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Conformal covariance and invariant formulation of scalar wave equations

by

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ABSTRACT. — A new formulation of the scalar field equation or the Klein-Gordon equation in the presence of external tensor $g^{ij}(x)$, vector $A^i(x)$ and scalar $c(x)$ field is given, which is covariant with respect to gauge transformations $A_i(x) \to A_i(x) + \nabla_i \nu(x)$ and conformal transformations of the tensor field $g^{ij}(x) \to \exp[-\theta(x)]g^{ij}$, $\nu(x)$ and $\theta(x)$ are arbitrary functions of $x = (x^1, x^2, \ldots, x^n)$. In this case the rest mass square $m^2(x)$ defined as the function of given fields $m^2(x) = c - \frac{n-2}{4(n-1)} R - \frac{1}{4} A^i A_i - \frac{1}{2} \nabla_i A^i$, transforms as follows : $m^2(x) \to \exp[-\theta(x)]m^2(x)$. Here R is the scalar curvature, ∇_i-covariant derivatives in the Riemann space V_n with metric tensor $g_{ij}(x)$, the reciprocal of $g^{ij}(x)$, $A_i = g_{ij}A^j$. It is shown that in considering the class of the scalar wave equations with a constant mass one deals with the metrics $\bar{g}_{ij}(x) = m^2(x)g_{ij}(x)$ which depend implicitly on given tensor, vector and scalar fields. For a potential vector field $\nabla_i A_j = \nabla_j A_i$ one has the wave equation obtained earlier in ref. [1]

$$g^{ij}\nabla_i\nabla_j\phi + \frac{n-2}{4(n-1)} R\phi + m_0^2\phi = 0 \qquad (m_0 = \text{const})$$

which describes the free motion [2] in the Riemann space V_n and possesses correct group properties (contrary to the equation $g^{ij}\nabla_i\nabla_j\phi + m_0^2\phi = 0$ considered usually in extension of the relativistic quantum mechanics and quantum field theory for the case of space-time with nonvanishing curvature). As an example, the scalar field geometrization problem is considered for the relativistic spinless particle wave equation.

1. INTRODUCTION

In the present paper a new formulation of the equation of scalar field ϕ or the Klein-Gordon equation is considered in the presence of external tensor $g^{ij}(x)$, vector $A^i(x)$ and scalar $c(x)$ fields, $x = (x^1, x^2, \dots, x^n)$, $n \geqslant 2$.

Assuming the total covariance and homogeneity of the wave equations, we consider the linear homogeneous partial differential equations of second order

$$F(\phi) \equiv g^{ij}(x)\nabla_i\nabla_j\phi + A^i(x)\nabla_i\phi + c(x)\phi = 0, \tag{1}$$

where ∇_i are the covariant derivatives in the Riemann space V_n with metric

$$ds^2 = g_{ij}(x)dx^i dx^j \quad , \quad g_{jk}g^{ji} = \delta^i_k$$

(under the assumption of nondegeneracy of the metric tensor $g^{ij}(x)$). Eq. (1) may be written in the gauge invariant form (see Sec. 2)

$$g^{ij}\left(\nabla_i + \frac{1}{2}A_i\right)\left(\nabla_j + \frac{1}{2}A_j\right)\phi + \frac{n-2}{4(n-1)}R\phi + m^2(x)\phi = 0, \tag{1'}$$

putting the rest mass square of the particle $m^2(x)$ as the function of given tensor, vector and scalar fields:

$$m^2(x) \equiv c - \frac{n-2}{4(n-1)}R - \frac{1}{4}A^iA_i - \frac{1}{2}\nabla_iA^i.$$

The mass *vs* the scalar curvature R of the Riemann space V_n provides covariance of Eq. (1′) under the group C_g of all conformal transformations of the metric tensor $g_{ij}(x) \to \bar{g}_{ij}(x) = e^{\theta(x)}g_{ij}(x)$ (Sec. 3), with $m^2(x)$ transforming as $m^2(x) \to \bar{m}^2(x) = e^{-\theta(x)}m^2(x)$. The mapping

$$g_{ij} \to \bar{g}_{ij} = m^2(x)g_{ij}$$

gives the wave equation with a constant mass $\bar{m}^2(x) = \text{const} = m_0^2$

$$\bar{\Delta}_2\phi + \frac{n-2}{4(n-1)}\bar{R}\phi + \bar{A}^i\frac{\partial\phi}{\partial x^i} + \left(\frac{1}{4}\bar{A}^i\bar{A}_i + \frac{1}{2}\bar{\nabla}_i\bar{A}^i\right)\phi + m_0^2\phi = 0, \tag{2}$$

where $\bar{\Delta}_2 \equiv \bar{g}^{ij}\bar{\nabla}_i\bar{\nabla}_j$ is the Laplace-Beltrami operator and $\bar{R}$ is the scalar curvature [3] determined by the metric $\bar{g}_{ij} = m^2(x)g_{ij}$ of the Riemann space $\bar{V}_n$ conformal to V_n, $\bar{\nabla}_i$ are the covariant derivatives in $\bar{V}_n$. The form (2) is called the canonical form of Eq. (1). Thus, considering the class of wave equations with constant mass one deals with the metrics $\bar{g}_{ij}$ which depend implicitly on given tensor $g^{ij}(x)$, vector $A^i(x)$ and scalar $c(x)$ fields. As a particular case in Sec. 6 the scalar field geometrization problem is considered by an example of the relativistic spinless particle wave equation.

For a potential vector field $A_i(x)$ ($\nabla_i A_j = \nabla_j A_i$) form (2) provides the wave equation obtained earlier in ref. [*1*]

$$\bar{\Delta}_2\phi + \frac{n-2}{4(n-1)}\bar{R}\phi + m_0^2\phi = 0, \tag{3}$$

which corresponds [*2*] to the free motion in the Riemann space $\bar{V}_n$ with metric tensor $\bar{g}_{ij} = m^2(x)g_{ij}$ (Eq. (3) is invariant under the group of motion of $\bar{V}_n$ [*1*], see Sec. 4). In the case of conformally flat space-time $\bar{V}_4$ for the classical problem corresponding to the wave equation (3) one obtains the geodesic equation of motion in $\bar{V}_4$ (Sec. 6). Eq. (3) differs from the equation $\Delta_2\phi + m_0^2\phi = 0$ (commonly used on the flat space-time analogy as the Klein-Gordon equation in the space-time with a nonvanishing curvature R) by the term $(n-2)R/4(n-1)$. In fact, from the mathematical point of view the only natural extension of the Laplace-Beltrami operator Δ_2 for the case of an arbitrary Riemann space should be the invariant Laplace operator defined as follows [*1*]

$$\Delta \equiv \Delta_2 + (n-2)R/4(n-1).$$

The equation of scalar field with zero mass $m^2(x) \equiv 0$ (e. g. the equation describing the light propagation in a gravitational field g_{ij}, $n = 4$)

$$\Delta_2\phi + \frac{n-2}{4(n-1)}R\phi = 0 \tag{4}$$

is considered in Sec. 5. Eq. (4) is covariant with respect to the group C_c of conformal coordinate transformations and covariant under the group C_g of conformal transformations of the metric tensor (Maxwell's equations are known to be C_c-invariant and C_g-covariant, see Ref. [*4*]).

Our treatment of scalar wave equations (*) is mathematically based on the simple structure of finite transformations of symmetry groups allowed by Eq. (1) [*5*], which will be briefly outlined in what follows.

(*) The results of this work were previously presented in Ref. [*6*].

2. THE CANONICAL FORM OF Eq. (1)

Let us consider the equations equivalent to equation (1) (see Ref. [2] [5])

$$\overline{\mathrm{F}}(\phi) \equiv e^{-\theta(x)-\nu(x)}\mathrm{F}[e^{\nu(x)}\phi] = \overline{\Delta}_2\phi + \overline{\mathrm{A}}^i(x)\frac{\partial\phi}{\partial x^i} + c(x)\phi$$

$$= \overline{\Delta}_2\phi + \frac{n-2}{4(n-1)}\overline{\mathrm{R}}\phi + \overline{\mathrm{A}}^i\frac{\partial\phi}{\partial x^i} + \left(\frac{1}{4}\overline{\mathrm{A}}^i\overline{\mathrm{A}}_i + \frac{1}{2}\overline{\nabla}_i\overline{\mathrm{A}}^i\right)\phi + e^{-\theta(x)}m^2(x)\phi = 0 \quad (5)$$

Here $\overline{\Delta}_2$ is the Laplace-Beltrami operator and $\overline{\mathrm{R}}$ is the scalar curvature determined by the metric

$$\bar{g}_{ij} = \exp\,[\theta(x)]\cdot g_{ij} \qquad (6)$$

of the Riemann space $\overline{\mathrm{V}}_n$ conformal to V_n. $\theta(x)$ and $\nu(x)$ are arbitrary differentiable functions of x,

$$\begin{aligned} \overline{\mathrm{A}}^i &= e^{-\theta}\left[\mathrm{A}^i + g^{ij}\partial\left(2\nu + \frac{2-n}{2}\theta\right)\Big/\partial x^j\right], \\ \overline{\mathrm{A}}_i &= \bar{g}_{ij}\overline{\mathrm{A}}^j = \mathrm{A}_i + \partial\left(2\nu + \frac{2-n}{2}\theta\right)\Big/\partial x^i. \end{aligned} \qquad (7)$$

The function $m^2(x)$ is given by the tensor, vector and scalar fields associated with initial Eq. (1)

$$m^2(x) \equiv c - \frac{n-2}{4(n-1)}\mathrm{R} - \frac{1}{4}\mathrm{A}^i\mathrm{A}_i - \frac{1}{2}\nabla_i\mathrm{A}^i, \qquad (8)$$

where R is the scalar curvature of V_n, $\nabla_j\mathrm{A}^i$ and $\overline{\nabla}_j\overline{\mathrm{A}}^i$ are the covariant derivatives in V_n and $\overline{\mathrm{V}}_n$, respectively. Thus the form

$$ds^2 = g_{ij}(x)dx^i dx^j \qquad (9)$$

associated with Eq. (1) transforms into

$$\overline{ds}^2 = e^{\theta(x)}g_{ij}(x)dx^i dx^j \qquad (10)$$

Geometrically this means a one-to-one mapping of V_n onto $\overline{\mathrm{V}}_n$. According to (5) and (8) the function

$$\overline{m}^2(x) \equiv \bar{c} - \frac{n-2}{4(n-1)}\overline{\mathrm{R}} - \frac{1}{4}\overline{\mathrm{A}}^i\overline{\mathrm{A}}_i - \frac{1}{2}\overline{\nabla}_i\overline{\mathrm{A}}^i$$

characterizing the equivalent Eq. (5) will be

$$\overline{m}^2(x) = e^{-\theta(x)} m^2(x). \tag{11}$$

If V_n remains the same $m^2(x)$ does not change either.

Thus, contrary to scalar $c(x)$ the function $m^2(x)$ (obtained from $c(x)$ by substracting the scalars constructed by given tensor and vector fields) behaves as a scalar function not only under transformations $x' = x'(x)$ which evidently preserve V_n. The most general transformation

$$x' = x'(x), \qquad \phi' = e^{\nu(x)}\phi, \qquad F'(\phi) = e^{-\nu(x)}F[e^{\nu(x)}\phi]$$

preserving linearity, homogeneity and the Riemann space V_n, associated with Eq. (1), does not also affect $m^2(x)$ either (gauge invariance).

It enables us to suppose that $m^2(x)$ may be thought as a rest mass square and general Eq. (1) may be considered as the equation of scalar field with mass $m(x)$ or the Klein-Gordon equation in the Riemann space V_n with metric tensor $g_{ij}(x)$ in the presence of vector field with vector-potential $\frac{1}{2}A_i(x)$ ($h = c = e = 1$)

$$\Delta_2\phi + \frac{n-2}{4(n-1)} R\phi + A^i \frac{\partial\phi}{\partial x^i} + \left(\frac{1}{4} A^i A_i + \frac{1}{2} \nabla_i A^i\right)\phi + m^2(x)\phi = 0. \tag{12}$$

In particular the Klein-Gordon equation for a charged particle in a gravitational field g_{ij} (that is in a Riemann space V_4 with signature $+ 2$) will be

$$\left[\square + \frac{1}{6} R\right]\phi + A^i \frac{\partial\phi}{\partial x^i} + \left(\frac{1}{4} A^i A_i + \frac{1}{2} \nabla_i A^i\right)\phi + m^2(x)\phi = 0, \tag{13}$$

where $\square$ is the d'Alembert operator in a four-dimensional Riemann space V_4.

Providing $m^2(x)$ is a nonvanishing function of x, transformation (5) at $\theta = \log | m^2(x) |$ leads to the form, which will be referred to as a canonical form of Eq. (1)

$$\overline{\Delta}_2\phi + \frac{n-2}{4(n-1)} \overline{R}\phi + \overline{A}^i \frac{\partial\phi}{\partial x^i} + \left(\frac{1}{4} \overline{A}^i \overline{A}_i + \frac{1}{2} \overline{\nabla}_i \overline{A}^i\right)\phi = (-\operatorname{sign} m^2)\phi \tag{14}$$

given by tensor field $m^2(x)g_{ij}(x)$ and vector field $\overline{A}_i(x)$, the latter is defined (7) with an accuracy to a gradient of an arbitrary differentiable function $\nu(x)$ (gauge transformation).

From (7) it follows that if

$$F_{ij} \equiv \nabla_i A_j - \nabla_j A_i = 0 \qquad (i, j = 1, 2, \ldots, n) \tag{15}$$

transformation (5) at

$$v(x) = (n - 2)\theta(x)/4 - \frac{1}{2}\int A_i(x)dx^i$$

leads to an equation with $\overline{A}_i \equiv 0$. In this case the form (14) coincides with Eq. (3). Thus if the vector field $A_i(x)$ is a potential one, Eq. (1) is reduced to the invariant Klein-Gordon equation (3) given entirely by the tensor field $m^2(x)g_{ij}(x)$ [1] [2].

At $m(x) \equiv 0$ we shall obtain form (14) with the right-hand part equal to zero in an arbitrary conformal to V_n Riemann space $\overline{V}_n$ with metric $\bar{g}_{ij} = e^{\theta}g_{ij}$, θ is an arbitrary differentiable function of x. In a particular case $F_{ij} \equiv 0$ one has the following property of the invariant Laplace operator $\Delta \equiv \Delta_2 + (n - 2)R/4(n - 1)$.

If the Riemann spaces V_n and $\overline{V}_n$ are conformal, that is $\bar{g}_{ij} = e^{\theta(x)}g_{ij}$, than

$$\overline{\Delta}\phi \equiv \exp\left[-\frac{n+2}{4}\theta(x)\right]\Delta\left[\exp\left(\frac{n-2}{4}\theta(x)\right)\phi\right], \tag{16}$$

where Δ and $\overline{\Delta}$ are invariant Laplace operators in V_n and $\overline{V}_n$, respectively. As we shall see in what follows, this property is directly related to the fact that the equation $\Delta\phi = 0$ is invariant under the group of conformal coordinate transformations of a Riemann space V_n.

3. CONFORMAL COVARIANCE OF EQUATION (1)

Let us show that Eq. (1) given by a tensor field $g_{ij}(x)$, vector field $A^i(x)$ and scalar field $c(x)$ is covariant (form-invariant) under the group C_g of conformal transformations of the metric tensor

$$g_{ij}(x) \rightarrow \bar{g}_{ij}(x) = e^{\theta(x)}g_{ij}(x) \tag{17}$$

This question is of interest in view of the conformal covariance of the basic equations of classical physics [4]. The group C_g gives a one-to-one mapping of V_n with metric (9) onto $\overline{V}_n$ with metric (10).

Now let us condier a one-to-one correspondance between the mapping (17) and the equivalence transformation (5) with the same function $\theta(x)$ at $v(x) = (n - 2)\theta(x)/4$, that is,

$$F(\phi) \rightarrow \overline{F}(\phi) \equiv e^{-\frac{n+2}{4}\theta(x)}F\left[e^{\frac{n-2}{4}\theta(x)}\phi\right]. \tag{18}$$

Then according to (7) for the contravariant and covariant components of the vector field A we have

$$\overline{A}^i(x) = e^{-\theta(x)}A^i(x), \qquad \overline{A}_i(x) = A_i(x), \qquad (i = 1, 2, \ldots, n) \tag{19}$$

i. e., one of the two possible ways [4] of the vector field transformation under the group C_g of conformal transformations (17). Scalar $m^2(x)$ transforms like (11)

$$m^2(x) \to \overline{m}^2(x) = e^{-\theta(x)}m^2(x). \tag{20}$$

This result has its full counterpart in classical mechanics. Schouten and Haantjes [7] have shown that the Lorentz equation of motion of charged particle is covariant under conformal transformations, providing the rest mass is not invariant but transforms as (20). In the quantum theory this assumption leads to conformal covariance (18) of fundamental Eq. (1) (since conformal covariance is intimately connected with relativity, we have in mind here relativistic quantum mechanics and relativistic quantum field theory). On the other hand, the set of Eq. (1) on the totality of spaces $\overline{V}_n$ conformal to V_n and with the same tensor F_{ij} is a set of equivalent equations. The evidence of this theorem follows from (5), (7) and (20). From this point of view reduction of Eq. (1) to the canonical form (14) can be considered as a conformal point mapping of V_n onto such a $\overline{V}_n$ where the mass $m(x)$ becomes constant. The physical meaning of this mapping will be discussed in sec. 6.

4. INVARIANCE GROUPS OF EQUATIONS (1)

Definition. — The invariance group G/T of an equation (1) $F(x, \phi) = 0$ is the set of those substitutions of independent and dependent variables $x' = x'(x)$ and $\phi' = e^{\nu(x)}\phi$ that does not affect the appearance of the equation under consideration with an accuracy to some nonvanishing multiplier $\lambda(x)$, say $e^{-\theta(x)-\nu(x)}$. That is, the equation $F(x, \phi) = 0$ written in new variables x', ϕ' must be $F'(x', \phi') = \lambda(x')F(x', \phi')$.

Such a definition is obviously equivalent to the requirement that Eq. (1) written in new independent variables $x' = x'(x)$ coincide with one of Eq. (5) (on replacing x by x' in the latter). On substitution $x' = x'(x)$ Eq. (1) written as Eq. (5) at $\theta = \nu = 0$ will be

$$\Delta_2'\phi + \frac{n-2}{4(n-1)}R'\phi + A'^i\frac{\partial\phi}{\partial x'^i} + \left(\frac{1}{4}A'^iA'_i + \frac{1}{2}\nabla'_iA'^i\right)\phi + m^2(x(x'))\phi = 0 \tag{21}$$

Here Δ'_2 and R' are defined by the V_n-space metric

$$ds'^2 = g'_{ij}(x')dx'^i dx'^j$$

transformed to new variables $x' = x'(x)$. Comparison of the coefficients at the second-order derivatives in Eq. (21) and (5) gives

$$g'^{ij}(x') = e^{-\theta(x')}g^{ij}(x') \qquad (i, j = 1, 2, \ldots, n).$$

Thus the substitution $x' = x'(x)$ belonging to G/T must be conformal coordinate transformation (*), i. e.

$$ds'^2 = e^{\theta(x')}g_{ij}(x')dx'^i dx'^j \tag{22}$$

Thus Eq. (21) in the coordinate system (x') of the Riemann space V_n with metric (22) can be formally treated as Eq. (5) associated with the Riemann space $\overline{V}_n$ conformal to V_n with metric $\overline{ds}^2 = e^{\theta}g_{ij}dx^i dx^j$ (one should only substitute x' for x in Eq. (5). Then comparison of the coefficients of Eq. (21) and (5) gives the theorem [*2*] [*5*]:

The invariance group G/T of Eq. (1) is isomorphic to a subgroup of the group C_c of conformal coordinate transformations $x' = x'(x)$ of the Riemann space V_n with metric $g_{ij}(x)$ associated with Eq. (1), for which

$$m^2(x(x'))g'_{ij}(x') = m^2(x')g_{ij}(x')$$

and

$$A'_i(x') = A_i(x') + \partial\left(2\nu + \frac{2-n}{2}\theta\right)\Big/\partial x'^i,$$

that is

$$F'_{ij}(x') = F_{ij}(x').$$

Providing $m^2(x) \neq 0$, the set of those transformations $x' = x'(x)$ that leave tensor $m^2(x)g_{ij}(x)$ invariant is automatically a set of conformal transformations. Therefore, in this case the invariance group of Eq. (1) is the group of isometry of a Riemann space $\overline{V}_n$ with metric tensor $m^2 g_{ij}$. The case $m^2 \equiv 0$ will be discussed in the following section.

(*) The connection of group C_c of conformal coordinate transformations $x' = x'(x)$ in V and group C_g of conformal transformations (17) of the metric tensor is considered in Ref. [*4*]. The customary notations used in physics are employed here; x and x' refer to the same point seen by different coordinates (observers) S and S'. Note that the maximum number of parameters of group C_c at $n \geqslant 3$ in not in excess of $(n+1)(n+2)/2$ and achieved only for conformally flat spaces V_n.

5. THE EQUATION OF SCALAR FIELD WITH ZERO MASS

If $m^2 = 0$ Eq. (5) has the form

$$\Delta_2 \phi + \frac{n-2}{4(n-1)} R\phi + A^i \frac{\partial \phi}{\partial x^i} + \left(\frac{1}{4} A^i A_i + \frac{1}{2} \nabla_i A^i\right)\phi = 0. \tag{23}$$

In what follows we shall limit ourselves to the case of potential vector fields $F_{ij} = 0$. Then Eq. (23) is equivalent to the equation

$$\left[\Delta_2 + \frac{n-2}{4(n-1)} R\right]\phi = 0 \tag{4}$$

which will be termed as the invariant Laplace equation in the Riemann space V_n. Equation $\Delta_2 \phi = 0$ commonly thought [8] to be the wave equation in V_n, will be further referred to as the Laplace-Beltrami equation. Replacing $\Delta_2 \phi = 0$ by Eq. (4) for a wave equation in V_n is founded on the equivalence property (16) of Eq. (4) on a set of conformal spaces. Thus in a particular case of conformally flat spaces V_n the necessity to use Eq. (4) instead of the Laplace-Beltrami equation in V_n is apparent and related to the fact that usual Laplace-Beltrami equation in a flat space S_n proves to be equivalent to Eq. (4) in any space V_n conformal to S_n. With an arbitrary space V_n, the necessity to use Eq. (4) is accounted for by its invariance under the group C_c of all conformal coordinate transformations in V_n. Indeed, on substitution $x' = x'(x)$ belonging to C_c, Eq. (4) constructed from the metric $ds^2 = g_{ij}dx^i dx^j$ of V_n transforms to an equation, which can be considered to be the invariant Laplace equation in a Riemann space $\overline{V}_n$ with metric $\overline{ds}^2 = e^{\theta} ds^2$. By the definition of the conformal coordinate transformations C_c, in a primed coordinate system $(x')V_n$ has the metric

$$ds'^2 = e^{\theta(x')} g_{ij}(x') dx'^i dx'^j. \tag{24}$$

According to (16) this equation is equivalent to the original Eq. (4) in V_n (one should only substitute x' for x). Thus under the conformal coordinate transformation $x' = x'(x)$ (24) accompanied by the substitution

$$\phi' = \exp\left[\frac{n-2}{4}\theta(x)\right]\phi$$

Eq. 4 retains its appearance to within the factor exp $[-(n+2)\theta(x)/4]$, i.e. is invariant under Cc. Consequently, the linear homogeneous equation (4) is the natural extension of the equation $\Delta_2\phi = 0$ given in flat spaces S_n for an arbitrary Riemann space V_n, which retains the total covariance and invariance under the group C_c of conformal coordinate transformations. Note, that equation $\Delta_2\phi = 0$ in space V_n with metric g_{ij} is invariant only under the group of motions of the Riemann space $\bar{V}_n$ with metric

$$\bar{g}_{ij} = -\frac{n-2}{4(n-1)}\mathrm{R}g_{ij} \tag{25}$$

(providing that scalar curvature R of V_n is not zero). This note follows from the theorem of Sec. 4. Such transformations are evidently included in the transformations corresponding to (24). With $\mathrm{R} = \text{const} \neq 0$ the invariance groups of the invariant Laplace equation (4) and the Laplace-Beltrami equation $\Delta_2\phi = 0$ in the same Riemann space V_n are readily intercompared because in this case the latter is invariant under the group of motions of the Riemann space $\bar{V}_n$.

Note that while the suggested invariant Laplace equation (4) in any arbitrary Riemann space V_n is covariant under the group C_g of the conformal transformations of the metric tensor g_{ij}, the Laplace-Beltrami equation, say, in a Minkowski space ($\mathrm{R} = 0$), is covariant only under 15-parameter Lie group of the restricted conformal transformations considered by Cunningham and Bateman [*9*] [*10*]. These transformations transform flat space into flat space. Therefore the term $(n-2)\mathrm{R}/4(n-1)$ appearing under the equivalence transformations (16) (the only ones transforming equation $\Delta_2\phi = 0$ into $\bar{\Delta}_2\phi + \bar{c}(x)\phi = 0$ [*2*]) may be omitted. Note finally that in the case being considered ($m \equiv 0$) covariance of Eq. (4) under C_g is not related to any appropriate mass transformation (20).

6. THE KLEIN-GORDON EQUATION IN A RIEMANN SPACE V_n

On the analogy with the flat space-time case the Klein-Gordon equation in the Riemann space V_4 is commonly assumed to be

$$\Box\phi + m_0^2\phi = 0 \tag{26}$$

$c = h = 1$; $m_0 = \text{const}$ (see, e. g., Ref. [*4*] [*11*] [*12*]). We do suggest [*2*] [*5*] that the canonical form (3) of Eq. (1) at $F_{ij} = 0$ $(i, j = 1, 2, \ldots, n)$

$$\Delta_2\phi + \frac{n-2}{4(n-1)} R\phi + m_0^2\phi = 0 \tag{3}$$

should be used as the equation of scalar field or the Klein-Gordon equation in a Riemann space V_n with metric tensor g_{ij}. In the following Eq. (3) will be referred to as the invariant Klein-Gordon equation. In gravitational fields, i. e., in Rieman spaces V_4 with signature $+2$, the invariant Klein-Gordon equation (3) is

$$\Box\phi + \frac{1}{6} R\phi + m_0^2\phi = 0. \tag{3'}$$

This suggestion is based on the group properties of Eq. (3) which is invariant under the group of motions of V_n and covariant under group C_g of conformal transformations of the metric tensor g_{ij}. Generally speaking, the commonly used Eq. (26) does not possess either the first or the second, contrary to some claims [*4*], property.

Besides, the statement [*2*] that Eq. (3) is that of motion for a free particle in a Rieman space V_n agrees (as well as the requirement that the Klein-Gordon equation be conformally covariant) with the character of a classical particle motion in the scalar field. As a matter of fact, the relativistic wave equation for a particle with mass m_0 in a scalar field $\theta(x)$ is (see Ref. [*13*])

$$\left[\Box + m_0^2 e^{\frac{2\lambda}{m_0}\theta(x)}\right]\phi = 0 \tag{27}$$

where $\Box$ is the d'Alembert operator in the Minkowski space S_4 with metric g_{ij}, $h = c = 1$, λ is the coupling constant or the charge of a particle. The corresponding classical particle moves on a geodesic of the geometry $\bar{V}_4$ with metric [*14*]

$$\bar{g}_{ij} = \exp\left[\frac{2\lambda}{m_0}\theta(x)\right] g_{ij}$$

On the other hand, reduction (18) of Eq. (27) to the canonical form gives the invariant Klein-Gordon equation in the same Riemann space $\bar{V}_4$.

This example enables us to give some physical interpretation of scalar (8) which we identify with the rest mass square of a particle.

1. The rest mass $m(x)$ relating to Eq. (27) is by Eq. (8) equal to

$$m(x) = m_0 e^{\frac{\lambda}{m_0}\theta(x)}, \tag{28}$$

i. e., is the function of the scalar field $\theta(x)$.

2. Any scalar field $\theta(x)$ changes the space-time metric. This statement can be developed by reduction of Eq. (27) to the canonical form (in $\bar{V}_4$ with metric tensor $\bar{g}_{ij} = \exp\left[\frac{2\lambda}{m_0}\theta(x)\right]g_{ij}$) for which, by definition, $\bar{m}(x) = m_0 = \text{const}$. According to (28) this means that the scalar field vanishes and that the original Minkowski space-time S_4 conformally transforms to $\bar{V}_4$.

3. The physical reason of such a behaviour of the rest mass of a particle may be clarified in a particular case of the restricted conformal group C_0 transforming flat space into flat space. The group C_0 contains the acceleration transformations

$$x'^i = (1 + 2a^k x_k + x^2 a^2)^{-1}(x^i + a^i x^2) \qquad (i = 1, 2, 3, 4) \tag{29}$$

where $x^2 = x_k x^k$, $a^2 = a_k a^k$, a_k — the parameters, corresponding to the transitions from an inertial frame of reference to a uniformly accelerating frame of reference. Let us consider the Klein-Gordon equation in the Minkowski space S_4

$$\Box\phi + m_0^2\phi = 0 \tag{30}$$

On substitution (29) the metric tensor of S_4 transforms as follows (see, e. g., Ref [15])

$$ds'^2 = e^{\theta(x')}(dx_1'^2 + dx_2'^2 + dx_3'^2 - dx_4'^2), \quad e^{\theta(x')} = (1 + 2a^k x_k' + x'^2 a^2)^2. \tag{31}$$

In the primed frame of reference S' Eq (30) due to conformal covariance is equivalent to

$$[\Box + m_0^2 e^{-\theta(x')}]\phi = 0. \tag{32}$$

Then considering S' as an inertial frame of reference one obtains the following interesting application of the equivalence principle. An acceleration transformation (29) is equivalent to switching of some scalar field $\theta(x)$ (see Eq. (28)). Let us limit ourselves to a special choice of a^k in (29)), namely: $a^k = (0; 0, 0, g/2)$. Then

$$\bar{m}(x') = m_0 e^{-\frac{1}{2}\theta(x')} = m_0[1 - gz' + (g^2/4)(z'^2 - t'^2)]^{-1}$$

Providing that $gz' \gg (g^2/4)(z'^2 - t'^2)$, we have $\bar{m}(z') \simeq m_0(1 + gz')$.

Thus, a conformal transformation (29) corresponds to a change of the apparent force field acting on the particle and the mass transformation represent the corresponding change in the rest energy which takes account of the change in potential energy (cf. Ref. [4]).

7. CONCLUSION

The invariant Klein-Gordon equation (3) was originally obtained [*1*] as a canonical form of the generalized Schrœdinger equation

$$\Delta_2\phi + c(x)\phi = 0$$

in V_n. Transformation of the equation to the canonical form is essentially equivalent to some replacement of the dependent variable. It enables us to investigate the equation properties irrespective of the coordinate system, as the equation has the same form in all of them. In this case to study the Riemann space $\overline{V}_n$ properties becomes very important.

Knowledge of $\overline{V}_n$ geometry allows us by substituting independent variables (along with the dependent variable transformation accomplished in reducing the equation to the canonical form) to transform the equation to a more convenient form. This approach proves to be effective in solving concrete problems. Some solutions of the invariant Klein-Gordon equation have been investigated in multidimensional spaces of constant curvature (by an example of the discrete spectrum and the continuous spectrum of the multidimensional Coulomb problem [*16*]) and in the subprojective Riemann space V_4 (an exact solution of the « ladder approximation » Bethe-Salpeter equation for two scalar particles interacting by a massless scalar field [*17*]). Transformation to the canonical form can be also helpful in solving some other problems dealing with linear homogeneous partial differential equations of second order.

The above analysis of the group properties of general Eq. (1) enables us to state that the scalar wave equations usually introduced in extension of the relativistic quantum mechanics and quantum field theory for the case of an arbitrary space are not correct [*18*]. Such generalization should be treated in the light of the concept that real space-time geometry somehow affects the properties of elementary processes [*19*]. Suggested equations (3) and (4) possess the desired symmetry groups and are conformally covariant. The physical consequences of these properties agree, as far as we know, with the well-known results of the classical theory.

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