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Quantum Logic and Evolution

by

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ABSTRACT. — The quantum propositional system is interpreted as a set of filters used to select a beam of particles. Since new filters are discovered from time to time the « quantum logic » is not static but undergoes a process of evolution. We prove that every non-distributive logical system may be extended so that: 1) the resulting system is distributive; 2) all implications and the operation of negation in the initial system remain unchanged.

1. INTRODUCTION

It is generally believed that the crucial difference between classical and quantum physics can be expressed in the language of mathematical logic and lattice theory. This idea was introduced in 1932 by Birkhoff and von Neumann and continued by Jauch and Piron [2] [3] [5]. They postulated the existence of a certain non-classical logical system reflecting the nature of quantum phenomena and called *quantum logic*. While classical logic represents the properties of selection processes applied to ensembles of macro objects, quantum logic represents the structure of selection processes applied to ensembles of microparticles.

The physical meaning of quantum logic can be illustrated on the following

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model. Consider a stationary beam of particles and a class of filters which can be used to select the beam. Suppose the beam passes through various sequences of filters and we observe the resulting sub-beams. Assume, however, that we possess only specific detectors. They do not yield any numerical measure of the beam intensity. They only allow the comparison of intensities: we can observe two sub-beams and recognize the « more intense » one. One can ask what sort of physics can be constructed on the basis of these experiments? The answer is that we shall arrive precisely at the « quantum logic ». First we define *vacuum* (absence of beam): this is the beam of the smallest intensity possible. Next we discover the existence of the relations of equivalence ($\equiv$), inequality ($\leq$), and orthogonality ($\perp$) for some pairs of filters. More precisely we call two filters a, b equivalent ($a \equiv b$) if the substitution of a by b (and of b by a) in any chain of filters selecting the beam does not affect the intensity of the resulting sub-beam. For two filters a, b we say that a is *contained in* b ($a \leq b$) if any beam energing from a passes through b without being partially absorbed. We call two filters a, b *orthogonal* if the successive application of a and b (and b and a) produces the vacuum. By observing the structure of the set $\mathcal{Q}$ of all known filters we notice that:

I. The inequality ($\leq$) is a partially ordering relation in $\mathcal{Q}$.

II. For any $a, b \in \mathcal{Q}$ the subclass of all filters containing both a and b contains the smallest element. We call this element the *union* of a and b and we denote it by $a \cup b$. Similarly for any $a, b \in \mathcal{Q}$ the subclass of filters contained in both a and b contains the greatest element. We call it the *intersection* of a and b and we denote it by $a \cap b$.

III. For any $a \in \mathcal{Q}$ the subclass of all filters orthogonal to a contains the greatest element a'. The correspondence $a \rightarrow a'$ obeys the rules:

$$I' \equiv 0 \tag{1.1}$$

$$(a')' \equiv a \tag{1.2}$$

$$(a \cup b)' \equiv a' \cap b' \tag{1.3}$$

Points I, II, III imply that the set $\mathcal{Q}$ with the relation $\leq$ and with the napping $a \rightarrow a'$ is an orthocomplemented lattice. Provided that the properties I, II, III hold, we can introduce the analogy between the set of filters and a logical system as follows. We call the set $\mathcal{Q}$ the *logic* of the beam of particles. Any filter is called a *proposition*. The inequality $a \leq b$ means « a implies b ». The operations $a \cup b$, $a \cap b$ and $a \rightarrow a'$, are interpreted as the alternative, conjunction and negation of the logic respectively. If the beam in question is a beam of microparticles (such as photons) the resulting logical system is non-distributive, *i. e.*, the following distributive law does not hold:

$$(1.4) \qquad a \cap (b \cup c) \equiv (a \cap b) \cup (a \cap c)$$

The absence of (1.4) implies the existence of *incompatible* propositions in $\mathfrak{Q}$. Two propositions are called *compatible* if the smallest orthocomplemented sub-lattice of $\mathfrak{Q}$ containing both a and b is distributive, otherwise they are caled incompatible. Many authors consider the non-distributive character of quantum logic and the existence of incompatible propositions as the most important manifestation of the quantum nature of microphenomena. Thus, *e. g.*, Jauch and Piron opine that the appearance of incompatible propositions in $\mathfrak{Q}$ does not allow a classical theory of microphenomena. On this basis they predict the impossibility of discovering a new classical theory of microparticles [5]. One has to admit that, in fact, the return to the classical theory seems improbable at present. However, the reasons for this are not found in the structure of quantum logic. Future physical theories will depend rather on the evolution of quantum logic and not on its present structure. *A priori* one cannot exclude the possibility that the future structure of quantum logic will differ from the present one. Since the development of physics is not yet complete, the set $\mathfrak{Q}$ is extended from time to time by adding new elements. Examples of such extensions are given by the discovery of the crystal of tourmaline, the construction of the device used in Stern Gerlach experiment, etc., and we have no reasons to think that this process has terminated. Thus we should consider not a fixed class of filters $\mathfrak{Q}$, but rather an increasing sequence of classes $\mathfrak{Q}$, $\mathfrak{Q}'$, $\mathfrak{Q}''$, ..., corresponding to successive steps in the progress of experimental physics. One can assume various hypothesis about the nature of this process. A reasonable assumption is:

A) Although new elements are added to the set of filters, all relations of equivalence ($\equiv$), inequality ($\leq$) and orthogonality ($\perp$) which have been previously discovered for old filters remain valid. Also the operation $a \to a'$ for previously known filters is not affected by the evolution.

The above assumption is a hypothesis about « soft » changes. Note that even if the meaning of symbols $\leq$, $\equiv$ and $'$ is conserved by the extension process in agreement with the hypothesis A), the meaning of $\cup$ and $\cap$ may not be. The point is that the union and intersection are « relative » concepts. The elements $a \cup b$ and $a \cap b$ cannot be determined merely by observing the physical properties of a and b. They are determined by the whole partially ordered set to which a and b belong. When this set is extended, certain new filters x^* such that $a \leq x^*$ and $b \leq x^*$ may appear and the meaning of $a \cup b$ may change. It is quite possible that two elements which were incompatible in the initial orthocomplemented lattice will be compatible in an extended one. It may even happen that a non-distributive orthocomplemented lattice is completely converted as a result of an extension into a distributive lattice. We shall prove below that there is no lattice with a structure excluding the possibility of such an extension.

2. THEOREM OF EMBEDDING

DEFINITION. — Embedding of an orthocomplemented lattice (X, $\leq$, $'$) in an orthocomplemented lattice (Y, $\leq$, $'$) is a mapping $x \to y(x)$ of X into Y such that:

$$x_1 \leq x_2 \Rightarrow y(x_1) \leq y(x_2) \tag{2.1}$$

$$x_1 \neq x_2 \Rightarrow y(x_1) \neq y(x_2) \tag{2.2}$$

$$y(x') = y(x)' \tag{2.3}$$

THEOREM. — Each orthocomplemented lattice can be embedded in a distributive orthocomplemented lattice (*).

PROOF. — Let (X, $\leq$, $'$) be an orthocomplemented lattice. We call two elements $x, y \in \mathrm{X}$ *orthogonal* ($x \perp y$) if $x \leq y'$. The relation ($\perp$) is symmetric since $x \leq y' \Leftrightarrow y \leq x'$. A subset $\mathrm{Z} \subset \mathrm{X}$ will be called *skew* if Z does not contain any pair of orthogonal elements (thus the empty set is skew). A skew set $\mathrm{Z} \subset \mathrm{X}$ will be called *maximal* if for every $x \in \mathrm{X}$ either x or x' belongs to Z. Each skew set which is not maximal is a proper subset of another skew set. In fact let $\mathrm{Z} \subset \mathrm{X}$ be skew and let x_0 be an element of X such that neither x_0 nor x_0' belongs to Z. Then one of the subsets $\mathrm{Z} \cup (x_0)$, $\mathrm{Z} \cup (x_0')$ must be skew. If none of them were skew, this would imply the existence of two elements $z_1, z_2 \in \mathrm{Z}$ such that $z_1 \perp x_0$ and $z_2 \perp x_0'$. This would means $x_0 \leq z_1'$ and $z_2 \leq x_0$. Hence, $z_2 \leq z_1'$ and $z_1 \perp z_2$ which contradicts the assumption that Z is skw. It follows now by applying Zorn's lemma that each skew subset is contained in a certain maximal skew subset.

We shall call each maximal skew subset a *point*. Let $\mathfrak{Z}$ denote the set of all points. We shall define the mapping $x \rightarrow \xi(x)$ of the lattice X into the orthocomplemented distributive lattice of all subsets of $\mathfrak{Z}$ as follows: for each $x \in \mathrm{X}$ the symbol $\xi(x)$ means the set of all points $\mathrm{Z} \subset \mathrm{X}$ which contain x (and do not contain x'). We shall show that this mapping is an embedding. Let $x, y \in \mathrm{X}$ and $x \leq y$. Then $y' \perp x$ and any skew subset containing x must not contain y'. Hence, any point Z containing x must also contain y, *i. e.*, $\xi(x) \subset \xi(y)$. Now let $x, y \in \mathrm{X}$ and $x \neq y$. Thus either $x \nleq y$ or $y \nleq x$. Suppose that $x \nleq y$. The pair $\{ x, y' \}$ is then a skew subset of X. The maximal skew subset Z_0 containing $\{ x, y' \}$ belongs to $\xi(x)$ but does not belong to $\xi(y)$. Hence, $\xi(x) \neq \xi(y)$. Finally, since for any $x \in \mathrm{X}$ each point must contain either x or x' (but not both of them), we have:

$$\xi(x) \cup \xi(x') = \mathfrak{Z} \tag{2.4}$$

which means that $\xi(x')$ is the complement of $\xi(x)$. This completes the proof.

The above theorem shows that the current non-distributive structure of quantum logic does not imply its own stability. We thus conclude that the existence of incompatible propositions in $\mathfrak{Q}$ is not the main obstacle preventing a return to the classical theory. The basic reason why a return to the classical theory does not appear probable lies in empirical facts which cannot be expressed in the language of the lattice theory (more detailed discussion of this point is found in [7]). We also conclude that the non-distributive character of $\mathfrak{Q}$ cannot be considered as the most important manifestation of the quantum nature of microphenomena.

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(*) *Note added to proof :*

When this article was in print, the author was notified that the theorem of embedding was previously proved by N. Zierler and M. Schlesinger (*Duke Math. J.*, **32**, 251-262 (1965)).

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