parts respectively.

Thus in working towards a statistical theory we shall use, as a guide, the obvious classical relativistic equations for solute and solvent:

$$\left(\frac{E_1 - W_1}{c}\right)^2 = p_1^2 + (m_{10}c)^2 \tag{7}$$

and

$$\left(\frac{E_2 - W_2}{c}\right)^2 = p_2^2 + (m_{20}c)^2, \tag{8}$$

respectively. We have not included a vector potential simply because it is only our purpose to derive the usual Schrödinger equation for a scalar potential W.

Our first objective is to construct mass distributions on a velocity space and so (7) and (8) are not suitable as they stand. However, if we divide (7) and (8) by the magnitudes of their masses (not rest masses), we obtain

$$\frac{(E_1 - W_1)^2}{| m_1 c |^2} = v_1^2 \, | m_1 | + \frac{(m_{10}c)^2}{| m_1 |} \tag{9}$$

and

$$\frac{(E_2 - W_2)^2}{| m_2 c^2 |} = v_2^2 \, | m_2 | + \frac{(m_{20}c)^2}{| m_2 |}. \tag{10}$$

we now take the special case

$$| m_1 | = | m_2 | = | m |, \text{ say,} \tag{11}$$

but still keeping $m_{10} \neq m_{20}$. Thus in this relativistic situation v_1 will not necessarily be equal to v_2. We can subtract (10) from (9) to get

$$\frac{(E_1 - W_1) - (E_2 - W_2)^2}{| mc^2 |} = v_1^2 \, | m | - v_2^2 \, | m | + \beta \tag{12}$$

where

$$\beta = \frac{(m_{10}^2 - m_{20}^2)}{| m |} c^2. \tag{13}$$

Let us now consider positive and negative energy solutions to (12) separately. These can be extracted by taking,

$$E_1 - W_1 + E_2 - W_2 = +2 \, | mc^2 | \tag{14}$$

for the positive case and,

$$E_1 - W_1 + E_2 - W_2 = -2 \, | mc^2 | \tag{15}$$

for the negative case. We shall use plus and minus subscripts to distinguish the two different sets of variables arising from these two cases. Thus for the positive case (12), (13) and (14) yield,

$$E_{1+} - E_{2+} = \frac{1}{2}(v_{1+}^2 - v_{2+}^2) \, | m | + \frac{\beta}{2} + W_{1+} - W_{2+} \tag{16}$$

and for the negative case (12) (13) and (15) yield,

$$E_{1-} - E_{2-} = -\frac{1}{2}(v_{1-}^2 - v_{2-}^2)\,|\,m\,| - \frac{\beta}{2} + W_{1-} - W_{2-} \tag{17}$$

If we now define,

$$\begin{aligned} E\,(\text{solute}) &= E_{1+} + E_{1-}, & (a)\\ W\,(\text{solute}) &= W_{1+} + W_{1-}, & (b)\\ E\,(\text{solvent}) &= E_{2+} + E_{2-}, & (c)\\ W\,(\text{solvent}) &= W_{2+} + W_{2-} & (d) \end{aligned} \tag{18}$$

and W, the potential energy function of the solute relative to that of the solvent, by

$$W = W\,(\text{solute}) - W\,(\text{solvent}) \tag{19}$$

we obtain from (16) and (17), the equation,

$$\begin{aligned} &E\,(\text{solute}) - E\,(\text{solvent})\\ &= \frac{1}{2}(v_{1+}^2\,|\,m\,| - v_{1-}^2\,|\,m\,|) - \frac{1}{2}(v_{2+}^2\,|\,m\,| - v_{2-}^2\,|\,m\,|) + W. \end{aligned} \tag{20}$$

We remark here that (20) has been obtained from (7) and (8) without making any approximations. Certainly (11), (12), (14) and (15) represent a special case but there is no question of velocities necessarily being small.

STATISTICAL CONSIDERATIONS

Equation (20) can now be regarded as being a framework giving a basis onto which our statistical structure is to be moulded. We shall now regard E (solute) and E (solvent) as energy densities and we must replace the m variable by a mass distribution on the two dimensional velocity space (v_1, v_2). From (20) it is clear that our distribution function $m(x, t \mid v_1, v_2)$ should be negative for some regions of the joint velocity space. The physical idea involved in introducing the single function, $m(x, t \mid v_1, v_2)$ with the solute-solvent decomposition is that the very existence of the particle causes the vacuum to react (the vacuum carries the « weight » of the particle).

Thus the various separate distributions $m_{1\pm}$, $m_{2\pm}$, positive and negative for solute and positive and negative for solvent are not independent. This is expressed by the condition that these four distributions can be obtained from the single information carrying function $m(x, t \mid v_1, v_2)$ defined on the joint velocity space of bare particle and vacuum. We remark here that another quantity of interest which can be obtained from $m(x, t \mid v_1, v_2)$ is the total effective mass,

$$m_0 = \iint m(x, t \mid v_1, v_2) dv_1 dv_2. \tag{21}$$

Thus we are led inevitably to expressions for energy density in configuration space of the form,

$$\mathrm{E}(x, t \mid \text{solute}) = \iint \frac{v_1^2}{2} m(x, t \mid v_1, v_2) dv_1 dv_2 + \mathrm{W}, \tag{22}$$

and

$$\mathrm{E}(x, t \mid \text{solvent}) = \iint \frac{v_2^2}{2} m(x, t \mid v_1, v_2) dv_1 dv_2, \tag{23}$$

where now $m(x, t \mid v_1, v_2)$ is definitely negative for some regions of the velocity space and, in fact, we would recover the forms in the round brackets on the right hand side in (20), under the integrands in (22) and (23), if the positive and negative contributions from $m(x, t \mid v_1, v_2)$ were separated out by step functions. Identification of the local quantum energy density,

$$\mathrm{E_Q} = Re \cdot \left(i\hbar \frac{\partial \log \psi}{\partial t} \right), \tag{24}$$

as

$$\mathrm{E_Q} = \mathrm{E}\,(\text{solute}) - \mathrm{E}\,(\text{solvent}) \tag{25}$$

now completes the structure. We remark that $\mathrm{E_Q}$ as expressed by (24) is in general a function of x and t. It would only be a constant if ψ were a quantum steady state.

THERMAL ENERGY

Having admitted negative kinetic energy into our structure it is consistent that negative thermal energy should also occur. Thus we can use the two mean velocities

$$\bar{v}_1 = \frac{1}{m_0} \int v_1 m(x, t \mid v_1, v_2) dv_1 dv_2, \tag{26}$$

and

$$\bar{v}_2 = \frac{1}{m_0} \int v_2 m(x, t \mid v_1, v_2) dv_1 dv_2, \tag{27}$$

to help define the two local thermal energy densities,

$$\mu\,(\text{solute}) = \iint \frac{(v_1 - \bar{v}_1)^2}{2} m(x, t \mid v_1, v_2) dv_1 dv_2, \tag{28}$$

$$\mu\,(\text{solvent}) = \iint \frac{(v_2 - \bar{v}_2)^2}{2} m(x, t \mid v_1, v_2) dv_1 dv_2. \tag{29}$$

Both of these will contain some negative contributions from the integration ranges. The physical principle which then leads to Schrödinger quantum mechanics is that there should be local thermal equilibrium

$$\mu\,(\text{solute}) + \mu\,(\text{solvent}) \equiv 0. \tag{30}$$

The mass distribution which gives all the correct information is closely related to the Wigner distribution [7],

$$\mathrm{F}(p, q, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi^*\left(q - \frac{\hbar\tau}{2}, t\right) \psi\left(q + \frac{\hbar\tau}{2}, t\right) e^{-ip\tau} d\tau. \tag{31}$$

It has the form

$$m(x, t \mid v_1, v_2)$$
$$= \rho^{-1} \frac{m_0^3}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi^*\left(x - \frac{\hbar\tau}{2}, t\right) \psi\left(x + \frac{\hbar\tau^*}{2}, t\right) e^{-i(v_1\tau_1 + v_2\tau_2)m_0} d\tau_1 d\tau_2, \tag{32}$$

where

$$\tau = \tau_1 + i\tau_2 \tag{33}$$

and

$$\rho = \psi^*(x, t)\psi(x, t) \tag{34}$$

The details [6] of this last part of the argument leading to the Schrödinger equation will not be repeated here. However, it is not difficult to confirm that from (22), (23), (25), (30) and (32), we get

$$\mathrm{E_Q} = \frac{\bar{v}_1^2}{2} m_0 - \frac{\bar{v}_2^2}{2} m_0 - \frac{\hbar}{4m_0} \frac{\partial^2 \log \rho}{\partial x^2} + \mathrm{W} \tag{35}$$

with

$$m_0(\bar{v}_1 + i\bar{v}_2) = -i\hbar \frac{\partial \log \psi}{\partial x} \tag{36}$$

and that (35) and (36) correspond to the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = - \frac{\hbar^2}{2m_0} \frac{\partial^2 \psi}{\partial x^2} + W\psi \tag{37}$$

for the wave function ψ.

CONCLUSIONS

The work in this paper now seems to completely clarify the nature of the basic distributions necessary to derive the usual quantum structure from a classical basis at the hydrodynamical statistical level. One very interesting feature of this work is that the Schrödinger equation arises without the usual small velocity limit. This implies the qualitative conclusion that Schrödinger quantum mechanics is much more « relativistic » than has previously been thought. Another interesting result is the resolution of the problem of why Wigner's phase space distributions have negative regions. Although we have expressed the work in this paper in terms of a velocity space, it is clear that the negative regions of Wigner's $F(p, q)$ are a consequence of negative mass contributions. There are clearly great possibilities for further study along the lines suggested here and in reference [*6*]. It will be seen that our approach to the quantum problem is essentially similar to the usual methods from the statistical theory of fluids [*22*] [*23*].

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REFERENCES

[*1*] J. G. Gilson, Functional Viewpoint in Quantum Mechanics. *Il Nuovo Cimento*, Serie X, t. **40**, 1965, p. 993.
[*2*] J. G. Gilson, On Stochastic Theories of Quantum Mechanics. *Proc. Camb. Phil. Soc.*, t. **63**, 1968.
[*3*] J. G. Gilson, Quantum Probability Weighted Paths, to be published.
[*4*] J. G. Gilson, Feynman Integral and Phase Space Probability. *J. App. Prob.*, t. **5**, August 1968.
[*5*] J. G. Gilson, Expansion of Quantum Phase Space Probabilities Into Classically Deterministic Parts, *Journal Institute Mathematics*, t. **4**, 1968.

[6] J. G. Gilson, Thermal Content of Quantum States. *Z. Naturforsch.*, t. **24a**, 1969, p. 198.
[7] E. Wigner, On the Quantum Correction for Thermodynamic Equilibrium. *Phys. Rev.*, t. **40**, 1932, p. 749.
[8] J. E. Moyal, Quantum Mechanics as a Statistical Theory. *Proc. Camb. Phil. Soc.*, t. **45**, 1949, p. 99.
[9] M. S. Bartlett and J. E. Moyal, The Exact Transition Probabilities of Quantum Mechanical Oscillators calculated by the Phase-Space Method. *Proc. Camb. Phil. Soc.*, t. **45**, 1949, p. 545.
[10] B. Leaf, Weyl Transformations and the Classical Limit of Quantum Mechanics. *J. Math. Phys.*, t. **9**, n° 1, janvier 1968.
[11] E. Madelung, Quanten theorie in hydrodynamischer Form. *Z. Physik*, t. **40**, 1926, p. 332.
[12] J. P. Vigier, Model of the Causal Interpretation of Quantum Theory in terms of a Fluid with Irregular Fluctuations. *Phys. Rev.*, t. **96**, n° 1, October 1954, p. 208.
[13] R. P. Feynman, Space-Time Approach to Quantum Electrodynamics. *Phys. Rev.*, t. **76**, 1949, p. 749.
[14] J. Schwinger, *Phys. Rev.*, t. **93**, 1954, p. 615.
[15] A. Salam and P. T. Matthews, *Phys. Rev.*, t. **90**, 1953, p. 690.
[16] J. G. Gilson, Independent Electron Positron Pair Production Events and Energy Loss. *Fortschritte der Physik*, t. **15**, 1967, p. 537.
[17] P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford, 1947.
[18] E. Nelson, Derivation of the Schrödinger equation from Newtonian Mechanics. *Phys. Rev.*, t. **150**, n° 4, October 1966, p. 1079.
[19] E. Nelson, *Dynamical Theories of Brownian Motion*, Princeton, 1967.
[20] D. Kershaw, Theory of Hidden Variables. *Phys. Rev.*, t. **136**, n° 6, December 1964, p. B1850.
[21] D. Bohm, *Causality and Chance in Modern Physics*, Routledge, 1957.
[22] S. A. Rice and P. Gray, *The Statistical Mechanics of Simple Liquids*, Interscience, 1965.
[23] D. Massignon, *Mécanique Statistique de Fluides*. Dunod, 1957.
[24] J. G. Gilson, Quantum Wave Functionals and Some Related Topics. *Z. Naturforsch.*, t. **23a**, 1968, p. 1452.

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