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JOHN MADORE

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The Equations of Motion of an Extended Body in General Relativity

par

John MADORE

(Laboratoire de Physique Théorique associé au C. N. R. S.
Institut Henri Poincaré, Paris).

ABSTRACT. — The exponential map exp is used to associate with an arbitrary extended body an energy-momentum vector P^i and an angular momentum tensor M^{ij}. A center-of-mass line C is defined and the equations of evolution of P^i and M^{ij} along C are given.

RÉSUMÉ. — L'application exponentielle exp est utilisée pour associer à un corps arbitraire étendu un vecteur énergie-impulsion P^i et un tenseur de moment angulaire M^{ij}. Une ligne d'univers de centre de masse C est définie et les équations d'évolution de P^i et M^{ij} le long de C sont données.

INTRODUCTION

Consider U, the world-tube of an isolated material body described by a matter tensor T^{ij} and let $\tilde{U}$ be its convex hull. That is, $\tilde{U}$ is the smallest set which satisfies the following three conditions: (*i*) $U \subset \tilde{U}$; (*ii*) if γ is any space-like geodesic and x and y are any two points of γ which belong to $\tilde{U}$, then the segment xy of γ belongs also to $\tilde{U}$; (*iii*) the boundary $\partial\tilde{U}$ of $\tilde{U}$ is of class $\mathcal{C}^1$ as a hypersurface. Assume that the field equations are satisfied:

$$G_{ij} = -8\pi T_{ij} \qquad \text{in U}$$

and

$$G_{ij} = 0$$

in a region surrounding U; and that the solution is of class $\mathcal{C}^3$ in the interior of U and of class $\mathcal{C}^1$ in a neighbourhood of ∂U. G_{ij} is the Einstein tensor. Latin indices take the values (0, 1, 2, 3).

We shall define in $\tilde{U}$ an energy-momentum vector field P^i and an angular momentum tensor field M^{ij} and use them to construct a time-like curve C which always lies in $\tilde{U}$ and may be considered as the world-line of the center of mass of the body. We shall then define an energy-momentum vector and an angular momentum tensor as the restriction of these two fields to C.

In Special Relativity one defines the energy-momentum vector P^i, for example, as an integral of $T^{ij}n_j$ over a space-like hypersurface σ where n_i is the normal to σ. From the conservation laws $T^{ij}{}_{,j} = 0$ one finds that P^i does not depend on the hypersurface σ and since the space is flat there is no problem in integrating the vector $T^{ij}n_j$. A straightforward generalization of this definition to General Relativity encounters two main difficulties: how to choose σ and how to define the integral of $T^{ij}n_j$ over it. Also P^i is no longer in general conserved and cannot as in the flat space of Special Relativity be considered as a « free vector ». We have no longer a vector associated with the body but a vector field defined along a curve C·

Let x be any point in $\tilde{U}$. To define $P^i(x)$, we shall choose as hypersurface of integration a geodesic hypersurface through x such that its normal at x is parallel to $P^i(x)$. We must first of all show that such a hypersurface exists. The problem of defining the integral may be then conveniently solved by the use of the exponential map exp which maps the tangent space at x onto the space-time manifold V_4. The inverse of this mapping always exists in a neighbourhood of x. We assume this neighbourhood sufficiently large to include all points of σ where T_{ij} does not vanish and define the integral of $T^{ij}n_j$ over σ as the integral of the image of $T^{ij}n_j$ in the tangent space at x over the image of σ. By our choice of σ its image is the linear hypersurface perpendicular to $P^i(x)$. By construction, $P^i(x)$ is an element of the tangent space at x; that is, it is a vector. We shall construct the angular momentum tensor field M^{ij} similarly.

Apart from the standard restriction on T^{ij} that $T^{ij}u_iu_j > 0$ for all x in U and for all u_i time-like, we make several convenient restrictions on the field strength. These are listed at the end of section I. Solutions excluded by these restrictions would probably be of no interest in macrophysics. We refer to two articles by Dixon [*1*], [*2*] for a description of an alternate way of defining P^i and M^{ij} as well as for references to the previous literature.

I

Let T_x^* be the cotangent space at a point x of $\tilde{U}$ and let n_i be a unit time-like vector in T_x^*. Let $\sigma(n)$ be the geodesic hypersurface through x perpendicular to n_i. That is, $\sigma(n)$ is the hypersurface formed by the geodesics through x with tangent at x perpendicular to n_i. Let n_i' denote the unit vector field normal to σ. Therefore

$$n_i' = n_i \qquad \text{at } x.$$

The prime will be dropped when there is no risk of confusion. We shall assume that each point $x \in \tilde{U}$ is the origin of a normal coordinate system defined in a neighbourhood of x which includes the sets $\sigma(n) \cap U$ where n_i varies over the time-like unit vectors in T_x^*. That is, we assume that the exponential map is invertible in this neighbourhood.

At each point $x \in \tilde{U}$ and for each time-like $n_i \in T_x^*$, let $P^i(x; n)$ be the vector whose components in a normal coordinate system with origin x are given by

$$P^i(x\,;n) = \int_{\sigma(n)} T^{ij} n_j' d\sigma. \tag{1}$$

Here and in the following $d\sigma$ denotes the measure determined by the metric induced on σ.

We assume that all of these vectors are time-like and that they remain bounded away from the light cone as n_i varies over the time-like elements of T_x^*. This is based on the interpretation of $P^i(x; n)$ as the energy-momentum vector of a macroscopic body. We can then conclude by the Brouwer fixed-point theorem that the map $n^i \to P^i/|P|$ where $|P|$ is the norm of the vector P^i, has a fixed point. We shall assume that this fixed point is unique and designate by $P^i(x)$ the vector at x so obtained. $P^i(x)$ will be called the energy-momentum vector field. In what follows σ will always designate the geodesic hypersurface through x whose normal at x is parallel to $P^i(x)$.

At each point $x \in \tilde{U}$, we define the 2^m-pole moment $P_{(x)}^{i_1 \ldots i_m j}$ as the tensor whose components in a normal coordinate system with origin x are given by

$$P(x)^{i_1 \ldots i_m j} = \int_\sigma x^{i_1} \ldots x^{i_m} T^{jk} n_k' d\sigma, \tag{2}$$

where x^i are the coordinates in the normal coordinate system of the point of integration.

The energy-momentum vector and the multipole moment tensors may be also defined in a coordinate-free manner as follows. Let $\alpha \in T_x^*$ and set $\alpha^* = \exp^{*-1}\alpha$. Then the energy-momentum vector $P(x)$ is given by

$$P(x)(\alpha) = \int_\sigma T(\alpha^*, n)d\sigma. \tag{3}$$

Let $\alpha_1, \ldots, \alpha_{m-1}, \beta \in T_x^*$ and set $\beta^* = \exp^{*-1}\beta$, $X = \exp^{-1}x'$ where x' is the point of integration in σ. Then the 2^m-pole moment is given by

$$P(x)(\alpha_1, \ldots, \alpha_m, \beta) = \int_\sigma X(\alpha_1) \ldots X(\alpha_m)T(\beta^*, n)d\sigma. \tag{4}$$

We shall not discuss this notation further since we do not use it in what follows.

Define the linear combinations of the 2^m-pole moments $M^{i_1\ldots i_m j}$ by

$$M^{i_1\ldots i_m j} = 2P^{i_1\ldots i_m j} - P^{(i_1\ldots i_m j)}, \tag{5}$$

where, for example $P^{(ij)} = P^{ij} + P^{ji}$. M^{ij} is the angular momentum tensor.

We have defined the energy-momentum vector at x to be $P^i(x)$ derived from the contravariant form of the matter tensor. We could equally well have defined it to be $Q_i(x)$ derived from the mixed form. That is, $Q_i(x)$ is the vector whose components in a normal coordinate system with origin x are given by

$$Q_i(x) = \int_\sigma T_i^j n_j' d\sigma. \tag{6}$$

Since in general the components of the metric tensor in a normal coordinate system are not constants, the contravariant form of Q_i will not in general be equal to P^i.

If P^i (or Q_i) is to be the energy-momentum vector, then $| P |$ (or $| Q |$) will have to have an interpretation as the rest energy of the body. We have in fact three possible definitions of energy: $| P |$, $| Q |$ and $\sqrt{g(P, Q)}$ and no *a priori* reason for prefering one over the other two. In the limit of flat space, all three of these quantities reduce to the usual expression for rest energy in Special Relativity. For a discussion of this point in the case of the Schwarzschild solution see reference [*3*].

We shall now use the vector field P^i and the tensor field M^{ij} which we have defined everywhere in $\tilde{U}$ to construct a time-like curve C which lies in $\tilde{U}$ and which may be interpreted as the world-line of the center of mass.

To this end we consider the inner product $M^{ij}P_j$ at a point $x \in \tilde{U}$. We have in a normal coordinate system with origin x

$$M^{ij}P_j = \int_\sigma x^{[i}T^{j]k}n'_kP_j d\sigma. \qquad (7a)$$

Since P_j is parallel to the normal to σ at the point x and since x^i is a vector tangent to σ at x

$$P_i x^i = 0.$$

Therefore we have

$$M^{ij}P_j = \int_\sigma x^i T^{jk} n'_k P_j d\sigma. \qquad (7b)$$

So the vector $M^{ij}P_j$ at the point x is space-like and tangent to the hypersurface σ.

Consider the expression $T^{jk}n'_kP_j$ which appears in the integral in (7b). The functions n'_k are the components of a vector at a point of integration x' of σ and P_j are the components of a vector at x. Consider the vector at x' whose components in a normal coordinate system with origin x are P_j. Then P_j and n'_j are parallel. In fact, if the normal coordinate system is choosen such that σ is given by $t = 0$, both P_j and n'_j have only the zero component non null. We may therefore conclude that the expression $T^{jk}n'_kP_j$ is never negative:

$$T^{jk}n'_kP_j = cT^{jk}n'_jn'_k \geqslant 0,$$

where c is some positive constant.

Consider now $M^{ij}P_j$ at a boundary point $x \in \partial\tilde{U}$. Let l be the geodesic

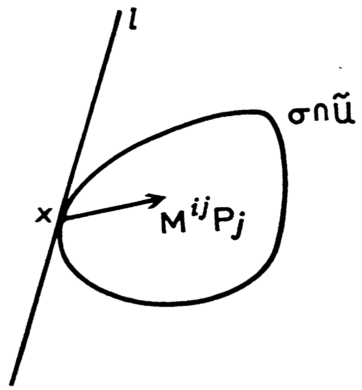

FIG. 1.

plane in σ tangent to $\partial\tilde{U}$ at x. By construction $\tilde{U} \cap \sigma$ lies to one side of l. We see from (7b) that the vector $M^{ij}P_j$ always lies on the same side of l as $\tilde{U} \cap \sigma$ and that it can never lie in l.

From this fact and from the fact that $M^{ij}P_j$ is always space-like, one deduces, again using the Brouwer fixed-point theorem, that any space-like section of $\tilde{U}$ contains a zero of $M^{ij}P_j$. We shall assume that this zero is unique. Beiglebӧck [4] has shown that this is so, if one places additional restrictions on the matter tensor. Denote by C the time-like curve formed by the zeros of $M^{ij}P_j$. This criteria for the center of mass was introduced by Pryce [5]. Since P^i and M^{ij} are of class $\mathcal{C}^1$ so is C. The curve C will be called the world-line of the center of mass. One sees from (7b) that in the Newtonian limit it reduces in fact to this line.

Apart from the assumptions relative to the positivity of the matter density: that $T^{ij}n_i n_j \geqslant 0$ and that $P^i(x; n)$ is a time-like vector for n_i time-like, we have made three physical assumptions in this section. They may be formulated as follows: if x and y are two points of $\tilde{U}$ which can be joined by a space-like geodesic, then this geodesic is unique; the map $n^i \rightarrow P^i/|P|$ has a unique fixed point; the vector $M^{ij}P_j$ has a unique zero on any space-like section of $\tilde{U}$. In general one could not expect these three assumptions to hold. Each constitutes a restriction on the strength of the field.

II

We shall now derive differential equations for P^i and M^{ij} along the curve C. As we shall never explicity use the defining relation for C, the same method may be used to derive differential equations for P^i and M^{ij} along any time-like curve in $\tilde{U}$. The basis for these equations is the integrability condition for the Einstein field equations, that is the conservation laws

$$T^{ij}{}_{;j} = 0. \tag{8}$$

For each point x of C we have the geodesic hypersurface σ through x determined in Section I such that its normal at x is parallel to P^i.

Let t be the geodesic parameter along C, chosen such that $t = 0$ is the point x, and let y be a point of C with $t > 0$. Let σ and σ' be respectively the geodesic hypersurfaces through x and y and V the part of $\tilde{U}$ between them. Choose a normal coordinate system x^i with center x and a normal

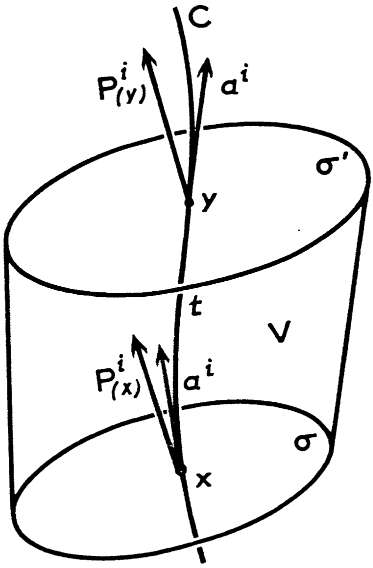

FIG. 2.

coordinate system y^i with center y. Choose the basis of y^i the basis of x^i transported parallely along C to y. This choice of basis at y gives

$$\frac{\partial y^i}{\partial x^j} = \delta^i_j + 0(t^2) \qquad \text{on C,}$$

a fact which will be useful later. Denote by a^i the unit tangent vector field to the curve C. Components of tensors with respect to the coordinate system x^i are unprimed; those with respect to the coordinate system y^i are primed. For convenience set

$$\lim_{t\to 0} \frac{dv}{t} = d\bar{\sigma}.$$

The coordinate transformation $y^i = f^i(x^j)$ contains t as a parameter. For $t = 0$ we have $f^i(x^j) = x^i$. Define μ^i by

$$y^i = x^i - ta^i - t\mu^i + 0(t^2), \tag{9}$$

where a^i are the components of the tangent to C at x in the normal coordinate system with origin x. One sees that μ^i is the part of $- \left.\frac{\partial f^i}{\partial t}\right|_{t=0}$ which depends on the curvature of the space. In a flat space $y^i = x^i - ta^i$ exactly. Differentiation with respect to x^i gives

$$\frac{\partial y^i}{\partial x^j} = \delta^i_j - t\mu^i_{,j} + O(t^2), \tag{10}$$

with

$$\mu^i_{,j} = 0 \qquad \text{on C.}$$

By definition, we have at y

$$P'^i(y) = \int_{\sigma'} T'^{ij} n'_j d\sigma.$$

Therefore we have

$$\frac{\partial y^i}{\partial x^j} P^j(y) = \int_{\sigma'} \frac{\partial y^i}{\partial x^k} T^{kj} n_j d\sigma. \tag{11}$$

Substituting the above expansion (10) for $\frac{\partial y^i}{\partial x^j}$ gives

$$\int_{\sigma'} T^{ij} n_j d\sigma = P^i(y) + t\int_{\sigma} \mu^i_{,k} T^{kj} n_j d\sigma + O(t^2). \tag{12}$$

Since the metric has been supposed to be of class $\mathcal{C}^1$ we have from the joining conditions $T^{ij}\nu_j = 0$, where ν_i is the unit normal to ∂U. The conservation laws (8) may be written as

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}T^{ij})_{,j} + \Gamma^i_{jk}T^{jk} = 0. \tag{13}$$

Integrating this over V and applying Green's theorem to the first term on the left-hand side gives

$$\int_{\sigma'} T^{ij} n_j d\sigma - P^i(x) + \int_V \Gamma^i_{jk} T^{jk} dV = 0. \tag{14}$$

Substituting the expression (12) for $\int_{\sigma'} T^{ij} n_j d\sigma$ in this equation gives

$$P^i(y) - P^i(x) + t\int_{\sigma} \mu^i_{,k} T^{kj} n_j d\sigma + \int_V \Gamma^i_{jk} T^{jk} dV = O(t^2). \tag{15}$$

Dividing by t and taking the limit $t \to 0$ gives the following differential equation for P^i:

$$\frac{dP^i}{dt} + \int_{\sigma} \mu^i_{,k} T^{kj} n_j d\sigma + \int_{\sigma} \Gamma^i_{jk} T^{jk} d\bar{\sigma} = 0. \tag{16}$$

This differential equation describes the evolution of P^i along the curve C.

An exactly similar proceedure leads to a differential equation for M^{ij}. The details are as follows. By definition, we have at y

$$P'(y)^{ij} = \int_{\sigma'} y^i T'^{jk} n'_k d\sigma. \tag{17}$$

Therefore

$$\frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^l}P^{kl} = \int_{\sigma'} y^i \frac{\partial y^j}{\partial x^l} T^{kl} n_k d\sigma. \tag{18}$$

Substituting the expansion of y^i and $\dfrac{\partial y^i}{\partial x^j}$ in this expression gives

$$\int_{\sigma'} x^i T^{jk} n_k d\sigma = P^{ij}(y) + ta^i P^j(y) + t\int_\sigma \mu^i T^{jk} n_k d\sigma + t\int_\sigma x^i \mu^j_{,l} T^{lk} n_k d\sigma + 0(t^2). \tag{19}$$

From (13) we arrive at the identity

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}x^i T^{jk})_{,k} - T^{ij} + x^i \Gamma^j_{kl} T^{kl} = 0. \tag{20}$$

Integrating this over V and applying Green's theorem to the first term on the left-hand side gives

$$\int_{\sigma'} x^i T^{jk} n_k d\sigma - P^{ij}(x) - \int_V T^{ij} dV + \int_V x^i \Gamma^j_{kl} T^{kl} dV = 0. \tag{21}$$

Therefore, using (19) we have

$$P^{ij}(y) - P^{ij}(x) + ta^i P^j(y) + t\int_\sigma \mu^i T^{jk} n_k d\sigma + t\int_\sigma x^i \mu^j_{,l} T^{lk} n_k d\sigma - \int_V T^{ij} dV + \int_V x^i \Gamma^j_{kl} T^{kl} dV = 0(t^2). \tag{22}$$

Dividing by t and taking the limit $t \to 0$ gives the following differential equation for P^{ij}:

$$\frac{dP^{ij}}{dt} + a^i P^j + \int_\sigma \mu^i T^{jk} n_k d\sigma + \int_\sigma x^i \mu^j_{,l} T^{lk} n_k d\sigma - \int_\sigma T^{ij} d\bar{\sigma} + \int_\sigma x^i \Gamma^j_{kl} T^{kl} d\bar{\sigma} = 0. \tag{23}$$

Taking the antisymmetrical part of this equation yields

$$\frac{dM^{ij}}{dt} + a^{[i}P^{j]} + \int_\sigma \mu^{[i} T^{j]k} n_k d\sigma + \int_\sigma x^{[i} \mu^{j]}_{,l} T^{lk} n_k d\sigma + \int_\sigma x^{[i} \Gamma^{j]}_{kl} T^{kl} d\bar{\sigma} = 0. \tag{24}$$

This differential equation describes the evolution of M^{ij} along the curve C.

III

In this section we derive a multipole expansion for the various integrals in equations (16) and (24), retaining terms up to the quadrupole order. We shall make constant use of the fact that the Γ^i_{jk} are the components of the affine connection in a normal coordinate system. Hence we have

$$\Gamma^i_{(jk, \ldots, l_n)} = 0 \text{ at } x,$$

and the following equations can be immediately seen to be satisfied at x:

$$\Gamma^i_{jk,n}\xi^{jk} = \frac{2}{3}\,\mathrm{R}^i_{jnk}\xi^{jk}, \tag{25a}$$

$$\Gamma^i_{jk,ln}\xi^{jkl} = \frac{1}{2}\,\mathrm{R}^i_{jnk;l}\xi^{jkl}, \tag{25b}$$

where ξ^{jk} and ξ^{jkl} are the components of arbitrary completely symmetric tensors.

First of all we calculate a multipole expansion for the integrals involving μ^i and $\mu^i{}_{,j}$. Let x' be a point in a neighbourhood of x (fig. 3) and let C′ be the geodesic joining x' and y with affine parameter s normalized in such a way that y is the point $s = 0$ and x' is the point $s = 1$. For all points x' near enough to σ, t can be chosen small enough that the geodesic is space-like and hence unique by the assumption made in Section I. Let x^i be the coordinates of the point x' in the normal coordinate system with origin x. We should logically designate the coordinates of this point by x'^i but since the coordinates of the point x are (0, 0, 0, 0) we drop the prime to alleviate the formulæ. Since the curve C′ is parameterized by s the coordinates $x^i(x'')$ of any point x'' on C′ may be considered as functions $x^i(s)$ of s. Since the metric has been assumed to be of class C^3 we may expand $x^i(s)$ in a Taylor series expansion about the point y, which is given by $s = 0$. We have therefore

$$x^i(s) = x^i(y) + s\frac{dx^i}{ds}(y) + \frac{s^2}{2\,!}\frac{d^2x^i}{ds^2}(y) + \frac{s^3}{3\,!}\frac{d^3x^i}{ds^3}(y) + f^i(s), \tag{26a}$$

where $f^i(s)$ is a function which vanishes as s^4 at $s = 0$. In particular for S = 1 we have the coordinates of the point x' :

$$x^i = x^i(y) + \frac{dx^i}{ds}(y) + \frac{1}{2\,!}\frac{d^2x^i}{ds^2}(y) + \frac{1}{3\,!}\frac{d^3x^i}{ds^3}(y) + f^i(1). \tag{26b}$$

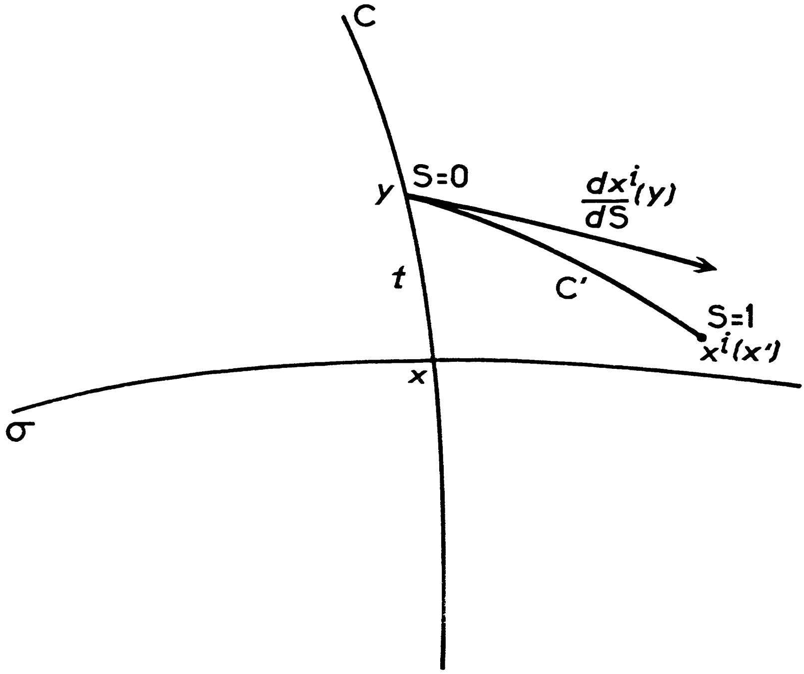

FIG. 3.

The coordinates $x^i(y)$ of the point y are given by

$$x^i(y) = a^i t + 0(t^2). \qquad (27)$$

By definition the functions $\frac{dx^i}{ds}(y)$, the components of the tangent to the curve C′ at the point y, are the coordinates of the point x' in the normal coordinate system with origin y. We have therefore

$$\frac{dx^i}{ds}(y) = y^i.$$

Since C′ is a geodesic, we have at the point y

$$\frac{d^2x^i}{ds^2}(y) = -\Gamma^i_{jk}(y)y^i y^j. \qquad (28)$$

But the $\Gamma^i_{jk}(y)$ are the components of the affine connection in a normal coordinate system with origin x. From (27) we have therefore

$$\Gamma^i_{jk}(y) = t\Gamma^i_{jk,l}a^l + 0(t^2).$$

Here and in the rest of this section if the point of valuation of a function is not given it is to be considered as the point x. Since we have

$$x^i = y^i + 0(t),$$

equation (28) may be written as

$$\frac{d^2x^i}{ds^2}(y) = -t\Gamma^i_{jk,l}x^jx^ka^l + 0(t^2).$$

A similar expression for $\frac{d^3x^i}{ds^3}(y)$ is obtained by differentiating the equation

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

and setting $s = 0$. Substituting these expressions in equation (26*b*) yields

$$y^i = x^i - ta^i + \frac{t}{2\,!}\Gamma^i_{jk,n}x^jx^ka^n + \frac{t}{3\,!}\Gamma^i_{jk,ln}x^jx^kx^la^n - f^i(1) + 0(t^2).$$

The functions $f^i(1)$ are of fourth order in the coordinates x^i and do not contribute to the quadrupole terms. Therefore in the following we shall neglect them.

Using the expressions (25) for the derivatives of the components of the affine connection at the point x one derives from its definition by formula (9), the following expression for μ^i:

$$\mu^i = \frac{2}{3\,!}R^i_{jkn}x^jx^ka^n + \frac{2}{4\,!}R^i_{jkn;l}x^jx^kx^la^n. \tag{29}$$

Using this we see that the second term in equation (16) is given by

$$\int_\sigma \mu^i_{,k}T^{kj}n_jd\sigma = \frac{2}{3\,!}R^i_{jkn}a^nP^{(jk)} + \frac{2}{4\,!}R^i_{jkn;l}a^nP^{(jkl)}. \tag{30}$$

The sum of the third and fourth terms in equation (24) is given by

$$\int_\sigma \mu^{[i}T^{j]k}n_kd\sigma + \int_\sigma x^{[i}\mu^{j]}_{,l}T^{lk}n_kd\sigma = \frac{2}{3\,!}M^{kl[j}R^{i]}_{kln}a^n. \tag{31}$$

Next we calculate a multipole expansion for the integrals involving the components of the affine connection. As with the previous calculations, we here retain only those terms which contribute to the multipoles up to and including the quadrupole. This means that we neglect all terms containing more than two factors x^i. We set for convenience

$$\omega^{i_1 \ldots i_n jk} = \int x^{i_1} \ldots x^{i_n} \mathrm{T}^{jk} d\bar{\sigma}.$$

Expanding Γ^i_{jk} about the point x, gives, for the last term in equation (16) the expression

$$\int_\sigma \Gamma^i_{jk} \mathrm{T}^{jk} d\bar{\sigma} = \Gamma^i_{jk,l} \omega^{ljk} + \frac{1}{2} \Gamma^i_{jk,lm} \omega^{lmjk}. \tag{32}$$

For the last term in equation (24), we obtain the expression

$$\int_\sigma x^{[i} \Gamma^{j]}_{kl} \mathrm{T}^{kl} d\bar{\sigma} = \Gamma^{[j}_{kl,mn} \omega^{i]mkl}. \tag{33}$$

We shall now find expressions for the terms on the right-hand side of equations (32) and (33) as functions of the linear combinations of multipole moments M^{ij} and M^{ijk}. For this we must use again the conservation laws. We find using (13) the following identity

$$\frac{1}{\sqrt{-g}} (\sqrt{-g} x^i x^j \mathrm{T}^{kl})_{,l} - x^{(i} \mathrm{T}^{j)k} + x^i x^j \Gamma^k_{lm} \mathrm{T}^{lm} = 0. \tag{34}$$

Integrating this over V, applying Green's theorem to the first term, dividing by t and taking the limit $t \to 0$ gives the following equation for $\omega^{(ij)k}$:

$$\frac{d\mathrm{P}^{ijk}}{dt} + a^{(i} \mathrm{P}^{j)k} - \omega^{(ij)k} = 0. \tag{35}$$

We do not give in detail the intermediate steps since they are exactly the same as those which lead from equation (20) to equation (23). Since ω^{ijk} is symmetric in its last two indices, it may be expressed in terms of $\omega^{(ij)k}$:

$$\omega^{jki} = \frac{1}{2} (\omega^{(ij)k} + \omega^{(jk)i} - \omega^{(ki)j}).$$

Therefore we have from (35) the following expression for ω^{kij}:

$$2\omega^{kij} = a^k \mathrm{P}^{(ij)} - a^{(i} \mathrm{M}^{j)k} - \frac{d\mathrm{M}^{ijk}}{dt}. \tag{36}$$

From this we find the following expression for the first term on the right-hand side of (32):

$$\Gamma^i_{jk,l}\omega^{ljk} = \frac{1}{2\,!}\mathrm{R}^i_{njk}a^n\mathrm{M}^{jk} + \frac{2}{3\,!}\mathrm{R}^i_{jkn}\frac{d\mathrm{M}^{jkn}}{dt} - \frac{2}{3\,!}\mathrm{R}^i_{jkn}a^n\mathrm{P}^{(jk)}. \tag{37}$$

Proceeding in a similar manner, from (13) we obtain the identity

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}x^ix^jx^k\mathrm{T}^{lm})_{,m} - x^{(i}x^j\mathrm{T}^{k)l} + x^ix^jx^k\Gamma^l_{mn}\mathrm{T}^{mn} = 0. \tag{38}$$

Just as (34) yielded us the equation (35), from this identity we obtain the following equation for $\omega^{(ijk)l}$:

$$\omega^{(ijk)l} = a^{(i}\mathrm{P}^{jk)l}. \tag{39}$$

This may be written as an equation for $\omega^{l(ijk)}$:

$$a^{(i}\mathrm{M}^{jk)l} - a^l\mathrm{P}^{(ijk)} + 2\omega^{l(ijk)} = 0. \tag{40}$$

We have here a situation which is more complicated than that above. Whereas the identity (34) was sufficient to completely determine ω^{ijk} in terms of the multipole moments, this is not true of (38). The quantity ω^{ijkl} is not completely determined. However to find the expression for the second term on the right-hand side of (32), the limited amount of information contained in equation (40) will be sufficient. In fact using the symmetry properties of the components of the affine connection, we have the following equalities:

$$\frac{1}{2}\Gamma^i_{jk,lm}\omega^{lmjk} = -\frac{1}{2}\Gamma^i_{lk,mj}\omega^{l(mjk)} = \frac{6}{4\,!}\mathrm{R}^i_{klm;j}\omega^{l(mjk)}. \tag{41}$$

Therefore from (40) we find the following expression for the second term on the right-hand side of (32):

$$\frac{1}{2}\Gamma^i_{jk,lm}\omega^{lmjk} = \frac{1}{3\,!}\mathrm{R}^i_{njk;l}a^{(n}\mathrm{M}^{l)jk} - \frac{2}{4\,!}\mathrm{R}^i_{jkn;l}a^n\mathrm{P}^{(jkl)}. \tag{42}$$

Placing (42) and (37) in (32) and then placing (32) and (30) in equation (16) of Section II yields the following differential equation for P^i correct to within terms containing multipole factors of order higher than the quadrupole:

$$\frac{d\mathrm{P}^i}{dt} + \frac{1}{2\,!}\mathrm{R}^i_{njk}a^n\mathrm{M}^{jk} + \frac{1}{3\,!}\mathrm{R}^i_{njk;l}a^{(n}\mathrm{M}^{l)jk} + \frac{2}{3\,!}\mathrm{R}^i_{njk}\frac{d\mathrm{M}^{njk}}{dt} = 0. \tag{43}$$

This differential equation describes the evolution of P^i along the curve C.

We have performed all of the calculations in a particular coordinate system, a normal coordinate system with origin x where x is an arbitrary given point of the curve C. In this coordinate system, since the components of the affine connection vanish at x, covariant differentiation and ordinary differentiation are equal at this point. Therefore all of the terms occuring in equation (43) are components of tensors if we stipulate that d/dt be covariant differentiation along the curve C. Equation (43) is thus a tensorial equation.

To find the corresponding equation for M^{ij} we must make an additional assumption if we wish to obtain a differential equation which contains only the multipole moments of the matter distribution. We found above that we were unable to obtain the quantity ω^{ijkl} in terms of a^i and the quadrupole moment P^{jkl}. The multipole moments involve the projection of the tensor T^{ij} onto the normal to the surface σ. That is, it is the vector $T^{ij}n_j$ which appears in the integrals in equations (1) and (2) defining the multipole moments. The quantity ω^{ijkl} is defined as an integral over σ containing the full tensor T^{ij}. The fact that $T^{ij}n_j$ does not in general determine T^{ij} is the reason why we cannot express ω^{ijkl} in terms of a^i and P^{jkl}.

If we multiply both sides of equation (39) by n_k we find the following equation:

$$(\omega^{ijlk} - P^{ijl}a^k)n_k = 0. \tag{44}$$

We recall that n_k is the unit normal to σ at the point x. What we shall assume is that the difference

$$\omega^{ijlk} - P^{ijl}a^k, \tag{45}$$

which by equation (44) is normal to n_k, is of higher order than the quadrupole. Taub was led to make an equivalent assumption when he considered the motion of a point quadrupole (see [*6*], formula (3.3)).

To find the equation for M^{ij} we need to calculate $\Gamma^j_{kl,m}\omega^{imkl}$, which appears on the right-hand side of equation (33). We write this expression as the sum of three terms:

$$\begin{aligned}\Gamma^j_{kl,m}\omega^{imkl} &= \Gamma^j_{kl,m}(\omega^{imkl} - \omega^{klim}) \\ &\quad + \Gamma^j_{kl,m}(\omega^{klim} - P^{kli}a^m) \\ &\quad + \Gamma^j_{kl,m}P^{kli}a^m.\end{aligned} \tag{46}$$

The first term on the right-hand side of this equation is easily calculated. Using (39) we have

$$\Gamma^{j}_{kl,m}(\omega^{imkl} - \omega^{klim}) = -\Gamma^{j}_{kl,m}\omega^{(ikl)m} = \frac{2}{3}\,R^{j}{}_{klm}a^{(i}P^{kl)m}. \qquad (47)$$

The second term is of higher order by assumption. Therefore we find the following expression for the term on the right-hand side of (33):

$$\Gamma^{[j}_{kl,m}\omega^{i]mkl} = \frac{2}{3\,!}\,R^{[i}{}_{klm}M^{j]n(k}a^{l)} - \frac{2}{3\,!}\,M^{kl[j}R^{i]}{}_{kln}a^{n}. \qquad (48)$$

We have introduced the extra term on the right-hand side of equation (46) because of the simplification thus obtained in the final formula. We remark that if the ratio v/c is much less than one where v is a typical velocity of the sources of the field with respect to the body whose movement we are considering, then the introduction of the extra term makes the error we have made by assuming that the difference (45) is of higher order, correspondingly small. In fact, if we consider a normal coordinate system at x such that σ is given by $t = 0$, then what the assumption we have made means is that to within higher order multipoles, it is only the $(k, l) = (0,0)$ components of ω^{ijkl} which do not vanish. Therefore the quantity

$$\Gamma^{j}_{kl,m}(\omega^{klim} - P^{kli}a^{m}) \approx \Gamma^{j}_{kl,0}(\omega^{kli0} - P^{kli})$$

is in general smaller by a factor v/c than the quantity

$$\Gamma^{j}_{kl,m}(\omega^{imkl} - P^{imk}a^{l}) \approx \Gamma^{j}_{00,\alpha}(\omega^{i\alpha 00} - P^{i\alpha 0})$$

because of the extra time derivative.

Substituting (48) into (33) and then substituting (33) and (31) into equation (24) of Section II yields the following differential equation for M^{ij} correct up to quadrupole terms:

$$\frac{dM^{ij}}{dt} + a^{[i}P^{j]} + \frac{2}{3\,!}\,R^{[i}{}_{kln}M^{j]n(k}a^{l)} = 0. \qquad (49)$$

This equation describes the evolution of M^{ij} along the curve C. As before with equation (43) it may be considered as tensorial with d/dt covariant differentiation along C.

We have ten equations, describing the evolution of the energy-momentum vector and the angular momentum tensor along the curve C. In addition we have a condition on C. The curve C was chosen such that the inner product of P^i and M^{ij} vanishes:

$$M^{ij}P_j = 0. \tag{50}$$

We have therefore in all 14 equations.

If we neglect the quadrupole moments we have exactly 14 unknowns: M^{ij}, P^i, a^i. In principle therefore we have a unique solution with given initial conditions [7]. In general if we cannot neglect the quadrupole terms then our system of equations is underdetermined.

If we multiply both sides of equation (49) by a_i we obtain the following relation between a^i and P^i:

$$P^i - P\cdot aa^i = \frac{dM^{ij}}{dt} a_j + \frac{2}{3\,!} R^i{}_{kln} M^{jn(k} a^{l)} a_j - \frac{2}{3\,!} R_{jkln} M^{ink} a^l a^j. \tag{51}$$

This relation was remarked by Papapetrou [8] in the case of vanishing quadrupole moment.

Although formally we have included the case where the body contributes significantly to the field in which it is moving, the series expansion in terms of multipoles we have given would probably be of little use in this case. The multipole expansion in the case of a body of dimension d would be a good approximation only if the wave lengths λ of the field in which the body is moving are much longer than d. That is, if the inequality $d/\lambda \ll 1$ is satisfied. The exact equations (16) and (24) of course remain valid and eventually these equations could be used to consider self-acceleration effects.

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