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Section A :
Physique théorique.

On the existence of gravitational fields which are stationary initially and finally

par

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SUMMARY. — We consider gravitational fields which are initially stationary and then become variable (and radiative) during a certain time interval. We list the conditions which must be satisfied in order that the field become again stationary. For comparison we consider the same problem in Maxwell theory (in special relativity).

We discuss in detail the special case in which 1) the two characteristic hypersurfaces Σ and Σ' marking the beginning and the end of the variable field have both equations of the form u = const in the same Bondi frame $x^\mu = (u, r, \theta, \varphi)$ and 2) in the region of the variable field the quantity σ^0 contains a finite number of time-dependent terms. We arrive at the following result: If the field has additionally axial symmetry, it is impossible to satisfy the first of the necessary conditions and consequently such a field cannot exist. On the contrary, it is possible to satisfy this condition as well as some of the subsequent conditions if the field has no axial symmetry.

RÉSUMÉ. — Nous considérons des champs gravitationnels qui sont initialement stationnaires et deviennent variables (et radiatifs) pendant un certain intervalle de temps. Nous établissons la liste des conditions qu'on doit satisfaire pour que le champ devienne de nouveau stationnaire. Nous considérons le même problème dans la théorie de Maxwell (en relativité restreinte).

Nous discutons en détail le cas spécial dans lequel 1) les deux hyper-

surfaces caractéristiques Σ et Σ' qui marquent le commencement et la fin du champ variable ont des équations de la forme u = const dans le même repère de Bondi $x^\mu = (u, r, \theta, \varphi)$ et 2) dans le domaine du champ variable la quantité σ^0 contient un nombre fini de termes variables. Nous arrivons au résultat suivant : si le champ est à symétrie axiale, il est impossible de satisfaire la première des conditions nécessaires, ce qui signifie qu'un tel champ n'existe pas. Par contre on peut satisfaire cette conditon, ainsi que quelques-unes des conditions suivantes, quand le champ n'est pas à symétrie axiale.

I. INTRODUCTION

In this paper we shall examine gravitational fields containing (gravitational) radiation. We shall consider also electromagnetic fields in special relativity, but this only for the purpose of obtaining suggestions how to approach the questions concerning gravitational radiation.

There are two possible ways to discuss electromagnetic radiation:

1) Direct calculation based on the general formula for the retarded solution, when the sources are given.

2) Asymptotic discussion of the field far from its sources for which we shall assume that they are confined permanently in a limited region of the 3-dimensional space. The discussion is based on the development of the field components in power series of $1/r$.

The second method of discussion is of little interest for the electromagnetic field itself. Its importance lies in the fact that it constitutes the model of the only « exact » method we have at present for studying gravitational radiation. A direct calculation of the gravitational radiation emitted by given material sources is at present possible only in the frame of the « linearised » Einstein theory. But we don't know yet whether or how the linearised theory could be made the first step of a satisfactory approximation method.

The asymptotic discussion of the field has drawn attention to a special class of solutions which are non-stationary and at the same time contain no radiation. The electromagnetic case has been discussed first [*1*] and it has been found that the coefficients $F^n_{\mu\nu}$ entering into the development of the field components,

$$F_{\mu\nu} = \sum_n \frac{1}{r^n} F^n_{\mu\nu}, \qquad n = 1, 2, 3 \ldots,$$

are in the non-radiative case polynomials of degree $n - 2$ in the retarded time $t - r$. The sources corresponding to fields of this type can be determined easily and it has been found that the source multipole of order n is a polynomial in t of degree $n - 1$.

In the gravitational case the discussion of non-radiative fields has lead to an essentially identical result [2, 3]: the coefficients of the development in power series of $1/r$ of $g_{\mu\nu}$ as well as of $R_{\mu\nu\rho\sigma}$ are polynomials in the retarded time u. In this case it is of course not possible to determine the corresponding sources exactly. However it seems very plausible that the sources of a gravitational non-radiative field will have non-radiative motions of a type similar to that found in the electromagnetic case.

Non-radiative motions of electromagnetic sources—e. g. an electric dipole increasing linearly with time—can in principle be obtained with the help of non-electromagnetic forces. But this could be achieved only during a limited time-internal. The simplest way to arrive at this conclusion is by remarking that with the time interval increasing indefinitely the non-radiative motion would require an infinite amount of energy. Non-radiative fields are of some interest but only for theoretical considerations.

The fields which are physically interesting and important are of the following type. The field is non-stationary and radiative during a certain time-interval, being stationary before and after this interval (fig. 1). The aim of the present paper is a preliminary discussion of the question whether gravitational fields of this type can exist in General Relativity.

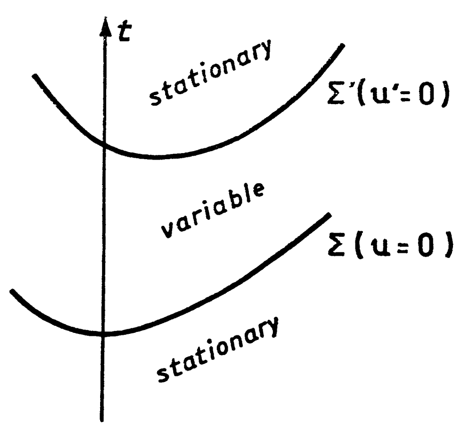

FIG. 1.

2. SOME GENERAL REMARKS

On figure 1 one sees at once that there are two regions of space-time which we must consider separately. Firstly we have the region A containing the initial stationary field and the neighbouring part of the variable one. The second region B contains the final stationary field and the part of the variable field neighbouring to it.

Evidently we shall have to use in A a Bondi frame [*4*] $x^\mu = (u, r, \theta, \varphi)$ adapted to the initial stationary field. Let us assume that the hypersurface Σ is determined by the equation $u = 0$, with $u < 0$ in the region of the initial stationary field. Then all field quantities will be independent of the retarded time coordinate u when $u \leqslant 0$. On the hypersurface Σ we have the propagation of the shock wave which represents the transition from the stationary to the variable field. Therefore Σ will depend on the details of the distribution of the material sources of the field as well as of the physical perturbation which causes the transition from the stationary to the variable state of these sources. This remark shows that the introduction of special simplifying assumptions conceining Σ shoud be avoided. Indeed such assumptions may be equivalent to the exclusion of physically interesting types of motion of the sources.

For a better understanding of the situation let us consider the following example. In the Minkowski space there are characteristic hypersurfaces —the light cones—with vanishing shear,

$$\sigma = 0. \tag{2, 1}$$

Similarly in a Riemannian space representing a stationary solution of the field equations

$$R_{\mu\nu} = 0$$

there are characteristic hypersurfaces with the property that the coefficient of the first term in the development of σ,

$$\sigma = \frac{\sigma^0}{r^2} + \ldots,$$

vanishes:

$$\sigma^0 = 0. \tag{2, 2}$$

Imposing on Σ a condition of the type (2, 1) of (2, 2) would be equivalent to restrictions imposed on the sources of the field. Indeed in the electromagnetic case (in Minkowski space) imposing the condition (2, 1) on Σ

would mean essentially that we consider the field of a single point charge. In the gravitational case the exact physical meaning of the condition (2, 2) is not clear. But there is no doubt that this or any other condition imposed on Σ would be equivalent to more or less strong restrictions imposed on the sources of the field.

In the region B we shall have to use a Bondi frame $x'^{\mu} = (u', r', \theta', \varphi')$ adapted to the stationary character of the field above the hypersurface Σ'. On this hypersurface we have the propagation of the shock wave corresponding to the transition from the variable to the final stationary field. The hypersurface Σ' will in general not belong to the family of characteristic hypersurfaces determined by the equation $u = \text{const}$. We shall consider only the case in which there are no other shock waves on any hypersurface between Σ and Σ'. The two frames (u, r, θ, φ) and $(u', r', \theta', \varphi')$ will then be connected by a Bondi-Metzner transformation.

In the gravitational case the transformation $x^{\mu} \rightarrow x'^{\mu}$ will contain necessarily a Lorentz part if the radiation emitted in the interval between Σ and Σ' has a non-vanishing total momentum (because of the recoil of the sources). When this momentum vanishes the transformation $x^{\mu} \rightarrow x'^{\mu}$ will reduce to a supertranslation and in special cases it may reduce to a simple translation or to the identical transformation. In the last case the hypersurface Σ' will belong to the family of hypersurfaces $u = \text{const}$ its equation being $u = u_1$ where u_1 is a positive constant.

Finally we have to mention the possibility of the surface Σ' being at infinite distance from Σ. In such a case the field would become again stationary asymptotically for $u \rightarrow \infty$. Since a shock wave on an infinitely distant hypersurface doesn't make sense, the transition from the variable to the final stationary field will in this case occur without any discontinuities. Therefore in this case the transformation $x^{\mu} \rightarrow x'^{\mu}$ will either contain a Lorentz part, if the total momentum of the radiation is non-zero, or it will be reducible to the identical transformation if the total momentum vanishes.

In this paper we shall examine in detail the special case of the hypersurface Σ' belonging to the family $u = \text{const}$, with the constant u_1 finite or infinite. The other cases will be considered only qualitatively in § 8.

3. THE NECESSARY AND SUFFICIENT CONDITIONS

For the description of the gravitational field we shall use the Newman-Penrose formalism [*5*, *6*]. We take over the notation as well as the results obtained in these two papers. Further we choose the two angular coordinates so that they reduce to the ordinary polar coordinates θ, φ at $r \to \infty$; i. e. we take as the asymptotic form of $g_{\mu\nu}$ for $r \to \infty$ the Minkowski metric

$$(ds^2)_{\mathrm{Mink}} = du^2 + 2dudr - r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

In our discussion we shall need explicitly the quantities σ and Ψ_A, $A = 0, 1, \ldots, 4$. They have developments of the form:

$$\sigma = \sum_n \frac{\sigma^n}{r^{n+2}}, \qquad n = 0, 1, 2\ldots, \tag{3, 1}$$

$$\Psi_A = \sum_n \Psi_A^n / r^{5-A+n}, \qquad n = 0, 1\ldots \tag{3, 2}$$

The coefficients σ^n and Ψ_A^n are functions of u, θ, φ.

The detailed discussion of the characteristic initial value problem has shown that the field equations do determine completely the field when we have chosen the coefficient σ^0 as a function of u, θ, φ and the coefficients Ψ_2^0, Ψ_1^0 and Ψ_0^n as functions of θ, φ on the hypersurface $u = 0$. The dependence of Ψ_2^0, Ψ_1^0 and Ψ_0^n on u will be determined by a set of equations of motion deduced from the field equations [7]:

$$\dot{\Psi}_2^0 = -\frac{1}{2}\eth\eth\dot{\bar{\sigma}}^0 - \sigma^0\ddot{\bar{\sigma}}^0, \tag{3, 3a}$$

$$\dot{\Psi}_1^0 = -\frac{1}{\sqrt{2}}\eth\Psi_2^0 + \sqrt{2}\sigma^0\eth\dot{\bar{\sigma}}^0, \tag{3, 3b}$$

$$\dot{\Psi}_0^0 = -\frac{1}{\sqrt{2}}\eth\Psi_1^0 + 3\sigma^0\Psi_2^0, \quad \dot{\Psi}_0^1 = -\frac{1}{\sqrt{2}}\bar{\eth}\eth\Psi_0^0 + 2\sqrt{2}\,\bar{\eth}(\sigma^0\Psi_1^0), \ldots \tag{3, 3c}$$

In these equations (1) a dot denotes differentiation with respect to u and the symbol $\eth$ is the angular differential operator defining the spin spherical harmonics [7]:

$$\dot{a} = \frac{\partial a}{\partial u},$$

$$\eth\chi = -\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} - s\frac{\cos\theta}{\sin\theta}\right)\chi, \tag{3, 4}$$

(1) In our formulae (3, 3) there is an extra factor $1/\sqrt{2}$ associated with each operator $\eth$ or $\bar{\eth}$, compared with the formulae of [7]. This is so because in [7] the Minkowski metric has been written in the form

$$(ds^2)_{\mathrm{Mink}} = 2du^2 + 2dudr - \frac{1}{2}r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

χ being of spin weight s. The right hand side of the equations of motion (3, 3c) for Ψ_0^n is well defined for every value n, but becomes very long and complicated with increasing n. All remaining field quantities will be determined by the other field equations in terms of σ^0, Ψ_2^0, Ψ_1^0 and Ψ_0^n. The equations (3, 3) are to be solved one after the other in the sequence indicated in (3, 3). This is so because the quantity Ψ_2^0 enters into the right-hand side of equation (3, 3b), the quantities Ψ_2^0 and Ψ_1^0 enter into the right-hand side of the first of equations (3, 3c) and so on.

The fact that the field is stationary below Σ will be expressed in the frame x^μ by the relations

$$(3, 5') \qquad \dot{\sigma}^0 = 0 \rightarrow \dot{\Psi}_2^0 = 0 \,;$$

$$(3, 5) \qquad \dot{\Psi}_1^0 = 0 \,; \qquad \dot{\Psi}_0^0 = \dot{\Psi}_0^1 = \ldots = 0$$

which are valid for $u \leqslant 0$. In order to obtain a field which will be stationary above Σ' we have to take, in the frame x'^μ and for $u' > 0$

$$(3, 6') \qquad \sigma'^0 = 0 \rightarrow \dot{\Psi}_2'^0 = 0.$$

and then we must demand, again for $u' > 0$, the following infinite set of conditions:

$$(3, 6a) \qquad \dot{\Psi}_1'^0 = 0 \,;$$

$$(3, 6b) \qquad \dot{\Psi}_0'^0 = \dot{\Psi}_0'^1 = \ldots = 0.$$

Actually we have to start with some initial stationary field which we have chosen and consequently the frame x^μ is to be considered as given *a priori*. On the contrary we have to determine the frame x'^μ and to satisfy the conditions (3, 6) at the same time. This is what makes the problem we are interested in so difficult in its general form.

In this paper we shall discuss in detail the special case in which the hypersurface Σ' belongs to the family $u = \text{const}$ and has the equation $u = u_1$. The conditions (3, 6) become then identical in form with (3, 5) which now must be demanded also for $u > u_1$.

For comparison we recapitulate briefly the corresponding situation in the Maxwell theory (in special relativity). Instead of the Ψ_A we now have the scalars Φ_B, $B = 0, 1, 2$, representing the electromagnetic field [5]. They have the developments

$$\Phi_B = \sum_n \Phi_B^n / r^{3-B+n}, \qquad n = 0, 1, 2, \ldots$$

The role of σ^0 is now played by Φ_2^0. The field will be determined by the field equations when we have chosen Φ_2^0 as a function of u, θ, φ and the coefficients Φ_1^0 and Φ_0^n $(n = 0, 1, 2, \ldots)$ as functions of θ, φ for $u = 0$. This is so because the Maxwell equations contain equations of motion for Φ_1^0 and Φ_0^n :

$$(3, 7a) \quad \dot{\Phi}_1^0 = -\frac{1}{\sqrt{2}}\eth\Phi_2^0,$$

$$(3, 7b) \quad \dot{\Phi}_0^0 = -\frac{1}{\sqrt{2}}\eth\Phi_1^0 + \sigma^0\Phi_2^0, \quad \dot{\Phi}_0^1 = -\frac{1}{\sqrt{2}}\eth\Phi_1^1 - \Phi_0^0 + \sqrt{2}\,\bar{\eth}\sigma^0\cdot\Phi_1^0, \ \ldots$$

The stationary character of the field below Σ is expressed in the frame x^μ by the following relations, valid for $u < 0$:

$$(3, 8') \qquad \Phi_2^0 = 0 \ \rightarrow \ \dot{\Phi}_1^0 = 0;$$

$$(3, 8) \qquad \dot{\Phi}_0^0 = \dot{\Phi}_0^1 = \ldots = 0.$$

In order to obtain a field which is stationary above Σ' we shall have to take, in the frame x'^μ and for $u' > 0$,

$$(3, 9') \qquad \Phi_2'^0 = 0 \rightarrow \dot{\Phi}_1'^0 = 0$$

and then we must demand, again for $u' > 0$, the infinite set of conditions

$$(3, 9) \qquad \dot{\Phi}_0'^0 = \dot{\Phi}_0'^1 = \ldots = 0.$$

In the special case of a hypersulface Σ' having the equation $u = u_1$ the conditions (3, 9) become identical with (3, 8), which must then be demanded also for $u > u_1$.

If we have only satisfied explicitly the first N of the conditions (3, 6) or (3, 9) we must in general expect that the field will be non-stationary above Σ', with non-radiative motions of the sources appearing in the multipoles of order $n \geqslant$ N. However the following example will show that the situation is essentially simpler in the electromagnetic case.

4. A SIMPLIFIED ELECTROMAGNETIC PROBLEM

We consider an electromagnetic field which is stationary for $u \leqslant 0$ as well as for $u \geqslant u_1$. We simplify further by assuming that the hypersurfaces $u =$ const have vanishing shear, $\sigma = 0$. The scalar Φ_2

has spin weight -1. Therefore the coefficient Φ_2^0 will be of the form

$$\Phi_2^0 = \bar{\eth} \sum_{lm} \alpha_{lm}(u) \cdot Y_l^m.$$

The Y_l^m are spherical harmonics and each term in the sum has spin weight zero. The definition of the operator $\bar{\eth}$ is

$$\bar{\eth}\chi = -\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + s\frac{\cos\theta}{\sin\theta}\right)\chi$$

if χ has spin weight s.

The Maxwell equations are linear and allow superposition of solutions. Therefore it will be sufficient to consider the case of a Φ_2^0 containing just one term:

$$\Phi_2^0 = \alpha_{lm}(u) \cdot \bar{\eth} Y_l^m \tag{4, 1}$$

with any given values of l and m satisfying $|m| \leqslant l$. The coefficient α_{lm} will be a function of u in the interval $0 \leqslant u \leqslant u_1$; it will vanish for $u \leqslant 0$ and for $u \geqslant u_1$. Besides specifying Φ_2^0 as a function of u, θ, φ we have to specify the quantities Φ_1^0 and Φ_0^n ($n = 0, 1, 2, \ldots$) as functions of θ, φ for $u \leqslant 0$. Again because of the linearity of the field equations one sees at once that it is sufficient to consider the case

$$\Phi_1^0 = 0 = \Phi_0^n \quad (n = 0, 1, \ldots) \qquad \text{for} \qquad u \leqslant 0. \tag{4, 2}$$

With $\sigma = 0$ the equations of motion (3, 7) are simplified as follows:

$$\dot{\Phi}_1^0 = -\frac{1}{\sqrt{2}}\eth\Phi_2^0; \quad \dot{\Phi}_0^0 = -\frac{1}{\sqrt{2}}\eth\Phi_1^0; \quad \dot{\Phi}_0^n = -\frac{1}{\sqrt{2}}\eth\Phi_1^n - \frac{n+1}{2}\Phi_0^{n-1} \quad \text{for} \quad n \geqslant 1. \tag{4, 3}$$

We shall need also the field equation determining Φ_1^n, $n \geqslant 1$ which is (when $\sigma = 0$):

$$n\Phi_1^n = \frac{1}{\sqrt{2}}\bar{\eth}\Phi_0^{n-1}. \tag{4, 4}$$

These equations enable us to determine the coefficients Φ_1^n and Φ_0^n. From the first of (4, 3) we have

$$\dot{\Phi}_1^0 = -\frac{1}{\sqrt{2}}\alpha_{lm}\eth\bar{\eth}Y_l^m.$$

We shall have to use the relations (see e. g. [7]):

(4, 5) $\eth\bar{\eth}_s Y_l^m = -(l+s)(l-s+1)_s Y_l^m, \qquad \bar{\eth}\eth_s Y_l^m = -(l-s)(l+s+1)_s Y_l^m$

where ${}_sY_l^m$ is the spin s spherical harmonic. Remembering that $Y_l^m = {}_0Y_l^m$ we find

$$\eth\bar{\eth}Y_l^m = -l(l+1)Y_l^m.$$

Therefore

$$\dot{\Phi}_1^0 = \frac{l(l+1)}{\sqrt{2}}\alpha_{lm}Y_l^m$$

and after integration

(4, 6) $$\Phi_1^0(u) = \frac{l(l+1)}{\sqrt{2}}\beta_{lm}(u)Y_l^m, \qquad \beta_{lm}(u) = \int_0^u \alpha_{lm}(u)du.$$

The second of (4, 3) takes now the form

$$\dot{\Phi}_0^0 = -\frac{l(l+1)}{2}\beta_{lm}(u)\cdot\eth Y_l^m$$

and so we find after integration

(4, 7) $$\Phi_0^0(u) = -\frac{l(l+1)}{2}\gamma_{lm}(u)\cdot\eth Y_l^m, \qquad \gamma_{lm}(u) = \int_0^u \beta_{lm}(u)du.$$

Now we get from (4, 4) with $n = 1$:

(4, 8) $$\Phi_1^1 = \frac{1}{\sqrt{2}}\bar{\eth}\Phi_0^0 = -\frac{l(l+1)}{2\sqrt{2}}\gamma_{lm}(u)\bar{\eth}\eth Y_l^m = \frac{l^2(l+1)^2}{2\sqrt{2}}\gamma_{lm}(u)Y^m$$

because of (4, 5).

The last equation (4, 3) gives for $n = 1$

$$\dot{\Phi}_0^1 = -\frac{1}{\sqrt{2}}\eth\Phi_1^1 - \Phi_0^0 = \frac{l(l+1)}{4}[1\cdot 2 - l(l+1)]\gamma_{lm}\eth Y_l^m.$$

Therefore after integration

(4, 9) $$\Phi_0^1(u) = \frac{l(l+1)}{4}[1\cdot 2 - l(l+1)]\delta_{lm}(u)\cdot\eth Y_l^m, \quad \delta_{lm}(u) = \int_0^u \gamma_{lm}(u)du.$$

In the same manner we calculate Φ_1^2 from Φ_0^1 using (4, 4) and then $\dot{\Phi}_0^2$ from the last of (4, 3). After integration we find

$$\Phi_0^2(u) = -\frac{l(l+1)}{16}[1\cdot 2 - l(l+1)][2\cdot 3 - l(l+1)]\varepsilon_{lm}(u)\eth Y_l^m, \quad \varepsilon_{lm}(u) = \int_0^u \delta_{lm}(u)du.$$

Proceeding in the same way we find that the expression for Φ_0^n with $n \geqslant 1$ contains as a factor the product

$$\prod_{n'}[n'(n'+1) - l(l+1)], \qquad n' = 1, 2, \ldots, n.$$

This result shows that when Φ_2^0 is of the form (4, 1) with given values of l and m, the conditions (3, 9) are satisfied automatically for $n \geqslant l$. Thus in order to obtain a field which is again stationary for $u \geqslant u_1$ we have to satisfy only the conditions

$$(4,\ 10) \qquad \dot{\Phi}_0^0 = \dot{\Phi}_0^1 = \ldots = \dot{\Phi}_0^{l-1} = 0 \qquad \text{for} \qquad u \geqslant u_1.$$

The conditions (4, 10) are immediately seen to be equivalent to the following l conditions on the function $\alpha_{lm}(u)$:

$$(4,\ 11) \quad \beta_{lm}(u_1) = \int_0^{u_1} \alpha_{lm} du = 0, \qquad \gamma_{lm}(u_1) = \int_0^{u_1} \beta_{lm} du = 0, \ \ldots$$

When $\alpha_{lm}(u)$ satisfies these conditions the field will be again stationary for $u \geqslant u_1$.

It is to be expected that a similar situation will present itself in the more general electromagnetic problem with the hypersurfaces $u =$ const having $\sigma \neq 0$: the infinite number of conditions (3, 9) will reduce to a finite number N (with N depending on l, m and the structure of σ^0), the remaining conditions (3, 9) being satisfied automatically. We did not check this point directly.

There is no doubt however that the situation will be basically different in the gravitational case. Because of the non-linearity of the field equations the conditions (3, 6) when written in detail become more and more complicated with increasing n and it is certain that none of them will be satisfied automatically. Thus it is inevitable to think that the totality of the conditions (3, 6) will only be satisfied in a restricted number of cases, if this will be possible at all.

5. THE GRAVITATIONAL CASE. GENERAL CONSIDERATIONS

In the remaining part of this work we shall consider the gravitational case. We start with the detailed discussion of the equations of motion (3, 3*a*) and (3, 3*b*).

In the region of the stationary field we shall have

$$\dot{\sigma}^0 = 0.$$

The equation of motion (3, 3*b*) has the conséquence that the condition (3, 6*a*) reduces to

$$\eth\Psi_2^0 = 0.$$

This equation shows that Ψ_2^0 will be independent of θ, φ. According to (3, 3*a*) Ψ_2^0 is also independent of u and therefore it will be a (complex) constant:

$$\Psi_2^0 = -\mathrm{M} + i\mathrm{M}', \tag{5, 1}$$

M and M′ two real constants. This is not the final form of Ψ_2^0 in the case of a stationary field. Indeed the field equations give for Ψ_2^0 the following additional relation:

$$\overline{\Psi}_2^0 - \Psi_2^0 = \frac{1}{2}(\eth\eth\bar{\sigma}^0 - \bar{\eth}\bar{\eth}\sigma^0) + \sigma^0\dot{\bar{\sigma}}^0 - \bar{\sigma}^0\dot{\sigma}^0. \tag{5, 2}$$

In the stationary case (5, 2) reduces to

$$\overline{\Psi}_2^0 - \Psi_2^0 = \frac{1}{2}(\eth\eth\bar{\sigma}^0 - \bar{\eth}\bar{\eth}\sigma^0). \tag{5, 2a}$$

The quantity σ^0 has spin weight 2 and therefore it will be of the form

$$\sigma^0 = \eth\eth\sum_{lm}\alpha_{lm}(u)\mathrm{Y}_l^m \quad ; \quad l \geqslant 2, \ -l \leqslant m \leqslant l. \tag{5, 3}$$

(Terms with $l < 2$ do not contribute to σ^0.) The coefficients α_{lm} are functions of u in general, reducing to constants in the stationary case.

Let us calculate the quantity appearing in the right-hand side of (5, 2*a*):

$$\bar{\eth}\bar{\eth}\sigma^0 = \bar{\eth}\bar{\eth}\eth\eth\sum_{lm}\alpha_{lm}Y_l^m, \qquad l \geqslant 2.$$

Taking into account that

$$\eth Y_l^m \sim {}_1Y_l^m, \qquad Y_l^m = {}_0Y_l^m$$

and using (4, 5) we find

$$(5,\ 4) \quad \bar{\eth}\bar{\eth}\sigma^0 = -\bar{\eth}\eth\sum_{lm}(l-1)(l+2)\alpha_{lm}Y_l^m = \sum_{lm}(l-1)l(l+1)(l+2)\alpha_{lm}Y_l^m.$$

The conjugate-complex of this equation is

$$(5,\ 4a) \qquad \eth\eth\bar{\sigma}^0 = \sum_{lm}(l-1)l(l+1)(l+2)\bar{\alpha}_{lm}\bar{Y}_l^m.$$

The validity of (5, 4) and (5, 4*a*) is not restricted to stationary fields.

Introducing (5, 4), (5, 4*a*) and (5, 1) in (5, 2*a*) we find

$$(5,\ 5) \qquad -4iM' = \sum_{lm}(l-1)l(l+1)(l+2)\left(\bar{\alpha}_{lm}\bar{Y}_l^m - \alpha_{lm}Y_l^m\right).$$

N·w $\bar{Y}_l^m = Y_l^{-m}$ and consequently

$$(5,\ 5a) \qquad 4iM' = \sum_{lm}(l-1)l(l+1)(l+2)(\alpha_{lm} - \bar{\alpha}_{l-m})Y_l^m.$$

Remembering that in the right-hand side of (5, 5*a*) we have $l \geqslant 2$ we conclude:

$$(5,\ 6) \qquad M' = 0 \quad ; \quad \alpha_{lm} = \bar{\alpha}_{l-m}.$$

Thus for a stationary field we have

$$(5,\ 7a) \qquad \Psi_2^0 = -M,$$

$$(5,\ 7b) \qquad \sum_{lm}\alpha_{lm}Y_l^m = \overline{\left(\sum_{lm}\alpha_{lm}Y_l^m\right)}:$$

Ψ_2^0 is a real constant and the generating function of σ^0 is real. We recall that M is the total energy of the stationary field.

The next equation of motion is the first of (3, 3c). For a stationary field it reduces to the relation

$$\eth\Psi_1^0 - 3\sqrt{2}\sigma^0\Psi_2^0 = 0.$$

This relation determines, by an argument analogous to the preceeding one, the structure of Ψ_1^0. Similarly the second of the equations (3, 3c) determines the structure of Ψ_0^0 and so on. We shall not discuss these equations in the present work.

We shall now determine the quantity $\Psi_2^0(u)$ in the interval $0 < u < u_1$ by integrating the equation of motion (3, 3a). We first rewrite this equation in the form

$$(5, 8) \qquad \dot{\Psi}_2^0 = -\left(\frac{1}{2}\eth\eth\bar{\sigma}^0 + \sigma^0\dot{\bar{\sigma}}^0\right)^{\cdot} + \dot{\sigma}^0\dot{\bar{\sigma}}^0.$$

Integrating this equation from $u = 0$ to any $u > 0$ we find:

$$\Psi_2^0(u) - \Psi_2^0(0) = \int_0^u \dot{\sigma}^0\dot{\bar{\sigma}}^0 du + \left(\frac{1}{2}\eth\eth\bar{\sigma}^0 + \sigma^0\dot{\bar{\sigma}}^0\right)_{u=0} - \left(\frac{1}{2}\eth\eth\bar{\sigma}^0 + \sigma^0\dot{\bar{\sigma}}^0\right)_{u=u}.$$

Remembering that $\dot{\sigma}^0 = 0$ for $u = 0$ and putting

$$\Psi_2^0(0) = -\mathrm{M}_0$$

we find finally:

$$(5, 9) \quad \Psi_2^0(u) + \mathrm{M}_0 = \int_0^u \dot{\sigma}^0\dot{\bar{\sigma}}^0 du + \left(\frac{1}{2}\eth\eth\bar{\sigma}^0\right)_{u=0} - \left(\frac{1}{2}\eth\eth\bar{\sigma}^0\right)_{u=u} - (\sigma^0\dot{\bar{\sigma}}^0)_{u=u}.$$

For $u \geqslant u_1$ the field will be again stationary. Therefore

$$\Psi_2^0(u_1) = -\mathrm{M}_1,$$

M_1 being the total energy of the final stationary field. Putting $u = u_1$ in (5, 9) we get

$$(5, 10) \qquad \mathrm{M}_0 - \mathrm{M}_1 = \int_0^{u_1} \dot{\sigma}^0\dot{\bar{\sigma}}^0 du + \left(\frac{1}{2}\eth\eth\bar{\sigma}^0\right)_{u=0} - \left(\frac{1}{2}\eth\eth\bar{\sigma}^0\right)_{u=u_1}.$$

The left-hand side of this relation is a physically meaningful constant: it is the loss of energy of the system because of the gravitational radiation emitted in the interval $0 < u < u_1$. The quantities appearing in the right-hand side of (5, 10) will in general depend on θ, φ. Therefore in order to have (5, 10) satisfied we must demand that the terms of (5, 10) which depend on θ, φ vanish. This will give us the first necessary conditions for the existence of an initially and finally stationary gravitational field of the special type we have described at the end of § 2.

In the next two sections we shall discuss these conditions in detail, first for an axially symmetric field and then for a field without any symmetry.

6. THE AXIALLY SYMMETRIC GRAVITATIONAL FIELD

All field quantities are now independent of the angle φ. Therefore the sum in the right-hand side of (5, 3) will contain only terms with $m = 0$:

$$\sigma^0 = \eth\eth \sum_l \alpha_l(u) Y_l^0 = \eth\eth \sum_l \alpha_l(u) P_l(\cos\theta), \tag{6, 1}$$

P_l being the Legendre polynomials.

We are interested in a field which is non-stationary in the interval $0 < u < u_1$. We must therefore assume that

$$\dot{\alpha}_l \neq 0 \qquad \text{for} \qquad 0 < u < u_1 \tag{6, 2}$$

for at least one of the coefficients α_l appearing in (6, 1). If there is in (6, 1) a term $\alpha_{l'}$ such that

$$\dot{\alpha}_{l'} = 0 \text{ everywhere,} \tag{6, 3}$$

one sees at once that this term will not give any contribution to the right-hand side of equation (5, 10). It follows that this equation will not give any restriction on the terms of the type (6, 3) and consequently σ^0 can contain any number of them. We shall write

$$\sigma^0 = \eth\eth\Big(\sum_l \alpha_l P_l + \ldots\Big), \tag{6, 4}$$

the terms in $\sum_l$ being of the type (6, 2) and the omitted terms of the type (6, 3). Finally we shall introduce a last restrictive assumption: we assume that the sum $\sum_l$ in (6, 4) contains a finite number of terms;

i. e. that the terms of the type (6, 2) have values l with a maximum :

$$l \leqslant l_{\max} \equiv \mathrm{L}. \tag{6, 5}$$

In the axisymmetric case the discussion of equation (5, 10) will be simplified if we use the functions $\cos^l\theta$ instead of the Legendre polynomials P_l. Putting $\cos\theta = z$ we find immediately from Legendre's equation

$$\eth\bar{\eth}\mathrm{P}_l = (1 - z^2)\frac{d^2}{dz^2}\,\mathrm{P}_l. \tag{6, 6}$$

Since P_l is a polynomial in z of degree l, it follows from (6, 6) that $\eth\bar{\eth}\mathrm{P}_l$ will be again a polynomial of degree l in z. We can therefore write σ^0 in the form

$$\sigma^0 = \sum_l \beta_l(u)\cos^l\theta + \ldots \qquad \text{with} \qquad l \leqslant \mathrm{L}. \tag{6, 7}$$

The new coefficients β_l are linear combinations of the coefficients α_l in (6, 1). In particular the coefficent β_{L} is equal to α_{L} multiplied by a non-vanishing numerical factor (depending on L). A similar reasoning leads to the formula

$$\bar{\eth}\bar{\eth}\sigma^0 = \sum_l \gamma_l(u)\cos^l\theta + \ldots \qquad \text{with} \qquad l \leqslant \mathrm{L}. \tag{6, 8}$$

The coefficients $\gamma_l(u)$ will be linear combinations of the β_l. We shall not need the explicite relations between γ_l and β_l.

Introducing (6, 7) and (6, 8) in (5, 10) we see at once that the right-hand side of (5, 10) will be of the form $\sum_l \mathrm{A}_l\cos^l\theta$ with $0 \leqslant l \leqslant 2\mathrm{L}$. Therefore equation (5. 10) will lead to the relation

$$\mathrm{M}_0 - \mathrm{M}_1 = \mathrm{A}_0 \tag{6, 9}$$

and to the conditions

$$\mathrm{A}_l = 0 \qquad \text{for} \qquad 0 < l \leqslant 2\mathrm{L}. \tag{6, 10}$$

Let us consider the last of these conditions,

$$\mathrm{A}_{2\mathrm{L}} = 0. \tag{6, 11}$$

One sees immediately that a term proportional to $\cos^{2L}\theta$ is contained only in the first term of the right-hand side of (5, 10) and that

$$A_{2L} = \int_0^{u_1} \dot{\beta}_L \dot{\bar{\beta}}_L du. \tag{6, 12}$$

Therefore from (6, 11) we conclude that

$$\dot{\beta}_L = 0 \text{ everywhere.} \tag{6, 13}$$

But this is contrary to our hypothesis: if we accept (6, 13) we shall have $\dot{\sigma}^0 = 0$ and consequently the field will contain no radiation.

We have thus proved the non-existence of axially symmetric fields of the type we are considering, i. e. of fields having the following 3 properties:

1) The field is axially symmetric.

2) In a global frame x^μ the field is time-dependent and radiative in the interval $0 < u < u_1$ and stationary for $u \leqslant 0$ and $u \geqslant u_1$.

3) The generating function of σ^0 contains a finite number of time-dependent terms.

7. GRAVITATIONAL FIELDS WITHOUT AXIAL SYMMETRY

We shall now consider a field without axial symmetry, but again satisfying the last two assumptions we listed at the end of § 6. We may announce beforehand the puzzling result to which we shall arrive: when we drop the property of axial symmetry we can construct gravitational fields satisfying (5, 10).

The general case with σ^0 of the form (5, 3) would be very tiresome. Since we are here interested in the question of existence of fields of this type, it will be sufficient to consider a special case requiring simpler calculations. The special case we shall discuss in detail is the one in which the formula (5, 3) for σ^0 contains time dependent terms corresponding to $l = 2$ only. I. e. we shall consider a field in which

$$\sigma^0 = \eth\eth \sum_m \alpha_m(u) \cdot Y_2^m + \ldots, \qquad -2 \leqslant m \leqslant 2. \tag{7, 1}$$

The omitted terms, which are independent of u, may correspond to any values of l: We noticed already that time-independent terms do not give any contribution to the right-hand side of equation (5, 10). The coefficients α_m in (7, 1) are complex fonctions of u. In the intervals $u \leqslant 0$ and $u \geqslant u_1$, where the field is stationary, the coefficients α_m will be constants satisfying the second of equations (5, 6):

$$\alpha_m = \bar{\alpha}_{-m} = \text{const.} \tag{7, 2}$$

From (5, 4a) and (7, 1) we get

$$\eth\eth\bar{\sigma}^0 = 24 \sum_m \bar{\alpha}_m \bar{Y}_2^m + \ldots, \tag{7, 3}$$

the omitted terms corresponding to time-independent coefficients α_{lm}. Consequently the last two terms in (5, 10) give

$$\frac{1}{2}(\eth\eth\bar{\sigma}^0)_{u=0} - \frac{1}{2}(\eth\eth\bar{\sigma}^0)_{u=u_1} = -12 \sum_m [\bar{\alpha}_m] \bar{Y}_2^m \tag{7, 4}$$

with

$$[\bar{\alpha}_m] \equiv (\bar{\alpha}_m)_{u=u_1} - (\bar{\alpha}_m)_{u=0}. \tag{7, 5}$$

We now have to calculate the first term in the right-hand side of (5, 10). This requires some less simple calculations. We shall describe them briefly, giving only the more important intermediate results.

Remembering that

$$Y_l^m = P_l^{|m|} e^{im\varphi}, \quad P_l^{|m|} = (1-z^2)^{|m|/2} \frac{d^{|m|}}{dz^{|m|}} P_l, \quad P_l = \frac{1}{2^l l!} \frac{d^l}{dz^l} (z^2-1)^l \tag{7, 6}$$

we find immediately that $\eth\eth Y_l^m$ will be of the form

$$\eth\eth Y_l^m = A_l^m e^{im\varphi}, \tag{7, 7}$$

where A_l^m is a real function of θ. Here we need the quantities $A_2^m \equiv A^m$ which are to be found by direct calculation. The result is

$$\left\{ \begin{aligned} A^0 &= 3(1 - z^2), \\ A^{\pm 1} &= \pm 6(1 \mp z)\sqrt{1 - z^2}, \\ A^{\pm 2} &= 6(1 \mp z)^2. \end{aligned} \right. \tag{7, 8}$$

From (7, 1) and (7, 7) we find

(7, 9) $$\dot{\sigma}^0 = \sum_m \dot{\alpha}_m A^m e^{im\varphi}, \quad -2 \leqslant m \leqslant 2.$$

It follows:

(7, 9a) $$\dot{\bar{\sigma}}^0 = \sum_m \dot{\bar{\alpha}}_m A^m e^{im\varphi},$$

(7, 10) $$\int_0^{u_1} \dot{\sigma}^0 \dot{\bar{\sigma}}^0 du = \sum_{m,m'} A^m A^{m'} e^{i(m-m')\varphi} \int_0^{u_1} \dot{\alpha}_m \dot{\bar{\alpha}}_{m'} du.$$

The products $A^m A^{m'}$ will be calculated from (7, 8). It is easy to see that they are of the form

(7, 11) $$A^m A^{m'} = \sum_{l'} a_{l'}^{mm'} P_{l'}^{|m-m'|}, \qquad |m - m'| \leqslant l' \leqslant 4,$$

the $a_{l'}^{mm'}$ being numerical factors. With (7, 11) we get from (7, 10):

(7, 12) $$\int_0^{u_1} \dot{\sigma}^0 \dot{\bar{\sigma}}^0 du = \sum_{mm'l'} a_{l'}^{mm'} Y_{l'}^{m-m'} \int_0^{u_1} \dot{\alpha}_m \dot{\bar{\alpha}}_{m'} du.$$

Introducing now (7, 4) and (7, 12) into (5, 10) we get an equation of the form

(7, 13) $$M_0 - M_1 = \sum_{ml'} b_{ml'} Y_{l'}^m ; \qquad 0 \leqslant l' \leqslant 4, \ -l' \leqslant m \leqslant l'.$$

The $b_{ml'}$ are linear combinations of the integrals $\int_0^{u_1} \dot{\alpha}_m \dot{\bar{\alpha}}_{m'} du$ and the differences $[\alpha_m]$ with numerical coefficients. From (7, 13) we deduce the conditions

(7, 14) $$b_{ml'} = 0 \quad \text{for} \quad 0 < l' \leqslant 4 \quad \text{and} \quad -l' \leqslant m \leqslant l$$

and the relation which determines M_1:

(7, 15) $$M_0 - M_1 = b_{00}.$$

In order to arrive at the explicite form of the conditions (7, 14) we have to calculate the coefficients $a_{l'}^{mm'}$. For this it is sufficient to calculate the products $A^m A^{m'}$, starting from (7, 8) and expressing the result in terms of the Legendre functions $P_l^{|m|}$. The final results are listed in the appendix.

With the values of $a_{l'}^{mm'}$ determined in this way we can calculate the quantities $b_{ml'}$ and thus arrive at the explicite form of the conditions (7, 14). We shall give here directly the final results.

For $l' = 1$, 3 and 4 and all possible values of m $(-l' \leqslant m \leqslant l')$ we get 8 complex conditions containing the integrals

$$\int_0^{u_1} \dot{a}_m \dot{\bar{a}}_{m'} du \equiv \mathrm{B}_{mm'} \tag{7, 16}$$

with $m \neq m'$ and 3 real conditions containing the same integrals with $m = m'$. These equations are:

$$\left\{\begin{array}{llll} \mathrm{B}_{2-2} = \mathrm{B}_{2-1} = \mathrm{B}_{1-2} = 0\,; & & & \\ \mathrm{B}_{20} \;= \mathrm{B}_{0-2} = -\,\mathrm{B}_{1-1}, & & & \\ \mathrm{B}_{10} \;= \mathrm{B}_{0-1} = 2\mathrm{B}_{21} = 2\mathrm{B}_{-1-2}\,; & & & \\ \mathrm{B}_{11} \;= \mathrm{B}_{-1-1}, & \mathrm{B}_{22} = \mathrm{B}_{-2-2}, & \mathrm{B}_{00} = 8(\mathrm{B}_{11} - \mathrm{B}_{22}). & \end{array}\right. \tag{7, 17}$$

For $l' = 2$ we get 2 complexe and 1 real conditions containing the differences $[\bar{\alpha}_m]$ and integrals $\mathrm{B}_{mm'}$. These conditions are:

$$[\bar{\alpha}_{-2}] = 2\mathrm{B}_{20}, \quad [\bar{\alpha}_{-1}] = -8\mathrm{B}_{21}, \quad [\bar{\alpha}_0] = 24\mathrm{B}_{11} - 4\mathrm{B}_{00}. \tag{7, 18}$$

Finally we get for $l' = 0$ the relation

$$\mathrm{M}_0 - \mathrm{M}_1 = \frac{24}{5}(\mathrm{B}_{00} + 12\mathrm{B}_{11} + 48_{22}) \tag{7, 19}$$

expressing the total energy of the radiation emitted in the interval $0 < u < u_1$ in terms of the (real, positive) quantities B_{00}, B_{11} and B_{22}.

These conditions are not prohibitive: one can easily construct as an example a system of 5 functions $\alpha_m(u)$ satisfying all these conditions. We thus have the result that the condition (5, 10) can be satisfied when the axial symmetry has been dropped.

We close this section with some remarks on the results we have obtained. The first remark is that we can simplify the field we considered in (7, 1), and still have the possibility to satisfy (5, 10). Indeed if we take

$$\dot{\alpha}_2 = \dot{\alpha}_{-2} = 0 \text{ everywhere}$$

the conditions (7, 17), (7, 18) reduce to

$$\left\{\begin{array}{l} \mathrm{B}_{1-1} = \mathrm{B}_{10} = \mathrm{B}_{0-1} = 0, \quad \mathrm{B}_{00} = 8\mathrm{B}_{11} = 8\mathrm{B}_{-1-1}\,; \\ \qquad [\bar{\alpha}_{-1}] = 0, \quad [\bar{\alpha}_0] = 24\mathrm{B}_{11} - 4\mathrm{B}_{00} = -\,\mathrm{B}_{00}. \end{array}\right. \tag{7, 20}$$

These conditions, imposed now on the 3 fonctions α_{-1}, α_0 and α_1, can also be satisfied.

No other simplification of (7, 1) is possible. Indeed, if we take

$$\dot{\alpha}_1 = \dot{\alpha}_{-1} = 0,$$

we find from the last of (7, 17)

$$B_{00} = -8B_{22}.$$

Since B_{00} and B_{22} are non-negative quantities, it follows $B_{00} = B_{22} = 0$. But then we shall have $\dot{\sigma}^0 = 0$ and the field will contain no radiation. Similarly if we take $\dot{\alpha}_0 = 0$ then we have from the last of equations (7, 17) and (7, 18):

$$B_{11} = B_{22} = 0,$$

i. e. again the field will contain no radiation.

8. CONCLUDING REMARKS

The next question to be examined is whether in the case of a field without axial symmetry we can satisfy more of the conditions (3, 6). This has been done by Hallidy [8] for a slightly more special case. Hallidy assumes that $\sigma^0 = 0$ for $u < 0$ and $u > u_1$. In the interval $0 \leqslant u \leqslant u_1$ he assumes that σ^0 is of a form similar to (7, 1):

$$\sigma^0 = \eth\eth \sum_m \alpha_{lm}(u) \cdot Y_l^m$$

with an arbitrarily chosen value of l ($l \geqslant 2$). He then imposes the conditions (3, 6*a*) and the first two of (3, 6*b*) in the stronger form

$$\Psi_1^0 = \Psi_0^0 = \Psi_0^1 = 0 \quad \text{for} \quad u < 0 \quad \text{and} \quad u > u_1.$$

I. e. he demands the spherical symmetry of the initial as well as of the final stationary field in the approximation he is considering. When these conditions have been worked out in detail they lead to a number of conditions on the functions $\alpha_{lm}(u)$. Hallidy proves, by constructing a special example, that these conditions can be satisfied. Remembering that the quantity Ψ_0^1 is the one entering into the conservation law of Newman and Penrose [7] we see that in the example treated by Hallidy this law has been satisfied explicitly [1].

Let us return to the case of axial symmetry. Evidently the question of the existence of axially symmetric gravitational fields which are stationary initially and finally is an important one. The negative result we found in § 6 does not give the complete answer to this question. Indeed the case we have discussed was a special one. In order to answer the question generally we have to complete the discussion by examining the following possibilities.

1) The hypersurface Σ' (fig. 1) does not belong to the family of hypersurfaces u = const.

2) The expression (6, 4) or (6, 7) for σ^0 contains an infinite number of time-dependent terms.

Actually it is almost certain that it will be impossible to satisfy the infinite set of conditions (3, 6) with the hypersurface Σ' at a finite distance from Σ. It seems more reasonable to expect that the field will become again stationary—if this is possible at all—only at an infinite distance from Σ. But then the field would tend to the stationary state in the asymptotic manner we described briefly in § 2. If the radiation carries away a non-vanishing total momentum, the frame $x'^\mu = (u', r', \theta', \varphi')$ will be obtained from the initial frame $x^\mu = (u, r, \theta, \varphi)$ by a Bondi-Metzner transformation containing a Lorentz part. Because of the necessity of such a transformation this case will be rather difficult to discuss.

However there is one special case in which a transformation of this type will not be needed: if the field is axially symmetric (independent of φ) and is also symmetric with respect to the hyperplane $\theta = \pi/2$ the total momentum of the radiation will be necessarily zero and consequently there will be no recoil of the sources. Therefore in this case—which is evidently interesting from the physical point of view—the field will become asymptotically stationary in the initial frame (u, r, θ, φ). It is easy to see that it will be impossible to find a field of this type when one starts with a σ^0 containing a finite number of time-dependent terms. The reason is the following. The proof we gave in § 6 for the case in which Σ' had the equation $u = u_1$ remains valid when $u_1 \to \infty$. Therefore what is left for discussion is the case of a field having the two symmetries we mentionned and a σ^0 which contains an infinite number of time-dependent terms. This

(1) Hallidy has discussed also the case of an axially symmetric gravitational field, again under the restriction $\sigma^0 = 0$ for $u < 0$ and $u > u_1$, and arrived at the result we established at the end of § 6. A similar result has been obtained by Unt [9].

case seems to be relatively easier to discuss, but it has not been discussed yet.

We conclude by stressing once again the important difference between the gravitational and the electromagnetic problem (in special relativity). If we start with an initially stationary gravitational field and wish to make it stationary again, we have to satisfy an infinite number of non-linear conditions. On the contrary in the electromagnetic case the conditions are linear and their number essentially finite. Therefore we must expect to meet, in the gravitational case, with situations which may appear strange if we compare them with what we know in the electromagnetic case. A first example of this kind is given by the negative result we found in § 6 about axially symmetric gravitational fields.

APPENDIX

We give here the formulae obtained from (7, 8) for the products $A^m A^{m'}$ where

$$-2 \leqslant m, m' \leqslant 2:$$

$$A^{\pm 2}A^{\pm 2} = \frac{288}{35}(\ P_4 \mp 7P_3 + 20P_2 \mp 28P_1 + 14),$$

$$A^{\pm 1}A^{\pm 1} = \frac{144}{35}(-2P_4 \pm 7P_3 - 5P_2 \mp 7P_1 + 7),$$

$$A^0 A^0 = \frac{24}{35}(\ 3P_4 - 10P_2 + 7);$$

$$A^{\pm 2}A^{\pm 1} = \frac{72}{35}(-P_4^1 \pm 7P_3^1 - 20P_2^1 \pm 28P_1^1),$$

$$A^{\pm 1}A^0 = \frac{12}{35}(\ 3P_4^1 \mp 7P_3^1 - 10P_2^1 \pm 42P_1^1);$$

$$A^{\pm 2}A^0 = \frac{12}{35}(\ P_4^2 \mp 7P_3^2 + 20P_2^2),$$

$$A^1 A^{-1} = \frac{24}{35}(\ P_4^2 - 15P_2^2);$$

$$A^{\pm 2}A^{\mp 1} = \frac{12}{35}(\ P_4^3 \mp 7P_3^3);$$

$$A^2 A^{-2} = \frac{12}{35} P_4^4.$$

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