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ABSTRACT. — The kinetic theory of particles distributed in velocity
space is formulated in terms of stochastic theory of continuous parametric
systems. Using the multiple-point product densities, the kinetic equations
are derived for particles obeying different types of statistics.

1. INTRODUCTION

The equilibrium properties of ideal gases both classical and quantum
are described by different distribution functions for the number of particles
in each energy region or velocity interval in the phase-space. In the classical
case, the well known Maxwell-Boltzmann distribution which is independent
of the character of the system depend only on the parameters like the
temperature and density of the system. In quantum statistics there are
two possible forms of the equilibrium distribution dictated by the symmetry
of the wave functions of the particles involved. The experimental results
decisively prove that electrons, protons and neutrons are described by

Fermi-Dirac distribution and assemblies of photons, pions and He^4 must follow the Bose-Einstein distribution. There are a number of alternate ways in which the choice of the statistics of a particle may be expressed. A system of N particles distributed over the energy space can be described by a set of numbers n_i corresponding to each energy level. Anti-symmetry condition on the wave function demands that n_i can be either 0 or 1 only and symmetric wave functions correspond to any of the values 0, 1, 2, ... Of course, the Schrödinger equation for the N-body system allows a host of solutions other than symmetric or anti-symmetric. In terms of occupation numbers, some of the solutions correspond to intermediate possibility that n_i assum only one of the values 0, 1, 2, ..., or p. These are called intermediate statistics. Ingenious combinatorial methods [1]-[3] have been devised to derive the equilibrium distribution corresponding to these situations.

However, there is an altogether different approach to arrive at the equilibrium distribution reflecting the different symmetry conditions by considering the detailed dynamics of the problem. This is achieved by considering the collision term in the Boltzmann equation, which yields the change in the occupation number of a given state in detail according to the type of the particles. Thus we can derive the equilibrium distributions assuming microscopic reversibility for the cross-sections for collisions. Such an attempt has been made by Moyal [4] who has obtained the energy spectrum of the particles on the assumption that the energy levels over which the particles are distributed are discrete. Moyal attempted this problem by assuming that the particles are distributed in discrete levels of energy and took into account the full dynamics of the problem. However, as pointed out in a recent contribution by one of us [5] the E-space or velocity space over which the particles are distributed is a continuum and as such there does not exist any comprehensive probability distribution function capable of describing the system adequately. Thus we are naturally led to the product density [6] formulation for describing the system. Moyal himself has pointed out in one of the discussions in [4], that the approach of the type used here should be more appropriate to discuss the case where the energy spectrum is continuous. This is possible now due to introduction of multiple product densities which come naturally in the description of different types of statistics. However, a slight departure from the mode of description of reference [5] is necessary to introduce the statistics of the physical systems. This is done by dealing with the random variables $dN(v, t)$ themselves rather than the product densitie s which stand for the expectation value of one or more of the variables $dN(v, t)$.

In section 2, we define the product densities and multiple densities that are necessary for the description of different occupancy conditions relating to our problem. The characteristic functionals of the stochastic variables are then introduced. In section 3, the Maxwell distribution is derived using the characteristic functionals. Other types of equilibrium distribution for different occupancy conditions are also arrived at using the Boltzmann equations for the stochastic variables $dN(v, t)$ themselves and the dynamics of the collision processes. The final section deals with ome general results and the possible connection to parastatistics.

2. PRODUCT DENSITIES

As has been emphasised in reference [5], the technique of product densities is a powerful method of description of statistical mechanical system. Though the importance of the notion of product densities has been amply realised in the cascade theory of cosmis ray showers [7], it has not been very much used in statistical mechanics until very recently.

The basic idea is that since the particles are distributed in a continuum, it is fairly easy to deal with the random variable $dN(x, t)$ denoting the number of particles with parametric values in the elemental range $(x, x + dx)$ at t. Here t stands for the continuous parameter with respect to which the process progresses or evolves. One of the characteristic features of the random variable $dN(x, t)$ is that the significant contributions to the probability magnitudes arise from only two of the values namely 0 and 1 assumed by $dN(x, t)$. This fact suggests that the description in terms of $dN(x, t)$ might be useful for dynamical systems obeying Fermi-Dirac statistics. To describe other types of occupancy it is found necessary to deal with an extended definition of the product densities formulated by us (see references [8] and [9]) in connection with the solution of the statistical problem of population growth in which « twins » and « multiples » arise. This leads us to the concept of multiple point product densities which can be defined with reference to the random variable $dN_k(x, t)$ representing the number of « k-tuples » in the range $(x, x + dx)$. We are generally interested in the total number of particles and denoting this by $dN(x, t)$ we find

$$(2.1) \qquad\qquad dN(x, t) = \Sigma k dN_k(x, t)$$

where the summation over k is to be extended upto the maximum order of the tuple that is allowed. It is to be understood that the different multiple points in the x-space are to be treated as different species and that

the variables $dN_k(x, t)$; $k = 1, 2, \ldots$, enjoy exactly the same properties as the original $dN(x, t)$ does. Thus we can define the multiple point product densities by

$$(2.2) \qquad \varepsilon\{ dN_j(x, t)\} = f_1^j(x, t)dx$$

$$(2.3) \qquad \varepsilon\{ dN_i(x_1, t)dN_j(x_2, t)\} = f_2^{i,j}(x_1, x_2, t)dx_1dx_2$$

where dx_1 and dx_2 do not overlap. We can also relate this to the total product densities defined by

$$(2.4) \qquad \varepsilon\{ dN(x, t)\} = f_1(x, t)dx$$

$$= \sum j f_1^j(x, t)dx$$

$$(2.5) \qquad \varepsilon\{ dN(x_1, t)dN(x_2, t)\} = f_2(x_1, x_2, t)dx_1dx_2$$

$$= \sum_{ij} ij f_2^{i,j}(x_1, x_2, t)dx_1dx_2$$

where the summation over i and j are over the admissible multiplicities. We wish to emphasise here that the occurrence of a j-tuple is to be viewed as a single multiple point and as such it has a probability proportional to dx while the occurrence of two multiple points whose orders of multiplicity add to j is of a smaller order of magnitude as compared to dx.

3. ENERGY SPECTRUM OF PARTICLES OBEYING DIFFERENT SYMMETRIES

We shall present in this section a dynamical method of arriving at the equilibrium distribution function corresponding to Maxwell-Boltzmann, Fermi-Dirac, Bose-Einstein and intermediate statistics. The distribution corresponding to the first three types of symmetries have been obtained by Moyal for discrete energy and velocity. We shall present a realistic method of arriving at the energy spectrum taking into account the continuous nature of the dynamical variables and also extend these results to particles obeying intermediate statistics. Let $dN(v, t)$ be the number of particles in the range dv at time t. Clearly $dN(v, t)$ is the random variable enjoying the properties mentioned in section 2. We can set up generalised Boltzmann equation for the product densities in an exactly the same manner as in reference [5]. The changes in the distribution are caused by binary collisions in which two particles with velocities in the ranges $(u, u + du)$, $(v, v + dv)$ acquire velocities in the ranges $(r, r + dr)$, $(s, s + ds)$ with the probability $\lambda_{uv, rs}drds$. If the particles obey *Maxwell-Boltzmann statistics*,

there is no restriction on the number of particles for the occupancy of each velocity range dv.   Thus n_i can take all positive integral values in the case of Maxwell-Boltzmann statistics and hence is composed of multiple points in v-space of *all* orders:

$$(3.1) \qquad dN(v, t) = \sum_{j=1}^{\infty} j dN_j(v, t).$$

Thus the probability of collision that carries over the two particles one of which is in the range $(v, v + dv)$ the other being in $(u, u + du)$ into the ranges $(s, s + ds)$, $(r, r + dr)$ is proportional to the product of the $dN(v, t)dN(u, t)$.   In the case of Bose-Einstein statistics, any number of particles can occupy a particular range dv as in the case of Maxwell-Boltzmann particles.   However, the collision probability will depend on the number of particles that occupy the ranges $(r, r + dr)$ and $(s, s + ds)$.

If, on the other hand, the particles obey *Fermi statistics*, $dN(v, t)$ can take the values 0 and 1 only and hence mean values of $dN(v, t)$ and their products will lead to simple product densities.   Apart from this, the collision probability will also contain additional factors to ensure that in a binary collision, the particles acquire velocities only in those ranges which are not already occupied.   We now proceed to write down the dynamical equations.

a) Maxwell-Boltzmann statistics.

It is clear that we have to deal with individual $dN_i(v, t)$ rather than the total $dN(v, t)$ and then compose $dN_i(v, t)$ with appropriate weights to form $dN(v, t)$.   Even though $dN_i(v, t)$'s are random variables, there is a certain regularity in the dynamics of collisions due to the conservation of probability.   Using the same notation of section 2 to denote the expectation values of the random variables $dN_i(v, t)$ and studying the changes that occur in the time interval $(t, t + \Delta)$, we obtain

$$(3.2) \qquad \frac{\partial}{\partial t} f_1(v, t) = -\sum_{i} \int_u \int_r \int_s i f_2^{i,1}(u, v ; t) \lambda_{uv,rs} dr ds du$$

$$+ \sum_{ij} \int_u \int_r \int_s ij f_2^{i,j}(r, s; t) \mu_{uv,rs} du dr ds$$

$$+ 2 \sum_{i} \int_u \int_r \int_s i f_2^{i,2}(u, v ; t) \mu_{uv,rs} dr ds du \quad (^1)$$

$(^1)$ $\lambda_{uv,rs} = \mu_{rs,uv}$

$$(3.3) \qquad \frac{\partial}{\partial t} f_1^2(v, t) = - 2 \sum_i i \int_u \int_r \int_s f_2^{i,2}(u, v ; t)\lambda_{uv,rs} dr ds du$$

$$+ 3 \sum_i i \int_u \int_r \int_s f_2^{i,3}(u, v ; t)\mu_{uv,rs} dr ds du$$

$$(3.4) \qquad \frac{\partial}{\partial t} f_1^3(v, t) = - 3 \sum_i i \int_u \int_r \int_s f_2^{i,3}(u, v ; t)\lambda_{uv,rs} dr ds du$$

$$+ 4 \sum_i i \int_u \int_r \int_s f_2^{i,4}(u, v ; t)\mu_{uv,rs} dr ds du.$$

From the above chain of equations, it is easy to obtain the equation for the total $f_1(v, t)$ as

$$(3.5) \qquad \frac{\partial}{\partial t} f_1(v, t) = - \int_u \int_r \int_s f_2(u, v ; t)\lambda_{uv,rs} dr ds du$$

$$+ \int_u \int_r \int_s f_2(r, s ; t)\lambda_{rs,uv} du dr ds.$$

We assume complete « Stosszahlansatz » and impose the principle of microscopic reversibility. These are expressed by

$$(3.6) \qquad f_m(v_1, v_2, \ldots, v_m, t) = f_1(v_1, t)f_1(v_2, t), \ldots, f_1(v_m, t)$$

$$(3.7) \qquad \lambda_{uv,rs} = \lambda_{rs,uv}.$$

Equation (3.6) expresses a valid assumption for rarefied systems or gases without any long range interaction while equation (3.7) is more deep and has something to do with the symmetry of the dynamics of collision processes. Thus we have

$$(3.8) \qquad \frac{\partial f_1(v, t)}{\partial t} = - \int \int \int f_1(u, t)f_1(v, t)\lambda_{uv,rs} dr ds du$$

$$+ \int \int \int f_1(r, t)f_1(s, t)\lambda_{rs,uv} dr ds du$$

or

$$\frac{\partial f_1(v, t)}{\partial t} = - \int \int \int [f_1(u, t)f_1(v. t) - f_1(r, t)f(s, t)]\lambda_{rs,uv} dr ds du.$$

The equilibrium configuration is obtained by setting $\frac{\partial f}{\partial t}$ equal to zero; we then find

$$(3.9) \qquad f_1(u, t)f_1(v, t) = f_1(r, t)f_1(s, t).$$

The above functional equation coupled with the energy conservation at each collision yields

$$(3.10) \qquad\qquad f_1(v, t) = A e^{-\beta v^2}$$

where A and β can be identified with the familiar constants of the Boltzmann distribution.

b) Fermi-Dirac statistics.

In a similar fashion, we can write the differential equation for product density $f_1(u, t)$ in the case of Fermi-Dirac particles assuming that $dN(u, t)$ can be only equal to $dN_1(u, t)$ which corresponds to the occupancy of 1 or 0 particles only of the du region in velocity space. To accord with the fact that the presence of a particle in a region dr inhibits other particles going into it the density of states is represented by $\left[1 - \dfrac{dN(r, t)}{dr}\right] dr$. By the same token for a Bose gas with unlimited occupancy of each velocity range to accord with the fact that the existence of particles in each region facilitates other particles going into the region, the increased density of states can be represented by $\left[1 + \dfrac{dN(r, t)}{dr}\right] dr$. For the Fermi-Dirac particles we obtain

$$(3.11)$$

$$\frac{\partial}{\partial t} \langle dN_1(v, t) \rangle = \int \langle dN_1(v, t) dN_1(u, t)[dr - dN_1(r, t)][ds - dN_1(s, t)] \rangle \lambda_{uv, rs}$$

$$+ \int \langle dN_1(r, t) dN_1(s, t) [dv - dN(v, t)][du - dN(u, t)] \rangle \lambda_{rs, uv}$$

where $\langle \ \rangle$ denotes the expectation symbol. Proceeding exactly as in the case of Maxwell-Boltzmann particles, we find

$$(3.12) \quad f_1(v, t) f_1(u, t)[1 - f_1(r, t) - f_1(s, t)]$$
$$= f_1(r, t) f_1(s, t)[1 - f_1(v, t) - f_1(u, t)].$$

If we impose energy conservation in each collision and take into account the total energy and particle number of the system, we arrive at

$$(3.13) \qquad\qquad f_1(v, t) = \frac{1}{A e^{\beta v^2} + 1}$$

which can be identified with the Fermi distribution by a proper choice of A and β.

c) Intermediate and Bose-Einstein statistics.

We next take the case of particles whose occupancy in any region of velocity space is restricted to 0 or 1 or 2. The density of states is taken to be of the Bose type. Hence the total number in each infinitesimal range dv is given by

$$(3.14) \qquad dN(v, t) = dN_1(v, t) + 2dN_2(v, t)$$

The equations for the rate of change of $dN_1(v, t)$ and $dN_2(v, t)$ are given by

$$(3.15) \quad \frac{\partial}{\partial t} \langle dN_1(v, t) \rangle$$

$$= -\iiint \langle \Sigma i dN_i(u, t) dN_1(v, t) \lambda_{rs,uv} drds \left[1 + \frac{\Sigma i dN_i(r, t)}{dr}\right]\left[1 + \frac{\Sigma i dN_i(s, t)}{ds}\right]$$
$$[1 - dN_2(r, t)][1 - dN_2(s, t)] \rangle$$

$$+ \iiint \Sigma \langle ij dN_j(s, t) dN_i(r, t) \lambda_{rs,uv} dudv \left[1 + \frac{\Sigma i dN_i(v, t)}{dv}\right]\left[1 + \frac{\Sigma i dN_i(u, t)}{du}\right]$$
$$[1 - dN_2(u, t)][1 - dN_2(v, t)] \rangle$$

$$+ \iiint \langle \Sigma 2 i dN_i(u, t) dN_2(v, t) \lambda_{uv,rs} drds \left[1 + \frac{\Sigma i dN_i(r, t)}{dr}\right]\left[1 + \frac{\Sigma i dN_i(s, t)}{ds}\right]$$
$$[1 - dN_2(r, t)][1 - dN_2(s, t)] \rangle$$

$$(3.16) \quad \frac{\partial}{\partial t} \langle dN_2(v, t) \rangle$$

$$= -\iiint \langle \Sigma 2 i dN_i(u, t) dN_2(v, t) \lambda_{uv,rs} drds \left[1 + \frac{\Sigma i dN_i(r, t)}{dr}\right]\left[1 + \frac{\Sigma i dN_i(s, t)}{ds}\right]$$
$$[1 - dN_2(r, t)][1 - dN_2(s, t)] \rangle$$

$$+ \iiint \langle \Sigma ij dN_i(r, t) dN_j(s, t) \lambda_{rs,uv} dudv \left[1 + \frac{dN_1(v, t)}{dv}\right]\left[1 + \frac{\Sigma i dN_i(u, t)}{du}\right]$$
$$[1 - dN_2(u, t)][1 - dN_2(v, t)] \rangle$$

where factors $[1 - dN_2]$ have been used to indicate that the total occupancy is limited to 2. Imposing microscopic reversibility, we obtain

$$(3.17) \quad \frac{\partial}{\partial t} \langle dN(v, t) \rangle = \iiint \langle dN(r, t) dN(s, t) \lambda_{uv,rs} dudv$$
$$\left[1 + \frac{dN_1(v, t)}{dv} - \frac{dN_2(v, t)}{dv}\right]\left[1 + \frac{dN_1(u, t)}{du} - \frac{dN_2(u, t)}{du}\right] \rangle$$
$$- \iiint \langle dN(u, t) dN(v, t) \lambda_{uv,s} dsdr$$
$$\left[1 + \frac{dN_1(r, t)}{dr} - \frac{dN_2(r, t)}{dr}\right]\left[1 + \frac{dN_1(s, t)}{ds} - \frac{dN_2(s, t)}{ds}\right] \rangle.$$

At the equilibrium configuration the left hand side of (3.17) is zero.   Proceeding as before, we obtain

$$(3.18) \quad f(r, t)f(s, t)[1 + f^1(v, t) - f^2(v, t)][1 + f^1(u, t) - f^2(u, t)]$$
$$= f(u, t)f(v, t)[1 + f^1(r, t) - f^2(s, t)][1 + f^1(s, t) - f^2(s, t)]$$

Taking into account energy conservation in each collision, we see that the above can be satisfied if

$$(3.19) \qquad\qquad f^1(v, t) = \frac{Ae^{+\beta v^2}}{1 + Ae^{+\beta v^2} + A^2 e^{+2\beta v^2}}$$

and

$$(3.20) \qquad\qquad f^2(v, t) = \frac{1}{1 + Ae^{+\beta v^2} + A^2 e^{+2\beta v^2}}$$

Thus the total product density is given by

$$(3.21) \qquad\qquad f(v, t) = \frac{Ae^{+\beta v^2} + 2}{1 + Ae^{+\beta v^2} + A^2 e^{+2\beta v^2}}$$

which is in accordance with the distribution for the intermediate statistics or Gentile statistics [2] with highest occupancy for each state being only 2.

The above result can be generalised to the case when the maximum number of particles that can occupy a state is p.   In this case, we have to write equations similar to (3.17) for $dN_1(v, t)$, $dN_2(v, t)$, ..., $dN_p(v, t)$ and the factors $[1 - dN_2(r, t)]$, $[1 - dN_2(s, t)]$, etc., which multiply the classical collision probability should be replaced by $[1 - dN_p(r, t)][1 - dN_p(s, t)]$, etc. to ensure that no more than p particles can occupy any elemental velocity range.   In this case, the total product density is given by

$$(3.22) \qquad\qquad f(v, t) = \frac{1}{Ae^{\beta v^2} - 1} - \frac{p + 1}{A^{p+1}e^{(p+1)\beta v^2} - 1}$$

If, however, there is no limit for the highest occupancy of any state and if then density of each state dv is given by $\left[1 + \dfrac{\Sigma i dN_i(v, t)}{dv}\right]$ as in equations (3.15), (3.16), we can write a sequence of equations for $dN_1 \ldots dN_i \ldots$ similar to (3.17) except that now the factor $[1 - dN_p(r, t)]$ no longer multiplies the transition probability since there is no limit or constraint for the

occupancy of each state. Adding up all these equations, we arrive at the equilibrium condition

(3.23) $f(r, t)f(s, t)[1 + f(u, t)][1 + f(v, t)]$
$$= f(u, t)f(v, t)[1 + f(r, t)][1 + f(s, t)]$$

This leads us (for the usual energy conserving collisions) to the Bose-Einstein distribution:

(3.24) $$f(v, t) = \frac{1}{Ae^{\beta v^2} - 1}$$

Alternatively we can obtain the Bose-Einstein distribution from (3.22) by letting p tend to infinity.

4. DISCUSSION

In conclusion, we have to point out that the equations (3.22) lead to parafermi statistics of order p; we naturally arrive at Bose statistics if we allow $p \rightarrow \infty$: if we restrict p to be unity only, we have in equations of (3.17) the factors $[1 - dN_1(v, t)]\left[1 + \dfrac{dN_1}{dv}\right]$ which because of the property $dN_1^2 = dN_1$ becomes $\left[1 - \dfrac{dN_1(v, t)}{dv}\right]$ which occurs in (3.11) in the density of states factor. It is interesting to note that an analogous idea has been put forward by Schweber [12]. The action of the annihilation and creation operators on an assembly of fermions is described by

$$a_i a_i^+ \mid n_1 \ldots n_i \ldots \rangle = (-1)^{2s}(1 - n_i)(1 + n_i) \mid \ldots n_i \ldots \rangle.$$

Thus the matrix element for increasing the occuration number in a given state by one is proportional to $(1 - n_i)(1 + n_i) = 1 - n_i$.

The principal aim of this short note is to point out that this type of kinetic approach which was initiated by Moyal [4] on the basis that the number of particles are distributed in discrete energy states can be made more realistic by taking into consideration the continuous nature of v. The description of the kinetic changes should be in terms of the stochastic variables themselves, leading to product densities under the averaging operations. However, the spectral distribution corresponding to different statistics including parastatistics, can be achieved elegantly by the explicit use of multiple product densities.

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