ANNALES DE L'I. H. P., SECTION A

G. C. MCVITTIE
ROLF STABELL

Gravitational motions in general relativity. The scale-function

Annales de l'I. H. P., section A, tome 9, nº 4 (1968), p. 371-393
<http://www.numdam.org/item?id=AIHPA_1968__9_4_371_0>

Ann. Inst. Henri Poincaré,
Vol. IX, n° 4, 1968, p. 371-393.

Section A :
Physique théorique.

Gravitational motions in general Relativity. The scale-function

by

G. C. McVITTIE and Rolf STABELL
University of Illinois Observatory
Urbana, Illinois, U. S. A.

ABSTRACT. — The radial motions of a spherically symmetric mass under the influence of gravitation and its pressure-gradient are investigated. Solutions of Einstein's equations for the interior of the mass distribution can be subdivided into four different classes. Detailed analysis of the scale-function S is performed for three of these classes. Possibilities of expansion, contraction (collapse) and oscillations are found in all three cases.

1. INTRODUCTION

The problem of the radial motion of a spherically symmetric mass in general relativity can be attacked by a method analogous to that amployed for similarity motions in classical gas dynamics (McVittie 1967; hereafter referred to as Paper I). The metric within the mass is assumed to be of the form

$$ds^2 = y^2dt^2 - R_0^2S^2e^{\eta}(dr^2 + f^2d\omega^2)/c^2 \qquad (1.1)$$

where R_0, c are constants; S and f are dimensionless functions of t and r, respectively; y and η are dimensionless functions of a variable z which is defined by

$$e^z = Q(r)/S(t), \qquad (1.2)$$

Q being still another function of r; and

$$d\omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2. \tag{1.3}$$

When $f(r)$ is either $\sin r$, r or $\sinh r$ and Q, η are both zero, the metric (1.1) is identical with that of a uniform model of the universe. The function S(t) is then called the scale-factor because its value determines the dimensions of the system at each instant. In the general case (1.1), S plays an important role in the determination of the scale, though it does not control it exclusively. In order to have a name for S, it will be called the scale-function. The solution of the problem in this case consists of finding the functions S, f, y, η and Q. Many different types of motion are obtained by imposing certain very general conditions on the energy tensor T_ν^μ. It is shown in Paper I that, if (r, θ, φ) are co-moving coordinates $(T_4^1, T_1^4 = 0)$, then

$$\gamma = 1 - \frac{1}{2} d\eta/dz, \tag{1.4}$$

and if the stress is isotropic $(T_1^1 = T_2^2 = T_3^3)$ then Q, f and y satisfy three ordinary differential equations of the second order. These three equations involve two arbitrary constants a and b and they are

$$Q_{rr}/Q - Q_r f_r/(fQ) = a(Q_r/Q)^2, \tag{1.5}$$

$$f_{rr}/f - f_r^2/f^2 + 1/f^2 = b(Q_r/Q)^2, \tag{1.6}$$

$$y_{zz} + (a - 3 + y)y_z + y\{a + b - 2 - (a - 3)y - y^2\} = 0, \tag{1.7}$$

where suffixes denote derivatives. It is shown in Paper I that (1.5) and (1.6) are always integrable in terms of elementary functions but that (1.7) is only so integrable in four special cases, denoted in Paper I by equations (A.26) to (A.29). For these, the first integral of (1.7) is

$$y_z = \mu(y + \sigma - \delta)(y + \sigma + \delta), \tag{1.8}$$

where μ, σ, δ are constants that are defined for each of the four cases at the end of Sec. 2 of this paper.

In Paper I, and in the present investigation, the component T_4^4 of the energy-tensor is denoted by ρ and it will be called the « density », while the components $T_1^1 = T_2^2 = T_3^3$ are denoted by $-p/c^2$, and p will be referred to as the « pressure ». But no further analysis of the functions ρ and p will be made.

The objects of the present paper are: (i) the establishment of an ordinary second order differential equation for the scale-function S, applicable to all cases included under (1.8), from the condition that the pressure shall vanish at the outer boundary of the spherical mass; (*ii*) the determination of the first integral of this differential equation; and (*iii*) the demonstration by means of this integral that a number of oscillatory motions, of finite amplitude, can exist.

It must be emphasized that this procedure sets up necessary, but not sufficient, conditions for physically acceptable motions of the spherical mass. It indicates what possibilities are open but each of them must be further examined with regard to the signs and magnitudes of the internal pressure and density, the presence of singularities, and so forth. These additional problems are left to be resolved in later investigations.

The possibility of oscillatory motions among those of Paper I was pointed out to us during 1967 by Nariai (1967) and by Bonnor and Faulkes (1967). The former writer specified his solution of Einstein's equations as corresponding, in our classification, to the case defined by $a = 3, b = 0, k = +1$, and it is presumably the same as the one discussed in Sec. 4 under Case (A.28). Bonnor and Faulkes state that their solution is a member of the class of Paper I but express it in a very different notation. Its relationship to our work will be discussed briefly at the end of Sec. 2 below.

2. ZERO BOUNDARY PRESSURE

The necessity of a zero pressure at the boundary arises because it is only in this way that the internal solution can be fitted to an external vacuum Schwarzschild solution of Einstein's equations. The equation (A.34) of Paper I gives, for the pressure.

$$\begin{aligned}8\pi G\frac{p}{c^2} = \frac{1}{y}[&-2S_{tt}/S-(3y-2)(S_t/S)^2\\&-\frac{c^2e^{-\eta}}{R_0^2S^2}\{y(1-f_r^2)f^{-2}+2(y^2-y-y_z)f_rQ_r/(fQ)\\&+(1-y)(y^2-y-2y_z)(Q_r/Q)^2\}]. \qquad (2.1)\end{aligned}$$

Boundary values will be denoted by the suffix b. It has been found by trial and error that there is no loss of generality if

$$Q_b = 1. \qquad (2.2)$$

Since r is a co-moving coordinate, the value of Q_b remains fixed throughout the motion and this is also true of the boundary value of any other function of r. If

$$x = S^{2\mu\delta} \quad , \quad e^z = Qx^{-1/(2\mu\delta)}, \tag{2.3}$$

then (1.8) and (1.4) yield, on integration,

$$y = -\{(\sigma - \delta)x + (\sigma + \delta)Q^{2\mu\delta}\} / \{x + Q^{2\mu\delta}\}, \tag{2.4}$$

$$e^\eta = e^{2(1+\sigma-\delta)z}(e^{2\mu\delta z} + 1)^{2/\mu}, \tag{2.5}$$

if constants of integration, of the nature of scale-factors, are suitably chosen. Hence the boundary values are

$$y_b = -\{(\sigma - \delta)x + (\sigma + \delta)\} /(1 + x), \tag{2.6}$$

$$\exp(\eta_b) = x^{-(1+\sigma+\delta)/\mu\delta}(1 + x)^{2/\mu}, \tag{2.7}$$

$$(y_z)_b = -4\mu\delta^2 x(1 + x)^{-2}. \tag{2.8}$$

Zero pressure at the boundary will be expressed by the vanishing of the factor in square brackets on the right hand side of equation (2.1). The equation so produced will involve the three constants

$$\left.\begin{aligned} B_1 &= \{(1 - f_r^2)f^{-2}\}_b \quad , \\ B_2 &= \{f_r Q_r/(fQ)\}_b \quad , \\ B_3 &= \{Q_r^2/Q^2\}_b \quad , \end{aligned}\right\} \tag{2.9}$$

whose values will be found in Sec. 3. The condition for the vanishing of the pressure at the boundary is then

$$2S_{tt}/S + (3y_b - 2)(S_t/S)^2 + \frac{c^2 \exp(-\eta_b)}{R_0^2 S^2} F(y_b) = 0, \tag{2.10}$$

where

$$F(y_b) = B_1 y_b + 2(y^2 - y - y_z)_b B_2 + (1 - y_b)(y^2 - y - 2y_z)_b B_3. \tag{2.11}$$

If S is replaced by x through (2.3), and (2.6) to (2.8) are also employed, equation (2.10) may be reduced to

$$\frac{d}{dx}\{(1 + x)^{3/\mu}x^{-(3\sigma+3\delta+4\mu\delta)/(2\mu\delta)}x_t^2\} + \frac{2\mu\delta c^2}{R_0^2}(1 + x)^{(1-3\mu)/\mu}x^{-(\sigma+\delta+2\mu\delta)/(2\mu\delta)}F(x) = 0, \tag{2.12}$$

where

$$\begin{aligned} F(x) &= (N_0 + N_1 x + N_2 x^2 + N_3 x^3) \\ &= (n_0 + n_1 x + n_2 x^2)(1 + x) - 8B_3\delta^3(1 - 2\mu)x^3, \end{aligned} \tag{2.13}$$

and the constants N_0 to N_3 and n_0 to n_2 have the following values:

$$N_0 = (\sigma + \delta)\{-B_1 + 2B_2(1 + \sigma + \delta) + B_3(1 + \sigma + \delta)^2\}, \tag{2.14}$$

$$N_1 = -B_1(3\sigma + \delta) + 2B_2\{3\sigma + \delta + 3\sigma^2 + 2\sigma\delta + (4\mu - 1)\delta^2\} + B_3(1 + \sigma + \delta)\{3\sigma + \delta + 3\sigma^2 + (8\mu - 3)\delta^2\}, \tag{2.15}$$

$$N_2 = -B_1(3\sigma - \delta) + 2B_2\{3\sigma - \delta - 2\sigma\delta + 3\sigma^2 + (4\mu - 1)\delta^2\} + B_3(1 + \sigma - \delta)\{3\sigma - \delta + 3\sigma^2 + (8\mu - 3)\delta^2\}, \tag{2.16}$$

$$N_3 = (\sigma - \delta)\{-B_1 + 2B_2(1 + \sigma - \delta) + B_3(1 + \sigma - \delta)^2\}, \tag{2.17}$$

$$n_0 = N_0, \quad n_1 = N_1 - N_0, \quad n_2 = N_2 - N_1 + N_0. \tag{2.18}$$

The equation (2.12) then becomes

$$x_t^2 = \frac{2\mu\delta c^2}{R_0^2} x^{(3\sigma+3\delta+4\mu\delta)/(2\mu\delta)}(1 + x)^{-3/\mu}\{E - I(x)\}, \tag{2.19}$$

where E is the constant of integration and

$$I(x) = \int (1 + x)^{(1-3\mu)/\mu} x^{-(\sigma+\delta+2\mu\delta)/(2\mu\delta)} F(x)dx. \tag{2.20}$$

It is always possible to adjust the constant R_0 in (1.1) so that S shall have the value unity at a pre-assigned instant. It will be assumed that $S = 1$, which means $x = 1$, corresponds to $x_t = 0$. This implies that, in (2.19),

$$E = I(1).$$

This initial condition will be employed in equation (2.19) for each of the four cases that are included under (1.8). In each case the constants N_0 to N_3 or n_0 to n_2 are to be calculated with the appropriate values of μ, σ, δ from equations (2.14) to (2.18).

CASE (A.26)

Definition :

$$\left.\begin{aligned} &\mu = 1/2, \quad b = -(6a^2 - 11a + 4)/25, \quad \text{any } a \neq 3 \\ &\sigma = \delta = (a - 3)/5. \end{aligned}\right\} \tag{2.21}$$

$$x = S^{(a-3)/5} \tag{2.22}$$

$$\left.\begin{aligned} n_0 &= 2\sigma\{-B_1 + 2B_2(1 + 2\sigma) + B_3(1 + 2\sigma)^2\} \\ 2n_1 + n_0 &= 2\sigma\{-3B_1 + 2B_2(3 + 4\sigma) + B_3(3 + 8\sigma + 4\sigma^2)\}, \\ n_2 &= 0. \end{aligned}\right\} \tag{2.23}$$

$$x_t^2 = \frac{a-3}{10}(2n_1 + n_0)(c^2/R_0^2)(1-x)(x-x_1)x^6(1+x)^{-6}, \quad (2.24)$$

$$x_1 = -n_0/(2n_1 + n_0). \quad (2.25)$$

CASE (A.28)

Definition:

$$\left.\begin{array}{lll} \mu = 1/2, & a = 3, & \text{any } b \neq -1 \\ \sigma = 0, & \delta^2 = b + 1 & \end{array}\right\} \quad (2.26)$$

$$x = S^\delta \quad (2.27)$$

$$\left.\begin{array}{l} n_0 = \delta\{-B_1 + 2B_2(1+\delta) + B_3(1+\delta)^2\}, \\ n_1 = 0, \\ n_2 = \delta\{B_1 - 2B_2(1-\delta) - B_3(1-\delta)^2\}. \end{array}\right\} \quad (2.28)$$

$$x_t^2 = \delta n_2(c^2/R_0^2)(1-x)(x-x_1)x^4(1+x)^{-6}, \quad (2.29)$$

$$x_1 = -n_0/n_2.$$

CASE (A.27)

Definition:

$$\left.\begin{array}{lll} \mu = -1, & b = 2 - a, & \text{any } a \neq 3, \\ \sigma = \delta = (a-3)/2. & & \end{array}\right\} \quad (2.30)$$

$$x = S^{3-a}. \quad (2.31)$$

The equation (2.19) is reducible to

$$x_t^2 = \frac{3-a}{8} M_2(c^2/R_0^2)(1-x)(x-x_1)(x+x_2)x^{-1} \quad (2.32)$$

where

$$\left.\begin{array}{l} x_2 = \dfrac{1}{M_2}\{(M_1^2 - 4M_0M_2)^{1/2} - (M_1 + M_2)\}, \\ x_2 = \dfrac{1}{M_2}\{(M_1^2 - 4M_0M_2)^{1/2} + M_1 + M_2\} \\ \quad = x_1 + 2(M_1 + M_2)/M_2, \end{array}\right\} \quad (2.33)$$

and

$$\left.\begin{array}{l} M_0 = 8\sigma^3 B_3, \\ M_1 = N_1 - 2N_2 + 8\sigma^3 B_3 = 8\sigma^2\{2B_2 + B_3(2-\sigma)\}, \\ M_2 = N_1 + 2N_2 + 8\sigma^3 B_3 = 8\sigma\{-B_1 + 2B_2(1-\sigma) \\ \qquad + (1 - 2\sigma - \sigma^2)B_3\}. \end{array}\right\} \quad (2.34)$$

It follows that $x_t^2 > 0$ and $x_1 > 0$ when $a < 3$ if

$$M_2 > 0, \qquad -4M_0 > 2M_1 + M_2, \tag{2.35}$$

and, when $a > 3$, if

$$M_2 < 0, \qquad -4M_0 < 2M_1 + M_2. \tag{2.36}$$

CASE (A.29)

Definition:

$$\left.\begin{aligned} &\mu = -1, \qquad \text{any } a \text{ and } b, \\ &\sigma = (a-3)/2, \qquad \delta^2 = (a-1)^2/4 + b. \\ &x = S^{-2\delta} \end{aligned}\right\} \tag{2.37}$$

This case is the most all-embracing of the four and includes (A.27). It differs from the other three in that x_t^2 is not necessarily expressible as a ratio of polynomials in x. Bonnor (1968) has informed us that the oscillatory motions found by himself and Faulkes (1967) were discovered by working out the special case of (A.29) in which $b = 0$ and $f = r$. This means that $\delta = (a-1)/2$ so that $1 + \sigma - \delta = 0$. Moreover it will be shown in Sec. 3 that $b = 0, f = r$ imply $B_1 = 0$ also. Hence, by (2.17), $N_3 = 0$ and $F(x)$ in (2.13) is reduced to a quadratic function of x. Clearly there are a large number of other motions besides these in Case (A.29); their detailed analysis will be left for a later investigation.

3. THE CONSTANTS B_1, B_2, B_3

These three constants are defined in (2.9) and occur in the expressions for N_0 to N_3 in equations (2.14) to (2.17). They are independent of the values of σ and δ and are known when the equations (1.5) and (1.6) have been solved. The solutions are most easily achieved when $b = 0$, a case that will be considered first. Apart from a constant of integration of the nature of a scale-factor for the coordinate r, it was shown in Paper I that the solutions of (1.6) are

$$f(r) = \sin r,\ r,\ \text{or} \sinh r, \tag{3.1}$$

according as $k = +1, 0$ or -1, where k is the constant which determines the nature of the curvature of space. The first integral of (1.5) is

$$f = AQ_r/Q^a, \tag{3.2}$$

where A is the constant of integration. It is convenient to introduce the auxiliary variable q where

$$q = \int f dr + \mathrm{C}, \tag{3.3}$$

where C is a constant of integration, and then Q is found from the equations

$$\left.\begin{aligned} (a \neq 1) \qquad \mathrm{Q}^{1-a} &= \frac{1-a}{\mathrm{A}}\, q, \\ (a = 1) \qquad \mathrm{Q} &= e^{q/\mathrm{A}}. \end{aligned}\right\} \tag{3.4}$$

The case $a \neq 1$ is the more general and, indeed, it does not appear that the $a = 1$ case presents any radical differences from the other one. Hence, it will be assumed in the rest of this paper that $a \neq 1$. Suppose that $k = +1$ so that $f = \sin r$. Then by (3.3) and (3.4)

$$q = -(\cos r + \mathrm{C}),$$

$$\mathrm{Q}^{1-a} = -\frac{1-a}{\mathrm{A}}(\cos r + \mathrm{C}).$$

The boundary value of Q is chosen to be unity and then

$$\cos r_b = -\left(\frac{\mathrm{A}}{1-a} + \mathrm{C}\right).$$

Instead of the constants of integration A and C it is convenient to use α and β where

$$\alpha = -\frac{\mathrm{A}}{1-a}, \qquad \beta = 1 - \mathrm{C}/\alpha, \tag{3.5}$$

so that

$$\cos r_b = \alpha\beta.$$

Hence

$$\left.\begin{aligned} q &= -\{\cos r + \alpha(1-\beta)\}, \\ \mathrm{Q}^{1-a} &= \{\cos r + \alpha(1-\beta)\}/\alpha = -q/\alpha. \end{aligned}\right\} \tag{3.6}$$

Thus the boundary value, and the central value, q_c, of q are, respectively,

$$q_b = -\alpha, \qquad q_c = -\{1 + \alpha - \alpha\beta\}. \tag{3.7}$$

These expressions show that it is impossible to have both $\mathrm{Q}_b = 1$ and $q_b = 1$ without limiting the value of α, and that the central value of q, unlike that of r, is not necessarily zero.

Since f and Q are now known, the values of B_1, B_2, B_3 are immediatly calculated from (2.9). A similar procedure may be used for the cases $k = 0$ and $k = -1$. The results are shown in Table I.

TABLE I

		Column 1	Column 2	Column 3
k	=	$+1$	0	-1
f	=	$\sin r$	r	$\sinh r$
$\alpha\beta$	=	$\cos r_b\, (\lvert\cos r_b\rvert < 1)$	$r_b^2 > 0$	$\cosh r_b > 1$
Q^{1-a}	=	$\{\cos r + \alpha(1-\beta)\}\,\alpha^{-1}$	$\{r^2 + \alpha(1-\beta)\}\,\alpha^{-1}$	$\{\cosh r + \alpha(1-\beta)\}\,\alpha^{-1}$
B_1	=	1	0	-1
B_2	=	$-\beta(1-a)^{-1}$	$2(1-a)^{-1}\alpha^{-1}$	$\beta(1-a)^{-1}$
B_3	=	$\dfrac{\alpha^{-2} - \beta^2}{(1-a)^2}$	$\dfrac{4\beta\alpha^{-1}}{(1-a)^2}$	$\dfrac{\beta^2 - \alpha^{-2}}{(1-a)^2}$

In all cases it assumed that the center of the distribution of matter lies at $r = 0$. It is clear that α and β must have the same sign when $k = 0$ or $k = -1$ and it will be sufficient, though not necessary, to impose this limitation on α and β when $k = +1$ also. To avoid complications it will be assumed that α and β are both positive.

However, it is not necessary that b should equal zero. One way of dealing with $b \neq 0$ was suggested by Professor W. B. Bonnor (1967). Suppose that a coordinate transformation were to be found which changed the (t, r) of equation (1.1) into (t^*, r^*) and which left the form of (1.1) unchanged. Then the operations by which equations (1.5) and (1.6) were established could be repeated in the (t^*, r^*) system to yield two new constants, a^* and b^*, which would be functions of a and b. The coordinate transformation could be chosen so that either $a^* = 0$ or $b^* = 0$. In practice, it turned out that there were exceptional cases, when either the coordinate transformation did not exist, or the form of (1.1) was destroyed. The first possibility occured when it was sought to produce $a^* = 0$, the exceptional case arising when $a = 3$. The destruction of the form of (1.1) took place when it was desired to produce $b^* = 0$ and occurred when $b = 2 - a$ in the original coordinate system. An example of this is to be found in McVittie and Stabell (1967). Since $a = 3$ and $b = 2 - a$ are parts of the definitions of the cases (A.28) and (A.27) respectively (see equations (2.26) and (2.30) above) this line of attack was abandoned.

The alternative is to solve the problem directly by means of the method of Paper I (Sec. 2 (*iii*), Appendix). If $a \neq 1$, $b \neq 0$, the equations (3.2) to (3.4) are still valid and f can be found in terms of q by means of the substitutions

$$q = e^{w}, \qquad f = e^{w/2}v(w),$$

which produce the solution of (1.5) and (1.6) through (3.2) to (3.4) together with

$$n(w - w_0) = \int \left\{ (v^2 - \gamma)^2 + \frac{4}{n^2} - \gamma^2 \right\}^{-1/2} d(v^2 - \gamma),$$

where

$$n^2 = 1 + 4b(1 - a)^{-2}, \tag{3.8}$$

and γ, w_0 are constants of integration. Three cases arise according as $\gamma^2 > 4/n^2$, $\gamma^2 < 4/n^2$ or $\gamma^2 = 4/n^2$. Define

$$\nu^2 = \{ 1 - 4(n\gamma)^{-2} \}/4, \qquad \varepsilon = +1 \qquad \text{if} \qquad \gamma^2 > 4/n^2;$$
$$\nu^2 = \{ 4(n\gamma)^{-2} - 1 \}/4, \qquad \varepsilon = -1 \qquad \text{if} \qquad \gamma^2 < 4/n^2;$$

with, in both cases,

$$e^{-nw_0} = -\lambda q_b{}^{-n},$$

λ being an arbitrary constant. It then follows that

$$f^2 = \gamma q \{ 1 - \nu\lambda(q/q_b)^n - \varepsilon(\nu/\lambda)(q_b/q)^n \}. \tag{3.9}$$

The case $\gamma^2 = 4/n^2$ may be obtained as the limit when ν tends to zero and λ to infinity in such a way that $\nu\lambda$ tends to a constant Δ. Thus

$$f^2 = (2/n)q \{ 1 - \Delta(q/q_b)^n \}. \tag{3.10}$$

This suggests that the constants ν, λ should be replaced by ξ, ζ where

$$\left. \begin{aligned} \xi &= \nu(\lambda + \varepsilon/\lambda), \\ \zeta &= \nu(\lambda - \varepsilon/\lambda), \end{aligned} \right\} \tag{3.11}$$

and then (3.9) becomes

$$f^2 = \gamma q \left\{ 1 - \frac{1}{2}(\xi + \zeta)(q/q_b)^n - \frac{1}{2}(\xi - \zeta)(q_b/q)^n \right\}. \tag{3.12}$$

The equation (3.10) may be formally obtained from (3.12) by setting $\gamma = 2/n$ and $\xi = \zeta = \Delta$.

Since f is thus explicitly known in terms of q, the relation (3.3) may be written in the alternative form

$$r = \int_{q_c}^{q} dq/f(q),$$

and therefore q can no longer be expressed as an elementary function of r for arbitrary n. Nevertheless it seems clear that by a suitable choice of the constant q_c, it would still be possible to choose $Q_b = 1$ provided that, in (3.4), A were chosen to be equal to $(1 - a)q_b$. This does not imply that $q_c = 0$, any more than it did in the case $b = 0$. Differentiation of (3.12) with respect to r and use of (3.3) in the form $f = q_r$, lead to the boundary values

$$f_b^2 = \gamma q_b(1 - \xi) > 0, \tag{3.13}$$

$$(f_r)_b = \frac{1}{2}\gamma(1 - \xi - n\zeta). \tag{3.14}$$

Again logarithmic differentiation of (3.4) leads to

$$Q_r/Q = (1 - a)^{-1}(q_r/q) = (1 - a)^{-1}(f/q). \tag{3.15}$$

With the aid of the last three formulae, it follows that the constants B_1, B_2, B_3 are given by (2.9) are:

$$B_1 = \left\{1 - \frac{\gamma^2}{4}(1 - \xi - n\zeta)^2\right\}\{\gamma(1 - \xi)\}^{-1}(1/q_b), \tag{3.16}$$

$$B_2 = \frac{1}{1 - a}\frac{\gamma}{2}(1 - \xi - n\zeta)(1/q_b), \tag{3.17}$$

$$B_3 = \frac{1}{(1 - a)^2}\gamma(1 - \xi)(1/q_b). \tag{3.18}$$

It will now be proved that the expressions (3.16) to (3.18) may be identified with those given in one or other of the three columns of Table I by suitable redefinitions of α and β. Consider for example the values in Col. 1 of Table I. If $B_1 = 1$ then γ, ξ, n, ζ and q_b are related by an equation that may be written as

$$1 - \frac{\gamma^2}{4}(1 - \xi - n\zeta)^2 = \left\{\gamma\frac{(1 - \xi)}{q_b}\right\}q_b^2. \tag{3.19}$$

Since ξ, ζ involve, by (3.11), the three constants n, γ and λ, the equation (3.19) expresses q_b in terms of the other three. This does not appear to be a serious limitation. If, next, $B_2 = -\beta/(1 - a)$, equation (3.17) gives

$$\beta = -\frac{\gamma}{2}(1 - \xi - n\zeta)(1/q_b). \tag{3.20}$$

Finally, if (3.19) and (3.20) are introduced into (3.18), there is obtained

$$B_3 = \frac{1}{(1-a)^2}\{q_b^{-2} - \beta^2\}.$$

Comparison with the expression for B_3 in Col. 1, Tab. I shows that α may be defined as in (3.7), namely

$$q_b = -\alpha. \qquad (3.21)$$

Equation (3.19) may also be written as

$$1 - (f_r^2)_b = f_b^2 > 0,$$

whence

$$|(f_r)_b| < 1. \qquad (3.22)$$

The equations (3.14), (3.20) and (3.21) yield

$$\alpha\beta = (f_r)_b. \qquad (3.23)$$

Hence, the final conclusion is that, if f is defined by (3.12), n, γ, λ and q_b satisfy (3.19) and α, β are defined by (3.20) and (3.21), respectively, then B_1, B_2, B_3 have the expressions shown in Col. 1, Table I, and α, β are again related by the condition $|\alpha\beta| < 1$.

The method also permits the establishment of the following results:

(*i*) If n, γ and λ are related by

$$1 = \frac{\gamma^2}{4}(1 - \xi - n\zeta)^2, \qquad (3.24)$$

and

$$\left.\begin{aligned} q_b = \alpha/2, \qquad \beta = \gamma(1-\xi)/2, \\ \alpha\beta = f_b^2 > 0, \end{aligned}\right\} \qquad (3.25)$$

then the three B_i have the expressions shown in Col. 2 of Table I;

(*ii*) If n, γ, λ and q_b are related by

$$1 - \frac{\gamma^2}{4}(1 - \xi - n\zeta)^2 = -\left\{\frac{\gamma(1-\xi)}{q_b}\right\} q_b^2 \qquad (3.26)$$

and

$$\left.\begin{aligned} q_b = \alpha, \qquad \beta = \frac{\gamma}{2}(1 - \xi - n\zeta)(1/q_b), \\ \alpha\beta = (f_r)_b > 0, \end{aligned}\right\} \qquad (3.27)$$

then the three B_i have the expressions shown in Col. 3 of Table I.

These results mean, as will be shown in the next section, that the three B_i, in the form in which they appear in Table I, are sufficient for any conclusion based on the constants N_0 to N_3 of equations (2.14) to (2.17). The proviso must be entered that the conclusions shall not imply too rigid restrictions on the values of α and β.

It is also easily verified that the three cases of Table I, from the point of view of the B_i values, correspond to three possibilities defined by the boundary value of the derivative of f with respect to the radial coordinate r. These are that $|(f_r)_b|$ is less than, equal to or greater than, unity. This statement is valid whether f is given by (3.1), (3.10) or (3.12).

4. THE BEHAVIOR OF THE SCALE-FUNCTION

The results of Sec. 2 and 3 in combination can now be used to find the characteristics of each type of motion. By this is meant that it is possible to decide which of the motions are of expansion, which of contraction (collapse) and which are oscillatory. Each of the cases (A.26) to (A.28) will be considered separately.

Case (A.26)

When $b = 0$, it follows from (2.21) that there are two possible values of a, namely, 1/2 and 4/3. These give rise to different types of motion and will be considered separately.

(*i*) $a = 1/2$, $b = 0$. These values lead, by (2.21) and (2.22), to

$$\sigma = -1/2, \qquad x = S^{-1/2}. \tag{4.1}$$

This value of σ is employed in equations (2.23), and the B_i listed in the three columns of Table I are used in turn. The three forms of equations (2.24) thus obtained are:

$$x_t^2 = \frac{3+4\beta}{4}(c/R_0)^2(x-1)\left(x+\frac{1}{3+4\beta}\right)x^6(1+x)^{-6}, \quad \text{(Col. 1)} \tag{4.2}$$

$$x_t^2 = \frac{1}{\alpha}(c/R_0)^2(1-x)x^7(1+x)^{-6}, \quad \text{(Col. 2)} \tag{4.3}$$

$$x_t^2 = \frac{3+4\beta}{4}(1-x)\left(x+\frac{1}{3+4\beta}\right)x^6(1+x)^{-6} \quad \text{(Col. 3)} \tag{4.4}$$

Since by (4.1) x increases as S decreases and *vice versa*, there is collapse from $S = 1$ in the case (4.2) and expansion from this value in the other two cases. Oscillatory motions are therefore not possible.

(*ii*) $a = 4/3$, $b = 0$. These values lead, by (2.21) and (2.22), to

$$\sigma = -1/3, \qquad x = S^{-1/3}. \tag{4.5}$$

Proceeding as for case (*i*) we find the following three forms of equation (2.24) for x_t^2:

$$x_t^2 = \frac{7}{9}A_2(c/R_0)^2(1-x)(x-x_1)x^6(1+x)^{-6}, \quad \text{(Col. 1)} \tag{4.6}$$

where

$$\left.\begin{aligned} x_1 &= A_1/(7A_2), \\ A_1 &= (1-\beta)^2 - \alpha^2, \\ A_2 &= \alpha^{-2} + 4/49 - (\beta - 5/7)^2; \end{aligned}\right\} \tag{4.7}$$

$$x_t^2 = \frac{4}{9\alpha}(7\beta - 5)(c/R_0)^2(1-x)(x-x_1)(1+x)^{-6}x^6, \quad \text{(Col. 2)} \tag{4.8}$$

where

$$x_1 = (1-\beta)/(7\beta - 5); \tag{4.9}$$

and

$$x_t^2 = \frac{7}{9}A_2'(c/R_0)^2(1-x)(x-x_1)x^6(1-x)^{-6}, \quad \text{(Col. 3)} \tag{4.10}$$

where

$$x_1 = A_1'/(7A_2'), \qquad A_1' = -A_1, \qquad A_2' = -A_2. \tag{4.11}$$

When α, β are chosen so that $x_1 \leq 0$, only motions of expansion or of contraction are possible. But when $x_1 > 0$ oscillatory motions are obtained. This is most easily seen from the case (4.8) which refers to Col. 2 of Table I. The only restriction on α and β is that they must satisfy $\alpha\beta > 0$. For any positive α and β in the range $1 > \beta > 5/7$, x_1 is positive. Then x_t is real only if x lies in the range from 1 to x_1, or *vice versa*, and thus oscillation must occur.

The equations (4.6) and (4.10) yield similar results that will be illustrated by (4.6), which refers to Col. 1 of Table I. Since $\alpha\beta = \cos r_b$, suppose that $r_b = 2\pi/3$ so that $\alpha\beta = 1/2$. The conditions that x_t^2 and x_1, should both be positive are satisfied by $A_1 > 0$, $A_2 > 0$ or $(1-\beta)^2 > 4\beta^2$, $4\beta^2 - (\beta - 5/7)^2 + 4/49 > 0$.

The first inequality is satisfied by $1/3 > \beta$ and the second by

$$\beta > \frac{5}{21}\{(3.52)^{1/2} - 1\} \simeq 0.21.$$

Therefore, if $\alpha = (2\beta)$ and $0.33 > \beta^{-1} > 0.21$, oscillations are possible, with a boundary value of r given by $r_b = 2\pi/3$. Similar conclusions follow for equation (4.10) in which α, β satisfy $\cosh r_b = \alpha\beta > 1$.

In summary, therefore, Case (A.26) can yield only motions of expansion or of contraction when $b = 0, a = 1/2$; but when $b = 0, a = 4/3$, oscillatory motions are possible, in addition to those of the other two kinds.

CASE (A.28)

When $b = 0$, it follows from (2.26) that δ is either $+1$ or -1. However, these two values do not lead to different equations for the scale-function. Therefore it is sufficient to consider $\delta = +1$, and it then follows from (2.27) that

$$x = \mathrm{S}. \tag{4.12}$$

The following three equations are then found from (2.29) according as the B_i are chosen from one or other of the three columns of Table I:

$$x_t^2 = (c/\mathrm{R}_0)^2(1 - x)(x - x_1)x^4(1 + x)^{-6}, \quad \text{(Col. 1)} \tag{4.13}$$

where

$$x_1 = (1 - \beta)^2 - \alpha^{-2}; \tag{4.14}$$

$$x_t^2 = (c/\mathrm{R}_0)^2 \frac{4}{\alpha}(\beta - 1)(1 - x)x^4(1 + x)^{-6}, \quad \text{(Col. 2)} \tag{4.15}$$

$$x_t^2 = (c/\mathrm{R}_0)^2(x - 1)(x - x_1)x^4(1 + x)^{-6}, \quad \text{(Col. 3)} \tag{4.16}$$

where

$$x_1 = (1 - \beta)^2 - \alpha^{-2}. \tag{4.17}$$

When equation (4.15) defines x_t^2 and α is positive, collapse from $x = \mathrm{S} = 1$ is the only possibility if $\beta > 1$, and expansion from the same value of S when $\beta < 1$. When x_t^2 is defined by (4.16), expansion from $\mathrm{S} = 1$ is the only possibility if $x_1 \leq 0$. If however $x_1 > 0$, either collapse or expansion is possible as the following example illustrates. Suppose that

$$\cosh r_b = \alpha\beta = 2.$$

Then (4.17) yields

$$x_1 = (1 - \beta)^2 - \beta^2/4 = (3/4)(\beta - 2)(\beta - 2/3).$$

Hence x_1 is positive for all values of β except those lying in the range

$$2 > \beta > 2/3. \tag{4.18}$$

If for example $\beta = 3$, than $x_1 = 7/4$, and x_t is real if either $x \geq 7/4$, which means an expansion from $x = 7/4$; or $x \leq 1$, which means collapse from $x = 1$. It is not possible for x to lie in the range

$$7/4 > x > 1,$$

if $\alpha\beta = 2$ and (4.18) is satisfied.

The case when x_i^2 is given by (4.13) is of most interest. Suppose that α and β are both positive; then by Col. 1 of Table I

$$\cos r_b = \alpha\beta = 1/m,$$

where $m > 1$ if $r_b > 0$. Thus (4.14) yields

$$x_1 = \{ 1 - (m + 1)\beta \} \{ 1 + (m - 1)\beta \}.$$

Therefore if $\beta > 1/(m + 1)$, the value of x_1 is negative and only collapse from $x = 1$ could occur. But if

$$1/(m + 1) > \beta > 0,$$

then x_1 is positive and oscillations may take place between the values 1 and x_1 of S. It is interesting to compare these oscillatory possibilities with those found through equation (4.6) of Case (A.26), in which $a = 4/3$, $b = 0$, $\delta = - 1/3$. It was shown that oscillations there occurred for $\alpha\beta = 1/2$, or $m = 2$, if β lay in the narrow range

$$1/3 > \beta > \frac{5}{21}\{ (3.52)^{1/2} - 1 \}.$$

But in Case (A.28), in which $a = 3$, $b = 0$, $\delta = + 1$, the permissible range of β is extended to

$$1/3 > \beta > 0.$$

In summary, therefore Case (A.28) with $b = 0$ can give rise to oscillatory motions only when the B_i from the first column of Table I are used. This means that the 3-space t = constant occupied by the material is a spherical space. For all three columns of Table I, however, motions of collapse or of expansion are also possible.

However, these conclusions are dependent on the choice $b = 0$, as the following illustration will show. When the B_i are chosen from Col. 2, Tab. I, we have for $a = 3$,

$$B_1 = 0, \qquad B_2 = - 1/\alpha, \qquad B_3 = \beta/\alpha$$

and $\alpha\beta > 0$. Then by the use of (2.28) with any $\delta \neq 1$, the constants in the equation (2.29) for x_t^2 are

$$x_1 = -n_0/n_2 = \frac{1+\delta}{1-\delta} \cdot \frac{2-(1+\delta)\beta}{2-(1-\delta)\beta},$$

$$\delta n_2 = \delta^2(1-\delta)\{2-(1-\delta)\beta\}\alpha^{-1}.$$

It is clear that both these quantities can be positive if $0 < \delta < 1$ and β is suitably chosen. For example, $\delta = 1/2$ and $0 < \beta < 4/3$ could be chosen; but it must be noticed that $\beta = 1$ has to be excluded because it corresponds to the statical case. The function f in such a case has the form (3.12) wherein, by (3.8), $n = \delta$, and (3.24), (3.25) are also valid. Since x_1 and δn_2 are now both positive, inspection of (2.29) shows that oscillations are possible between the values 1 and x_1 of $x = S^\delta$. A value of δ in the range $0 < \delta < 1$ corresponds by (2.26) to a negative value of b in the range $-1 < b < 0$. Thus whereas in Case (A.28) oscillations could not occur for the Col. 2, Tab. I, values of the B_i when $b = 0$ and $f = r$ from (3.1), they can occur when $-1 < b < 0$ and f is given by (3.12).

CASE (A.27)

When $b = 0$, equations (2.30), (2.31) show that

$$a = 2, \qquad \sigma = \delta = -1/2, \qquad x = S. \tag{4.19}$$

The formula (2.32) for x_t^2 is a cubic in x. There are many possibilities for motions of collapse or of expansion which will not be considered in detail. Attention will be concentrated on the question whether oscillations are possible. Since, in (2.32), $3 - a = 1$, oscillations could occur if the conditions (2.35) were satisfied, provided, of course, that $x_2 \geq 0$, where x_2 is given by (2.33). By (2.34)

$$M_0 = -B_3,$$
$$M_1 = 4B_2 + 5B_3,$$
$$M_2 = 4B_1 - 12B_2 - 7B_3.$$

Hence $M_2 > 0$ if

$$B_3 < \frac{4}{7}(B_1 - 3B_2) \tag{4.20}$$

and $-4M_0 > 2M_1 + M_2$ if

$$B_3 > 4(B_1 - B_2). \tag{4.21}$$

When the values of the three B_i found in Col. 1, Tab. I are employed, the inequalities (4.20), (4.21) are respectively

$$\alpha^{-2} < \left(\beta - \frac{6}{7}\right)^2 - \frac{8}{49}, \tag{4.22}$$

$$\alpha^{-2} > (\beta - 2)^2. \tag{4.23}$$

There is also the condition $\alpha\beta < 1$ or

$$\alpha^{-2} > \beta^2. \tag{4.24}$$

As usual, α and β will be assumed to be positive. It is shown in the Appendix that no pair of values of α, β satisfies all three inequalities. Therefore oscillations cannot occur. When the values of the B_i in Col. 2, Tab. I are employed, the conditions (2.35) lead, by means of (4.20) and (4.21), to $\beta < 6/7$ and $\beta > 2$ simultaneously. Therefore again oscillations are impossible. However, when the B_i of Col. 3, Tab. I are used, the three inequalities to be satisfied are

$$\alpha^{-2} > \left(\beta - \frac{6}{7}\right)^2 - \frac{8}{49}, \qquad \alpha^{-2} < (\beta - 2)^2, \qquad \alpha^{-2} < \beta^2. \tag{4.25}$$

It is shown in the Appendix that these three inequalities can be satisfied if β lies in the range $\frac{1}{3} < \beta < \frac{3}{2}$. For oscillatory motions to occur, the requirement $x_2 \geq 0$ must also be satisfied. Knowing that the value of x_1 is positive we only consider the difference between x_2 and x_1, which, by (2.33) and the use of (2.34) and Col. 3 Tab. I, is found to be

$$\begin{aligned} 2(M_1 + M_2)/M_2 &= 4(2B_1 - 4B_2 - B_3)/(4B_1 - 12B_2 - 7B_3) \\ &= \frac{4}{7}(\alpha^{-2} - (\beta - 2)^2 + 2)\left\{\alpha^{-2} - \left(\beta - \frac{6}{7}\right)^2 - \frac{8}{49}\right\}^{-1}. \end{aligned} \tag{4.26}$$

The denominator is necessarily positive. A sufficient, though not necessary, condition for $x_2 > 0$ is therefore that

$$\alpha^{-2} > (\beta - 2)^2 - 2. \tag{4.27}$$

It is shown in the Appendix that this inequality is satisfied for most of the pairs of values α, β, that satisfy the inequalities (4.25).

The conclusion is that in Case (A.27) with $b = 0$ oscillatory motions may occur only when $k = -1$.

The Case (A.27) is also a convenient one through which to examine the influence of the value of b on the possibility of oscillations. It has been shown that $b = 0$ is inconsistent with oscillations when $k = +1$ or $k = 0$. Since $b = 0$ inplies that $a = 2$ in Case (A.27), the relevant inequalities that had to be satisfied were given by (2.35). The alternative inequalities are (2.36) and they apply when $a > 3$. As an example, the case of $a = 5$, $b = -3$ is considered in which, by (2.30) and (2.31),

$$\sigma = \delta = 1, \qquad x = S^{-2}. \tag{4.28}$$

The equations (2.34) yield

$$M_0 = 8B_3, \qquad M_1 = 8(2B_2 + B_3), \qquad M_2 = -8(B_1 + 2B_3),$$

and the inequalities (2.36) are, respectively,

$$4B_3 > B_1 - 4B_2, \tag{4.29}$$

$$2B_3 > -B_1. \tag{4.30}$$

Moreover by (2.33) the difference between x_2 and x_1 is

$$2(M_1 + M_2)/M_2 = 2(B_1 - 2B_2 + B_3)/(B_1 + 2B_3). \tag{4.31}$$

The function f applicable in this case is given by (3.9) and, as was shown in the second part of Sec. 3, the expressions for the B_i in terms of α, β are still those found in Table I. Consider therefore the entries in Col. 1, for which $\alpha\beta < 1$. This inequality, together with (4.29), (4.30) lead to the three conditions

$$\alpha^{-2} > \beta^2, \qquad \alpha^{-2} > (\beta - 2)^2, \qquad \alpha^{-2} > \beta^2 - 8.$$

Clearly the first inequality implies the third; thus α^{-2} must be greater than β^2 or $(\beta - 2)^2$ whichever is numerically the larger for any chosen value of β. In fact, the permissible pairs of (α, β) correspond to points in Fig. 1 which lie in the region above the boundary KEJ. By (4.31)

$$2(M_1 + M_2)/M_2 = \{\alpha^{-2} - (\beta^2 + 8\beta - 16)\}/\{\alpha^{-2} - (\beta^2 - 8)\}. \tag{4.32}$$

By reading off the coordinates $y = (\alpha^{-2})$ and β of certain points in the region above KEJ in Fig. 1, it is easy to find (α, β) pairs that make the right-hand side of (4.32) positive. The point $(y, \beta) = (4, 0.2)$, for which which $(\alpha, \beta) = (0.5, 0.2)$ possesses the required property; on the other hand the point $(y, \beta) = (10, 3)$, for which $(\alpha, \beta) = (10^{-1/2}, 3)$, does not, though it lies in the region KEJ.

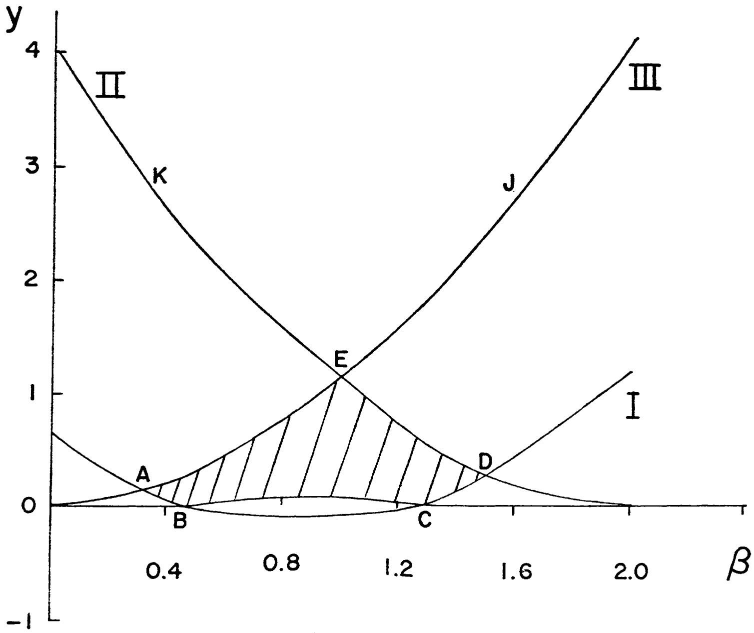

FIG. 1. — Inequalities Diagram. I is $y = (\beta - 6/7)^2 - 8/49$, II is $y = (\beta - 2)^2$ and III is $y = \beta^2$.

If the foregoing conditions on α, β are satisfied, it follows that, in (2.32), the constants $(3 - a)M_2$, x_1 and x_2 are all positive. If (4.28) is used to convert (2.32) from x to S, under the assumption that S is always positive (or zero), it is easy to show that

$$S_t^2 = (S - 1)(S_1 - S)F(S),$$

where $S_1 = + x_1^{-1/2}$ and F(S) is a positive function of S which includes factors $(S_1 + 1)(S_1 + S)$. Thus S_t is real only if S lies between 1 and S_1 so that oscillations are possible. A similar analysis with the B_i taken from either Col. 2 or Col. 3 of Table I again shows that oscillatory motions are possible. The general conclusion is that, in case (A.27), oscillatory motions can occur under certain conditions. It is however necessary to emphasize again the statement made in Sec. 1, namely, that the present analysis establishes only necessary conditions for oscillations. In each particular sub-case the properties of the density and pressure will have to be analyzed before a physically acceptable solution of Einstein's equations is reached.

The conclusion that oscillations may occur in case (A.27) is in apparent contradiction to that of McVittie and Stabell (1967), who did not find oscillations.

McVittie and Stabell dealt with the particular form (3.10) with $\Delta = 0$. But the most important reduction thus introduced was the disappearance of the term in q^{-n} in the general solution (3.12). They also assumed that the central value of Q, and therefore of q_c, was zero, and that the boundary value of Q was different from unity. These restrictions are implicitly incorporated in the constants that occur in their differential equation for S, which is the analogue of (2.32) above. In the present investigation it has been assumed that f may have the general form (3.12), that the central value of Q is not zero and that its boundary value is equal to unity. The two investigations together serve to emphasize the importance that must be attached to the constants of integration in the expressions for f and of Q, constants which a first sight might appear to have little significance.

APPENDIX

The inequalities that have to be satisfied in the (A.27) case, when Column 1 of Table I is used, are

$$\alpha^{-2} < \left(\beta - \frac{6}{7}\right)^2 - \frac{8}{49}, \tag{4.22}$$

$$\alpha^{-2} > (\beta - 2)^2, \tag{4.23}$$

$$\alpha^{-2} > \beta^2. \tag{4.24}$$

On the other hand, when Column 3 of Table I is employed, the inequalities are

$$\alpha^{-2} > \left(\beta - \frac{6}{7}\right)^2 - \frac{8}{49}, \quad \alpha^{-2} < (\beta - 2)^2, \quad \alpha^{-2} < \beta^2. \tag{4.25}$$

These two sets of inequalities are most easily dealt with by a graphical method. The three parabolas

$$y = \left(\beta - \frac{6}{7}\right)^2 - \frac{8}{49}, \tag{I}$$

$$y = (\beta - 2)^2, \tag{II}$$

$$y = \beta^2, \tag{III}$$

are drawn in a y, β diagram as shown in Figure 1. These parabolas have the property that, if taken in pairs, they intersect in one point only. In Fig. 1, A is the point where curves I and III intersect at $\beta = 1/3$, $y = 1/9$; D is the intersection of I and II at $\beta = 3/2$, $y = 1/4$; and E is the intersection of II and III at $\beta = 1$, $y = 1$. The points where curve I intersects the β-axis are B ($\beta = 2(3 - \sqrt{2})/7$, $y = 0$) and C ($\beta = 2(3 + \sqrt{2})/7$, $y = 0$).

Consider first the inequalities (4.22) to 4.24). Since α is real α^{-2} is positive. Hence $(\alpha^{-2}, \beta) = (y, \beta)$ are the coordinates of points in the (y, β) plane that lie above the β-axis. A representative point whose (α, β) values satisfy (4.23) and (4.24) must lie in the area whose lower boundary is the curve KEJ, where K and J are points on the curves I and II as remote as we please from E. But at the same time (4.22) shows that the same representative point must lie below the curve I on the diagram. This it impossible and therefore the inequalities (4.22) to (4.24) cannot be satisfied.

Next consider the inequalities (4.25). Inspection of Fig. 1 now shows that permissible pairs of (α, β) correspond to points with positive y that lie above curve I and below curves II and III. Therefore such points exist and lie in the shaded area bounded by the curves AB, CD, DE, EA and the portion BC of the β-axis. Therefore the inequalities (4.25) can be satisfied and the permissible values of β lie between $\beta_A = 1/3$ and $\beta_D = 3/2$.

Fig. 1 may also be used to find pairs of values of (α, β) that satisfy, in addition to (4.25), the inequality (4.27). The parabola $y = (\beta - 2)^2 - 2$ is similar in shape to curve II but is moved two units of y downwards. It is easy to see that most of the points in the shaded area, except for those lying in a small region towards its left-hand end, will lie above this new parabola. Thus their (α, β) will satisfy (4.27) also.

Manuscrit reçu le 15 juillet 1968.

REFERENCES

W. B. BONNOR, 1967. Private communication.
W. B. BONNOR, 1968. Private communication.
W. B. BONNOR and M. C. FAULKES, *Mon. Not. R. Astr. Soc*, t. **137**, 1967, p. 239.
G. C. MCVITTIE, *Ann. Inst. H. Poincaré*, t. A, **6**, 1, 1967.
G. C. MCVITTIE and R. STABELL, *Ann. Inst. H. Poincaré*, t. A, **7**, 103, 1967.
H. NARIAI, *Prog. Theor. Phys.*, t. **38**, 1967, p. 740.