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On an Inverse Problem in Potential Scattering Theory (*)

by

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ABSTRACT. — We consider the Schrödinger scattering of a particle of spin 0 by a real, rotationally invariant potential at fixed energy. We prove the following statement, due to Regge: if two such potentials, both belonging to a certain class $\mathscr{V}$ described in the text, are known to give rise to partial wave amplitudes having the same Regge interpolation in the complex angular momentum plane, then these two potentials are identical. As a first illustration, we obtain a uniqueness result about Yukawa-like potentials first derived by Martin and Targonski. As a second illustration, we show that if two potentials, both belonging to the class $\mathscr{V}$ and being of finite range, are known to generate the same scattering amplitude, then these potentials are identical. To conclude, we show that one can extract from results of Agranovich and Marchenko a complete characterization of the scattering amplitudes which can be generated by potentials of finite range belonging to the class $\mathscr{V}$.

CONTENTS

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1. INTRODUCTION

The present article deals with the scattering of spin zero particles by real valued rotation invariant potentials. It contains results on the following inverse problem: *Let a scattering amplitude* F *be given as a function of the scattering angle, the energy* E *being fixed. Find a potential* V *such that* V *generates the amplitude* F *via the Schrödinger equation taken at the energy* E.

In all what follows, we shall choose a system of units such that E will be equal to 1. Furthermore, the term « potential » will mean « real valued rotation invariant potential ».

Let us formulate the inverse problem in more precise terms. Taking F as a function of $z = \cos\theta$, where θ is the scattering angle, we know that we can write down the Legendre expansion

$$F(z) = C \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l P_l(z)$$

with real phase shifts δ_l. C is a number whose value depends on the particular normalization conventions we adopt. We shall assume that C is equal to 1. Put

$$s_l = e^{2i\delta_l} \qquad (l = 0, 1, \ldots).$$

We shall apply the term « scattering amplitude » both to F and to the sequence $s = \{ s_l \}$. We state our problem in the following manner:

Given a class $\mathscr{V}$ of « admissible » potentials and a sequence s of complex numbers, of modulus 1, answer the following questions:

A) *Does there exist in $\mathscr{V}$ a potential* V *which generates s via the Schrödinger equation?* (We shall refer to such a V as a *solution*).

B) *If there is such a solution in $\mathscr{V}$, is it unique?*

Ideally, we would like to answer these questions by giving necessary and sufficient conditions on s in order to ensure the existence of an admissible solution (i. e., a solution in $\mathscr{V}$), and by describing a method to construct all admissible solutions.

An interesting attack on the present inverse problem was started by a work of R. G. Newton (1). In his paper, Newton was led to the conjecture that many different potentials could generate the same scattering amplitude. The method proposed by Newton consists in deriving from the given sequence s a Gel'fand-Levitan type of integral equation (like Eqs. (42) and (200) below) for a function (denoted here by L_{10} or by L) which is closely related to a « candidate solution » V (by relations of the type of Eqs. (28) and (202)). The suspected non-uniqueness comes from the fact that many different integral equations can be derived from one given sequence s. However, Ref. 1 left many properties of the « candidate

(1) R. G. Newton, Construction of Potentials from the Phase Shifts at Fixed Energy. *J. Math. Phys.*, **3**, 1962, 75-82.

solutions » in the dark. P. C. Sabatier has recently improved the situation [2] [3]; he was able in particular to discuss the asymptotic behavior at large distances of the « candidates ». His results may be summarized as follows: *let the sequence s be such that, for some positive numbers* K *and* ε,

$$|s_l - 1| < K(1 + l)^{-3-\varepsilon}$$

($l = 0, 1, \ldots$); *then, for each real number a, there is a « candidate »* V_a *with the following properties:*

a) *it is meromorphic in the complex plane cut along the negative real axis; its poles are of multiplicity* 2;
b) *for x real and positive,* $V_a(x)$ *is real;*
c) *for* $x > 0$, $x \to 0$, V_a *is « well behaved »;*
d) *for any positive number* η,

$$V_a(x) = ax^{-3/2} \cos\left(2x - \frac{1}{4}\pi\right) + O(x^{-2+\eta})$$

when x real tends to $+\infty$. If V_a happens to have no poles on the positive real axis, then the scattering problem (formulated in terms of the radial Schrödinger equations at least) makes sense, and one has every reason to expect that V_a is a solution. Furthermore, Sabatier's construction makes it appear plausible that if K is small enough, then there should exist a positive number a_0 such that V_a has no poles for $x > 0$ as soon as $|a| < a_0$.

Thus, Sabatier's papers give an answer to both questions A) and B): it formulates sufficient conditions for s to be generated by a potential of a given class $\mathcal{V}$, and it gives a host of solutions when these conditions are fulfilled. It also teaches us the important fact that, unless one imposes some restrictions on the asymptotic behavior of the admissible potentials at large distances, question B) has to be answered negatively. However, the problem is not completely solved by Newton's method. As used by Sabatier, this method yields solutions with rather special analyticity properties only. Thus, it cannot be expected to give « all » potentials which generate a given sequence s (this remains true even after a generalization

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[2] P. C. SABATIER, Asymptotic Properties of the Potentials in the Inverse-Scattering Problem at Fixed Energy. *J. Math. Phys.*, 7, 1966, 1515-1531.

[3] P. C. SABATIER, Analytic Properties of a Class of Potentials and the Corresponding Jost Functions. *J. Math. Phys.*, 7, 1966, 2079-2091.

of Newton's scheme recently proposed by Sabatier) [4]. Moreover, if we restrict somewhat the class of admissible potentials, by imposing for example a mild condition of the kind

$$\int_1^\infty x\,|\,V(x)\,|\,dx < \infty,$$

it is not obvious from Sabatier's results that Newton's method will give any solution whatsoever.

If we go over to more restricted classes of potentials, we do get affirmative answers to question B). It has been known for some time that if $\mathscr{V}$ is the class of the Yukawa-like potentials (i. e., the potentials which are Laplace transforms and which vanish exponentially at large distances), then for any sequence s there is at most one solution in $\mathscr{V}$. This was shown by A. Martin and Gy. Targonski [5]. This result can be derived also from the papers of T. Regge [6] [7] (see Section 3 below). Furthermore, it will be shown in the present paper that if $\mathscr{V}$ is a certain class of *finite range* potentials (i. e., of potentials which vanish outside of some sphere), then the solution is again unique (see Section 3, Theorem 3).

The first part of the present work (Sections 2 and 3) deals with the uniqueness problem. Section 2 follows closely Regge's paper, Ref. 6 (especially its Section 4). The treatment is based on the method invented by I. M. Gel'fand and B. M. Levitan [8] to solve « inverse problems ». The main purpose of this section is to prove the following theorem for a broad class $\mathscr{V}$ described below: let V_1 and V_2 be two potentials in $\mathscr{V}$; assume that the Regge interpolations to complex values of the angular momentum of their scattering amplitudes are equal meromorphic functions; then $V_1 = V_2$ (Theorem 2 below). This theorem, due to Regge, has been proved by him in Ref. [6] under slightly more restrictive assumptions than mine. The methods I use are very nearly always his, except that instead of my Lemma 10, which allows to derive Theorem 2 from Theorem 1, he could

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[4] P. C. SABATIER, A General Method for the Inverse Scattering Problem at Fixed Energy. *J. Math. Phys.*, **8**, 1967, 905-918.

[5] A. MARTIN and Gy. TARGONSKI, On the Uniqueness of a Potential Fitting a Scattering Amplitude at a Given Energy, *Nuovo Cimento*, **20**, 1961, 1182-1190.

[6] T. REGGE, Introduction to Complex Orbital Momenta. *Nuovo Cimento*, **14**, 1959, 951-976.

[7] T. REGGE, Bound States, Shadow States and Mandelstam Representation. *Nuovo Cimento*, **18**, 1960, 947-956.

[8] I. M. GEL'FAND and B. M. LEVITAN, Ob opredelenii differentsial'nogo uravneniya po ego spektral'noi funktsii. *Izv. Akad. Nauk S. S. S. R.*, **15**, 1951, 309-360 ; translated in *Amer. Math. Soc. Transl.*, **1**, 1955, 253-304.

use a simpler argument. The class $\mathscr{V}$ is formed by potentials satisfying rather mild conditions which insure that :

a) they are not too rough, in order that the Schrödinger equation makes unambiguous sense;

b) they become weak at large distances, in order that the phase shifts can be defined;

c) they behave not too wildly at the origin in order that, in the radial Schrödinger operator, the « centrifugal » singularity dominates near the origin, in such a manner that we may treat this singular point in a way which does not depend on the particular choice of V in $\mathscr{V}$.

The first requirement is attained by assuming that the admissible potentials are locally integrable; the second, by imposing that

$$\int_1^\infty |\mathrm{V}(r)|\, dr < \infty;$$

the third, by assuming the existence of a positive number ε such that

$$\int_0^1 r^{1-2\varepsilon} |\mathrm{V}(r)|\, dr < \infty.$$

(Regge assumes moreover that the first moment $\int_0^\infty r|\mathrm{V}(r)|\, dr$ of the potential is less than infinity). [In view of the fact that the Jost function is closely related to a Carlson function (see Eq. (80) and the equation below Eq. (84)), and hence uniquely given by its values at the physical values of the angular momentum, Theorem 1 below, which asserts that two potentials in $\mathscr{V}$ are equal if the corresponding Jost functions are equal in the complex angular momentum plane, is interesting by itself].

Having thus obtained a fairly general uniqueness result for the step carrying us from the Regge interpolation σ of the sequence s to the potential, we realize that any non-uniqueness in the correspondence between the sequence s, i. e., the scattering amplitude, and the potential can be interpreted as a non-uniqueness in the step from s to its interpolation σ. In Section 3, I discuss, as illustrative examples, the two classes of potentials, already discussed above, for which we know that question B) has an affirmative answer. For both of these classes, the sequence s determines σ uniquely.

For the class of the Yukawa-like potentials, not much has been done to answer question A). It is not impossible that both the method of Gel'fand and Levitan and the method of Martin and Targonski can be useful. For the class of finite range potentials already mentioned, I have been able to solve the inverse problem rather completely. The results are expounded in Section 4 below. The goal is attained by using the solution found by Z. S. Agranovich and V. A. Marchenko (9) in their study of a related problem (10). I can formulate necessary and sufficient conditions on the sequence s in order that it be generated by a potential of finite range (Theorem 5). The trick is rather simple: it is sufficient to make a change of variable in our Schrödinger operator to transform our problem, *in the case of a finite range potential,* into the problem exhaustively studied in Ref. (10). My Theorem 4 is very closely related to the results of the authors mentioned. From it, Theorem 5 follows without too much difficulty.

I have gathered for the convenience of the reader a few facts on the Mellin transformation in a first appendix. In a second appendix, I give an example to illustrate Theorems 4 and 5.

I found many hints of a general character in the review article by L. D. Faddeyev (11) and in the work of I. Kay and H. E. Moses (12) which are not otherwise acknowledged in the text.

2. REGGE'S UNIQUENESS THEOREMS

2.1 Preliminaries

We start from the radial Schrödinger equation, which we put in the following form ($x > 0$):

$$-\frac{d}{dx}\left(x^2\frac{d\varphi}{dx}\right)(x) + \left(\nu^2 - \frac{1}{4}\right)\varphi(x) + x^2(\mathrm{V}(x) - 1)\varphi(x) = 0 \tag{1}$$

$\nu^2 - \dfrac{1}{4}$ stands for $l(l+1)$, where l is the usual angular momentum quantum

(9) Z. S. Agranovich and V. A. Marchenko, Bosstanovlenie potentsiala po matritse rasseyaniya dlya sistemy differentsial' nykh uravnenii. *Dokl. Akad. Nauk S. S. S. P.*, **113**, 1957, 951-954.

(10) The proofs are given in the following book : Z. S. Agranovich and V. A. Marchenko. *The Inverse Problem of Scattering Theory* (Gordon and Breach, New York, 1963).

(11) L. D. Faddeyev, Obratnaya zadacha kvantovoi teorii rasseyaniya. *Uspekhi Mat. Nauk*, **14**, vyp. 4, 1959, 57-119 ; transl. in *J. Math. Phys.*, **4**, 1963, 72-104.

(12) I. Kay and H. E. Moses, The Determination of the Scattering Potential from the Spectral Measure Function I. *Nuovo Cimento*, **2**, 1955, 917-961.

number. We have chosen our units of time, length and mass in such a way that $\hbar = E = 2m = 1$ (E is the energy, m is the mass of the particle).

Let us list a few conventions and facts we shall refer to in what follows:

a) R_+ denotes the open half-line $\{ x \mid x > 0 \}$.

b) τ denotes the formally self-adjoint differential operator

$$-\frac{d}{dx}\left(x^2 \frac{d}{dx}\right) - \frac{1}{4} + x^2(V(x) - 1).$$

c) V denotes a real (Lebesgue-) measurable function on R_+ ; V is assumed to be locally integrable on R_+, i. e. (absolutely Lebesgue-) integrable on each compact interval contained in R_+.

d) Let φ and ψ be two differentiable functions on R_+. We denote their Wronskian multiplied by x^2 by $W(\varphi, \psi)$:

$$W(\varphi, \psi)(x) = x^2(\varphi(x)\varphi'(x) - \varphi'(x)\psi(x))$$

(φ', resp. ψ': derivative of φ, resp. ψ).
If φ and ψ are two solutions of (1) for the same value of ν^2, then $W(\varphi, \psi)$ is independent of x.

e) Let $-\infty \leq a < b \leq +\infty$; we denote by $\mathscr{L}_2(a, b)$ the Hilbert space of the square integrable functions on the (open) interval (a, b) of the real line.

f) Let I be an interval of the real line (which can be bounded or unbounded, open, half-open, or closed) ; we denote by $\mathscr{C}_2(I)$ the set of all absolutely continuous complex valued functions on I which have an absolutely continuous first derivative (and which therefore possess almost everywhere in I a locally integrable second derivative). The formal differential operator τ gives rise in a natural way to a well defined operator which maps $\mathscr{C}_2(R_+)$ into the set of the locally integrable functions on R_+.

g) a. e. means « almost everywhere ».

h) Let u and v be in $\mathscr{C}_2(R_+)$; then $v(x)\tau(u)(x)$ is locally integrable. Let a and b be in R_+. We have Green's formula:

$$\int_a^b u(x)\tau(v)(x)dx - \int_a^b v(x)\tau(u)(x)dx = -(W(u, v)(b) - W(u, v)(a)) \quad (2)$$

In particular, if $\tau(u) = \lambda u$ (a. e.) and if $\tau(v) = \mu v$ (a. e.), then

$$(\lambda - \mu) \int_a^b u(x)v(x)dx = W(u, v)(b) - W(u, v)(a) \quad (2')$$

2.2 The Regular Solution

If Lemma 1 were the mere statement of the existence and uniqueness properties of the so-called regular solution of Eq. (1) we would just quote it without bothering to reproduce a too predictable proof. However, it contains more, namely estimates related to the asymptotic behavior of the regular solution for large values of $|\nu|$.

LEMMA 1. — *If* V (*locally integrable on* R_+) *is such that there exists a positive* ε *for which*

$$\int_0^1 u^{1-2\varepsilon}|V(u)-1|\,du<\infty,$$

then Eq. (1) *has for every* ν *with* $\operatorname{Re}\nu \geq -\varepsilon$ *a solution* φ *in* $\mathscr{C}_2(R_+)$. *The function* φ *and its x-derivative* φ' *have the following properties* (*we use the following abbreviations:* $\rho(\varepsilon,\nu)=\min\{(2\varepsilon)^{-1},\ |\nu|^{-1}\}$,

$$F(\varepsilon,x)=\int_0^x u^{1-2\varepsilon}|V(u)-1|\,du,\ E(\varepsilon,x,\nu)=\rho(\varepsilon,\nu)F(\varepsilon,x)x^{2\varepsilon}):$$

a) *For* $\operatorname{Re}\nu\geq-\varepsilon$, *and* $x>0$:

$$|\varphi(x,\nu)|\leq x^{\operatorname{Re}\nu-\frac{1}{2}}\exp(E(\varepsilon,x,\nu)) \tag{3}$$

$$|x^{-\nu+\frac{1}{2}}\varphi(x,\nu)-1|\leq E(\varepsilon,x,\nu)\exp(E(\varepsilon,x,\nu)) \tag{4}$$

(*so that in particular* $\lim\limits_{x\to 0}\varphi(x,\nu)x^{-\nu+\frac{1}{2}}=1$). *Furthermore, if* ν *is not zero:*

$$\left|x^{-\nu+\frac{1}{2}}\varphi(x,\nu)-1-(2\nu)^{-1}\int_0^x u(V(u)-1)(1-(u/x)^{2\nu})\,du\right|$$
$$\leq\frac{1}{2}(E(\varepsilon,x,\nu))^2\exp(E(\varepsilon,x,\nu)). \tag{5}$$

b) *For* $\operatorname{Re}\nu\geq-\varepsilon$, *and* $x>0$

$$|\varphi'(x,\nu)|\leq x^{\operatorname{Re}\nu-3/2}\left(\left|\nu-\frac{1}{2}\right|+3x^{2\varepsilon}F(\varepsilon,x)\right)\exp(E(\varepsilon,x,\nu)) \tag{6}$$

$$\left|x^{-\nu+3/2}\varphi'(x,\nu)/\left(\nu-\frac{1}{2}\right)-1\right|\leq\left(|\nu|^{-1}+3\left|\nu-\frac{1}{2}\right|^{-1}\right)$$
$$x^{2\varepsilon}F(\varepsilon,x)\exp(E(\varepsilon,x,\nu)) \tag{7}$$

$\left(\textit{in particular } \lim\limits_{x\to 0}x^{-\nu+3/2}\varphi'(x,\nu)=\nu-\frac{1}{2}\right)$.

c) *For* $x > 0$, φ *and* φ' *are holomorphic functions of* v *for* $\operatorname{Re} v > -\varepsilon$, *continuous for* $\operatorname{Re} v \geq -\varepsilon$. *We have*

$$\varphi(x, v^*) = (\varphi(x, v))^* \tag{8}$$

$$\varphi'(x, v^*) = (\varphi'(x, v))^* \tag{8'}$$

(an asterisk means complex conjugation). These two functions have (for each fixed positive x*) a finite number of zeroes for* $\operatorname{Re} v \geq 0$; *these are all simple and real. If* $\varphi(x, v) = 0$, *then* $\varphi'(x, v) \neq 0$; *if* $\varphi'(x, v) = 0$, *then* $\varphi(x, v) \neq 0$.

Proof: α) Let a be a positive real number. Let $\mathscr{C}$ be the Banach space of the complex valued functions f which are continuous and bounded on $\mathrm{D} = \{(x, v) | x \text{ real}, 0 \leq x \leq a, v \text{ complex}, \operatorname{Re} v \geq -\varepsilon\}$, and whose norm is $\| f \| = \sup \{ | f(x, v) | | (x, v) \text{ in } \mathrm{D} \}$. Define k for $x > 0$, $0 < u \leq x$, v complex by

$$k(x, u, v) = \begin{cases} (2v)^{-1}(1 - (u/x)^{2v}) & \text{if} \quad v \neq 0 \\ ln(x/u) \quad \text{if} \quad v = 0 \end{cases}$$

The function k is continuous in its domain of definition. For x and u fixed, k is an entire function of v. We have for $x > 0$:

$$| k(x, u, v) | \leq (2\varepsilon)^{-1}(x/u)^{2\varepsilon} \quad \text{in} \quad \{ 0 < u \leq x, \operatorname{Re} v \geq -\varepsilon \}$$

and

$$| k(x, u, v) | \leq | v |^{-1}(x/u)^{2\varepsilon} \quad \text{in} \quad \{ 0 < u \leq x, \operatorname{Re} v \geq -\varepsilon, v \neq 0 \}$$

(Proof: For $0 < u \leq x$, $k(x, u, v)$ is holomorphic and bounded in $\operatorname{Re} v \geq -\varepsilon$. Therefore,

$$\sup \{ | k(x, u, v) | | \operatorname{Re} v \geq -\varepsilon \} = \sup \{ | k(x, u, -2\varepsilon + i\tau) | | \tau \text{ real} \}.$$

Now

$$k(x, u, v) = \int_{u/x}^{1} v^{2v-1} dv,$$

so that

$$| k(x, u, -2\varepsilon + i\tau) | \leq \int_{u/x}^{1} v^{-2\varepsilon-1} dv = (2\varepsilon)^{-1}((x/u)^{2\varepsilon} - 1) \leq (2\varepsilon)^{-1}(x/u)^{2\varepsilon}.$$

Furthermore,

$$| k(x, u, v) | \leq (2 | v |)^{-1}(1 + (u/x)^{2\operatorname{Re} v}) \leq (2 | v |)^{-1}(1 + (x/u)^{2\varepsilon}) \leq | v |^{-1}(x/u)^{2\varepsilon}).$$

We define q by

$$q(x) = x^2(\mathrm{V}(x) - 1).$$

Since by assumption $u^{-1-2\varepsilon}q(u)$ is integrable on the interval $(0, x)$ for $x > 0$, and since for all v and $x > 0$, k is continuous for $0 < u \leq x$, it follows from the two estimates for k that $u^{-1}q(u)k(x, u, v)$ is integrable on $\{u \mid 0 < u < x\}$ for $x > 0$ and $\operatorname{Re} v \geq -\varepsilon$. Moreover, we have

$$\int_0^x |k(x, u, v)q(u)| u^{-1} du \leq \mathrm{E}(\varepsilon, x, v)$$

for $x > 0$, $\operatorname{Re} v \geq -\varepsilon$. Therefore, for f in $\mathscr{C}$, we can define a function $\mathrm{B}(f)$ on D in the following way: for (x, v) in D,

$$\mathrm{B}(f)(x, v) = \int_0^x k(x, u, v)q(u)f(u, v)u^{-1}du.$$

For (x, v) in D, we have

$$|\mathrm{B}(f)(x, v)| \leq \mathrm{E}(\varepsilon, x, v) \sup \{ |f(y, v)| \mid 0 \leq y \leq x \} \leq \mathrm{E}(\varepsilon, a, v) \| f \|. \tag{9}$$

The function $\mathrm{B}(f)$ is continuous. We show this in the following way: we have

$$\mathrm{B}(f)(y, \mu) = \int_0^a \mathrm{G}(y, u, \mu)du$$

where

$$\mathrm{G}(y, u, \mu) = \begin{cases} k(y, u, \mu)f(u, \mu)q(u)u^{-1} & \text{for } u \leq y \\ 0 & \text{for } u > y \end{cases}$$

For fixed $u, 0 < u \leq a, u \neq x$, $\mathrm{G}(y, u, \mu)$ tends to $\mathrm{G}(x, u, v)$ when $(y, \mu) \to (x, v)$. On the other hand, $|\mathrm{G}(y, u, \mu)|$ is majorized for $0 < y \leq a$ and $\operatorname{Re} \mu \geq -\varepsilon$ by the integrable function $|q(u)| u^{-1-2\varepsilon}(2\varepsilon)^{-1}a^{2\varepsilon} \| f \|$. It follows now from Lebesgue's (bounded convergence) theorem that $\mathrm{B}(f)(y, \mu)$ tends to $\mathrm{B}(f)(x, v)$ whenever (y, μ) tends to (x, v). We see therefore that $f \to \mathrm{B}(f)$ is a continuous linear mapping B of $\mathscr{C}$ into itself, and that its norm $\| \mathrm{B} \|$ does not exceed $(2\varepsilon)^{-1}a^{2\varepsilon}\mathrm{F}(\varepsilon, a)$. Since

$$k(x, u, v) = u^{2v} \int_u^x v^{-2v-1}dv,$$

we can write

$$\mathrm{B}(f)(x, v) = \int_0^x du u^{-1}q(u)f(u, v)u^{2v} \int_u^x v^{-2v-1}dv.$$

This double integral is absolutely convergent in the given order. Therefore, by Fubini's theorem, we can write

$$\mathrm{B}(f)(x, v) = \int_0^x dv v^{-2v-1} \int_0^v u^{-1} q(u) u^{2v} f(u, v) du.$$

$\mathrm{B}(f)$ is thus seen to be absolutely continuous in x for fixed v, and to have the following partial derivative with respect to x:

$$\partial_x \mathrm{B}(f)(x, v) = x^{-1} \int_0^x u^{-1} q(u)(u/x)^{2v} f(u, v) du$$

or

$$\partial_x \mathrm{B}(f)(x, v) = x^{-1}\left(\int_0^x u^{-1} q(u) f(u, v) du - 2v\mathrm{B}(f)(x, v) \right).$$

From this, we easily obtain the inequality

$$|x\partial_x \mathrm{B}(f)(x, v)| \leq 3x^{2\varepsilon}\mathrm{F}(\varepsilon, x) \sup \{ |f(y, v)| \,|\, 0 \leq y \leq x \}. \tag{10}$$

Moreover, we see that $\partial_x \mathrm{B}(f)$ is itself an absolutely continuous function of x (v fixed); for its derivative we obtain (a. e.)

$$x\partial_x^2 \mathrm{B}(f)(x, v) = x^{-1} q(x) f(x, v) - (2v + 1)\partial_x \mathrm{B}(f)(x, v). \tag{11}$$

We remark that, if f is for each fixed x a holomorphic function in $\mathrm{Re}\, v > -\varepsilon$, the same is true of $\mathrm{B}(f)$ and of $\partial_x \mathrm{B}(f)$. (To prove it, compute any closed contour integral $\oint \mathrm{B}(f)(x, v) dv$, resp. $\oint \partial_x \mathrm{B}(f)(x, v)$, and show that it vanishes by using the holomorphy of f and of k and Fubini's theorem. Then apply Morera's theorem).

For the powers of B we have, for (x, v) in D,

$$\begin{aligned} |\mathrm{B}^n(f)(x, v)| &\leq (n!)^{-1}(\mathrm{E}(\varepsilon, a, v))^n \sup \{ |f(y, v)| \,|\, 0 \leq y \leq x \} \\ &\leq (n!)^{-1}(\mathrm{E}(\varepsilon, a, v))^n \, \| f \|. \end{aligned} \tag{12}$$

To show this, we remark that (12) is the same as (9) if $n = 1$. If now (12) is assumed to be true for $n = k - 1$, we have

$$\begin{aligned} |\mathrm{B}^k(f)(x, v)| &= \left| \int_0^x k(x, u, v) u^{-1} q(u) \mathrm{B}^{k-1}(f)(u, v) du \right| \\ &\leq ((k-1)!)^{-1}(\rho(\varepsilon, v))^k x^{2\varepsilon k} \sup \{ |f(y, v)| \,|\, 0 \leq y \leq x \} \\ &\int_0^x u^{-1-2\varepsilon} |q(u)| (\mathrm{F}(\varepsilon, u))^{k-1} du \\ &= (k!)^{-1}(\mathrm{E}(\varepsilon, x, v))^k \sup \{ |f(y, v)| \,|\, 0 \leq y \leq x \} \end{aligned}$$

since

$$F(\varepsilon, x) = \int_0^x u^{-1-2\varepsilon} |q(u)| \, du.$$

Thus,

$$\| B^n \| \leq (n!)^{-1}((2\varepsilon)^{-1} a^{2\varepsilon} F(\varepsilon, a))^n.$$

Therefore, for $\lambda \neq 0$, the Neumann series

$$\lambda^{-1}(1 + \lambda^{-1}B + \lambda^{-2}B^2 + \dots)$$

converges in the operator norm topology to a bounded linear operator on $\mathscr{C}$, which is equal to $(\lambda - B)^{-1}$. We get for $\lambda = 1$

$$\| (1 - B)^{-1} \| \leq \exp((2\varepsilon)^{-1} a^{2\varepsilon} F(\varepsilon, a)) \tag{13}$$

and, by using (12)

$$|(1 - B)^{-1}(f)(x, v)| \leq \sup \{ |f(y, v)| \, | \, 0 \leq y \leq x \} \exp(E(\varepsilon, x, v)). \tag{14}$$

Since the series

$$f(x, v) + B(f)(x, v) + B^2(f)(x, v) + \dots$$

tends to $(1 - B)^{-1}(f)(x, v)$ uniformly in D, we draw from previous remarks the conclusion that if f is holomorphic in v for every fixed x, then the same is true for $(1 - B)^{-1}(f)$.

β) From what has been said above, it follows that for $a > 0$ the integral equation

$$f = f_0 + B(f) \tag{15}$$

where $f_0(x, v) = 1$ in D, has a unique solution in $\mathscr{C}$, which is given by

$$f = (1 - B)^{-1}(f_0) = f_0 + B(f_0) + \dots \tag{16}$$

Now, the expression for $B(g)$, g in $\mathscr{C}$, does not explicitly depend on a, and neither does f_0; given any two positive numbers a_1 and a_2, the corresponding D-domains D_1 and D_2 have a non empty intersection; the corresponding solutions of (15), f_1 and f_2, say, are equal on $D_1 \cap D_2$ (take f_0 to be equal to 1 everywhere!). From now on, when we speak of the solution of (15), we mean that continuous function on $\{ (x, v) \, | \, x \geq 0, \operatorname{Re} v \geq -\varepsilon \}$ which agrees, for every positive a, with the corresponding solution (16) of (15) in the corresponding D-domain. From (15), it follows that f is for fixed v a function from $\mathscr{C}_2(R_+)$, since it is the sum of two such functions. From (15) and (11) we get that a. e.,

$$x\partial_x^2 f(x, v) + (2v + 1)\partial_x f(x, v) - x^{-1}q(x) f(x, v) = 0. \tag{17}$$

Since f_0 is holormorphic for x fixed, the same is true for f as long as Re $\nu > \varepsilon$; this is also true for $\partial_x f$, because, by (15), $\partial_x f = \partial_x \mathrm{B}(f)$; moreover, f and $\partial_x f$ are both real when ν is real. Furthermore, for $x \geq 0$, Re $\nu \geq -\varepsilon$, we obtain the following inequalities: from (16) and (14), we get

$$|f(x, \nu)| \leq \exp(\mathrm{E}(\varepsilon, x, \nu)); \tag{18}$$

from $\partial_x f = \partial_x \mathrm{B}(f)$ again, and from (10) and (18), we get

$$|x\partial_x f(x, \nu)| \leq 3x^{2\varepsilon}\mathrm{F}(\varepsilon, x) \exp(\mathrm{E}(\varepsilon, x, \nu)); \tag{19}$$

from (15), by (9) and (18), we get

$$|f(x, \nu) - 1| \leq \mathrm{E}(\varepsilon, x, \nu) \exp(\mathrm{E}(\varepsilon, x, \nu)); \tag{20}$$

and finally, because

$$(1 - \mathrm{B})^{-1} = 1 + \mathrm{B} + \mathrm{B}^2(1 - \mathrm{B})^{-1}$$

(see the Neumann series), which implies

$$f = f_0 + \mathrm{B}(f_0) + \mathrm{B}^2(f),$$

we see, using (12) for $n = 2$ and (18), that

$$\left| f(x, \nu) - 1 - (2\nu)^{-1} \int_0^x u^{-1} q(u)(1 - (u/x)^{2\nu}) du \right| \leq \frac{1}{2}(\mathrm{E}(\varepsilon, x, \nu))^2 \exp(\mathrm{E}(\varepsilon, x, \nu)). \tag{21}$$

γ) Define φ for $x > 0$ and Re $\nu \geq -\varepsilon$ by

$$\varphi(x, \nu) = x^{\nu - \frac{1}{2}} f(x, \nu) \tag{22}$$

f being the unique solution of Eq. (15) described above. For fixed ν, φ belongs to $\mathscr{C}_2(\mathrm{R}_+)$. From (22) and (17), we see that

$$-\partial_x(x^2\partial_x\varphi)(x, \nu) + \left(q(x) - \frac{1}{4}\right)\varphi(x, \nu) = -\nu^2\varphi(x, \nu),$$

i. e., φ is a solution of (1). The estimates (3), (4), (5), (6) and (7) for φ and for $\varphi' = \partial_x\varphi$ follow easily from definition (22) and inequalities (18), (19), (20) and (21). The first part of *c*) (including (8) and (8′)) is also an easy consequence of the properties of f and of definition (22).

δ) To prove the last part of statement *c*), we remark that, for fixed $x > 0$, φ and φ' are holomorphic for Re $\nu \geq -\varepsilon(\varepsilon > 0)$. Therefore, the zeros of these functions in Re $\nu \geq 0$ have no finite accumulation point. Furthermore, it follows from (4), resp. (7), that there exist a real positive number r

such that $|\varphi(x, \nu)|$, resp. $|\varphi'(x, \nu)|$, is positive for $|\nu| \geq r$ in Re $\nu \geq 0$(E(ε, x, ν) tends to 0 uniformly in Re $\nu \geq 0$ as $|\nu|$ tends to ∞). Thus, the set of the zeros of φ and φ' in Re $\nu \geq 0$ is bounded. Therefore, it must be finite. Let τ be real; $\varphi(\cdot, i\tau)$ and $\varphi(\cdot, -i\tau)$ are then solutions of Eq. (1) for $\nu^2 = -\tau^2$. Therefore, $\mathrm{W}(\varphi(\cdot, i\tau), \varphi(\cdot, -i\tau))(x)$ is independent of x. With the help of (4) and (7), we get

$$\mathrm{W}(\varphi(\cdot, i\tau), \varphi(\cdot, -i\tau))(x) = -2i\tau.$$

On the other hand,

$$\mathrm{W}(\varphi(\cdot, i\tau), \varphi(\cdot, -i\tau))(x) = 2ix^2 \operatorname{Im} (\varphi(x, i\tau)(\varphi'(x, i\tau))^*).$$

Thus, if $\varphi(x, i\tau) = 0$ and/or $\varphi'(x, i\tau) = 0$, then $\tau = 0$. Assume that $\varphi(x, 0) = 0$. Since φ is holomorphic in ν near 0,

$$\varphi(x, i\tau) = i\tau\partial_\nu\varphi(x, 0) + \mathrm{O}(\tau^2).$$

Since φ' is holormorphic in ν near 0,

$$\varphi'(x, i\tau) = \varphi'(x, 0) + \mathrm{O}(\tau).$$

Therefore,

$$\mathrm{W}(\varphi(\cdot, i\tau), \varphi(\cdot, 0i\tau)) = -2i\tau = 2ix^2 \operatorname{Im} \tau(i\partial_\nu\varphi(x, 0)\varphi'(x, 0) + \mathrm{O}(\tau)),$$

i. e.,

$$1 = -x^2 \operatorname{Im} (i\partial_\nu\varphi(x, 0)\varphi'(x, 0)).$$

Thus, we see that if $\varphi(x, 0) = 0$, then $\partial_\nu\varphi(x, 0)$ is not zero (and neither is $\varphi'(x, 0)$). In other words, if $\varphi(x, 0) = 0$, then $\nu = 0$ is a simple zero. Interchanging in the previous argument φ and φ', we see that if $\varphi'(x, 0) = 0$, then $\nu = 0$ is a simple zero. Take now a ν with Re $\nu > 0$; with $\mu = \nu^*$, we get by using (2′) in conjunction with (8) and (8′) (and with (3) and (6)):

$$2 \operatorname{Re} \nu \operatorname{Im} \nu \int_0^x |\varphi(y, \nu)|^2 dy = 2ix^2 \operatorname{Im} ((\varphi(x, \nu))^* \varphi'(x, \nu)).$$

Thus, if $\varphi(x, \nu) = 0$, it follows that Im $\nu = 0$. Assume now that ν (real and positive) is a zero of $\varphi(x, \nu)$. Then formula (2′) reads (for Re $\mu \geq 0$)

$$(\mu^2 - \nu^2) \int_0^x \varphi(y, \nu)\varphi(y, \mu)dy = -x^2\varphi'(x, \nu)\varphi(x, \mu).$$

Since (3) implies

$$|\varphi(y, \nu)\varphi(y, \mu)| \leq y^{\nu-1} \exp (\varepsilon^{-1}x^{2\varepsilon}\mathrm{F}(\varepsilon, x)),$$

where the right hand side is integrable on $(0, x)$, and since for each $y > 0$,

$\varphi(y, v)\varphi(y, \mu)$ tends to $(\varphi(y, v))^2$ when μ tends to v, it follows from Lebesgue's theorem that $\int_0^x \varphi(y, v)\varphi(y, \mu)dy$ tends to $\int_0^x (\varphi(y, v))^2 dy$ when μ tends to v. Thus, for $\mu = v + i\tau$, τ real and near 0, we have

$$i\tau\left(2v \int_0^x (\varphi(y, v))^2 dy + O(1)\right) = - i\tau(x^2\varphi'(x, v)\partial_v\varphi(x, v) + O(\tau));$$

i. e.

$$2v \int_0^x (\varphi(y, v))^2 dy = - x^2\varphi'(x, v)\partial_v\varphi(x, v).$$

Thus, from our assumption that $v > 0$ is a zero of φ, it follows that $\partial_v\varphi(x, v)$ is not zero (and that neither is $\varphi'(x, v)$). Therefore, if φ has a zero for $\operatorname{Re} v > 0$, this zero is real, and simple. By interchanging the role of φ and φ' in the previous argument, we get the same result for φ'. We have thus proven all of statement *c*). Q. e. d.

From the preceding proof we extract the following relation, valid for $x > 0$ and $|\operatorname{Re} v| \leq \varepsilon$:

$$W(\varphi(\cdot, v), \varphi(\cdot, - v))(x) = - 2v \tag{23}$$

The function φ described in Lemma 1 will be called the *regular solution* of Eq. (1) in what follows.

We shall need the following result:

LEMMA 2. — *Let* V *be as in Lemma* 1. *For every complex* v *with* $\operatorname{Re} v \geq -\varepsilon$, *there is a function* φ_1 *in* $\mathscr{C}_2(\mathrm{R}_+)$, *which forms with* φ *a fundamental system of solutions of Eq.* (1), *and which behaves near* $x = 0$ *in the following way:*

$$\varphi_1(x, v) = - (2v)^{-1}x^{-v-\frac{1}{2}}(1 + O(x^{2\varepsilon}))\ (v \neq 0) \tag{24a}$$

and

$$\varphi_1(x, 0) = x^{-\frac{1}{2}}(ln x + O(1)) \tag{24b}$$

Proof: If $|\operatorname{Re} v| \leq \varepsilon$, and if $v \neq 0$, we put $\varphi_1(x, v) = - (2v)^{-1}\varphi(x, - v)$. It then follows from Lemma 1 and from Eq. (23) that φ_1 has the asserted properties.

If $v = 0$, or if $\operatorname{Re} v > \varepsilon$, take a positive number a; we can choose b and c in such a way that

$$a^2(\varphi(a, v)b - \varphi'(a, v)c) = 1.$$

Let $\varphi_1(x, v)$ be the solution of Eq. (1) in $\mathscr{C}_2(\mathrm{R}_+)$ which satisfies

$$\varphi_1(a, v) = c, \qquad \varphi_1'(a, v) = b.$$

We have for all positive x:

$$W(\varphi(\cdot, \nu), \varphi_1(\cdot, \nu))(x) = 1. \tag{25}$$

Therefore φ and φ_1 form a fundamental system of solutions of Eq. (1).

Write $\varphi(x, \nu) = x^{\nu-\frac{1}{2}}(1 + g(x))$. From (4), we conclude that $g(x) = O(x^{2\varepsilon})$ and that there is a positive h (depending on ν) such that $|g(x)| \leq \frac{1}{2}$ for $0 < x \leq h$. Defining χ for $0 < x \leq h$ by

$$\chi(x) = \varphi_1(x, \nu)/\varphi(x, \nu)$$

we obtain from (25) the relation

$$\chi'(x) = (x\varphi(x, \nu))^{-2}$$

i. e.,

$$\chi'(x) - x^{-2\nu-1} = -g(x)(1 + g(x))^{-1}x^{-2\nu-1}.$$

Thus, there is a positive M such that

$$|\chi'(x) - x^{-2\nu-1}| \leq Mx^{-2(\mathrm{Re}\,\nu-\varepsilon)-1}$$

for $0 < x \leq h$. From this estimate, we obtain by integration from x to h, x tending to 0:

$$\left|\chi(x) + \frac{1}{2\nu}x^{-2\nu} + O(1)\right| \leq \frac{1}{2}M\,(\mathrm{Re}\,\nu - \varepsilon)^{-1}(x^{-2(\mathrm{Re}\,\nu-\varepsilon)} + O(1))$$

when $\mathrm{Re}\,\nu > \varepsilon$, and

$$|\chi(x) + \ln x + O(1)| \leq M(2\varepsilon)^{-1}x^{2\varepsilon}$$

when $\nu = 0$. Therefore, for x near 0,

$$\chi(x) = -\frac{1}{2\nu}x^{-2\nu}(1 + O(x^{2\varepsilon}))$$

if $\mathrm{Re}\,\nu > \varepsilon$, and

$$\chi(x) = \ln x + O(1)$$

if $\nu = 0$. From these last estimates, and from (4) again, we find that φ_1 has the announced asymptotic properties. Q. e. d.

We notice the following consequence of Lemmata 1 and 2: *φ has the following uniqueness property: if* $\mathrm{Re}\,\nu \geq 0$, *and if φ_2 is a solution in $\mathscr{C}_2(\mathrm{R}_+)$ of Eq.* (1) *with*

$$\lim_{x\to 0} x^{-\nu+\frac{1}{2}}\varphi_2(x, \nu) = 1,$$

then, $\varphi_2(x, \nu) = \varphi(x, \nu)$ for all positive x.

2.3 The Transformation Kernel

For the proof of Lemma 3, which states the existence and lists certain properties of the *transformation kernel*, we need the following results, due for the main part to R. Paley and N. Wiener ([13]) ([14]). Call $\mathscr{H}$ the set of the functions f which are holomorphic for Re $\nu > 0$, and such that there exists a number M (depending on f) with

$$\int_{-\infty}^{+\infty} |f(\sigma + i\tau)|^2 d\tau \leq \mathrm{M}$$

for $\sigma > 0$. Then the following is true: $\mathscr{H}$ is identical with the set of the Laplace transforms $\int_0^\infty e^{-p\nu} g(p)dp$ of the functions g of $\mathscr{L}_2(0, \infty)$. From this fact, we deduce, using the Fourier-Plancherel theorem, that the functions $\tau \to f(\sigma + i\tau)$ converge in $\mathscr{L}_2(-\infty, \infty)$ when σ goes to zero to a limit $\tau \to f(i\tau)$; this limiting function is also the $\mathscr{L}_2(-\infty, \infty)$-limit of $\tau \to \int_0^\omega e^{-i\tau p} g(p)dp$, when ω goes to ∞; g itself, which we identify in a natural way to an element of $\mathscr{L}_2(-\infty, \infty)$, can be viewed as an $\mathscr{L}_2(-\infty, \infty)$-limit of $p \to (2\pi)^{-1} \int_{-\omega}^{\omega} e^{i\tau p} f(i\tau)d\tau$ as ω tends to ∞.

That these results are useful for the proof of Lemma 3 was noticed by Regge in Ref. ([6]), who uses the well-known similarity between the Mellin and the Fourier transformations (see Appendix I for details).

LEMMA 3 ([15]). — *Let* V *be as in Lemma* 1. *There exists a real valued function* L_{10}, *continuous on* $\{(x, y) | 0 < y \leq x\}$, *with the following properties:*

$$(xy)^{\frac{1}{2}} | \mathrm{L}_{10}(x, y) | \leq (xy)^{\varepsilon} \mathrm{K}(x), \tag{26}$$

where

$$\mathrm{K}(x) = \frac{1}{2} \mathrm{F}(\varepsilon, x)(1 + \varepsilon^{-1} x^{2\varepsilon} \mathrm{F}(\varepsilon, x) \exp((2\varepsilon)^{-1} x^{2\varepsilon} \mathrm{F}(\varepsilon, x)));$$

([13]) R. E. A. C. PALEY and N. WIENER, *Fourier Transforms in the Complex Domain* (Amer. Math. Soc. Coll. Publ. XIX, New York, 1934), Introduction.

([14]) See also G. DOETSCH, *Handbuch der Laplace-Transformation* (Birkhäuser, Basel, 1950), I. Bd. 12. Kap.

([15]) The *transformation kernel* L_{01} transforms the eigenfunctions of the differential operator τ_0 (i. e. τ with V = 1) into eigenfunctions of the differential operator τ (with « arbitrary » V).

for Re $\nu > -\varepsilon$,

$$\varphi(x, \nu) = x^{\nu-\frac{1}{2}} + \int_0^x \mathrm{L}_{10}(x, y) y^{\nu-\frac{1}{2}} dy, \tag{27}$$

where φ *is the regular solution of* (1); *and*

$$\lim_{y \to x-0} \mathrm{L}_{10}(x, y) = (2x)^{-1} \int_0^x u(\mathrm{V}(u) - 1) du. \tag{28}$$

Proof: For $x > 0$, define the functions f_{x0} and f_x on $\{\nu \mid \mathrm{Re}\, \nu \geq 0\}$ by

$$f_{x0}(\nu + \varepsilon) = (2\nu)^{-1} \int_0^x u^{-1} q(u)(1 - (u/x)^{2\nu}) du$$

and

$$f_x(\nu + \varepsilon) = x^{-\nu+\frac{1}{2}} \varphi(x, \nu) - 1 - f_{x0}(\nu + \varepsilon)$$

(as before, $q(u) = u^2(\mathrm{V}(u) - 1)$). From Lemma 1, we know that f_{x0} and f_x are holomorphic for Re $\nu > 0$, and continuous and bounded for Re $\nu \geq 0$, and that they satisfy

$$f_{x0}(\nu^*) = (f_{x0}(\nu))^* \quad \text{and} \quad f_x(\nu^*) = (f_x(\nu))^*.$$

We know also that they are subjected to the following inequalities ($\sigma > 0$):

$$| f_{x0}(\sigma + i\tau) | \leq | f_{x0}(i\tau) | \leq \rho(\varepsilon, -\varepsilon + i\tau) \mathrm{M}_x$$

and

$$| f_x(\sigma + i\tau) | \leq | f_x(i\tau) | \leq \frac{1}{2} (\rho(\varepsilon, -\varepsilon + i\tau))^2 \mathrm{N}_x,$$

where

$$\begin{aligned} \mathrm{M}_x &= x^2 \mathrm{F}(\varepsilon, x) \exp((2\varepsilon)^{-1} x^{2\varepsilon} \mathrm{F}(\varepsilon, x)), \\ \mathrm{N}_x &= x^2 \mathrm{F}(\varepsilon, x) \mathrm{M}_x \end{aligned}$$

and, as before

$$\begin{aligned} \rho(\varepsilon, \nu) &= \min \{ (2\varepsilon)^{-1}, |\nu|^{-1} \} \\ \mathrm{F}(\varepsilon, x) &= \int_0^x u^{-1-2\varepsilon} | q(u) | du. \end{aligned}$$

Since both functions $\tau \to \rho(\varepsilon, -\varepsilon + i\tau)$ and $\tau \to (\rho(\varepsilon, -\varepsilon + i\tau))^2$ are in $\mathscr{L}_2(-\infty, \infty)$, we can apply the result of Paley and Wiener to f_{x0} and to f_x: f_{x0}, resp., f_x, is the Laplace transform of a function g_{x0}, resp. g_x, in $\mathscr{L}_2(0, \infty)$. We know that in $\mathscr{L}_2(-\infty, \infty)$,

$$g_{x0}(p) = \lim_{\omega \to \infty} (2\pi)^{-1} \int_{-\omega}^{\omega} e^{i\tau p} f_{x0}(i\tau) d\tau;$$

we can therefore take

$$g_{x0}(p) = \begin{cases} 0, & p < 0 \\ \dfrac{1}{2} e^{\varepsilon p} \displaystyle\int_0^{x\exp(-p/2)} u^{-1} q(u) du, & p \geq 0. \end{cases}$$

Notice that

i)
$$\lim_{p \to +0} g_{x0}(p) = \frac{1}{2} \int_0^x u^{-1} q(u) du,$$

ii) $(x, p) \to g_{x0}(p)$ is continuous in $\{(x, p) \mid x \geq 0, p \geq 0\}$,

iii)
$$|g_{x0}(p)| \leq \frac{1}{2} x^{2\varepsilon} \mathrm{F}(\varepsilon, x e^{-p/2}),$$

iv) $g_{x0}(p)$ is real.

Since the function $\tau \to f_x(i\tau)$ is not only in $\mathscr{L}_2(-\infty, \infty)$, but also in $\mathscr{L}_1(-\infty, \infty)$, we can take

$$g_x(p) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{i\tau p} f_x(i\tau) d\tau;$$

since $f_x(-i\tau) = \bar{f_x}(i\tau)$, $g_x(p)$ is real. Noticing that

$$(\rho(\varepsilon, -\varepsilon + i\tau))^2 \leq (\tau^2 + \varepsilon^2)^{-1},$$

we get

$$|g_x(p)| \leq \frac{1}{2} \varepsilon^{-1} \mathrm{N}_x.$$

Since for each τ, $f_y(i\tau)e^{i\tau q}$ tends to $f_x(i\tau)e^{i\tau p}$ when (y, q) tends to (x, p), and since, by choosing $a > x$, we get

$$|f_y(i\tau)e^{i\tau q}| \leq (\rho(\varepsilon, -\varepsilon + i\tau))^2 \mathrm{N}_a$$

as soon as y is close enough to x, we find using Lebesgue's theorem that the function $(x, p) \to g_x(p)$ is continuous in

$$\{(x, p) \mid x \geq 0, \quad \text{all} \quad p\}.$$

By the result of Paley and Wiener, we also know that $g_x(p)$ vanishes for $p < 0$, and thus $\lim_{p \to 0} g_x(p) = 0$. Put

$$h_x(p) = g_{x0}(p) + g_x(p).$$

We have for $\operatorname{Re} \nu > -\varepsilon$:

$$\varphi(x, \nu) = x^{\nu - \frac{1}{2}} + x^{\nu - \frac{1}{2}} \int_0^{\infty} h_x(p) e^{-p(\nu + \varepsilon)} dp;$$

or, with

$$L_{01}(x, y) = (xy)^{-\frac{1}{2}}(y/x)^{\varepsilon}h_x(ln(x/y))$$

for $0 < y \leq x$:

$$\varphi(x, \nu) = x^{\nu-\frac{1}{2}} + \int_0^x L_{10}(x, y)y^{\nu-\frac{1}{2}}dy.$$

All the stated properties of L_{10} are now easily established. Q. e. d.

The function L_{10} is what we call the *transformation kernel.* We notice the two following consequences of (26):
for $a > 0$,

$$\int_0^a dx \int_0^x dy\,|L_{10}(x, y)|^2 < \infty; \tag{29}$$

for $\sigma > -\varepsilon$,

$$\int_0^x |L_{10}(x, y)|\,y^{\sigma-\frac{1}{2}}dy < \infty. \tag{30}$$

2.4 The Inverse of the Transformation Kernel

For $a > 0$, the mapping $f \to \Lambda_0(f)$, defined for f in $\mathscr{L}_2(0, a)$ by

$$\Lambda_0(f)(x) = \int_0^x L_{01}(x, y)f(y)dy, \tag{31}$$

is a Hilbert-Schmidt operator of $\mathscr{L}_2(0, a)$ into itself, since Λ_0 is an integral operator with a square integrable kernel.

LEMMA 4. — *If* V *is as in Lemma* 1, *then for* $a > 0$, *the mapping* $f \to f + \Lambda_0(f)$ *of* $\mathscr{L}_2(0, a)$ *into itself has a bounded inverse* $f \to f + \Lambda_1(f)$, *where* Λ_1 *is a Hilbert-Schmidt operator. There is a real valued continuous function* L_{01} *on* $\{(x, y)\,|\,0 < y \leq x\}$ *such that*

$$\Lambda_1(f)(x) = \int_0^x L_{01}(x, y)f(y)dy. \tag{32}$$

Beside satisfying the condition

$$\int_0^a dx \int_0^x |L_{01}(x, y)|^2dy < \infty, \tag{33}$$

L_{01} *has the following properties:*

$$(xy)^{\frac{1}{2}}\,|L_{01}(x, y)| \leq (xy)^{\varepsilon}K(x) \exp\left((2\varepsilon)^{-1}x^{2\varepsilon}K(x)\right) \tag{34}$$

(K *as in Lemma* 2); *for* Re $\nu > -\varepsilon$,

$$x^{\nu-\frac{1}{2}} = \varphi(x, \nu) + \int_0^x \mathrm{L}_{01}(x, y)\varphi(y, \nu)dy. \tag{35}$$

Proof: For k integer, ≥ 1, Λ_0^k is a Hilbert-Schmidt operator, the kernel of which we denote by $l^{(k)}$. We have the following relations

$$l^{(1)}(x, y) = \begin{cases} 0 & 0 < x < y \leq a \\ \mathrm{L}_{01}(x, y) & 0 < y \leq x \leq a \end{cases}$$

(a. e.) $$l^{(k+1)}(x, y) = \int_0^a l(x, z)l^{(k)}(z, y)dz \qquad (k \geq 1)$$

One can easily show by induction that $l^{(k)}(x, y)$ is real and vanishes for $x < y$; since then

$$l^{(k+1)}(x, y) = \begin{cases} 0 & x < y \\ \displaystyle\int_y^x l(x, z)l^{(k)}(z, y)dz & x \geq y \end{cases}$$

we see that $l^{(k)}$ does not depend on a. Put

$$\mathrm{M}_k(a) = \int_0^a dx \int_0^a dy\, |\, l^{(k)}(x, y)\,|^2.$$

It is easy to show, again by induction, that

$$\mathrm{M}_k(a) \leq (k!)^{-1}(\mathrm{M}_1(a))^k$$

(The above inequality is true for $k = 1$. Assume that it is true for $k = n - 1$. Then

$$\mathrm{M}_n(a) = \int_0^a dx \int_0^a dy\, |\, l^{(n)}(x, y)\,|^2 = \int_0^a dx \int_0^x dy\, |\, l^{(n)}(x, y)\,|^2$$
$$= \int_0^a dx \int_0^x dy \left| \int_y^x l(x, z)l^{(n-1)}(z, y)dz \right|^2$$

But, by Schwartz inequality,

$$\left| \int_y^x l(x, y)l^{(n-1)}(z, y)dz \right|^2 \leq \int_y^x |\, l(x, z)\,|^2 dz \int_y^x |\, l^{(n-1)}(z, y)\,|^2 dz.$$

Also,

$$\int_y^x |\, l(x, z)\,|^2 dz \leq \int_0^x |\, l(x, z)\,|^2 dz = \partial_x\left(\int_0^x dy \int_0^y dz\, |\, l(y, z)\,|^2 \right) = \partial_x \mathrm{M}_1(x).$$

Thus,

$$M_n(a) \leq \int_0^a dx \partial_x M_1(x) \int_0^x dy \int_y^x dz |l^{(n-1)}(z, y)|^2 = \int_0^a dx M_{n-1}(x) \partial_x M_1(x).$$

With

$$M_{n-1}(x) \leq ((n-1)!)^{-1}(M_1(x))^{n-1},$$

we get

$$M_n(x) \leq (n!)^{-1}(M_1(x))^n.)$$

As the norm of Λ_0^k does not exceed $(M_k(x))^{\frac{1}{2}}$, we see that the Neumann series

$$1 - \Lambda_0 + \Lambda_0^2 - \Lambda_0^3 + \ldots$$

converges in the operator norm topology. Its sum is the inverse $(1 + \Lambda_0)^{-1}$ of $1 + \Lambda_0$. This inverse is bounded. One can write

$$(1 + \Lambda_0)^{-1} = 1 + \Lambda_1,$$

with

$$\Lambda_1 = - \Lambda_0(1 + \Lambda_0)^{-1}$$

Since Λ_0 is a Hilbert-Schmidt operator and $(1 + \Lambda_0)^{-1}$ is bounded, Λ_1 is itself a Hilbert-Schmidt operator.

The series

$$- l^{(1)}(x, y) + l^{(2)}(x, y) - l^{(3)}(x, y) + \ldots$$

converges in $\mathscr{L}_2(\{(x, y) | 0 \leq x \leq a, 0 \leq y \leq a\})$ to the kernel which corresponds to Λ_1. For $0 < x < y$, it converges to 0. Let us show that, for $0 < y \leq x \leq a$, it converges uniformly to a continuous function $L_{01}(x, y)$. We assert that for $k \geq 1$, $l^{(k)}(x, y)$ is continuous for $0 < y \leq x \leq a$, and moreover, that in the same region, the following estimate holds (K as in Lemma 2):

$$(xy)^{\frac{1}{2}} | l^{(k)}(x, y) | \leq (xy)^{\varepsilon} K(x)((k-1)!)^{-1}((2\varepsilon)^{-1} x^{2\varepsilon} K(x))^{k-1}.$$

Both statements can be proved by induction, the first by a repeated use of Lebesgue's theorem, the other by straightforward calculations starting from (26) (notice that $K(z) \leq K(x)$ follows from $z \leq x$). The continuity of $L_{01}(x, y)$ and inequality (34) follow from these facts. Notice that $L_{01}(x, y)$ does not depend on a and that it is real.

For Re $\nu > 0$, $x^{\nu - \frac{1}{2}}$ is in $\mathscr{L}_2(0, a)$; thus, it follows from (27) that, for these values of ν,

$$x^{\nu - \frac{1}{2}} = \varphi(x, \nu) + \int_0^x L_{01}(x, y) \varphi(y, \nu) dy.$$

But, because of (34) and of (3), the integrals converge uniformly for Re $\nu \geq -\eta$ if η is less than ε, thus representing an holomorphic function for Re $\nu > -\varepsilon$. As a consequence, the above relation is valid for Re $\nu > -\varepsilon$, as stated in the Lemma. Q. e. d.

Notice that for any positive a, the adjoint Λ_0^+ of Λ_0 is given by

$$\Lambda_0^+(f)(x) = \int_x^a f(y)\mathrm{L}_{10}(y, x)dy; \tag{36}$$

that the adjoint Λ_1^+ of Λ_1 is given by

$$\Lambda_1^+(f)(x) = \int_x^a f(y)\mathrm{L}_{01}(y, x)dy; \tag{37}$$

and that the mapping $1 + \Lambda_0^+$ has an inverse, which is $1 + \Lambda_1^+$.

2.5 The Gel'fand-Levitan Symmetric Kernel

Let us agree on the following: in $\{(x, y) \mid x > 0,\ y > 0\} = \mathrm{R}_+ \times \mathrm{R}_+$, we define a function which is identical with L_{10} in $\{(x, y) \mid 0 < y \leq x\}$ and which is zero otherwise; we shall again denote this function by L_{10}. We do the same with L_{01}. We then define on $\mathrm{R}_+ \times \mathrm{R}_+$ a function L_{10}^+ by putting

$$\mathrm{L}_{10}^+(x, y) = \mathrm{L}_{10}(y, x); \tag{38}$$

we define L_{01}^+ from L_{01} in an analogous way. We shall denote by $\mathrm{L}_{01}\mathrm{L}_{01}^+$ the function defined by

$$\mathrm{L}_{01}\mathrm{L}_{01}^+(x, y) = \int_0^\infty \mathrm{L}_{01}(x, z)\mathrm{L}_{01}^+(z, y)dz = \int_0^{\min\{x,y\}} \mathrm{L}_{01}(x, z)\mathrm{L}_{01}(y, z)dz \tag{39}$$

(the integral converges by inequality (29)). Since $\mathrm{L}_{01}\mathrm{L}_{01}^+$ is continuous (which we could prove by using Lebesgue's theorem), since L_{01}, resp. L_{01}^+, vanishes for $x < y$, resp. $x > y$, and is continuous for $x \geq y$, resp. $x \leq y$, and since

$$\mathrm{L}_{01}(x, x) = \mathrm{L}_{01}^+(x, x),$$

there exists on $\mathrm{R}_+ \times \mathrm{R}_+$ one and only one continuous function, which we call F, and which agrees with

$$\mathrm{L}_{01} + \mathrm{L}_{01}^+ + \mathrm{L}_{01}\mathrm{L}_{01}^+$$

on $\{(x, y) \mid (x, y) \text{ in } \mathrm{R}_+ \times \mathrm{R}_+, x \neq y\}$. We notice that

$$\mathrm{F}(x, y) = \mathrm{F}(x, y).$$

F will be called the *Gel'fand-Levitan symmetric kernel.* We mention two other properties of F. Since for $a > 0$ L_{01}, L_{01}^{+}, and (by (39)) $L_{01}L_{01}^{+}$ are in $\mathscr{L}_2(\{(x, y) | 0 < x \leq a, 0 < y \leq a\})$, the same is true of F. From the estimate (34) for L_{01}, and from analogous estimates for L_{01}^{+} and $L_{01}L_{01}^{+}$ which follow from their definitions, we get for $x \geq y$

$$(xy)^{\frac{1}{2}} |F(x, y)| \leq (xy)^{\varepsilon} K'(x)(1 + (2\varepsilon)^{-1}(xy)^{\varepsilon} K'(y)) \tag{40}$$

$[K'(x) = K(x) \exp((2\varepsilon)^{-1} x^{2\varepsilon} K(x))]$.

It can easily be checked that the Hilbert-Schmidt operator

$$\Phi = \Lambda_1 + \Lambda_1^{+} + \Lambda_1 \Lambda_1^{+},$$

mapping $\mathscr{L}_2(0, a)$ into itself, is given by

$$\Phi(f)(x) = \int_0^a F(x, y) f(y) dy, \tag{41}$$

(use (32) (37), the analog of (38) for L_{01}, and (39)). One sees that the bounded mapping $1 + \Phi$ has a bounded inverse; this follows from the fact that $1 + \Phi$ is the product of $1 + \Lambda_1$ by its adjoint $1 + \Lambda_1^{+}$, which both have a bounded inverse (namely $1 + \Lambda_0$, resp. $1 + \Lambda_0^{+}$). We have the following relation

$$(1 + \Lambda_0)(1 + \Phi) = 1 + \Lambda_1^{+}$$

i. e.,

$$\Phi + \Lambda_0 + \Lambda_0 \Phi = \Lambda_1^{+}.$$

Thus, for $a > 0$, we have in $\mathscr{L}_2(\{x, y) | 0 < x \leq a, 0 < y \leq a\})$ (i. e., a. e.).

$$F(x, y) + L_{10}(x, y) + \int_0^x L_{10}(x, z) F(z, y) dz = L_{01}^{+}(x, y)$$

(use (31), (37), (41), and the definition of L_{01}^{+}). Since the left hand side is continuous for $0 < y \leq x$ (prove the continuity of the integral by using the estimates (26) and (40) and Lebesgue's theorem), and since the right hand side is zero for $0 < y < x$, we obtain for $0 < y \leq x$ the *fundamental equation* (introduced in Ref. [8]):

$$F(x, y) + L_{10}(x, y) + \int_0^x L_{10}(x, z) F(z, y) dz = 0. \tag{42}$$

Looking at (42) as an integral equation depending on the parameter x, we get the following result (use the symmetry of F, Eq. (41), and the fact that $1 + \Phi$ has a bounded inverse as a mapping of $\mathscr{L}_2(0, x)$ into itself):

LEMMA 5. — *For $x > 0$, the function $y \to L_{10}(x, y)$ is the unique solution in $\mathscr{L}_2(0, x)$ of Eq.* (42).

Let us write down the functions L_{10}, L_{01} and F in the case where $V(x)=0$ for all x. For $0 < y \leq x$, we find

$$L_{10}(x, y) = -(xy)^{-\frac{1}{2}} \sum_{k=1}^{\infty} (\Gamma(k))^{-1} J_k(x)\left(\frac{1}{2} y\right)^k$$

and

$$L_{01}(x, y) = (xy)^{-\frac{1}{2}} \sum_{k=1}^{\infty} (\Gamma(k))^{-1}\left(\frac{1}{2} x\right)^k J_k(y),$$

where $(xy)^{-\frac{1}{2}}$ is taken to be positive. The definition of the Bessel functions J_ν can be found in Ref. ([16]). These series can be rearranged to give

$$L_{10}(x, y) = -\frac{1}{2} y^{\frac{1}{2}}(x-y)^{-\frac{1}{2}} J_1(x^{\frac{1}{2}}(x-y)^{\frac{1}{2}})$$

and

$$L_{01}(x, y) = \frac{1}{2} x^{\frac{1}{2}}(y-x)^{-\frac{1}{2}} J_1(y^{\frac{1}{2}}(y-x)^{\frac{1}{2}}),$$

where $x^{\frac{1}{2}}$ and $y^{\frac{1}{2}}$ are taken to be positive (the signs of $(x-y)^{\frac{1}{2}}$ and $(y-x)^{\frac{1}{2}}$ do not matter, since J_1 is an odd function). For $x > 0$, $y > 0$, we find

$$F(x, y) = -\frac{1}{2} i J_1(i(xy)^{\frac{1}{2}})$$

($(xy)^{\frac{1}{2}}$ positive).

In the same case, the regular solution φ_0 of Eq. (1) is given by

$$\varphi_0(x, \nu) = 2^\nu \Gamma(\nu+1) x^{-\frac{1}{2}} J_\nu(x)$$

From the expressions given above and from Eqs. (27) and (35), we obtain the following formulæ:

$$2^\nu \Gamma(\nu+1) J_\nu(x) = x^\nu - \frac{1}{2} x^{\frac{1}{2}} \int_0^x (x-y)^{-\frac{1}{2}} J_1(x^{\frac{1}{2}}(x-y)^{\frac{1}{2}}) y^\nu dy$$

and

$$x^{\nu-\frac{1}{2}} = \varphi_0(x, \nu) + \frac{1}{2} x^{\frac{1}{2}} \int_0^x (y-x)^{-\frac{1}{2}} J_1(y^{\frac{1}{2}}(y-x)^{\frac{1}{2}}) \varphi_0(y, \nu) dy;$$

both are valid for $x > 0$ and $\mathrm{Re}\, \nu > -1$.

([16]) Bateman Manuscript Project, *Higher Transcendental Functions* (McGraw-Hill Book Co., New York, 1953), vol. 2, Chap. VII.

2.6 The Outgoing and Ingoing Solutions

Having thus obtained all the results we need about the regular solution of Eq. (1), we turn now our attention to the *outgoing* and *ingoing* solutions of the same equation.

LEMMA 6. — *If* V (*locally integrable on* R_+) *is such that for* $x > 0$,

$$\int_x^\infty |V(u)|\,du < \infty,$$

then for every complex ν, *Eq.* (1) *has a fundamental system of solutions* $\psi_+(x, \nu)$ *and* $\psi_-(x, \nu)$ *in* $\mathscr{C}_2(R_+)$ *with the following properties*

$$\left(G(x, \lambda) = \int_x^\infty |\lambda u^{-2} + V(u)|\,du\right):$$

a)
$$\psi_-(x, \nu^*) = (\psi_+(x, \nu))^*; \tag{43}$$
$$\psi'_-(x, \nu^*) = (\psi'_+(x, \nu))^*; \tag{43'}$$

b)
$$|\psi_+(x, \nu)| \le x^{-1} \exp\left(G\left(x, \nu^2 - \frac{1}{4}\right)\right) \tag{44}$$

$$|x\psi_+(x, \nu) - e^{ix}| \le G\left(x, \nu^2 - \frac{1}{4}\right) \exp\left(G\left(x, \nu^2 - \frac{1}{4}\right)\right) \tag{45}$$

(*and analogous estimates for* ψ_-);

c)
$$|\psi'_+(x, \nu)| \le \left(G\left(x, \nu^2 - \frac{1}{4}\right) + 2x^{-1}\right) \exp\left(G\left(x, \nu^2 - \frac{1}{4}\right)\right) \tag{46}$$

$$|x\psi'_+(x, \nu) - ie^{ix}| \le \left(G\left(x, \nu^2 - \frac{1}{4}\right) + x^{-1}\right) \exp\left(G\left(x, \nu^2 - \frac{1}{4}\right)\right) \tag{47}$$

d) For fixed $x > 0$, ψ_+, ψ_-, ψ'_+ *and* ψ'_- *are entire functions of* ν^2.

Proof: α) If ψ in $\mathscr{C}_2(R_+)$ is a solution of Eq. (1) for a given value of ν, then χ, defined by

$$\chi(x) = x\psi(x),$$

is a function in $\mathscr{C}_2(R_+)$ which satisfies (a. e.)

$$\partial_x^2\chi(x) + \chi(x) = p\left(x, \nu^2 - \frac{1}{4}\right)\chi(x) \tag{48}$$

with

$$p(x, \lambda) = V(x) + \lambda x^{-2}.$$

Let a and r be positive numbers; let $\mathscr{C}$ be the Banach space of the complex valued functions χ which are continuous and bounded on $D = \{(x, \lambda) \mid x \text{ real}, x \geq 0, \lambda \text{ complex}, |\lambda| \leq r\}$, and whose norm is

$$\|\chi\| = \sup \{|\chi(x, \lambda)| \mid (x, \lambda) \text{ in } D\}.$$

For (x, λ) in D, and for χ in $\mathscr{C}$, the functions $u \to p(u, \lambda)\chi(u, \lambda) \sin u$ and $u \to p(u, \lambda)\chi(u, \lambda) \cos u$ are integrable on $x \leq u < \infty$, so that we can define

$$B(\chi)(x, \lambda) = \cos x \int_x^\infty p(u, \lambda)\chi(u, \lambda) \sin u du - \sin x \int_x^\infty p(u, \lambda)\chi(u, \lambda) \cos u du \quad (49)$$

or

$$B(\chi)(x, \lambda) = \int_x^\infty \sin (u - x)p(u, \lambda)\chi(u, \lambda)du. \quad (50)$$

We obviously have

$$|B(\chi)(x, \lambda)| \leq G(x, \lambda) \sup \{|\chi(y, \lambda)| \mid y \geq x\} \quad (51)$$

To show the continuity of $B(\chi)$, we write

$$B(\chi)(x, \lambda) = \int_a^\infty K(x, \lambda, u)du$$

with $$K(x, \lambda, u) = \begin{cases} \sin (u - x)p(u)\chi(u, \lambda) & \text{for} \quad u \geq x \\ 0 & \text{for} \quad u < x \end{cases}$$

and we remark that for almost all u the function $(x, \lambda) \to K(x, \lambda, u)$ is continuous, and that, given (x, λ), there is a neighborhood of (x, λ) and an integrable function of u such that, for (y, μ) in this neighborhood, $K(x, \lambda, u)$ is bounded by this function; the continuity of $B(\chi)$ follows then by Lebesgue's theorem. Together with inequality (51), this fact shows that $B : \chi \to B(\chi)$ is a bounded linear mapping of $\mathscr{C}$ into itself, the norm of which satisfies

$$\|B\| \leq \int_a^\infty |V(u)| du + r/a$$

(see definition of G). From (49), we see that $B(\chi)$ is an absolutely continuous function of x for λ fixed; we have

$$\partial_x B(\chi)(x, \lambda) = - \sin x \int_x^\infty p(u, \lambda)\chi(u, \lambda) \sin u du - \cos x \int_x^\infty p(u, \lambda)\chi(u, \lambda) \cos u du \quad (52)$$

or

$$\partial_x B(\chi)(x, \lambda) = -\int_x^\infty \cos(u - x)p(u, \lambda)\chi(u, \lambda)du.$$

Thus,

$$|\partial_x B(\chi)(x, \lambda)| \leq G(x, \lambda) \sup\{|\chi(x, \lambda)| \mid y \geq x\}. \tag{53}$$

From (52), it follows that $\partial_x B(\chi)$ is itself an absolutely continuous function of x, and we get (a. e.)

$$\partial_x^2 B(\chi)(x, \lambda) = -B(\chi)(x, \lambda) + p(x, \lambda)\chi(x, \lambda). \tag{54}$$

If, for fixed x, χ is holomorphic in λ for $|\lambda| < r$, then the same is true of $B(\chi)$. (To see this, consider in $|\lambda| < r$ an arbitrary closed rectifiable contour of finite length and compute $\oint B(\chi)(x, \lambda)d\lambda$ on this contour. Since

$$\oint B(\chi)(x, \lambda)d\lambda = \oint d\lambda \int_x^\infty \sin(u - x)p(u, \lambda)\chi(u, \lambda)du,$$

since the second integral converges absolutely in the order in which it is written, and since its integrand is jointly measurable in u and λ (λ being on the contour) we can interchange the order of integration by Fubini's theorem, so that

$$\oint B(\chi)(x, \lambda)d\lambda = 0.$$

This last result, together with the fact that $B(\chi)$ is continuous in λ, implies the holomorphy of $B(\chi)$ by Morera's theorem.) The same is also true for $\partial_x B(\chi)$ (analogous argument!). We remark further that if χ satisfies the *reality condition*

$$\chi(x, \lambda^*) = (\chi(x, \lambda))^*,$$

then $B(\chi)$ and $\partial_x B(\chi)$ will also satisfy it. For the powers of B, one finds by iteration that

$$|B^n(\chi)(x, \lambda)| \leq (n!)^{-1}(G(x, \lambda))^n \sup\{|\chi(y, \lambda)| \mid y \geq x\} \tag{55}$$

thus,

$$\| B^n \| \leq (n!)^{-1}\left(\int_a^\infty |V(u)|\,du + r/a\right)^n.$$

Therefore, the Neumann series

$$1 + B + B^2 + \ldots$$

converges in the operator norm topology to the bounded inverse of $1 - \mathrm{B}$. From (55), we get

$$|(1 - \mathrm{B})^{-1}(\chi)(x, \lambda)| \leq \exp(\mathrm{G}(x, \lambda)) \sup \{ |\chi(y, \lambda)| \mid y \geq x \}. \tag{56}$$

It also follows from (55) that if χ is, for fixed x, holomorphic in $|\lambda| < r$, then the same will be true of $(1 - \mathrm{B})^{-1}(\chi)$ (since the last function is the limit of a uniformly convergent series of holomorphic functions); furthermore, if χ satisfies a reality condition, the same will be true of $(1 - \mathrm{B})^{-1}(\chi)$.

β) From what was said above, it follows that the integral equation

$$\chi = \chi_0 + \mathrm{B}(\chi) \tag{57}$$

where

$$\chi_0(x, \lambda) = e^{ix}$$

has for every pair of positive numbers (a, r) a unique solution in the corresponding Banach space $\mathscr{C}$; we call it χ_+; it is given by

$$\chi_+ = (1 - \mathrm{B})^{-1}(\chi_0) = \chi_0 + \mathrm{B}(\chi_0) + \mathrm{B}^2(\chi_0) + \ldots \tag{58}$$

Since neither B nor χ_0 depend explicitly on the pair (a, r), two solutions χ_+ corresponding to two different pairs (a, r) are identical where they are both defined. From now on, χ_+ will denote the unique continuous function on $\{ (x, \lambda) \mid x \text{ real}, x > 0, \lambda \text{ complex} \}$ which agrees with the solutions (58) of (57) in the particular domain of definition of each of them. From (57), it follows that for fixed λ, χ_+ is in $\mathscr{C}_2(\mathrm{R}_+)$, since it is the sum of two such functions. From

$$\partial_x^2(\chi_0) = -\chi_0$$

and from (57) and (54), we see that χ_+ is a solution of Eq. (48) $\left(\text{with } v^2 = \lambda + \dfrac{1}{4}\right)$. Since χ_0 is entire in λ for x fixed, the same is true for χ_+; this is also true for $\partial_x \chi_+$, as one sees from (57). Using (58), (56) and the fact that

$$|\chi_0(x, \lambda)| = 1,$$

we obtain

$$|\chi_+(x, \lambda)| \leq \exp(\mathrm{G}(x, \lambda)). \tag{59}$$

From this inequality, from (51) and from (57), we get

$$|\chi_+(x, \lambda) - e^{ix}| \leq \mathrm{G}(x, \lambda) \exp(\mathrm{G}(x, \lambda)). \tag{60}$$

From (57), we furthermore get that

$$\partial_x \chi_+ = i\chi_0 + \partial_x \mathrm{B}(\chi_+);$$

thus, with (53) and (59), we get

$$|\partial_x \chi_+(x, \lambda) - ie^{ix}| \leq G(x, \lambda) \exp(G(x, \lambda)) \tag{61}$$

Define χ_- by

$$\chi_-(x, \lambda) = (\chi_+(x, \lambda^*))^*.$$

We convince ourselves easily that χ_- is the unique solution (in the sense explained just below Eq. (57)) of the integral Eq. (57) with

$$\chi_0(x, \lambda) = e^{-ix}.$$

From (59), (60) and (61), we derive easily analogous estimates for χ_- and $\partial_x \chi_-$.

γ) Finally, define ψ_+ and ψ_- by

$$\psi_+(x, \nu) = x^{-1}\chi_+\left(x, \nu^2 + \frac{1}{4}\right)$$

$$\psi_-(x, \nu) = x^{-1}\chi_-\left(x, \nu^2 + \frac{1}{4}\right).$$

Then, all the statements of Lemma 6 follow easily from the preceding arguments, except for the one which says that ψ_+ and ψ_- form a fundamental system of solutions of Eq. (1). But that this also is true follows from the fact that the constant

$$W(\psi_+(\cdot, \nu), \psi_-(\cdot, \nu))(x)$$

is equal to $-2i$. This is readily proved using the definition of ψ_+, ψ_- and χ_- in terms of χ_+, the estimates (60) and (61), as well as the fact that $G(x, \lambda) = o(1)$ when x goes to infinity. Q. e. d.

We take note of the relation

$$W(\psi_+(\cdot, \nu), \psi_-(\cdot, \nu))(x) = -2i. \tag{62}$$

Using again the fact that $G(x, \lambda)$ goes to 0 as x goes to infinity (λ fixed), we find that

$$\lim_{x\to\infty} W(\psi_+(\cdot, \nu), \psi_+(\cdot, \mu))(x) = 0 \tag{63}$$

(and the analogue with ψ_-), and

$$\lim_{x\to\infty} W(\psi_+(\cdot, \nu), \psi_-(\cdot, \mu))(x) = -2i. \tag{64}$$

2.7 The Jost Functions

We study now the coefficients of the linear combination of the ingoing and outgoing solutions of Eq. (1) which represents the regular solution. These coefficients, which are functions of ν, are related in a simple way to the scattering amplitude, i. e., to the sequence $\{s_l\}_{l=0}^{\infty}$ (cf. Section 1), and its Regge interpolation σ.

We assume that the potential V is a locally integrable function on $\mathbf{R}_+$, which satisfies

$$\int_1^{\infty} |V(x)|\, dx < \infty, \tag{65}$$

and is such that there exists a positive number ε with

$$\int_0^1 x^{1-2\varepsilon} |V(x) - 1|\, dx < \infty. \tag{66}$$

Thus, Lemmata 1 to 6 can be used.

Define $\overline{\varphi}$ as the solution of Eq. (1) which for Re $\nu > 0$ satisfies the condition

$$\lim_{x\to 0} x^{-\nu+\frac{1}{2}}\overline{\varphi}(x, \nu) = 2^{-\nu}(\Gamma(\nu+1))^{-1} \tag{67}$$

and which for fixed x is holomorphic for Re $\nu > -\varepsilon$ and continuous for Re $\nu \geq -\varepsilon$. From the remark at the end of Subsection 2.2, it follows that

$$\overline{\varphi}(x, \nu) = 2^{-\nu}(\Gamma(\nu+1))^{-1}\varphi(x, \nu) \tag{68}$$

for $x > 0$, Re $\nu \geq -\varepsilon$.

Define $\overline{\psi}_+$ and $\overline{\psi}_-$ as the solutions of Eq. (1) which satisfy for all ν

$$\lim_{x\to\infty} \left| xe^{-ix}\overline{\psi}_+(x, \nu) - (2/\pi)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\pi\left(\nu+\frac{1}{2}\right)i\right)\right| = 0$$

resp.

$$\lim_{x\to\infty} \left| xe^{ix}\overline{\psi}_-(x, \nu) - (2/\pi)^{\frac{1}{2}} \exp\left(+\frac{1}{2}\pi\left(\nu+\frac{1}{2}\right)i\right)\right| = 0$$

From Lemma 6, it follows that

$$\overline{\psi}_+(x, \nu) = (2/\pi)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\pi\left(\nu+\frac{1}{2}\right)i\right)\psi_+(x, \nu) \tag{69}$$

$$\overline{\psi}_-(x, \nu) = (2/\pi)^{\frac{1}{2}} \exp\left(+\frac{1}{2}\pi\left(\nu+\frac{1}{2}\right)i\right)\psi_-(x, \nu) \tag{70}$$

For $V = 0$, we have the well known relations between $\overline{\varphi}$, $\overline{\psi}_+$ and $\overline{\psi}_-$ and the usual Bessel functions (see Ref. ([16]))

$$\begin{aligned} \overline{\varphi}(x, \nu) &= x^{-\frac{1}{2}} J_\nu(x) \\ \overline{\psi}_+(x, \nu) &= x^{-\frac{1}{2}} H_\nu^{(1)}(x) \\ \overline{\psi}_-(x, \nu) &= x^{-\frac{1}{2}} H_\nu^{(2)}(x) \end{aligned}$$

The coefficients we spoke about are defined by ([17]):

$$\overline{\alpha}(\nu) = \frac{1}{4} \pi i \, W(\overline{\varphi}(\cdot, \nu), \overline{\psi}_-(\cdot, \nu))(x) \tag{71}$$

$$\overline{\beta}(\nu) = -\frac{1}{4} \pi i \, W(\overline{\varphi}(\cdot, \nu), \overline{\psi}_+(\cdot, \nu))(x) \tag{72}$$

(the fact that $\overline{\alpha}$ and $\overline{\beta}$ are independent of x stems from the fact that $W(u, v)(x)$ is constant for any two solutions u and v of Eq. (1) for the same value of ν^2). We have

$$\overline{\varphi}(x, \nu) = \overline{\alpha}(\nu)\overline{\psi}_+(x, \nu) + \overline{\beta}(\nu)\overline{\psi}_-(x, \nu) \tag{73}$$

From these definitions, and from Lemmata 1 and 6, we obtain easily that

a) $$\overline{\alpha}(\nu^*) = (\overline{\beta}(\nu))^*; \tag{74}$$

b) *$\overline{\alpha}$ and $\overline{\beta}$ are holomorphic for* $\mathrm{Re}\ \nu > -\varepsilon$, *and continuous for* $\mathrm{Re}\ \nu \geq -\varepsilon$;
c) *$\overline{\alpha}$ and $\overline{\beta}$ are not (identically) zero.*

We need some other, less immediate, results (cf. Regge, Ref. ([6])):

d) *For* $|\mathrm{Re}\ \nu| \leq \varepsilon$, *we have the relation*

$$e^{i\pi\nu}\overline{\beta}(\nu)\overline{\alpha}(-\nu) - e^{-i\pi\nu}\overline{\beta}(-\nu)\overline{\alpha}(\nu) = \frac{1}{4}(e^{i\pi\nu} - e^{-i\pi\nu}) \tag{75}$$

Proof: Use (73) for ν and $-\nu$, and apply (23). Remember that

$$\Gamma(\nu)\Gamma(1-\nu) = \pi (\sin \pi\nu)^{-1}$$

and

$$\Gamma(\nu + 1) = \nu\Gamma(\nu). \qquad \text{Q. e. d.}$$

e) *$\overline{\alpha}$ has no zeroes in the closed set* $\{ \nu \,|\, \mathrm{Re}\ \nu \geq 0,\ \mathrm{Im}\ \nu \geq 0 \}$.

Proof: α) Using the estimates (3) and (44) (for ψ_+ and for ψ_-), the definitions (68), (69) and (70), and the relation (73), we see that

$$\overline{\varphi}(x, \mu)\overline{\varphi}(x, \nu)$$

([17]) The functions $\overline{\alpha}$ and $\overline{\beta}$ are closely related to the so-called *Jost functions*.

is integrable on R_+ for Re $\nu \geq -\varepsilon$, Re $\mu \geq -\varepsilon$, provided Re $(\nu + \mu) > 0$. Inserting $\overline{\varphi}(x, \nu)$ for u_ν and $\overline{\varphi}(x, \mu)$ for u_μ in Eq. (2′), we get (use the definitions (68), (69) and (70), the estimates (4) and (7), and the relations (63), (64) and (73)):

$$(\nu^2 - \mu^2) \int_0^\infty \overline{\varphi}(x, \mu)\overline{\varphi}(x, \nu)dx = (4i/\pi)(e^{\frac{1}{2}\pi(\mu-\nu)i}\overline{\beta}(\mu)\overline{\alpha}(\nu) - e^{-\frac{1}{2}\pi(\mu-\nu)i}\overline{\alpha}(\mu)\overline{\beta}(\nu))$$

Put $\mu = \sigma - i\tau$, $\nu = \sigma + i\tau$ (σ and τ real, σ positive). Remembering (8), (68) and (74), and the fact that

$$\Gamma(\nu^*) = (\Gamma(\nu))^*,$$

we get from the previous relation

$$\pi\sigma\tau \int_0^\infty |\overline{\varphi}(x, \sigma + i\tau)|^2 dx = e^{\pi\tau}|\overline{\alpha}(\sigma + i\tau)|^2 - e^{-\pi\tau}|\alpha(\sigma - i\tau)|^2 \tag{76}$$

Since, in view of (67), $\overline{\varphi}(\cdot, \sigma + i\tau)$ is not (identically) zero for $\sigma > 0$, we see that $\overline{\alpha}$ has no zeroes in $\{ \nu \mid \text{Re } \nu > 0, \text{ Im } \nu > 0 \}$

β) If Im $\nu = 0$, we have, for $\nu \geq -\varepsilon$:

$$\overline{\varphi}(x, \nu) = \overline{\alpha}(\nu)\overline{\psi}_+(x, \nu) + (\overline{\alpha}(\nu))^*\overline{\psi}_-(x, \nu)$$

(use (73) and (74)). Since, because of (67), $\overline{\varphi}(\cdot, \nu)$ is not (identically) zero for ν real (except perhaps for $\nu = -1, -2, \ldots$), it follows that $\overline{\alpha}$ has no zeroes for ν real and $\geq -\varepsilon$ (except perhaps for $\nu = -1, -2, \ldots$).

γ) If Re $\nu = 0$, we combine (74) and (75) to obtain

$$e^{\pi\tau}|\overline{\alpha}(i\tau)|^2 - e^{-\pi\tau}|\overline{\alpha}(-i\tau)|^2 = \frac{1}{2}\,\text{Sh}\ \pi\tau \tag{77}$$

($\nu = i\tau$, τ real). Thus, $\overline{\alpha}$ has no zeroes for $\nu = i\tau$, τ real and positive. Q. e. d.

We notice the relation:

$$\pi\sigma\tau \int_0^\infty |\overline{\varphi}(x, \sigma + i\tau)|^2 dx = e^{\pi\tau}|\overline{\alpha}(\sigma + i\tau)|^2 - e^{-\pi\tau}|\overline{\alpha}(\sigma - i\tau)|^2 \tag{78}$$

For the proof of property f) below we need the following: *There is a positive number* M *such that, for* Re $\nu \geq 0$, $|\nu| \geq 1$,

$$|\overline{\alpha}(\nu)e^{-\frac{1}{2}\pi\nu i}| \leq M\,|2^\nu\Gamma(\nu + 1)|^{-1}|\nu|^{\text{Re}\,\nu}|\nu|^{5/2}\exp(3|\nu|/2) \tag{79}$$

Proof (Regge, Ref. (6)): From (71), (68) and (70), we obtain, for $x > 0$ and Re $\nu \geq 0$,

$$|\overline{\alpha}(\nu)e^{-\frac{1}{2}\pi\nu i}| \leq (\pi/8)^{\frac{1}{2}}(2^\nu\Gamma(\nu+1))^{-1}x^2(|\varphi(x, \nu)\psi'_-(x, \nu)| + |\varphi'(x, \nu)\psi_-(x, \nu)|)$$

Following Regge, we put $x = |\nu|$ in the right-hand side of this inequality. For φ and φ', we use the estimates (3) and (6). Noting that the assumptions of Lemma 1 are still true if one replaces ε by η with $0 < \eta \leq \varepsilon$, and that, for fixed x and ν, the right-hand side of inequalities (3) and (6) are then continuous functions of η for $0 \leq \eta \leq \varepsilon$, we put in them $\eta = 0$. We then remark that

$$\mathrm{E}(0, x, \nu) = |\nu|^{-1}\mathrm{F}(0, x)$$

and that, for $x \geq 1$

$$\left|\mathrm{F}(0, x) - \frac{1}{2}x^2\right| \leq x\int_1^\infty |\mathrm{V}(u)|\,du + \int_0^1 u\,|\mathrm{V}(u)|\,du$$

(we have namely

$$\mathrm{F}(0, x) = \int_0^x u\,|\mathrm{V}(u) - 1|\,du;$$

thus

$$\mathrm{F}(0, x) - \frac{1}{2}x^2 = \int_0^x u(|\mathrm{V}(u) - 1| - 1)du;$$

but obviously, we have

$$|\,|a - b| - |a|\,| \leq |b|,$$

so that

$$\left|\mathrm{F}(0, x) - \frac{1}{2}x^2\right| \leq \int_0^x u\,|\mathrm{V}(u)|\,du, \quad \text{etc.}).$$

Therefore, there are positive numbers M_1 and M_2 such that, for $|\nu| > 0$ and $\mathrm{Re}\,\nu \geq 0$:

$$|\varphi(|\nu|, \nu)| \leq \mathrm{M}_1\,|\nu|^{\mathrm{Re}\,\nu - \frac{1}{2}}e^{\frac{1}{2}|\nu|}$$
$$|\varphi'(|\nu|, \nu)| \leq \mathrm{M}_2\,|\nu|^{\mathrm{Re}\,\nu + \frac{1}{2}}e^{\frac{1}{2}|\nu|}.$$

For ψ_- and ψ'_-, we use the estimates of Lemma 6. We use the fact that, for $x \geq 1$,

$$\left|\mathrm{G}\left(x, \nu^2 - \frac{1}{4}\right) - |\nu|^2x^{-1}\right| \leq \frac{1}{4} + \int_1^\infty |\mathrm{V}(u)|\,du.$$

Thus, we find that there are positive numbers M_3 and M_4 such that, for $\mathrm{Re}\,\nu \geq 0$, and $|\nu| \geq 1$,

$$|\psi_-(|\nu|, \nu)| \leq \mathrm{M}_3\,|\nu|^{-1}e^{|\nu|}$$
$$|\psi'_-(|\nu|, \nu)| \leq \mathrm{M}_4\,|\nu|\,e^{|\nu|}$$

Inequality (79) follows easily from these estimates. Q. e. d.

f) *There are two positive numbers* M′ *and* R *such that* Re $\nu \geq 0$ *and* $|\nu| \geq$ R *imply*

$$|\bar{\alpha}(\nu)e^{-\frac{1}{2}\pi\nu i}| \leq \mathrm{M}' e^{\frac{1}{2}(\pi+3)|\nu|} \text{ [18]} \tag{80}$$

Proof: Using (79), and the well known asymptotic expansion of log Γ (ν), which is (uniformly in $|\arg \nu| \leq \pi - \delta$, δ positive but otherwise arbitrary):

$$\log \Gamma(\nu) = \nu \log \nu - \nu + 0\,(\log |\nu|),$$

we see that there are two positive numbers M′ and R such that Re $\nu \geq 0$ and $|\nu| \geq$ R imply

$$|\bar{\alpha}(\nu)e^{-\frac{1}{2}\pi\nu i}| \leq \mathrm{M}' \exp\,(|\nu|(\varphi \sin \varphi + \cos \varphi(1 - \log 2) + 3/2))$$

where $\nu = |\nu| e^{i\varphi}, |\varphi| \leq \frac{1}{2}\pi$. The expression in φ takes its maximum at $|\varphi| = \frac{1}{2}\pi$; this maximum is $\frac{1}{2}(\pi + 3)$. Q. e. d.

The *Regge interpolation* σ is then defined by

$$\sigma(\nu) = \bar{\alpha}(\nu)(\bar{\beta}(\nu))^{-1}. \tag{81}$$

From the properties of $\bar{\alpha}$ and $\bar{\beta}$, we deduce the following well known facts:

g) σ *is meromorphic for* Re $\nu > -\varepsilon$.
h) σ *has no poles in* $\{\nu \,|\, \mathrm{Re}\, \nu \geq 0,\ \mathrm{Im}\, \nu \leq 0\}$.
i) $\sigma(\nu^*)(\sigma(\nu))^* = 1$.

Let us remind that the scattering amplitude F, as a function of the cosine z of the center-of-mass scattering angle θ, is given by

$$\mathrm{F}(z) = \sum_{l=0}^{\infty} (2l + 1)a_l \mathrm{P}_l(z) \tag{82}$$

where

$$a_l = \frac{1}{2} i\left(1 - \sigma\left(l + \frac{1}{2}\right)\right) \qquad (l = 0, 1, \ldots)$$

((82) is a formal expression only, since we have not proven the convergence of the Legendre series. As a matter of fact, our assumptions on V are too

[18] $\frac{1}{2}(\pi + 3) < \pi$, so that Carlson's theorem can be applied to $\bar{\alpha}$.

weak to imply the convergence of this series). In what follows, by « scattering amplitude », we understand the sequence $s = \{ s_l \}$, defined by

$$s_l = \sigma\left(l + \frac{1}{2}\right), \qquad l = 0, 1, 2, \ldots \tag{83}$$

2.8 The Method of Gel'fand and Levitan. An Overall View

Up to this point of section 2, we have progressed from the potential V, a coefficient in the differential Eq. (1), to the functions related to the scattering amplitude, namely $\bar{\alpha}$ and σ. In other words, we were solving the « direct » scattering problem. In order to trace our way back from $\bar{\alpha}$ and σ to the coefficient V, we shall use, as many others before us, the method due to Gel'fand and Levitan. In this, we follow rather closely the treatment by Regge in Ref. [6].

Let us remind the reader of the steps in which this method may be decomposed, and let us indicate to him how the results of the next few subsections are related to its development.

In a first step (Subsections 2.9 and 2.10) we shall obtain an expansion theorem for « arbitrary » functions in terms of those distinguished solutions of Eq. (1) (here: the « regular » solutions described in Lemma 1) for which the existence of the transformation kernel has been established. We shall see that the « spectral function », or better, the *spectral data* [19] which appear in the expansion theorem *are known as soon as one is given* $\bar{\alpha}$ *for* Re $\nu \geq 0$. In order to derive the needed expansion theorem, which is formulated in Lemma 8, we shall associate to the formal differential operator τ a self-adjoint operator T on $\mathscr{L}_2(0, \infty)$ (see Lemma 7), and then use one of the many methods which allow to prove expansion theorems in eigenfunctions of a self-adjoint differential operator.

In a second step (Subsection 2.11), we take advantage of the relation between the Gel'fand-Levitan kernel F and the kernels L_{01} and L_{01}^{+} (see Subsection 2.5), as well as of the expansion theorem stated in Lemma 8 to obtain an expression for F in terms of the spectral data and of the regular solutions of Eq. (1), with τ *being replaced by* $\tau_0 = -\dfrac{d}{dx}\left(x^2 \dfrac{d}{dx}\right) - \dfrac{1}{4}$. This expression for F is given in Lemma 9.

[19] We imitate the phrase « scattering data » which is used by Agranovich and Marchenko in Ref. [10].

The third step carries us from the spectral data to the potential V. As soon as Lemma 9 is proved, all we need here is already in our hands: Lemma 9 gives us F from the spectral data, Lemma 5 tells us that the function L_{10} is uniquely defined by the kernel F through the fundamental Eq. (42); the potential V is then calculated according to Eq. (28) of Lemma 3.

2.9 The Method of Gel'fand and Levitan. First Step. The Self-Adjoint Extensions of the Differential Operator τ

We shall not reproduce here a detailed proof of the results collected in Lemma 9. However, we hope that the indications we shall give below will enable the interested reader to reconstruct our argument, with the help of classical results from the theory of ordinary differential equation, in particular Green's formula (2), and with the help of the book of Akhiezer and Glazman ([20]) (we also used a review by B. Malgrange ([21])).

We shall assume of course that our potential V satisfies conditions (65) and (66), so that all the results obtained so far can be used.

We prefer to use in the following subsections the functions

$$\alpha(\nu) = \frac{1}{2} i\mathrm{W}(\varphi(\cdot, \nu), \psi_-(\cdot, \nu))$$

and

$$\beta(\nu) = -\frac{1}{2} i\mathrm{W}(\varphi(\cdot, \nu), \psi_+(\cdot, \nu))$$

in lieu of $\bar{\alpha}$ and $\bar{\beta}$. We have therefore

$$\varphi(x, \nu) = \alpha(\nu)\psi_+(x, \nu) + \beta(\nu)\psi_-(x, \nu) \tag{84}$$

and

$$\alpha(\nu) = (2/\pi)^{\frac{1}{2}} e^{-\frac{1}{2}\pi(\nu+\frac{1}{2})i} 2^\nu \Gamma(\nu + 1)\bar{\alpha}(\nu)$$
$$\beta(\nu) = (2/\pi)^{\frac{1}{2}} e^{+\frac{1}{2}\pi(\nu+\frac{1}{2})i} 2^\nu \Gamma(\nu + 1)\bar{\beta}(\nu);$$

α and β are seen to be holomorphic in the same domain, Re $\nu > -\varepsilon$, as $\bar{\alpha}$ and $\bar{\beta}$, except perhaps for simple poles at $\nu = -1, -2, \ldots$, coming from the Γ-function. We also have

$$\alpha(\nu^*) = (\beta(\nu))^*.$$

([20]) N. I. ACHIESER und I. M. GLASMANN, *Theorie der linearen Operatoren im Hilbert-Raum* (Akademie-Verlag, Berlin, 1954).

([21]) B. MALGRANGE, Equations de Sturm-Liouville, Séminaire Bourbaki, exposé n° 65 (Secrétariat Mathématique, Paris, 1959), 2nd ed.

Eq. (76) may be rewritten as

$$2\sigma\tau \int_0^\infty |\varphi(x, \sigma + i\tau)|^2 dx = |\alpha(\sigma + i\tau)|^2 - |\beta(\sigma + i\tau)|^2 \tag{85}$$

(σ, τ real, σ positive). From Eq. (77), we obtain

$$\tau = |\alpha(i\tau)|^2 - |\beta(i\tau)|^2 \tag{86}$$

(τ real).

In this subsection, $\mathscr{L}_2$ will stand for $\mathscr{L}_2(0, \infty)$. For f and g in $\mathscr{L}_2$, we shall define

$$(f, g) = \int_0^\infty (f(x))^* g(x) dx$$

and

$$\| f \|^2 = (f, f).$$

If f denote a complex valued function on some set X, f^* will denote the function defined by

$$f^*(x) = (f(x))^*$$

(x in X).

Let θ be a real number. We define ψ and ψ_1 by

$$\psi(x, \lambda) = \frac{1}{2}(\psi_+(x, \nu)e^{+i\theta} + \psi_-(x, \nu)e^{-i\theta}) \tag{87}$$

and

$$\psi_1(x, \lambda) = -\frac{1}{2} i(\psi_+(x, \nu)e^{+i\theta} - \psi_-(x, \nu)e^{-i\theta}), \tag{88}$$

where $\lambda = -\nu^2$. From Lemma 6, we deduce that ψ and ψ_1 are entire functions of λ (for x fixed), that

$$\psi(x, \lambda^*) = (\psi(x, \lambda))^* \tag{89}$$

and

$$\psi_1(x, \lambda^*) = (\psi_1(x, \lambda))^*, \tag{90}$$

θ being real, and that

$$\lim_{x\to\infty} |\psi(x, \lambda) - x^{-1} \cos(x+\theta)| = \lim_{x\to\infty} |\psi_1(x, \lambda) - x^{-1} \sin(x+\theta)| = 0$$

(with analogous results for the x-derivatives).
From (62), (63) and (64), we obtain

$$W(\psi(\cdot, \lambda), \psi_1(\cdot, \lambda)) = 1, \tag{91}$$

which shows that ψ and ψ_1 form a fundamental system of solutions of

$$\tau(\varphi) = \lambda\varphi;$$

we also find that

$$\lim_{x\to\infty} \mathrm{W}(\psi(\cdot, \lambda), \psi_1(\cdot, \mu))(x) = 1 \tag{92}$$

and

$$\lim_{x\to\infty} \mathrm{W}(\psi(\cdot, \lambda), \psi(\cdot, \mu))(x) = \lim_{x\to\infty} \mathrm{W}(\psi_1(\cdot, \lambda), \psi_1(\cdot, \mu))(x) = 0 \tag{93}$$

We define further u and v for Re $\nu \geq -\varepsilon$ by

$$\varphi(x, \nu) = u(\nu)\psi(x, \lambda) + v(\nu)\psi_1(x, \lambda) \tag{94}$$

$(\lambda = -\nu^2)$. We have the following formulæ:

$$u(\nu) = \alpha(\nu)e^{-i\theta} + \beta(\nu)e^{+i\theta} \tag{95}$$

and

$$v(\nu) = i(\alpha(\nu)e^{-i\theta} - \beta(\nu)e^{+i\theta}) \tag{96}$$

We see that u and v are holomorphic for Re $\nu > -\varepsilon$; since θ is real, we also see that

$$u(\nu^*) = (u(\nu))^* \tag{97}$$

and

$$v(\nu^*) = (v(\nu))^* \tag{98}$$

From Eq. (85), we get (use Eqs. (95) to (98)),

$$2\sigma\tau \int_0^\infty |\varphi(x, \sigma + i\tau)|^2 dx = \mathrm{Im}\,(v(\sigma + i\tau)u(\sigma - i\tau))$$

$(\sigma, \tau$ real, σ positive); and from Eq. (86),

$$\tau = \mathrm{Im}\,(v(i\tau)u(-i\tau)) \tag{99}$$

(τ real). Therefore, the zeroes of u and v in Re $\nu \geq 0$ are real. These zeroes are simple, since $v(\sigma) = 0$ and $\sigma > 0$ imply

$$2\sigma \int_0^\infty |\varphi(x, \sigma)|^2 dx = v'(\sigma)u(\sigma) > 0 \tag{100}$$

(v': derivative of v), and $v(0) = 0$ implies

$$1 = v'(0)u(0) > 0$$

(and analogous results if $u(\sigma) = 0$).

For Im $\lambda \neq 0$, we define G_λ by

$$G_\lambda(x, y) = \begin{cases} -(v(\nu))^{-1}\varphi(x, \nu)\psi(y, \lambda), & 0 < x \leq y \\ -(v(\nu))^{-1}\psi(x, \lambda)\varphi(y, \nu), & 0 < y \leq x \end{cases} \tag{101}$$

(Re $\nu > 0$, $\nu^2 = -\lambda$). For the various values of θ, Green's functions G_λ will appear as the kernel of the resolvent of the various self-adjoint operators to be described in Lemma 7 below.

By $\mathscr{D}_1$, we denote the space of the functions f in $\mathscr{C}_2 \cap \mathscr{L}_2$ such that $\tau(f)$ is in $\mathscr{L}_2$. Here $\mathscr{C}_2$ stands for $\mathscr{C}_2(0, \infty)$.

For f in $\mathscr{L}_2$, we define $G_\lambda(f)$ by

$$G_\lambda(f)(x) = \int_0^\infty G_\lambda(x, y) f(y) dy$$

(the integral converges since $\varphi(x, \nu)$ is in $\mathscr{L}_2$ for Re $\nu > 0$, and since

$$\psi(x, \lambda) \in \mathscr{L}_2(a, \infty)$$

for all positive a, and all λ). It is easily seen that $G_\lambda(f)$ is in $\mathscr{C}_2$, that

$$(\lambda - \tau)(G_\lambda(f))(x) = f(x) \qquad \text{(a. e.)};$$

by showing, for f with its support compact in R_+, that

$$(\tau(G_\lambda(f)), G_\lambda(f)) = (G_\lambda(f), \tau(G_\lambda(f))),$$

and that, therefore,

$$\| G_\lambda(f) \| \leq | \mathrm{Im}\, \lambda |^{-1} \| f \|,$$

one sees that $\mathscr{D}_\lambda = G_\lambda(\mathscr{L}_2)$ is contained in $\mathscr{D}_1$, and that G_λ defines on $\mathscr{L}_2$ a bounded linear operator. To show that $\mathscr{D}_\lambda$ does not in fact depend on λ, we use Hilbert's relation

$$G_\lambda - G_\mu = (\mu - \lambda) G_\lambda G_\mu$$

(Im $\lambda \neq 0$, Im $\mu \neq 0$), which can be obtained in the following way: it is possible to show that, for f and g in $\mathscr{D}_\lambda + \mathscr{D}_\mu$, $(f, \tau(g)) = (\tau(f), g)$, and therefore, that $(f, \tau(f))$ is real; from this it follows that $\tau(f) = \kappa f$ for f in $\mathscr{D}_\lambda + \mathscr{D}_\mu$ and Im $\kappa \neq 0$ implies $f = 0$; Hilbert's relation is a consequence of this last fact. Thus $\mathscr{D}_\lambda$ is independent of λ; we shall put $\mathscr{D}_\lambda = \mathscr{D}$. It makes sense to define an operator $T: \mathscr{D} \to \mathscr{L}_2$ by $T(f) = \tau(f)$. It is not very difficult to show that T is self-adjoint (use in particular the fact that $G_\lambda^+ = G_{\lambda^*}$, because of the reality properties of φ and ψ). Let f_0 be defined on R_+ by $f_0(x) = x^{-1} \cos(x + \theta)$, and let h be a C^∞-function on R_+ with

$h(x) = 0$ for $x \leq 1$, and $h(x) = 1$ for $x \geq 2$, say. Then, for f in $\mathscr{D}_1$, Green's formula gives

$$\lim_{x\to\infty} \mathrm{W}(f, f_0)(x) = \int_0^\infty \tau(f)(x)h(x)f_0(x)dx - \int_0^\infty f(x)\tau(hf_0)(x)dx$$

so that we can define on $\mathscr{D}_1$ a linear form B by

$$\mathrm{B}(f) = \lim_{x\to\infty} \mathrm{W}(f, f_0)(x).$$

We see that there are positive numbers a and b such that

$$|\mathrm{B}(f)| \leq a \| \tau(f) \| + b \| f \|.$$

Let us show that $\mathscr{D}$ is identical with the kernel of B. The last inequality shows that, if $\mathrm{Im}\ \mu \neq 0$, $f \to \mathrm{B}(\mathrm{G}_\mu(f))$ is a continuous linear form on $\mathscr{L}_2$ (because of $\| \tau(\mathrm{G}_\mu(f)) \| = \| f - \mu \mathrm{G}_\mu(f)) \| \leq (1 + |\mu|\ |\mathrm{Im}\ \mu|^{-1} \| f \|)$. It is easily verified that $\mathrm{B}(\mathrm{G}_\mu(f)) = 0$ for f in $\mathscr{L}_2$ with its support compact in R_+. Since such f's form a dense set in $\mathscr{L}_2$, the continuity of $f \to \mathrm{B}(\mathrm{G}_\mu(f))$ implies $\mathrm{B}(\mathrm{G}_\mu(f)) = 0$ for all f in $\mathscr{L}_2$. Therefore, $\mathscr{D}$ is in the kernel of B. Conversely: it is easily shown that, for f in $\mathscr{D}_1$, there is a complex number γ such that

$$f = \mathrm{G}_\mu(g) + \gamma\varphi(\cdot, \nu)$$

where $g = \mu f - \mathrm{G}_\mu(f)$, and $\mathrm{Re}\ \nu > 0$, $\nu^2 = -\mu$. Thus, $\mathrm{B}(f) = 0$ implies $\gamma \mathrm{B}(\varphi(\cdot, \nu)) = 0$. But $\mathrm{B}(\varphi(\cdot, \nu)) = -v(\nu)$ (use Eqs. (94), (87), (88) and the asymptotic results of Lemma 6). Now, $v(\nu) \neq 0$, since $\mathrm{Im}\ \nu \neq 0$; so that $\gamma = 0$, and f is in $\mathscr{D}$.

We collect the results:

LEMMA 7. — *Suppose that* V *is locally integrable, that*

$$\int_1^\infty |\mathrm{V}(x)|\, dx < \infty$$

and that there is a positive number ε *such that*

$$\int_0^1 x^{1-2\varepsilon} |\mathrm{V}(x)|\, dx < \infty;$$

let θ *be real; let* $\mathscr{D} = \mathscr{D}(\theta)$ *be the subspace of those* f *in* $\mathscr{D}_1$ *(defined after Eq.* (101)), *such that*

$$\lim_{x\to\infty} \mathrm{W}(f, x^{-1} \cos(x + \theta)) = 0. \tag{102}$$

Then the operator $\mathrm{T} = \mathrm{T}(\theta)$, *defined on* $\mathscr{D}$ *by* $\mathrm{T}(f) = \tau(f)$, *is self-adjoint.*

Let us make two comments: *i)* let θ and θ_1 be two real numbers; then $T(\theta) = T(\theta_1)$ if and only if $\theta = \theta_1 \bmod \pi$; *ii)* call $\mathcal{D}_0$ the set of the functions of $\mathcal{D}_1$ whose support is compact in R_+; define T_0 on $\mathcal{D}_0$ by $T_0(f) = \tau(f)$; it is now possible to show that the symmetric operator T_0 has no self-adjoint extension other than those described in Lemma 7 ([22]).

To conclude this subsection, we discuss briefly the eigenvalues of $T(\theta)$. These are of course real. The real number λ is an eigenvalue if and only if the differential equation

$$\tau(\varphi) = \lambda\varphi$$

has a nonzero solution in $\mathcal{D}_1$ satisfying condition (102). From Lemmata 1 and 2, we know that the equation above has no nonzero solution in $\mathcal{D}_1$ for $\lambda \geq 0$; from the same Lemmata, and from Lemma 7, we know that, for $\lambda < 0$, we have nonzero solutions in $\mathcal{D}_1$, and that they all are of the form $\gamma\varphi(\cdot, \nu)$, with γ: a constant, and $\nu > 0$, $\nu^2 = -\lambda$. As we saw just before Lemma 7, $\varphi(\cdot, \nu)$ satisfies condition (102) if and only if $v(\nu) = 0$. Thus, the eigenvalues of $T(\theta)$ are negative; λ is an eigenvalue of $T(\theta)$ if and only if $\lambda = -\nu^2$, where ν is a positive zero of v; *the set* $S(\theta)$ *of the eigenvalues of* T *is at most countable, and has no finite accumulation point*, since v is holomorphic for $\operatorname{Re} \nu > -\varepsilon$. As an example, if $V = 0$, then

$$v(\nu) = (2/\pi)^{\frac{1}{2}} 2^{\nu}\Gamma(\nu + 1) \sin \frac{1}{2}\left(\pi\nu + \frac{1}{2}\pi + 2\theta\right)$$

and therefore, the eigenvalues of T are

$$\lambda_k = -\nu_k^2$$

with

$$\nu_k = 2k - \frac{1}{2} - 2\theta/\pi$$

where k is any integer such that ν_k is positive.

Let us show for completeness that *the set* $S(\theta)$ *is actually infinite for every real value of* θ. Put

$$v(\nu, \theta) = i(\alpha(\nu)e^{-i\theta} - \beta(\nu)e^{i\theta})$$

(see Eq. (96)); we are interested, as we saw above, in the pairs of real numbers (ν, θ), with $\nu > 0$, such that

$$v(\nu, \theta) = 0.$$

([22]) Hint: see N. DUNFORD and J. T. SCHWARTZ, *Linear Operators* (Interscience Publishers, New York, 1963), Part. II, Chap. XIII, Sect. 9, Exercise 15.

Let K be the set of the real numbers modulo π; K is differentiable manifold in a natural way. Because

$$\beta(v) = (\alpha(v))^*$$

for v real, there is an unique mapping $v \to \Theta(v)$ of R_+ into K such that

$$\upsilon(v, \Theta(v)) = 0.$$

Since $\upsilon(v, \theta) = 0$ implies that the partial derivatives of υ with respect to v and θ (which are continuous on $\mathrm{R}_+ \times \mathrm{K}$) are both different from 0 (see Eq. (100); use Eqs. (95) and (96)), Θ is locally a diffeomorphism, so that its derivative Θ' exists everywhere on R_+, and is continuous and different from 0; its value is

$$\Theta'(v) = -2v \int_0^\infty |\psi(x, -v^2)|^2 dx$$

(use Eqs. (94) and (100)). Let a be a positive number; we show that Θ maps $\{ v \mid v \geq a \}$ *onto* K. By the definition of ψ, Eq. (87), and Lemma 6, we have

$$|x\psi(x, -v^2) - \cos(x + \theta)| \leq \mathrm{G}\left(x, v^2 - \frac{1}{4}\right) \exp\left(\mathrm{G}\left(x, v^2 - \frac{1}{4}\right)\right),$$

where, for $x \geq bv^2\left(b > 0, v > \dfrac{1}{2}\right)$,

$$\mathrm{G}\left(x, v^2 - \frac{1}{4}\right) \leq b^{-1} + \int_{bv^2}^\infty |\mathrm{V}(y)|\, dy.$$

Take b large enough, in order that

$$b^{-1} \exp(b^{-1}) < \frac{1}{2}.$$

It is then possible to find μ large enough such that $v \geq \mu$ implies

$$\left(b^{-1} + \int_{bv^2}^\infty |\mathrm{V}(y)|\, dy\right) \exp\left(b^{-1} + \int_{bv^2}^\infty |\mathrm{V}(y)|\, dy\right) < \frac{1}{2};$$

thus, for this choice of b,

$$x\,|\psi(x, -v^2)| > |\cos(x + \theta)| - \frac{1}{2}$$

as soon as $\nu \geq \mu_0 = \max(\mu, a)$, and $x \geq b\nu^2$. The last inequality now implies the existence of a positive number M such that $\nu \geq \mu_0$ implies

$$\int_0^\infty |\psi(x, -\nu^2)|^2 dx > \int_{b\nu^2}^\infty |\psi(x, -\nu^2)|^2 dx > \frac{1}{2} M\nu^{-2},$$

and therefore,

$$\Theta'(\nu) < -M\nu^{-1}.$$

We thus obtain, for $\nu_2 > \nu_1 \geq \mu_0$:

$$\Theta(\nu_1) - \Theta(\nu_2) > M(\log \nu_2 - \log \nu_1).$$

This inequality shows that Θ maps each closed interval

$$\{\nu \mid \mu_0 \leq \nu_1 \leq \nu \leq \nu_1 \exp(\pi M^{-1})\}$$

onto K, and therefore (since $\mu_0 \geq a$), Θ maps $\{\nu \mid \nu \geq a\}$ *onto* K. This proves our assertion that $S(\theta)$ is an infinite set for each real value of θ.

2.10 The Method of Gel'fand and Levitan. First Step. The Expansion Theorem

To obtain the expansion theorem stated below in Lemma 9, we find it expedient to use a method of Krein, as described in Ref. ([20]) (Anhang II). In this method, one takes advantage explicitly, and repeatedly, of the spectral decomposition theorem for self-adjoint operators. This is what compelled us to present first Lemma 8, which associates a self-adjoint operator T to the boundary value problem posed by the differential equation

$$\tau(f) = \lambda f$$

and the boundary condition

$$\lim_{x \to \infty} W(f, x^{-1} \cos(x + \theta)) = 0.$$

Following Krein's method, we note first that ψ, as defined in Eq. (87), is the solution of this boundary value problem which satisfies the additional normalization condition

$$\lim_{x \to \infty} |\psi(x, \lambda) - x^{-1} \cos(x + \theta)| = 0$$

(and which is thereby uniquely defined). We then define for f in the space

$\mathscr{F} = \{ f \mid f$ in $\mathscr{L}_2(0, \infty)$; for some $a > 0$, $f(x) = 0$ (a. e.) for $0 < x < a \}$ a function $\Phi(f)$ on the complex plane by

$$\Phi(f)(\lambda) = \int_0^\infty f(x)\psi(x, \lambda)dx$$

One verifies that Φ is a linear map from $\mathscr{F}$ into the space $\mathscr{E}$ of the entire functions. It is called Krein's functional in Ref. ([20]).

Using Φ as in Ref. ([20]), we obtain the following expansion theorem in terms of ψ($\mathscr{L}_2(\rho)$ is the Hilbert space of the functions Φ on R such that

$$\| \Phi \|^2 = \int_{-\infty}^{+\infty} | \Phi(\lambda) |^2 d\rho(\lambda) < \infty):$$

V *and* θ *being as in Lemma* 7, *and* ψ *being given by Eq.* (87), *there is a positive Borel measure* ρ *on* R, *and an isometric isomorphism* U *from* $\mathscr{L}_2(\rho)$ *onto* $\mathscr{L}_2(0, \infty)$, *given by*

$$\mathrm{U}(\Phi)(x) = \lim_{a \to \infty} \int_{-a}^{a} \Phi(\lambda)\psi(x, \lambda)d\rho(\lambda) \tag{103}$$

for Φ *in* $\mathscr{L}_2(\rho)$; *the inverse of* U *is given by*

$$\mathrm{U}^{-1}(f)(\lambda) = \lim_{a \to 0} \int_a^\infty f(x)\psi(x, \lambda)dx \tag{104}$$

for f *in* $\mathscr{L}_2(0, \infty)$ (*the limits are taken in the strong topology in* $\mathscr{L}_2(\rho)$, *resp. in* $\mathscr{L}_2(0, \infty)$). *For the left continuous nondecreasing function* Σ *on* R *which is associated with* ρ (uniquely defined up to an additive constant), *we have the formula*

$$\frac{1}{2}(\Sigma(t + 0) + \Sigma(t)) = \frac{1}{2}(\Sigma(+0) + \Sigma(0)) + \pi^{-1} \lim_{s \to 0} \int_0^t \mathrm{Im}\,(m(u+is))du \tag{105}$$

where ($\mathrm{Im}\,\lambda \neq 0$)

$$m(\lambda) = u(v)/v(v) \tag{106}$$

($\mathrm{Re}\, v > 0$, $v^2 = -\lambda$), u *and* v *being as in Eqs.* (95) *and* (96).

Two remarks: *i*) In the proof of the statement above, the existence of the limits is explicitly derived; *ii*) the relation between Σ and ρ is as follows: for $a < b$,

$$\rho([a, b)) = \Sigma(b) - \Sigma(a).$$

Although the proof of Ref. ([20]) has to be modified in a few points to apply in our case, especially in what concerns the computation of Σ, we shall not give it here.

The following also is true:

U maps the set of those functions Φ of $\mathscr{L}_2(\rho)$ such that

$$\int_{-\infty}^{+\infty} \lambda^2 |\Phi(\lambda)|^2 d\rho(\lambda) < \infty$$

onto $\mathscr{D}$; if f is in $\mathscr{D}$, then $(\mathrm{T}(\theta) = \mathrm{T})$:

$$\mathrm{U}^{-1}(\mathrm{T}(f))(\lambda) = \lambda \mathrm{U}^{-1}(f)(\lambda);$$

if Δ is a Borel set of R, *then* U *maps the set of those functions Φ of $\mathscr{L}_2(\rho)$ which vanish ρ-almost everywhere outside of Δ onto the set* $\mathrm{E}(\Delta)(\mathscr{L}_2(0, \infty))$, *where* E *is the operator valued measure associated with* T *in the spectral decomposition theorem.*

These statements are derived at the same time as the preceding ones.

Combining what we learned in Subsection 2.9 with Eqs. (105) and (106), we obtain for $\lambda \geq 0$ the symbolic relation.

$$d\rho(\lambda) = \pi^{-1} \operatorname{Im} (u(-i\tau)/v(-i\tau)) d\lambda$$

$(\tau \geq 0, \tau^2 = \lambda)$; or, using (97), (98) and (99),

$$d\rho(\lambda) = \pi^{-1}\tau |v(i\tau)|^{-2} d\lambda.$$

For $\lambda < 0$, we get symbolically:

$$\rho(\lambda) = \sum_k (2v_k u(v_k)/v'(v_k))\delta(\lambda + v_k^2)$$

where $\{ v_k \}$ is an enumeration of the positive zeroes of v (notice that $2v_k u(v_k)/v'(v_k)$ turns out to be positive, in agreement with Eq. (100)).

From this point, it is easy to obtain an expansion theorem in terms of the regular solutions of the Schrödinger Eq. (1).

Define a continuous ([23]) function γ on R by

$$\gamma(\tau) = v(-i\tau)/v(i\tau); \tag{107}$$

then the linear operator Γ defined on $\mathscr{L}_2(-\infty, \infty)$ by

$$\Gamma(\Psi)(\tau) = \gamma(\tau)\Psi(-\tau)$$

([23]) The function γ even has a meromorphic continuation to $|\operatorname{Im} \tau| < \varepsilon$. For $-\varepsilon < \operatorname{Im} \tau < 0$, the poles of γ are real and simple, and so on.

is unitary (since $|\gamma(\tau)|^2 = 1$) and satisfies $\Gamma^2 = 1$ (since $\gamma(\tau)\gamma(-\tau) = 1$); therefore

$$P_1 = \frac{1}{2}(1 - \Gamma) \quad \text{and} \quad P_2 = \frac{1}{2}(1 + \Gamma)$$

are symmetric and mutually orthogonal projection operators on $\mathscr{L}_2(-\infty, \infty)$; their sum is 1.

E being as before the spectral measure associated with T, we put

$$E_1 = E([0, \infty)) \quad \text{and} \quad E_2 = E((-\infty, 0)).$$

Define also

$$n(x, \tau) = \frac{1}{2}(\varphi(x, -i\tau) - \gamma(\tau)\varphi(x, i\tau)). \tag{108}$$

Using these definitions and the relation

$$2v\psi(x, -v^2) = v(v)\varphi(x, -v) - v(-v)\varphi(x, v)$$

which follows for $|\operatorname{Re} v| \leq \varepsilon$ from Eq. (94), we obtain after a few unproblematic steps the following result (all limits in the strong topology of the corresponding space):

LEMMA 8A. — (V *and* θ *as in Lemma* 7). *There is a continuous linear mapping* Y_1 *from* $\mathscr{L}_2(-\infty, \infty)$ *into* $\mathscr{L}_2(0, \infty)$, *given by*

$$\begin{aligned} Y_1(\Psi)(x) &= \lim_{a\to\infty} \pi^{-\frac{1}{2}} \int_{-a}^{+a} \Psi(\tau)(n(x, \tau))^* d\tau \\ &= \lim_{a\to\infty} \pi^{-\frac{1}{2}} \int_{-a}^{+a} P_1(\Psi)(\tau)(n(x, \tau))^* d\tau \\ &= \lim_{a\to\infty} \pi^{-\frac{1}{2}} \int_{-a}^{+a} P_1(\Psi)(\tau)\varphi(x, i\tau) d\tau \end{aligned}$$

such that

a) $Y_1(\mathscr{L}_2(-\infty, \infty)) = E_1(\mathscr{L}_2(0, \infty))$;

b) The restriction Y_1' *of* Y_1 *to* $P_1(\mathscr{L}_2(-\infty, \infty))$ *is an isometric isomorphism onto* $E_1(\mathscr{L}_2(0, \infty))$;

c) $Y_1 \circ P_2 = 0$.

There is also a continuous mapping Y_2 *from* $\mathscr{L}_2(0, \infty)$ *into* $\mathscr{L}_2(-\infty, \infty)$, *given by*

$$\begin{aligned} Y_2(f)(\tau) &= \pi^{-\frac{1}{2}} \lim_{a\to 0} \int_a^\infty f(x)n(x, \tau)dx \\ &= \frac{1}{2}\pi^{-\frac{1}{2}} \lim_{a\to 0} \left(\int_a^\infty f(x)\varphi(x, -i\tau)dx - \gamma(\tau) \int_a^\infty f(x)\varphi(x, i\tau)dx \right) \end{aligned}$$

such that

a) $Y_2(\mathscr{L}_2(0, \infty)) = P_1(\mathscr{L}_2(-\infty, \infty))$

b) The restriction Y_2' *of* V_2 *to* $E_1(\mathscr{L}_2(0, \infty))$ *is the inverse of* Y_1',

c) $Y_2 \circ E_2 = 0$.

Let $\{ v_k \}$ be the sequence of the positive zeroes of v, arranged in increasing order. Put

$$d_k = 2v_k(u(v_k)v'(v_k))^{-1} \tag{109}$$

that is, by Eq. (100):

$$d_k^{-1} = \int_0^\infty |\varphi(x, v_k)|^2 dx.$$

Let $l_2(d)$ be the Hilbert space of the sequences $\{ \Psi_k \}$ of complex numbers such that

$$\sum_k d_k |\Psi_k|^2 < \infty$$

$$\left(\text{scalar product: } (\Phi, \Psi) = \sum_k d_k \Phi_k^* \Psi_k\right).$$

LEMMA 8B. — *There is a continuous linear mapping* W_1 *from* $l_2(d)$ *into* $\mathscr{L}_2(0, \infty)$, *given by*

$$W_1(\Psi)(x) = \lim_{n \to \infty} \sum_{k=1}^{n} d_k \Psi_k \varphi(x, v_k).$$

The mapping W_1 *is an isometric isomorphism onto* $E_2(\mathscr{L}_2(0, \infty))$. *There is also a continuous mapping* W_2 *from* $\mathscr{L}_2(0, \infty)$ *onto* $l_2(d)$, *given by*

$$W_2(f)_k = \int_0^\infty f(x)\varphi(x, v_k)dx.$$

We have $W_2 \circ E_1 = 0$. *The restriction* W_2' *of* W_2 *to* $E_2(\mathscr{L}_2(0, \infty))$ *is the inverse of* W_1.

The triple formed by the function γ and the two sequences $\{ v_k \}$ and $\{ d_k \}$ can be called the *spectral data associated with the potential* V *and the parameter* θ.

2.11 The Method of Gel'fand and Levitan. Second Step. The Symmetric Kernel F in Terms of the Spectral Data

LEMMA 9. — (V, θ *as in Lemma 7*). *Let* $(\gamma, \{\nu_k\}, \{d_k\})$ *be the spectral data associated with* V *and* θ; *then*

$$F_{01}(x, y) = -(2\pi)^{-1}\int_{-\infty}^{\infty} \gamma(\tau)(xy)^{i\tau-\frac{1}{2}}d\tau + \sum_{k=1}^{\infty} d_k(xy)^{\nu_k-\frac{1}{2}}. \tag{110}$$

The series converges uniformly for (x, y) *in any compact contained in* $R_+ \times R_+$. *The integral symbolizes a unitary operator* U *defined on* $\mathscr{L}_2(0, \infty)$ *by*

$$U(f)(x) = -(2\pi)^{-\frac{1}{2}} \lim_{a\to\infty} \int_{-a}^{+a} \gamma(\tau)x^{-i\tau-\frac{1}{2}}\Phi(\tau)d\tau$$

(*strong limit in* $\mathscr{L}_2(0, \infty)$), *where*

$$\Phi(\tau) = (2\pi)^{-\frac{1}{2}} \lim_{a\to 0,\ b\to\infty} \int_a^b x^{-i\tau-\frac{1}{2}}f(x)dx$$

(*strong limit in* $\mathscr{L}_2(-\infty, \infty)$).

Proof: Let a be a positive number. Let f be in $\mathscr{L}_2(0, a)$; then f_0, defined by

$$f_0(x) = \begin{cases} f(x) & x < a \\ 0 & \text{elsewhere} \end{cases}$$

is in $\mathscr{L}_2(0, \infty)$. In this natural way, $\mathscr{L}_2(0, a)$ can be considered as a subspace of $\mathscr{L}_2(0, \infty)$. We shall identify f and f_0.

Put $h = f + \Lambda_1^+(f)$, Λ_1^+ being the mapping of $\mathscr{L}_2(0, a)$ into itself defined by Eq. (37) in terms of the kernel L_{01}; we think of Λ_1^+ as being extended in the natural way to a mapping from $\mathscr{L}_2(0, a)$ into $\mathscr{L}_2(0, \infty)$. Put $H = Y_2(h)$, Y_2 being as in Lemma 8; thus H is in $\mathscr{L}_2(-\infty, \infty)$. Put $\Psi = M(f)$, where M is the unitary Mellin transformation of $\mathscr{L}_2(0, \infty)$ onto $\mathscr{L}_2(-\infty, \infty)$:

$$M(f)(\tau) = \Psi(\tau) = (2\pi)^{-\frac{1}{2}} \lim_{b\to 0} \int_b^a f(x)x^{-i\tau-\frac{1}{2}}dx$$

(strong limit in $\mathscr{L}_2(-\infty, \infty)$). We remark first that, for almost all real τ,

$$H(\tau) = \frac{1}{2}\sqrt{2}(\Psi(\tau) - \gamma(\tau)\Psi(-\tau)) \tag{111}$$

i. e., that on $\mathscr{L}_2(0, a)$,

$$Y_2 \circ (1 + \Lambda_1^+) = \sqrt{2}P_1 \circ M \tag{112}$$

(for P_1, see Lemma 8). Statement (111) is easily verified for all real τ if f in $\mathscr{L}_2(0, a)$ vanishes in a neighborhood of 0 by an obvious application of Fubini's theorem, with the help of Eq. (37), Lemma 8A, and Lemma 4. Thus, Eq. (112) is valid on a set dense in $\mathscr{L}_2(0, a)$; since both sides of this equation are continuous on $\mathscr{L}_2(0, a)$, it is verified on all of $\mathscr{L}_2(0, a)$. Since it follows from Lemma 8 that $Y_1 \circ Y_2 = E_1$, we obtain from Eq. (111) the following relation, valid on $\mathscr{L}_2(0, a)$:

$$E_1 \circ (1 + \Lambda_1^+) = \sqrt{2} Y_1 \circ P_1 \circ M = \sqrt{2} Y_1 \circ M. \tag{113}$$

Define for f in $\mathscr{L}_2(0, a)$ the sequence $N_1(f)$ by

$$N_1(f)_k = \int_0^a f(x) x^{\nu_k - \frac{1}{2}} dx. \tag{114}$$

Using Lemma 4 (Eqs. (34) and (35)), we obtain easily

$$N_1(f)_k = W_2((1 + \Lambda_1^+)(f))_k$$

(W_2 as in Lemma 8). Thus, N_1 is seen to be a continuous map from $\mathscr{L}_2(0, a)$ into $l_2(d)$:

$$W_2 \circ (1 + \Lambda_1^+) = N_1.$$

Since $W_1 \circ W_2 = E_2$, we can write

$$E_2 \circ (1 + \Lambda^+) = W_1 \circ N_1 \tag{115}$$

Combining Eqs. (113) and (115), we obtain on $\mathscr{L}_2(0, a)(E_1 + E_2 = 1$ on $\mathscr{L}_2(0, \infty)$, and thus on $\mathscr{L}_2(0, a))$:

$$1 + \Lambda_1^+ = \sqrt{2} Y_1 \circ M + W_1 \circ N_1. \tag{116}$$

Putting $f(x) = 1$ in Eq. (114), we obtain that

$$\sum_{k=1}^{\infty} d_k a^{2\nu_k + 1}/(2\nu_k + 1) < \infty \tag{117}$$

for all positive a; from this fact, we deduce that

$$\sum_{k=1}^{\infty} d_k (xy)^{\nu_k - \frac{1}{2}}$$

converges uniformly for (x, y) in any compact contained in $R_+ \times R_+$ (remember that ν_k tends to ∞ with k). We now turn our attention to the operator $1 + \Lambda_1$ on $\mathscr{L}_2(0, a)$ (see Lemma 4). Since $Y_1 \circ Y_2 = E_1$, and $W_1 \circ W_2 = E_2$ (see Lemma 8), we have $Y_1 \circ Y_2 + W_1 \circ W_2 = 1$. Denoting by P_a the orthogonal projection from $\mathscr{L}_2(0, \infty)$ onto $\mathscr{L}_2(0, a)$, we have on $\mathscr{L}_2(0, a)$:

$$1 + \Lambda_1 = (1 + \Lambda_1) \circ P_a \circ (Y_1 \circ Y_2 + W_1 \circ W_2). \tag{118}$$

We compute first $(1 + \Lambda_1) \circ P_a \circ Y_1 \circ Y_2$. We obtain easily from Lemma 8A the following result

$$(1 + \Lambda_1) \circ P_a \circ Y_1 = \sqrt{2} P_a \circ M^{-1} \circ P_1 \tag{119}$$

where M^{-1} is the inverse of M:

$$M^{-1}(\Phi)(x) = (2\pi)^{-\frac{1}{2}} \lim_{\omega \to \infty} \int_{-\omega}^{+\omega} \Phi(\tau) x^{i\tau - \frac{1}{2}} d\tau$$

(Φ in $\mathscr{L}_2(-\infty, \infty)$; strong limit in $\mathscr{L}_2(0, a)$). (Prove Eq. (119) first on the set $\mathscr{A}$ of those Φ in $\mathscr{L}_2(-\infty, \infty)$ which have compact support, with the help of the definitions and of the estimates of Lemmata 1, 4 and 8A. Use then the fact that the set $\mathscr{A}$ is dense in $\mathscr{L}_2(-\infty, \infty)$, and that the involved operators are continuous). Since $P_1 \circ Y_2 = Y_2$, we get on $\mathscr{L}_2(0, a)$:

$$(1 + \Lambda_1) \circ P_a \circ Y_1 \circ Y_2 = \sqrt{2} P_a \circ M^{-1} \circ Y_2. \tag{120}$$

We compute then $(1 + \Lambda_1) \circ P_a \circ W_1 \circ W_2$; defining the mapping N_2 from $l_2(d)$ into $\mathscr{L}_2(0, a)$ by

$$N_2(\Phi)(x) = \Sigma d_k \Phi_k x^{\nu_k - \frac{1}{2}}$$

$(x < a)$, we obtain

$$(1 + \Lambda_1) \circ P_a \circ W_1 = N_2 = P_a \circ N_2,$$

and therefore

$$(1 + \Lambda_1) \circ P_a \circ W_1 \circ W_2 = P_a \circ N_2 \circ W_2. \tag{121}$$

Introducing (120) and (121) into (118), we obtain

$$1 + \Lambda_1 = P_a \circ (\sqrt{2} \circ M^{-1} \circ Y_2 + N_2 \circ W_2) \tag{122}$$

on $\mathscr{L}_2(0, a)$. Combining now (116) and (122), and using Lemma 8 once more, we get on $\mathscr{L}_2(0, a)$

$$(1 + \Lambda_1) \circ (1 + \Lambda_1^+) = P_a \circ (2M^{-1} \circ P_1 \circ M + N_2 \circ N_1).$$

Remember now the definition of P_1 in term of Γ, and the definition of Φ (Subsection 2.5); we obtain

$$\Phi = P_a \circ (-M^{-1} \circ \Gamma \circ M + N_2 \circ N_1)$$

This last relation, together with Eq. (41), yields Eq. (110). Q. e. d.

2.12 Regge's First Uniqueness Theorem

Let $\mathscr{V}$ be the class of those real valued locally integrable potentials V which satisfy

$$\int_1^\infty |V(x)|\,dx < \infty \tag{65}$$

and

$$\int_0^1 x^{1-2\varepsilon}|V(x) - 1|\,dx < \infty \tag{66}$$

for some positive ε. We can prove the following uniqueness theorem (Regge, Ref. ([6])).

THEOREM 1. — *Let* V_1 *and* V_2 *be in* $\mathscr{V}$; *let* α_1 *and* α_2 *be the corresponding Jost functions. If*

$$\alpha_1(\nu) = \alpha_2(\nu)$$

for Re $\nu \geq 0$, *then*

$$V_1(x) = V_2(x)$$

for almost all positive x.

Proof: Choose a real number θ. From θ and $\alpha_j (j = 1, 2)$, compute u_j and v_j according to Eqs. (95) and (96). From

$$\alpha_1(\nu) = \alpha_2(\nu)$$

for Re $\nu \geq 0$, it follows that

$$u_1(\nu) = u_2(\nu) \quad \text{and} \quad v_1(\nu) = v_2(\nu)$$

in the same region. Therefore, the spectral data $\{ \nu_k \}$ (the sequence of the positive zeroes of v), $\{ d_k \}$ and γ (see Eqs. (107) and (109)) are the same for both potentials (given the value of θ). According to Lemma 9, this last fact implies that both potentials have the same Gel'fand-Levitan kernel F. The rest of the proof was indicated at the end of Subsection 2.8. Q. e. d.

2.13 An Auxiliary Lemma

To conclude Section 2, we show in this and in the next subsection that, if two potentials in the class $\mathscr{V}$ have for Re $\nu > 0$ the same Regge interpolation, then they are equal (almost everywhere). We prove first the following function theoretic lemma:

LEMMA 10. — *Let* $\overline{\alpha}_1$ *and* $\overline{\alpha}_2$ *be two continuous functions on* Re $v \geq 0$, *holomorphic in* Re $v > 0$; *assume that they have the three following properties* ($j = 1, 2$):

a) for real τ,

$$e^{\pi\tau}\,|\,\overline{\alpha}_j(i\tau)\,|^2 - e^{-\pi\tau}\,|\,\overline{\alpha}_j(-i\tau)\,|^2 = \frac{1}{2}\,\mathrm{Sh}\,\pi\tau;$$

b) $\overline{\alpha}_j(v) \neq 0$ *for* v *in* $\{v\,|\,\mathrm{Re}\,v \geq 0,\ \mathrm{Im}\,v \geq 0\}$;

c) $\overline{\alpha}_j$ *is of exponential type in* Re $v \geq 0$, i. e.,

$$\limsup_{r\to\infty} r^{-1} ln \mathrm{M}_j(r) < \infty$$

where

$$\mathrm{M}_j(r) = \sup\left\{\,\overline{\alpha}_j(re^{i\varphi})\,|\,\varphi \text{ real, } |\,\varphi\,| \leq \frac{1}{2}\pi\,\right\}.$$

Let σ_j *be the function defined in* Re $v \geq 0$ *by*

$$\sigma_j(v) = \alpha_j(v)/(\alpha_j(v^*))^* \tag{123}$$

wherever $\overline{\alpha}_j(v^*) \neq 0$. *If*

$$\sigma_2 = \sigma_1 \tag{124}$$

(wherever both are defined), then there are real numbers a and b, $b^2 = 1$, *such that*

$$\alpha_2(v) = be^{av}\alpha_1(v) \tag{125}$$

in Re $v \geq 0$.

Proof: α) We first show that it follows from our assumptions that the function g, defined for $\overline{\alpha}_1(v) \neq 0$ by

$$g(v) = \overline{\alpha}_2(v)/\overline{\alpha}_1(v) \tag{126}$$

has an analytic continuation, also denoted by g, which is an entire function without zeroes, such that

$$g(v^*) = (g(v))^*$$

and

$$|\,g(i\tau)\,| = 1$$

(τ real).

From *b*), it follows that the poles of $\sigma_j(j = 1, 2)$ in Re $v > 0$ (which are all in Im $v > 0$) are the same in location and multiplicity as the zeroes of $\overline{\alpha}_j(v^*)$; therefore, Eq. (124) implies that the zeroes of $\overline{\alpha}_2$ are the same in location and multiplicity as those of $\overline{\alpha}_1$. Thus, g, defined by Eq. (126), is seen to be holomorphic in Re $v > 0$; it has no zeroes in Re $v > 0$. Furthermore, still because of *b*), it can be extended to a continuous function on $\{ \nu \mid \text{Re } \nu \geq 0, \text{Im } \nu \geq 0 \}$, still denoted by g, and given for $\nu = i\tau (\tau \geq 0)$ by

$$g(i\tau) = \bar{\alpha}_2(i\tau)/\bar{\alpha}_1(i\tau); \tag{127}$$

from Eqs. (123), (124) and (126), we also get

$$g(\nu^*) = (g(\nu))^* \tag{128}$$

which shows that g can be extended to a continuous function on $\{ \nu \mid \text{Re } \nu \geq 0 \}$, which we still denote by g. It also follows from *b*) that σ_j can be extended to a continuous function on $\{ \nu \mid \text{Re } \nu \geq 0, \text{ Im } \nu \leq 0 \}$, given for $\nu = -i\tau (\tau \geq 0)$ by

$$\sigma_j(-i\tau) = \bar{\alpha}_j(-i\tau)/(\bar{a}_j(i\tau))^*.$$

Thus, Eq. (124) implies

$$|\bar{\alpha}_2(-i\tau)/\bar{\alpha}_2(i\tau)| = |\bar{\alpha}_1(-i\tau)/\bar{\alpha}_1(i\tau)|$$

for $\tau \geq 0$. Combined with property *a*), this relation gives

$$|\bar{\alpha}_2(i\tau)| = |\bar{\alpha}_1(i\tau)|$$

for τ real. Thus Eq. (127) gives

$$|g(i\tau)| = 1 \tag{129}$$

for $\tau \geq 0$; Eq. (128) shows that the last relation is valid for all real τ. We now use Schwartz' continuation principle: define g for Re $\nu \leq 0$ by

$$g(\nu) = 1/(g(-\nu^*))^*. \tag{130}$$

Since g has no zeroes for Re $\nu \geq 0$, this continuation of g is holomorphic for Re $\nu < 0$, and continuous for Re $\nu \leq 0$; Eq. (129) shows that it is the *analytic* continuation of g from Re $\nu > 0$ to Re $\nu \leq 0$.

Thus, g has all the announced properties, since Eq. (130) remains valid for Re $\nu < 0$, by analytic continuation.

β) Since g is an entire function without zeroes, which is real for real ν, there is a uniquely defined entire function f, real for real ν, such that

$$g(\nu) = g(0) \exp (f(\nu)). \tag{131}$$

The function f satisfies

$$f(\nu^*) = (f(\nu))^*. \tag{132}$$

Eq. (130) yields, because of Eqs. (132) and (129) (for $\tau = 0$):

$$f(-\nu) = -f(\nu) \tag{133}$$

γ) We now obtain bounds on $|ln|g(v)||$.

For $\text{Im } v \geq 0$, $0 \leq \arg v^{\frac{1}{2}} \leq \frac{1}{2}\pi$, we define

$$\zeta_j(v) = \alpha_j(v^{\frac{1}{2}})$$

$(j = 1, 2)$; ζ_j is holomorphic in $\text{Im } v > 0$, and continuous in $\text{Im } v \geq 0$; it has no zero in $\text{Im } v \geq 0$. Define N_j for $r \geq 0$ by

$$\mathrm{N}_j(r) = \sup \{ |\zeta_j(re^{i\theta})| \; | 0 \leq \theta \leq \pi \}.$$

We have

$$\mathrm{N}_j(r) = \mathrm{M}_j(r^{\frac{1}{2}}).$$

Because of property *c*),

$$\limsup_{r \to \infty} r^{-\frac{1}{2}} ln \mathrm{N}_j(r) < \infty \tag{134}$$

whence, *a fortiori*,

$$\limsup_{r \to \infty} r^{-1} ln \mathrm{N}_j(x) < \infty \tag{135}$$

If we define ln^+ for $x > 0$ by

$$ln^+ x = \begin{cases} 0\,, & \text{for} \quad x \leq 1 \\ lnx, & \text{for} \quad x \geq 1 \end{cases}$$

we see from Eq. (134) that

$$\int_{-\infty}^{+\infty} (1 + x^2)^{-1} ln^+ |\zeta_j(x)| \, dx < \infty. \tag{136}$$

Since ζ_j satisfies the general conditions imposed at the beginning of Section 6.3 of Ref. ([24]), it follows from (136) that

$$\int_{-\infty}^{+\infty} (1 + x^2)^{-1} | ln |\zeta_j(x)| \, | dx < \infty, \tag{137}$$

by Theorem 6.3.6 of Ref. ([24]). We can furthermore apply Theorem 6.5.4 of Ref. ([24]) to ζ_j; we obtain for $\text{Im } v > 0$:

$$ln|\zeta_j(v)| = \text{Im } v \left(\pi^{-1} \int_{-\infty}^{+\infty} |x - v|^{-2} ln|\zeta_j(x)| \, dx + c_j \right),$$

where c_j is some real number. Now from definition (126), and from Eq. (129), which shows that

$$|\zeta_2(x)| = |\zeta_1(x)|$$

([24]) R. P. Boas, Jr., *Entire Functions* (Academic Press, New York, 1954).

for $x \leq 0$, we obtain for $0 < \arg v^{\frac{1}{2}} < \frac{1}{2}\pi$

$$ln|g(v^{\frac{1}{2}})| = \text{Im } v\left(\pi^{-1}\int_0^\infty |x-v|^{-2}(ln|\zeta_2(x)| - ln|\zeta_1(x)|)dx + c_2 - c_1\right),$$

whence the estimate

$$|ln|g(v^{\frac{1}{2}})|| \leq \text{Im } v\left(\pi^{-1}\int_0^\infty |x-v|^{-2}\rho(x)dx + |c_2| + |c_1|\right), \tag{138}$$

with

$$\rho(x) = |ln|\zeta_2(x)|| + |ln|\zeta_1(x)||.$$

But, for $|v| \geq \sqrt{2}, \frac{1}{4}\pi \leq \arg v \leq \pi$,

$$|x-v|^2 \geq \begin{cases} (x-1)^2+1 & \text{for} \quad 0 \leq x \leq 2 \\ \frac{1}{2}x^2 & \text{for} \quad x \geq 2 \end{cases}$$

Thus, by Eqs. (137) and (138) there is a positive number C such that, in the same region of the v-plane,

$$|ln|g(v^{\frac{1}{2}})|| \leq C \text{ Im } v.$$

In other words

$$|ln|g(re^{i\theta})|| = |\text{Re } f(re^{i\theta})| \leq C_1 r^2 \sin 2\theta \tag{139}$$

for $r \geq 2, \frac{1}{8}\pi \leq \theta \leq \frac{1}{2}\pi$.

For Im $v \geq 0$, $0 \leq \arg v^{\frac{1}{2}} \leq \frac{1}{2}\pi$, we now define *new* functions ζ_j by

$$\zeta_j(v) = \overline{\alpha}_j(e^{-1/4\pi i}v^{\frac{1}{2}})$$

($j=1,2$). Again, ζ_j is holormorphic in Im $v>0$ and continuous in Im $v \geq 0$; its zeroes in Im $v \geq 0$ have no finite accumulation point. Defining N_j as above, we find again the estimates (134) and (135). Eqs. (136) and (137) remain valid, and, by Theorem 6.5.4 of Ref. ([24]), we obtain

$$ln|\zeta_j(v)| = \prod_n |1 - z/z_n| \,.\, |1 - z/z_n^*|^{-1}$$
$$+ \text{Im } v\left(\pi^{-1}\int_{-\infty}^{+\infty} |x-v|^{-2} ln|\zeta_j(x)|dx + c_j\right),$$

where c_j is some real number, and where the z_n are the zeroes of ζ_j in Im $v>0$. Since the zeroes of ζ_1 are the same, in location and multiplicity, as those of ζ_2, it follows with the help of Eq. (126):

$$ln|g(e^{-1/4\pi i}v^{\frac{1}{2}})| = \text{Im } v\left(\pi^{-1}\int_{-\infty}^{\infty} |x-v|^2(ln|\zeta_2(x)| - ln|\zeta_1(x)|)dx + c_2 - c_1\right),$$

whence we obtain, by arguments similar to those given above, that there is a positive number C_1 such that

$$|\,ln|g(e^{-1/4\pi i}v^{\frac{1}{2}})|\,| \leq C_1 \text{ Im } v$$

for $|v|$ big enough, $\frac{1}{2}\pi \leq \arg v \leq \frac{3}{4}\pi$. Thus, for r big enough, and $0 \leq \theta \leq \frac{1}{8}\pi$

$$|\,ln|g(re^{i\theta})|\,| = |\text{Re } f(re^{i\theta})| \leq C_1 r^2 \cos 2\theta \tag{140}$$

Combining Ineq. (140) with Ineq. (139), we find with the help of Eqs. (132) and (133) that there is a positive number C_3 such that, for $|v|$ large enough,

$$|\,ln|g(v)|\,| = |\text{Re } f(v)| \leq C_2 |v|^2$$

δ) Using now Theorem 1.3.4 of Ref. ([24]), we see that f is a polynomial of degree less than or equal to 2. Because of Eq. (133), we then find that

$$f(v) = av$$

where a is real, because of Eq. (132). Therefore, by Eq. (131),

$$g(v) = g(0)e^{av},$$

and so, by Eq. (126),

$$\overline{\alpha}_2(v) = g(0)e^{av}\overline{\alpha}_1(v).$$

Since $g(0)$ is a real number (Eq. (128)) whose modulus is 1 (Eq. (129)), the proof of the Lemma is completed. Q. e. d.

2.14 Regge's Second Uniqueness Theorem

The results obtained above (in particular Eq. (28) in Lemma 3, Eq. (42) and Lemma 5, Subsection 2.7; Lemma 9; Lemma 10) now allow us to prove the following uniqueness theorem (Regge, Ref. ([6])).

THEOREM 2. — *Let* V_1 *and* V_2 *be in the class* $\mathcal{V}$ *(defined in Subsection* 2.12). *If the corresponding Regge interpolations* σ_1 *and* σ_2 *satisfy*

$$\sigma_1(\nu) = \sigma_2(\nu)$$

for all ν *with* $\operatorname{Re} \nu > 0$ *where both are holomorphic, then*

$$V_1(x) = V_2(x)$$

for almost all positive x.

Proof: In Subsection 2.7, we have learned that the function $\bar{\alpha}_j(j = 1, 2)$, corresponding to V_j according to Eq. (71), is a holomorphic function in $\operatorname{Re} \nu > 0$, continuous in $\operatorname{Re} \nu \geq 0$ and which satisfies assumptions *a*), *b*) and *c*) of Lemma 10 (see Subsection 2.7, properties *a*) to *f*)). Since the Regge interpolation σ_j is given in terms of $\bar{\alpha}_j$ by Eqs. (81) and (74), and since

$$\sigma_2 = \sigma_1,$$

we find by Lemma 10 that

$$\bar{\alpha}_2(\nu) = be^{a\nu}\bar{\alpha}_1(\nu)$$

where a and b are real, and $b^2 = 1$; and therefore, defining α_j as in Eq. (84), we find

$$\alpha_2(\nu) = be^{a\nu}\alpha_1(\nu). \tag{141}$$

Let γ_j, $\{ \nu_k^{(j)} \}$ and $\{ d_k^{(j)} \}$ be the spectral data corresponding to V_j (see Subsection 2.10). From the definitions given in the Eq. (95), (96), (107) and (109), we obtain, using Eq. (141):

$$\begin{aligned} \gamma_2(\tau) &= e^{-2ia\tau}\gamma_1(\tau) && (\tau\text{: real}) \\ \nu_k^{(2)} &= \nu_k^{(1)} = \nu_k && (k\text{: positive integer}) \\ d_k^{(2)} &= e^{-2a\nu_k}d_k^{(1)} \end{aligned} \tag{142}$$

If F_j is the Gelfand-Levitan kernel corresponding to V_j, then by Lemma 9, we find, using Eqs. (142) and the continuity of F_j (see Subsection 2.5):

$$F_2(x, y) = cF_1(cx, cy) \qquad (x > 0, y > 0) \tag{143}$$

where $c = e^{-a}$.

Let L_j be the transformation kernel corresponding to V_j (we suppress the subscripts 10). Eq. (143), we find, using the fundamental Eq. (42) and Lemma 5, as well as the continuity of L_j (see Lemma 3)

$$L_2(x, y) = cL_1(cy, cy) \qquad (0 < y \leq x). \tag{144}$$

By Eq. (28) of Lemma 3, we obtain from Eq. (144)

$$V_2(x) - 1 = c(V_1(cx) - 1)$$

for almost all positive x; in orther words,

$$V_2(x) - cV_1(cx) = 1 - c.$$

Since V_1 and V_2 are in $\mathscr{V}$, they satisfy both conditions (65); therefore,

$$\int_1^\infty |1 - c| \, dx < \infty,$$

which implies

$$c = 1.$$

This in turn yields

$$V_2(x) = V_1(x)$$

for almost all positive x. Q. e. d.

3. ILLUSTRATIONS. TWO UNIQUENESS RESULTS

To use the uniqueness result formulated in Theorem 2, one needs to know the Regge interpolation σ (Eq. (81)) on all of the right half plane $\operatorname{Re} \nu > 0$ (and, of course, one needs to know that the relevant potentials belong to the class $\mathscr{V}$ defined in Subsection 2.12). However, observation of the scattering process gives us no more than the sequence $s = \{ s_l \}$

$$\left(s_l = \sigma\left(l + \frac{1}{2} \right), \; l = 0, 1, 2, \ldots \right)$$

(and perhaps even less). We describe in this section two classes of potentials, contained in $\mathscr{V}$, for which the knowledge of the sequence s is enough to fix the Regge interpolation σ uniquely, and even to construct it by a suitable series.

1. The first example is well known (see Ref. ([5])). Let $\mathscr{Y}$ be the class of the exponentially decreasing potentials V of the form

$$V(x) = \int_\mu^\infty \rho(t) e^{-tx} dt$$

($x > 0$), where μ is a positive number, and ρ is a real element in a suitable class $\mathscr{R}$ of (perhaps even « generalized ») functions; $\mathscr{R}$ must be chosen in such a way that in particular: *i)* the ρ's in $\mathscr{R}$ are smooth enough, such that the results of Regge which we quote below can be applied; *ii)* the ρ's in $\mathscr{R}$ do not grow too fast, in order that the V's in $\mathscr{Y}$ satisfy condition (66). The class $\mathscr{Y}$ is then contained in $\mathscr{V}$, so that Theorem 2 applies. The results of Regge we need here are the following ([25]) ([26]): if V is in $\mathscr{Y}$, then there is a real number ν_0 such that the corresponding Regge interpolation σ is *holomorphic* and *bounded* in Re $\nu \geq \nu_0$. This allows the use of Carlson's theorem (Ref. ([24]), Theorem 9.2.1): if σ_1 and σ_2 are the Regge interpolations corresponding to the potentials V_1 and V_2 in $\mathscr{Y}$, and if the sequence of their « physical values » are identical, i. e. if

$$\sigma_1\left(l + \frac{1}{2}\right) = \sigma_2\left(l + \frac{1}{2}\right)$$

for $l = 0, 1, 2, \ldots$, then

$$\sigma_1 = \sigma_2$$

in Re $\nu > 0$. Therefore, by Theorem 2,

$$V_1 = V_2.$$

2. The second example is as follows. Let $\mathscr{F}$ be the class of those potentials V in $\mathscr{V}$ such that

$$V(x) = 0 \quad \text{for} \quad x > a$$

where a is some positive number (depending on V). Let V be in $\mathscr{F}$ and let ξ be the logarithmic x-derivative of the corresponding regular solution φ of Eq. (1) (see Lemma 1):

$$\xi(x, \nu) = \varphi'(x, \nu)/\varphi(x, \nu) \tag{145}$$

($x > 0$, Re $\nu > -\varepsilon$, ε as in condition (66)).
Using the definition of σ:

$$\sigma(\nu) = \bar{\alpha}(\nu)/\bar{\beta}(\nu),$$

as well as the different relations given at the beginning of Subsection 2.7,

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([25]) See Refs. ([6]) and ([7]).

([26]) See also : A. Bottino, A. M. Longoni and T. Regge, Potential Scattering for Complex Energy and Angular Momentum. *Nuovo Cimento*, **23**, 1962, 954-1004 ; and A. Martin, On the Behaviour of the Partial-Wave Amplitude for Large Angular Momenta in Potential Scattering. *Nuovo Cimento*, **31**, 1964, 1229-1245.

we find, for all $x > a$, the following formula (use in particular Eqs. (71) and (72); use also the fact that the regular solution is in $\mathscr{C}_2(\mathrm{R}_+)$):

$$\sigma(\nu) = -\frac{(x^{-\frac{1}{2}}\mathrm{H}_\nu^{(2)}(x))' - \xi(x, \nu)x^{-\frac{1}{2}}\mathrm{H}_\nu^{(2)}(x)}{(x^{-\frac{1}{2}}\mathrm{H}_\nu^{(1)}(x))' - \xi(x, \nu)x^{-\frac{1}{2}}\mathrm{H}_\nu^{(1)}(x)} \tag{146}$$

for Re $\nu > -\varepsilon$. Solving for $\xi(x, \nu)$, we get ($x > a$, Re $\nu > -\varepsilon$):

$$\xi(x, \nu) = \frac{\sigma(\nu)(x^{-\frac{1}{2}}\mathrm{H}_\nu^{(1)}(x))' + (x^{-\frac{1}{2}}\mathrm{H}_\nu^{(2)}(x))'}{\sigma(\nu)x^{-\frac{1}{2}}\mathrm{H}_\nu^{(1)}(x) + x^{-\frac{1}{2}}\mathrm{H}_\nu^{(2)}(x)}. \tag{147}$$

From Lemma 1, we find that for each fixed positive x, ξ is a meromorphic function in Re $\nu > -\varepsilon$; furthermore, using Eqs. (4) and (7), we have for each positive x an estimate of the form

$$\left|\left(x\xi(x, \nu)/\left(\nu - \frac{1}{2}\right)\right) - 1\right| \leq \mathrm{C}\,|\,\nu\,|^{-1}$$

for $|\nu|$ big enough, Re $\nu \geq -\varepsilon$ (C: positive number, independent of ν). This estimate shows that, for each positive x, there are positive numbers ν_0 and C_1 such that ξ is holomorphic in the half-plane Re $\nu > \nu_0$, and satisfies therein

$$|\,\xi(x, \nu)\,| \leq \mathrm{C}_1\,|\,\nu\,|, \tag{148}$$

for $|\nu|$ big enough. If now V_1 and V_2 are two potentials in $\mathscr{F}$, and if the corresponding Regge interpolations σ_1 and σ_2 are equal at the « physical points »:

$$\sigma_1\left(l + \frac{1}{2}\right) = \sigma_2\left(l + \frac{1}{2}\right) \qquad (l = 0, 1, 2, \ldots),$$

then, by Eq. (147), used for a sufficiently large value of x, we find that the corresponding logarithmique derivatives ξ_1 and ξ_2 satisfy

$$\xi_1\left(x, l + \frac{1}{2}\right) = \xi_2\left(x, l + \frac{1}{2}\right) \tag{149}$$

for all integer l larger than a given positive number l_0. But, by Carlson's theorem, which can be applied because of estimate (148), Eq. (149) implies that

$$\xi_1(x, \nu) = \xi_2(x, \nu)$$

for Re $\nu > 0$; and therefore, by Eq. (146), that

$$\sigma_1 = \sigma_2$$

for Re $\nu > 0$. Again, Theorem 2 yields here:

THEOREM 3. — *Let* V_1 *and* V_2 *be in the class* $\mathscr{F}$, *and let* σ_1 *and* σ_2 *be the corresponding Regge interpolations. If*

$$\sigma_1\left(l + \frac{1}{2}\right) = \sigma_2\left(l + \frac{1}{2}\right)$$

for $l = 0, 1, 2, \ldots$, *then*

$$V_1(x) = V_2(x)$$

for almost all positive x.

For completeness, we remind the reader of how one can reconstruct a function, holomorphic in a half-plane, from the values it takes at integer points (our problem being to reconstruct σ, resp. ξ, from the values it takes at $l + \frac{1}{2}$ (l integer, $> \nu_0$), in the case of potentials in the class $\mathscr{Y}$, resp. $\mathscr{F}$). The following assertion is true: let the function f be holomorphic in Re $\nu \geq \nu_0$; suppose that there are positive numbers C and η such that

$$| f(\nu) | < C(1 + | \nu |)^{\eta}$$

in Re $\nu \geq \nu_0$; define F for positive integer n by

$$\begin{aligned} F(n) &= 0 \qquad \text{if} \quad n < \nu_0 \\ F(n) &= f(n) \qquad \text{if} \quad n \geq \nu_0; \end{aligned}$$

then f can be represented by the following Newton series, which converges uniformly on any compact contained in Re $\nu > \nu_0$:

$$f(\nu) = \sum_{n=0}^{\infty} a_n(\nu - 1) \ldots (\nu - n)/n!$$

where

$$a_n = \sum_{m=0}^{n} (-)^{n-m}\binom{n}{m}F(m + 1).$$

This statement is a special case of a more general theorem by Nörlund ([27]).

([27]) N.-E. NÖRLUND, *Leçons sur les séries d'interpolation* (Gauthier-Villars, Paris, 1926), Chap. V, n° 66.

4. POTENTIALS OF FINITE RANGE

In this section, we give necessary and sufficient conditions for the scattering amplitude (as defined at the end of Subsection 2.7) to be produced by a potential in the class $\mathscr{F}$ described in Section 3. Similar conditions have been discovered by Agranovich and Marchenko in their treatment of a related problem, Refs. ([9]) and ([10]). By means of a simple artifice which we shall describe in Subsection 4.2, these can be seen to be relevant for our problem.

The proofs of the statements made in this section will not all be given in full detail. Many of them are almost identical to those given by Agranovich and Marchenko in Ref. ([10]), to which we shall frequently refer.

The general method underlying our considerations is again Gel'fand-Levitan's, although we shall not explain in which way. The reader interested in this aspect may consult the work just referred to.

We shall make in this section a frequent use of the theory of the Mellin transformation. It is identical to the theory of the Fourier transformation. See Appendix I for a abstract.

4.1 Necessary and Sufficient Conditions for an Amplitude to be Produced by a Potential of Finite Range

For $\varepsilon > 0$, we denote by $\mathscr{L}(\varepsilon)$ the set of those functions l on $0 < y \leq 1$ such that there is a function l_1, real, measurable on $0 < y \leq 1$, with

$$\int_0^1 y^{-\varepsilon} |l_1(y)| dy < \infty,$$

and

$$y^{\frac{1}{2}} l(y) = \int_0^y l_1(u) du.$$

Equipped with the norm $\| l \| = \int_0^1 y^{-\varepsilon} |l_1(y)| dy$, $\mathscr{L}(\varepsilon)$ is a Banach space.

Let l be in $\mathscr{L}(\varepsilon)$. Then l is absolutely continuous, and

$$y^{-\varepsilon} y^{\frac{1}{2}} |l(y)| \leq \| l \|, \tag{150}$$

$$\lim_{y \to 0} y^{-\varepsilon} y^{\frac{1}{2}} l(y) = 0, \tag{151}$$

$$\int_0^1 y^{-\varepsilon - \frac{1}{2}} |l(y)| dy \leq \varepsilon^{-1} \| l \|, \tag{152}$$

and therefore

$$\int_0^1 | y^{-\varepsilon} l(y) |^2 dy \leq \varepsilon^{-1} \| l \|^2. \tag{153}$$

From Ineq. (152), we see that we can define for Re $v \geq -\varepsilon$ a function g by

$$g(v) = \int_0^1 l(y) y^{v-\frac{1}{2}} dy. \tag{154}$$

It will have the following property: it will be holomorphic on Re $v > -\varepsilon$, continuous and bounded on Re $v \geq -\varepsilon$, and for all σ with $\sigma \geq -\varepsilon$,

$$(2\pi)^{-1} \int_{-\infty}^{+\infty} | g(\sigma + i\tau) |^2 d\tau \leq \int_0^1 | y^{-\varepsilon} l(y) |^2 dy. \tag{155}$$

(We have used here Parseval's equality for Mellin transforms; see Appendix I). Furthermore, integrating by parts in Eq. (154), we get

$$g(v) = v^{-1}\left(l(1) - \int_0^1 l_1(y) y^v dy \right); \tag{156}$$

thus,

$$| v g(v) | \leq 2 \| l \| \tag{157}$$

on Re $v \geq -\varepsilon$.

We denote by $\mathscr{A}(\varepsilon)$ the set of those functions w on Re $v \geq -\varepsilon$ such that

(A1) there is an l in $\mathscr{L}(\varepsilon)$ with

$$w(v) = 1 + \int_0^1 l(y) y^{v-\frac{1}{2}} dy$$

(A2) the zeroes of w in Re ≥ 0 are all real and simple.

Notice that, if w is in $\mathscr{A}(\varepsilon)$, it is holomorphic on Re $v > -\varepsilon$, and

$$| w(v) - 1 | \leq 2 | v |^{-1} \| l \|,$$

so that in any half plane Re $v \geq -\eta$, where $\eta < \varepsilon$, w has at most a finite number of zeroes.

LEMMA 11. — *Let* V *be locally integrable; let there be a positive* ε *such that*

$$\int_0^1 y^{1-2\varepsilon} | \mathrm{V}(y) - 1 | dy < \infty;$$

let φ *be the regular solution corresponding to* V, *according to Lemma* 1.

Then, for any fixed x (>0), *and any* η *with* $0 < \eta < \varepsilon$, *the function* w, *defined by*

$$w(v) = x^{-v+\frac{1}{2}}\varphi(x, v),$$

is in $\mathscr{A}(\eta)$.

Proof: All we have to do is to refine somewhat the proof of Lemma 3, in order to show that L_{10} is for fixed x an absolutely continuous function of y, whose derivative has the required integrability properties.

Using the notations used in the proof of Lemma 1, we have

$$x^{-v+\frac{1}{2}}\varphi(x, v) = f(x, v), \tag{22}$$

where f is the solution of Eq. (15); we have

$$f = f_0 + \mathrm{B}(f_0) + \mathrm{B}^2(f), \tag{158}$$

where $f_0(x, v) = 1$, and where B is the operator defined just before Eq. (9). Inequality (12) for $n = 2$ gives, for fixed x,

$$|\mathrm{B}^2(f)(x, v)| < \mathrm{C}\,(\min\{(2\varepsilon)^{-1}, |v|^{-1}\})^2 \,\|f\|$$

for all v with $\mathrm{Re}\, v \geq -\varepsilon$; here,

$$\|f\| = \sup\{|f(y, v)| \;|\; 0 \leq y \leq x, \mathrm{Re}\, v \geq -\varepsilon\}, \tag{159}$$

and C is a number depending only on x and on V. Therefore, besides being continuous, the function $\tau \to \mathrm{B}^2(f)(x, i\tau)$ is in $\mathscr{L}_1(-\infty, \infty)$; also, there is a positive number M such that, for $\sigma \geq -\varepsilon$,

$$\int_{-\infty}^{+\infty} |\mathrm{B}^2(f)(x, \sigma + i\tau)|^2 d\tau < \mathrm{M}$$

$$\int_{-\infty}^{+\infty} |(\sigma + i\tau)\mathrm{B}^2(f)(x, \sigma + i\tau)|^2 d\tau < \mathrm{M}.$$

We can thus define on $0 < y < \infty$ a function l_x by

$$l_x(y) = \int_{-\infty}^{+\infty} \mathrm{B}^2(f)(x, i\tau) y^{-i\tau - \frac{1}{2}} d\tau$$

which is continuous, and satisfies $\lim_{y \to 0} y^{\frac{1}{2}} l_x(y) = 0$, by the Riemann-Lebesgue Lemma (used for Mellin transforms). From the $\mathscr{L}_2$-estimates above, we find furthermore that $l_x(y) = 0$ for $y \geq 1$ (Hille-Tamarkin, see Ref. ([14])), that

$$\int_{-\infty}^{+\infty} |y^{-\varepsilon} l_x(y)|^2 dy < \infty$$

(Parseval's equality), that l_x is absolutely continuous, and that $y \to y^{\frac{1}{2}}(y^{\frac{1}{2}}l_x(y))'$ satisfies

$$\int_0^\infty |y^{\frac{1}{2}}(y^{\frac{1}{2}}l_x(y))'y^{-\varepsilon}|^2 dy < \infty$$

(Parseval's equality). Thus, for all η with $\eta < \varepsilon$,

$$\int_0^1 |(y^{\frac{1}{2}}l_x(y))'|\, y^{-\eta} dy < \infty$$

(use Schwartz' inequality) and

$$y^{\frac{1}{2}}l_x(y) = \int_0^y (y^{\frac{1}{2}}l_x(y))'dy.$$

In other words, l_x in $\mathscr{L}(\eta)$ for any positive η with $\eta < \varepsilon$, and

$$\mathrm{B}^2(f)(x, \nu) = \int_0^1 l_x(y)y^{\nu - \frac{1}{2}}dy.$$

Consider now the term $\mathrm{B}(f_0)$. We have (see a few lines above Eq. (10))

$$\mathrm{B}(f_0) = \int_0^x dv v^{-2\nu-1} \int_0^v u^{-1}q(u)u^{2\nu}du, \tag{160}$$

where

$$q(u) = u^2(\mathrm{V}(u) - 1). \tag{161}$$

Therefore,

$$\mathrm{B}(f_0) = \int_0^1 m_x(y)y^{\nu - \frac{1}{2}}dy,$$

where m_x is defined by

$$m_x(y) = \frac{1}{4}y^{-\frac{1}{2}} \int_0^y v^{-1}q(xv^{\frac{1}{2}})dv$$

Using the assumptions we made on V, we see that m_x is in $\mathscr{L}(\varepsilon)$, and therefore in $\mathscr{L}(\eta)$ for all positive η with $\eta < \varepsilon$.

Thus, since

$$w(\nu) = f(x, \nu),$$

we see that w satisfies property (A1).

That w satisfies property (A2) follows from Lemma 1, c). Q. e. d.

The preceding Lemma gives a necessary condition for a function of ν to be the value at a given positive x of the regular solution (normalized as in

Lemma 1; see Ineq. (4)) corresponding to a locally integrable potential V with

$$\int_0^1 y^{1-2\varepsilon} | V(y) - 1 | dy < \infty$$

for some positive ε. Now, from the results of Agranovich and Marchenko, it follows that this condition is sufficient, in the following precise sense:

THEOREM 4. — *Let the following be given:*

a positive number x;

a positive number ε and a function w in $\mathscr{A}(\varepsilon)$;

for each positive zero of w (which we call $0 < v_1 < \ldots < v_k$), a positive number $b_j (1 \leq j \leq k)$.

Then, there is a (uniquely defined) locally integrable potential V on $0 < y < x$, satisfying

$$\int_0^x y^{1-2\eta} | V(y) - 1 | dy < \infty \tag{162}$$

for some positive η, such that: the corresponding regular solution φ (as defined in Lemma 1) satisfies

$$\varphi(x, v) = x^{v-\frac{1}{2}} w(v) \tag{163}$$

for Re $v \geq 0$, *and*

$$\int_0^x | \varphi(y, v_j) |^2 dy = b_j \tag{164}$$

for $1 \leq j \leq k$.

We shall not reproduce the details of the proof of this theorem, since most of them are identical to arguments spelled out in Agranovich and Marchenko. We shall limit ourselves to an outline and a few explanatory remarks which the reader will find in Subsection 4.2.

In what immediately follows, we translate Theorem 4 into an equivalent statement in terms of the logarithmic x-derivative of the regular solution. This will be Theorem 5, which is the main new result of this paper.

We turn first our attention to the x-derivative of the regular solution, considered at fixed x as a function of v.

LEMMA 12. — *Let* V *and φ be as in Lemma* 11; *let f be defined by*

$$f(x, v) = x^{-v+\frac{1}{2}} \varphi(x, v); \tag{165}$$

let f' be the partial x-derivative of f. Then there is a real measurable function l on $0 < y \leq 1$, such that

$$\int_0^1 y^{-\eta} y^{-\frac{1}{2}} | l(y) | dy < \infty$$

for all η with $\eta < \varepsilon$, and

$$f'(x, v) = \int_0^1 l(y) y^{v - \frac{1}{2}} dy.$$

Proof: As in the case of Lemma 11, the proof is just a refinement of the proof of Lemma 3. From (158), we find, in the notation of the proof of Lemma 1 (just above Eq. (10)):

$$f' = \partial_x \mathrm{B}(f_0) + \partial_x \mathrm{B}^2(f).$$

For the first term, Eq. (160) gives

$$\partial_x \mathrm{B}(f_0)(x, v) = x^{-1} \int_0^x q(u)(u/x)^{2v} u^{-1} du = \int_0^1 l_x(y) y^{v - \frac{1}{2}} dy$$

with

$$l_x(y) = (2x)^{-1} y^{-\frac{1}{2}} q(x y^{\frac{1}{2}}),$$

q as in Eq. (161). Thus,

$$\int_0^1 | l_x(y) | y^{-\frac{1}{2}} y^{-\varepsilon} dy = x^{2\varepsilon - 1} \int_0^x u^{-1-2\varepsilon} | q(u) | du, < \infty$$

Take now the second term. We have $\partial_x \mathrm{B}^2(f) = \partial_x \mathrm{B}(\mathrm{B}(f))$. Using the estimates (10) and (9), we get, for fixed x,

$$\begin{aligned} | x \partial_x \mathrm{B}^2(f)(x, v) | &\leq \mathrm{C} \max \{ | \mathrm{B}(f)(x, v) | \mid 0 \leq y \leq x \} \\ &\leq \mathrm{C}_1 \min \{ (2\varepsilon)^{-1}, | v |^{-1} \} \, \| f \|, \end{aligned}$$

$\| f \|$ as in Eq. (159). Therefore, by the theorem of Fourier-Plancherel (applied to Mellin transforms),

$$\int_{-a}^{+a} \partial_x \mathrm{B}^2(f)(x, i\tau) y^{-i\tau - \frac{1}{2}} d\tau$$

has a limit in $\mathscr{L}_2(0, \infty)$ as $a \to \infty$, which we shall denote by m_x; furthermore (Paley-Wiener)

$$m_x(y) = 0$$

for almost all y with $y > 1$, and

$$\int_0^1 | y^{-\varepsilon} m_x(y) |^2 dy < \infty$$

(Parseval equality). Therefore,

$$\int_0^1 y^{-\eta} y^{-\frac{1}{2}} | m_x(y) | \, dy < \infty$$

for all η with $\eta < \varepsilon$. We furthermore have

$$\partial_x \mathrm{B}^2(f)(x, v) = \int_0^1 m_x(y) y^{v - \frac{1}{2}} dy$$

for v with $\operatorname{Re} v > -\varepsilon$. Q. e. d.

We now go over to a more careful study of the logarithmic x-derivative of the regular solution.

For $\varepsilon > 0$, we denote by $\mathscr{B}(\varepsilon)$ the set of those functions w which are meromorphic on $\operatorname{Re} v > -\varepsilon$, and which furthermore have following properties:

(B1) $w(v^*) = (w(v))^*$;

(B2) in $\operatorname{Re} v \geq 0$, w has a finite number of poles, which are all real and simple;

(B3) there is some positive number α such that, for τ real and positive,

$$\operatorname{Im} w(i\tau) > \alpha\tau;$$

(B4) if v is such that $\operatorname{Re} v > 0$, $\operatorname{Im} v > 0$, then $\operatorname{Im} w(v) > 0$;

(B5) let $0 \leq v_0 < v_1 < \ldots < v_k$ be the poles of w in $\operatorname{Re} v \geq 0$; then there are positive numbers $b_0, \ldots, b_k$, a positive number η, and a real measurable function m on $0 < y \leq 1$ such that

$$\int_0^1 y^{-\eta - \frac{1}{2}} | m(y) | \, dy < \infty$$

and

$$w(v) = v - \frac{1}{2} - \sum_{j=0}^{k} b_j (v - v_j)^{-1} + \int_0^1 m(y) y^{v - \frac{1}{2}} dy.$$

LEMMA 13. — *Let* V *and* φ *be as in Lemma* 11; *define* ξ *by*

$$\xi(x, v) = x\varphi'(x, v)/\varphi(x, v); \tag{166}$$

then, for all positive x, ξ *is a function of* $\mathscr{B}(\eta)$, *for some positive* η.

***Remark.* — The function ξ defined by Eq. (166) is similar, but not identical, to the one defined by Eq. (145).**

Proof: Let x be a fixed positive number. From Lemma 1, it follows easily that $\nu \to \xi(x, \nu)$ is meromorphic in $\operatorname{Re} \nu > -\varepsilon$, and that (B1) and (B2) are satisfied. To prove (B3), we recall that, for τ real,

$$x^2(\varphi(x, i\tau)\varphi'(x, -i\tau) - \varphi'(x, i\tau)\varphi(x, -i\tau)) = -2i\tau \tag{167}$$

(see proof of Lemma 1, part δ), whence

$$x\,|\,\varphi(x, i\tau)\,|^2 \operatorname{Im} \xi(x, i\tau) = \tau. \tag{168}$$

But $|\,\varphi(x, i\tau)\,|$ is bounded; this implies (B3), α being such that

$$x\,|\,\varphi(x, i\tau)\,|^2 < \alpha^{-1}$$

for all real τ. To prove (B4), we compute

$$\int_0^x |\,\varphi(y, \nu)\,|^2 dy$$

for $\operatorname{Re} \nu > 0$, using Green's formula. We find

$$2 \operatorname{Re} \nu \operatorname{Im} \nu \int_0^x |\,\varphi(y, \nu)\,|^2 dy = x\,|\,\varphi(x, \nu)\,|^2 \operatorname{Im} \xi(x, \nu), \tag{169}$$

whence (B4) follows.

The proof of (B5) is more delicate:

α) From Lemma 1, it follows that the set of the poles of ξ in $\operatorname{Re} \nu \geq 0$ is identical with the set of the zeroes of φ in the same region.

β) With f given by (165), we have

$$\xi(x, \nu) = \nu - \frac{1}{2} + xf'(x, \nu)/f(x, \nu).$$

We know that $\nu \to f(x, \nu)$ is in $\mathscr{A}(\eta_1)$ where η_1 is some positive number which we take smaller than ε (Lemma 11). Using the remark just preceding Lemma 11, we see that we can find a positive number η_2 in such a way that the only zeroes of f in $\operatorname{Re} \nu \geq -\eta_2$ are real and non-negative. We take η_2 smaller than or equal to η_1. Let $0 \leq \nu_0 < \ldots < \nu_k$ be these zeroes. These are also the poles of $\nu \to \xi(x, \nu)$ in $\operatorname{Re} \nu \geq -\eta_2$. The corresponding residues $-b_0, \ldots, -b_k$ are negative; this follows from (B4) if $\nu_j \neq 0$, from (B1) and (B3) if $\nu_j = 0$. We have of course

$$-b_j = xf'(x, \nu_j)/\partial_\nu f(x, \nu_j), \tag{170}$$

where $\partial_v f$ denotes the partial derivative of f with respect to v. Thus g, defined by

$$g(v) = xf'(x, v)/f(x, v) + \sum_{j=0}^{k} b_j(v - v_j)^{-1}$$
$$= \xi(x, v) - \left(v - \frac{1}{2}\right) + \sum_{j=0}^{k} b_j(v - v_j)^{-1}$$

is holomorphic in Re $v \geq -\eta_2$.

γ) $v \to f(x, v)$ being in $\mathscr{A}(\eta_1)$, there is a positive number C such that

$$| f(x, v) - 1 | < | v |^{-1}C \tag{171}$$

for v in Re $v > -\eta_1$. According to Lemma 12, we know that $v \to f'(x, v)$ is the Mellin transform of some measurable function l such that

$$\int_0^1 y^{-\frac{1}{2}-\eta_2} | l(y) | \, dy < \infty.$$

If we could assume that this l is an $\mathscr{L}_2$-function, we would obtain (B5) quite easily by a simple application of Paley-Wiener's theorem. Since this is not the case, we have to go through the following arguments.

Let $\{ l^{(n)} \}$ be a sequence of measurable functions on $0 < y < 1$ with the following properties:

a) $$\int_0^1 | l^{(n)}(y) y^{-\eta_2} |^2 dy < \infty ; \int_0^1 y^{-\frac{1}{2}-\eta_2} | l^{(n)}(y) | \, dy < \infty$$

b) $$\lim_{n \to \infty} \int_0^1 y^{-\frac{1}{2}-\eta_2} | l^{(n)}(y) - l(y) | \, dy = 0$$

(we let the reader construct such a sequence). Then, the sequence $\{ \bar{f}_n \}$, defined by

$$\bar{f}_n(v) = \int_0^1 l^{(n)}(y) y^{v-\frac{1}{2}} dy$$

has the following properties:

a) $\bar{f}_n$ is holomorphic on Re $v > -\eta_2$, continuous and bounded on Re $v \geq -\eta_2$,

b) $\lim_{n \to 0} \sup \{ | \bar{f}_n(v) - f'(x, v) |, \text{Re } v \geq -\eta_2 \} = 0$,

c) there are positive numbers M_n such that

$$\int_{-\infty}^{+\infty} | \bar{f}_n(\sigma + i\tau) |^2 d\tau < M_n$$

for $\sigma \geq -\eta_2$. If the sequence $\{g_n\}$ is defined by

$$g_n(v) = x\bar{f}_n(v)/f(x, v) + \sum_{j=0}^{k} b_j^{(n)}(v - v_j)^{-1}$$

where

$$b_j^{(n)} = -x\bar{f}_n(v_j)/\partial_v f(x, v_j),$$

we find, using estimate (171) for f:

a) g_n is holomorphic and bounded for Re $v > -\eta_2$,
b) $\lim_{n\to\infty} \sup \{|g_n(v) - g(v)|, \text{Re } v > -\eta_2\} = 0$,
c) there are positive numbers M_n' such that

$$\int_{-\infty}^{+\infty} |g_n(\sigma + i\tau)|^2 d\tau < M_n'$$

for $\sigma > -\eta_2$. Therefore, by Hille-Tamarkin's theorem (Ref. ([14])), there is a sequence $\{m_n\}$ of measurable functions on $y > 0$ such that

$$\int_0^\infty |m_n(y)y^{-\eta_2}|^2 dy < \infty$$

$$m_n(y) = 0 \quad \text{for} \quad y > 1$$

with

$$g_n(v) = \int_0^\infty m_n(y)y^{v-\frac{1}{2}}dy$$

for Re $v > -\eta_2$.

We now show that, for any η with $0 < \eta < \eta_2$,

$$\lim_{n',n\to\infty} \int_0^\infty |m_{n'}(y) - m_n(y)|\, y^{-\frac{1}{2}-\eta}dy = 0.$$

Since

$$f(x, -\eta + i\tau) = 1 + \int_0^\infty y^{-\eta}l_1(y)y^{i\tau-\frac{1}{2}}dy$$

where

$$\int_0^\infty y^{-\frac{1}{2}-\eta}|l_1(y)|\,dy < \infty$$

and

$$l_1(y) = 0 \quad \text{for} \quad y > 1$$

(see Lemma 11), and since

$$f(x, -\eta + i\tau) \neq 0$$

whatever the real value taken by τ, it follows from a celebrated theorem by Wiener that there is a measurable function l_2 with

$$\int_0^\infty y^{-\frac{1}{2}} |l_2(y)| dy < \infty$$

and

$$(f(x, -\eta + i\tau))^{-1} = 1 + \int_0^\infty l_2(y) y^{i\tau - \frac{1}{2}} dy$$

(see Appendix I). Using well known theorems on convolution (see again Appendix I), we find (almost everywhere on $y > 1$)

$$m_n(y) y^{-\eta} = l^{(n)}(y) y^{-\eta} + (l^{(n)}(y) y^{-\eta} * l_2)(y) + \sum_{j=0}^{k} b_j^{(n)} h_j(y)$$

where h_j is defined by

$$h_j(y) = \begin{cases} 0 & y < 1 \\ y^{-\eta - \nu_j - \frac{1}{2}} & y \geq 1 \end{cases}$$

When n goes to ∞, the right-hand side converges in $\mathscr{L}_1(0, \infty, y^{-\frac{1}{2}} dy)$; therefore, the same is true of the left-hand side. Denote the limit by $m(y) y^{-\eta}$. We have

$$\int_0^\infty |m(y)| y^{-\frac{1}{2} - \eta} dy < \infty$$

$$m(y) = 0 \quad \text{for} \quad y > 1$$

and, for Re $\nu \geq -\eta$

$$g(\nu) = \int_0^\infty m(y) y^{\nu - \frac{1}{2}} dy.$$

This concludes the proof of (B5). Q. e. d.

Theorem 5, to be formulated later, states that the converse of Lemma 13 is true. The proof of this theorem is based on the fact that the regular solution at fixed x and its logarithmic x-derivative at the same x are closely related analytic functions of ν; this circumstance permits a simple derivation of Theorem 5 starting from Theorem 4.

We study now these relations between φ and ξ. We denote by $\mathscr{P}$ the set of those pairs of functions (w_1, w_2), where w_1 is in $\mathscr{A}(\eta)$ and w_2 is in $\mathscr{B}(\eta)$ (for some positive η), which are such that:

(P1) the set of the poles of w_2 in $\operatorname{Re} v \geq 0$ is identical with the set of the zeroes of w_1 in the same region,

(P2) for all real τ with $\tau \neq 0$,

$$|w_1(i\tau)|^2 \operatorname{Im} w_2(i\tau) = \tau.$$

LEMMA 14. — *Let* V *and* φ *be as in Lemma* 11; *let* ξ *be as in Eq.* (166); *let* f *be given by*

$$f(x, v) = x^{-v+\frac{1}{2}}\varphi(x, v).$$

Then, for all positive x, *the pair* (f, ξ) *is in* $\mathscr{P}$.

Proof: As already seen in the proof of Lemma 13 (P1) is an easy consequence of Lemma 1. Property (P2) follows immediately from Eq. (168). Q. e. d.

We have the following uniqueness result:

LEMMA 15. — *Let* (w_1, w_2) *and* (w_1, w_3) *be in* $\mathscr{P}$. *Let* $0 < v_1 < \ldots < v_k$ *be the zeroes of* w_1 *in* $\operatorname{Re} v > 0$. *Then there are real numbers* $c_1, \ldots, c_k$ *such that*

$$w_2(v) - w_3(v) = \sum_{j=1}^{k} c_j(v - v_j)^{-1}$$

for all v *with* $\operatorname{Re} v > 0$.

Proof: Denote by u the map $v \to v^{\frac{1}{2}}$ $(0 < \arg v < \pi)$ of the open upper half plane onto the quadrant $\{ v \mid \operatorname{Re} v > 0, \operatorname{Im} v > 0 \}$. Define $u_n (n = 2, 3)$ by

$$u_n = w_n \circ u.$$

Since w_n is in some $\mathscr{B}(\eta)$, $\operatorname{Im} v > 0$ implies $\operatorname{Im} u_n(v) > 0$, by (B4); in other words, u_n is a Herglotz function. Nevanlinna's theorem for Herglotz functions ([28]) tells us that there are real numbers α_n and μ_n, μ_n non-negative, such that

$$u_n(v) = \alpha_n + \mu_n v + \int_{-\infty}^{+\infty} \frac{1 + \tau v}{\tau - v} d\sigma_n(\tau)$$

where σ_n is any real valued, nondecreasing, bounded, left continuous function satisfying

$$\frac{1}{2}(\sigma_n(\tau + 0) + \sigma_n(\tau)) - c_n = \lim_{\varepsilon \to +0} \int_{a_n}^{\tau} (1 + \rho^2)^{-1} \operatorname{Im} u_n(\rho + i\varepsilon) d\rho$$

([28]) See for example Ref. ([20]), Chap. VI, § 59.

for some fixed real numbers c_n and a_n. Since w_n is in some $\mathscr{B}(\varepsilon)$, u_n can be extended to a function, also denoted by u_n, which is continuous on the closed half plane from which some finite set Δ of the non-negative real axis, corresponding to the zeroes of w_1 in $\operatorname{Re} v \geq 0$, has been removed. For $v > 0$, $\operatorname{Im} u_n(v) = 0$ (where it makes sense); this follows from (B1). Furthermore, for $v < 0$,

$$\operatorname{Im} u_2(v) = \operatorname{Im} u_3(v)$$

since (w_1, w_2) and (w_1, w_3) are in $\mathscr{P}$ (use (P2)). Thus, whatever the choice of c_n and a_n, the function $\sigma_3 - \sigma_2$ is piece-wise constant; a discontinuity can occur at most at $\tau = 0$, and at $\tau \in \Delta$. Therefore,

$$u_3(v) - u_2(v) = \alpha_3 - \alpha_2 + (\mu_3 - \mu_2)v + \sum_{j=0}^{k} d_j(1 + \tau_j v)(v - \tau_j)^{-1}$$

where $0 = \tau_0 < \ldots < \tau_k$ and $d_1, \ldots, d_k$ are real numbers. We can rewrite this expression as (A, M, $D_1, \ldots, D_k$: real numbers)

$$u_3(v) - u_2(v) = A + Mv + \sum_{j=0}^{k} D_j(v - \tau_j)^{-1}$$

Since w_n is in some $\mathscr{B}(\eta)$, we have

$$u_n(v) = v^{\frac{1}{2}} - \frac{1}{2} + o(1)$$

as $|v| \to \infty$ (use (B5)); therefore,

$$A = M = 0.$$

From (B2), it follows that

$$u_n(v) = O(|v|^{-\frac{1}{2}})$$

as $|v| \to 0$; therefore

$$D_0 = 0.$$

Q. e. d.

We have furthermore the following:

LEMMA 16. — *Let w_2 be in $\mathscr{B}(\varepsilon)$ (some positive ε). Then there is one and only one w_1 in $\mathscr{A}(\eta)$ (some positive η) such that (w_1, w_2) is in $\mathscr{P}$.*

Proof: Uniqueness: Let w_1 be in $\mathscr{A}(\eta)$, $\overline{w}_1$ be in $\mathscr{A}(\overline{\eta})$; assume that (w_1, w_2) and $(\overline{w}_1, w_2)$ are both in $\mathscr{P}$. Then w_1 and $\overline{w}_1$ are both holomorphic on $\mathrm{Re}\, v > 0$, continuous and bounded on $\mathrm{Re}\, v \geq 0$, and their zeroes on $\mathrm{Re}\, v > 0$ (which are all simple and finite in number) coincide; furthermore, for τ real,

$$|w_1(i\tau)| = |\overline{w}_1(i\tau)|$$

We can apply Theorem 6.5.4, from Ref. [24], getting

$$ln\,|w_1(v)| = ln\,|\overline{w}_1(v)| + c\ \mathrm{Re}\ v$$

for $\mathrm{Re}\, v > 0$. Using the fact that

$$\lim_{v\ \mathrm{real}, \to\infty} w_1(v) = \lim_{v\ \mathrm{real}, \to\infty} \overline{w}_1(v) = 1, \tag{172}$$

we get

$$c = 0.$$

Therefore, there is some real number α such that

$$w_1(v) = e^{i\alpha}\overline{w}_1(v)$$

for $\mathrm{Re}\, v > 0$. Using Eq. (172) again, we obtain

$$e^{i\alpha} = 1.$$

Existence: The function w_2 being in $\mathscr{B}(\varepsilon)$, we have according to (B5)

$$w_2(v) = v - \frac{1}{2} - b_0 v^{-1} - \sum_{j=1}^{k} b_j(v - v_j)^{-1} + \int_0^1 m(y) y^{v-\frac{1}{2}} dy$$

where b_0 is non-negative, $b_1, \ldots, b_j$ and $v_1, \ldots, v_j$ are positive, and the real valued function m satisfies

$$\int_0^1 |m(y)|\, y^{-\frac{1}{2}-\varepsilon_1} dy < \infty$$

for some positive ε_1, which we take to be less or equal to ε. Defining m_1 by

$$m_1(y) = \begin{cases} m(y) & \text{for} \quad y < 1 \\ \displaystyle\sum_{j=1}^{k} b_j y^{-v_j - \frac{1}{2}} & \text{for} \quad y \geq 1 \end{cases}$$

we see that m_1 is real valued, and that there is a positive $\varepsilon_2 (\leq \varepsilon_1)$ such that

$$\int_0^\infty |m_1(y)|\, y^{-\frac{1}{2}-\varepsilon_2} dy < \infty \quad , \quad \int_0^\infty |m_1(y)|\, y^{-\frac{1}{2}+\varepsilon_2} dy < \infty,$$

and

$$w_2(\nu) = \nu - \frac{1}{2} - b_0\nu^{-1} + \int_0^\infty m_1(y)y^{\nu-\frac{1}{2}}dy$$

for $|\operatorname{Re} \nu| \leq \varepsilon_2$. Putting $\nu = i\tau$, τ real, we find

$$\operatorname{Im} w_2(i\tau) = \tau + b_0\tau^{-1} + f(\tau)$$

where f is defined on $|\operatorname{Im} z| \leq \varepsilon_2$ by

$$f(z) = \int_0^\infty m_2(y)y^{iz-\frac{1}{2}}dy,$$

with m_2 given by

$$m_2(y) = (2i)^{-1}(m_1(y) - y^{-1}m_1(y^{-1}))$$

Notice that

$$y^{-1}m_2(y^{-1}) = (m_2(y))^*,$$

and that

$$\int_0^\infty |m_2(y)|\, y^{-\frac{1}{2}-\varepsilon_2}dy < \infty \quad , \quad \int_0^\infty |m_2(y)|\, y^{-\frac{1}{2}+\varepsilon_2}dy < \infty.$$

Notice furthermore that f is holomorphic and bounded on $|\operatorname{Im} z| < \varepsilon_2$, and that f is odd (i. e. $f(-z) = -f(z)$; in particular, $f(0) = 0$). Using (B3), we find that the function $z \to z + b_0z^{-1} + f(z)$ has no zeroes for real non-zero values of z, and that it has either a simple zero ($b_0 = 0$) or a simple pole ($b_0 \neq 0$) at $z = 0$. Therefore, there is a positive number $\varepsilon_3 (\leq \varepsilon_2)$ such that the function

$$z \to z(z + b_0z^{-1} + f(z))^{-1} \tag{173}$$

is holomorphic and even on $|\operatorname{Im} z| < \varepsilon_3$; it tends uniformly to 1 as $|z| \to \infty$ in this strip; it is non-negative for real z. If $b_0 = 0$, it has no zero whatsoever in the strip; therefore, there is a function g, holomorphic in the strip, real for z real, such that

$$g(z) = \frac{1}{2} \log z(z + f(z))^{-1}$$

If $b_0 \neq 0$, the only zero of the function (173) is at $z = 0$; this zero is double; therefore, there is a function g, holomorphic in the strip, real for z real, such that

$$g(z) = \frac{1}{2} \log (z(z + b_0z^{-1} + f(z))^{-1}z^{-2}(z^2 + a^2))$$

for any fixed a with $a > \varepsilon_3$. In both cases, g is even (i. e. $g(-z) = g(z)$), bounded on $|\operatorname{Im} z| < \varepsilon_3$, and it tends uniformly to zero as $|z| \to \infty$ in this strip. Using also the fact that f is bounded in the strip, it is not difficult to show, with the help of Maclaurin's formula for $\log(1 + x)$ and small x's, that the function g_1, defined by

$$g_1(z) = g(z) + \frac{1}{2} f(z)(z - ia)^{-1}$$

(a as above), besides being holomorphic in the strip, satisfies the following: for all positive numbers η with $\eta < \varepsilon_3$, there is a positive number M such that

$$\begin{aligned} &\int_{-\infty}^{+\infty} |g_1(\sigma + i\tau)|^2 d\sigma < \mathrm{M} \\ &\int_{-\infty}^{+\infty} |(\sigma + i\tau)g_1(\sigma + i\tau)|^2 d\sigma < \mathrm{M} \end{aligned} \tag{174}$$

for all real τ with $|\tau| \leq \eta$.

We show now that there is a measurable function m_3 on $y > 0$ such that

$$\int_0^\infty y^\eta |m_3(y)| dy < \infty \tag{175}$$

for all real η with $|\eta| < \varepsilon_3$, and such that, for τ real

$$g(\tau) = \int_0^\infty m_4(y) y^{i\tau - \frac{1}{2}} dy \tag{176}$$

where m_4 is given by

$$y^{\frac{1}{2}} m_4(y) = \int_0^y m_3(u) du. \tag{177}$$

In view of (174), the above statement is easily proven for g replaced by g_1, because of a theorem by Paley and Wiener (see Appendix I). It remains therefore to prove it for g replaced by $-\frac{1}{2} f(\tau)(\tau - ia)^{-1}$. Here, both $f(\tau)$ and $(\tau - ia)^{-1}$ are Mellin transforms of functions in $\mathscr{L}_1(0, \infty; y^{-\frac{1}{2}} dy)$; therefore, their product $f(\tau)(\tau - ia)^{-1}$ has the same property (see Appendix I): it is the Mellin transform of the convolution product m_4 given by

$$m_4(y) = i \int_y^\infty (y/u)^{a - \frac{1}{2}} m_2(u) u^{-1} du$$

(almost everywhere). It is easily checked, using the fact that $a > \varepsilon_3$, that m_4 has the properties expressed by (175) and (177). The function g being even, we find for the function m_4 in (176) the relation

$$y^{-1}m_4(y^{-1}) = m_4(y) \quad \text{(almost everywhere).} \tag{178}$$

We now define on $\operatorname{Im} z < 0$ a function ψ by

$$\psi(z) = \pi^{-1}i \int_{-\infty}^{+\infty} g(\tau)(\tau - z)^{-1}d\tau \tag{179}$$

(for the convergence of the integral, remember that $g(\tau)$ is in $\mathscr{L}_2(-\infty, \infty)$). Using the Lemma proved in Appendix I, we find

$$\psi(z) = 2\int_0^1 m_4(y)y^{-z-\frac{1}{2}}dy \tag{180}$$

in $\operatorname{Im} z < 0$. This shows (use (175)) that ψ is holomorphic on $\operatorname{Im} z < \varepsilon_3$; furthermore, an easy computation gives us for τ real:

$$\operatorname{Re} \psi(\tau) = g(\tau) \tag{181}$$

(use (178)). Another consequence of Eq. (180) is the estimate

$$|z\psi(z)| < C$$

uniformly in $\operatorname{Im} z \leq \eta$, for any η smaller than ε_3. Finally, from (179), we get for σ real, positive:

$$\operatorname{Im} \psi(-i\sigma) = \pi^{-1}\int_{-\infty}^{+\infty} \tau g(\tau)(\tau^2 + \sigma^2)^{-1}d\tau$$

($g(\tau)$ is real!); because g is even, it follows that

$$\operatorname{Im} \psi(-i\sigma) = 0.$$

Now, let $0 < v_1 < \ldots < v_k$ be the poles of w_2 in $\operatorname{Re} v > 0$. Define h by

$$h(v) = \begin{cases} \prod_{j=1}^{k}(1 - 2v_j(v + v_j)^{-1}) & \text{when} \quad b_0 = 0 \\ (1 - a(a + v)^{-1})\prod_{j=1}^{k}(1 - 2v_j(v + v_j)^{-1}) & \text{when} \quad b_0 \neq 0. \end{cases}$$

Define then w_1 by

$$w_1(v) = h(v) \exp(\psi(iv))$$

Let us show that w_1 is in $\mathscr{A}(\eta)$ for any η with $\eta < \varepsilon_4 = \min\{\varepsilon_3, \nu_1\}$. The function w_1 is clearly holomorphic in $\mathrm{Re}\,\nu > -\varepsilon_4$, and it obviously satisfies property (A2). For any η with $\eta < \varepsilon_4$, there is a positive number C such that, for ν in $\mathrm{Re}\,\nu \geq -\eta$,

$$|\psi(i\nu)| < |\nu|^{-1}C$$
$$|\exp(\psi(i\nu)) - 1 - \psi(i\nu)| < |\nu|^{-2}C$$
$$\left| h(\nu) - 1 + \sum_{j=1}^{k} 2\nu_j(\nu + \nu_j) \right| < |\nu|^{-2}C \qquad (b_0 = 0)$$
$$\left| h(\nu) - 1 + a(\nu + a)^{-1} + \sum_{j=1}^{k} 2\nu_j(\nu + \nu_j)^{-1} \right| < |\nu|^{-2}C \qquad (b_0 \neq 0)$$

Therefore, defining h_1 by

$$w_1(\nu) = \begin{cases} 1 + \psi(i\nu) - \sum_{j=1}^{k} 2\nu_j(\nu + \nu_j)^{-1} + h_1(\nu) & (b_0 = 0) \\ 1 + \psi(i\nu) - a(\nu + a)^{-1} - \sum_{j=1}^{k} 2\nu_j(\nu + \nu_j)^{-1} + h_1(\nu) & (b_0 \neq 0) \end{cases}$$

we see that it is a holomorphic function on $\mathrm{Re}\,\nu > -\varepsilon_4$, which has the property that, for any η with $\eta < \varepsilon_4$, there is a positive number N such that

$$\int_{-\infty}^{+\infty} |h_1(\sigma + i\tau)|^2 d\tau < N$$
$$\int_{-\infty}^{+\infty} |(\sigma + i\tau)h_1(\sigma + i\tau)|^2 d\tau < N$$

for all σ with $\sigma \geq -\eta$. Using Paley-Wiener's theorem for h_1 and formula (180) for ψ (remember (175) and (177)), it is easy to verify that w_1 satisfies (A1), i. e.

$$w(\nu) = 1 + \int_0^1 l(y)y^{\nu - \frac{1}{2}}dy,$$

where l is in $\mathscr{L}(\eta)$ for any η smaller than ε_4. Thus, we have shown that w_1 is in $\mathscr{A}(\eta)$ for the same values of η.

To check that the pair (w_1, w_2) is in $\mathscr{P}$, we notice first that (P1) is obviously satisfied. For τ real, we have

$$|w_1(i\tau)|^2 = \begin{cases} \exp(2\,\mathrm{Re}\,\psi(-\tau)) & \text{if } b_0 = 0 \\ |\tau(\tau - ia)^{-1}|^2 \exp(2\,\mathrm{Re}\,\psi(-\tau)) & \text{if } b_0 \neq 0; \end{cases}$$

therefore, by Eq. (181),

$$|w_1(i\tau)|^2 = \begin{cases} \exp(2g(\tau)) & \text{if } b_0 = 0 \\ \tau^2(\tau^2 + a^2)^{-1} \exp(2g(\tau)) & \text{if } b_0 \neq 0. \end{cases}$$

(remember that $g(-\tau) = g(\tau)$); therefore,

$$|w_1(i\tau)|^2 = \tau\,(\operatorname{Im} w_2(i\tau))^{-1}$$

in both cases. This shows that (P2) is also satisfied. Q. e. d.

In addition to Lemmata 15 and 16, we need one more link between the regular solution and its logarithmic x-derivative:

LEMMA 17. — *Let* V, φ *and* ξ *be as in Lemma* 13; *let* x *be fixed; let* $v > 0$ *be such that* $\varphi(x, v) = 0$. *Then*

$$\operatorname{Res} \xi(x, v) = -2vx^{-1}(\partial_v \varphi(x, v))^{-2} \int_0^x |\varphi(y, v)|^2 dy.$$

Proof: From Eq. (166), we obtain

$$\operatorname{Res} \xi(x, v) = x\varphi'(x, v)/\partial_v \varphi(x, v)$$

On the other hand, using Green's formula, we get

$$\int_0^x (\varphi(y, v))^2 dy = -(2v)^{-1} x^2 \varphi'(x, v) \partial_v \varphi(x, v)$$

Q. e. d.

Before formulating Theorem 5, we repeat for the convenience of the reader the definition of $\mathscr{F}$. $\mathscr{F}$ is the class of those real-valued, locally integrable potentials V such that

$$\int_0^1 |V(y) - 1|\, y^{1-2\varepsilon} dy < \infty$$

for some positive ε, and

$$V(y) = 0 \quad \text{for} \quad y > a,$$

where a is some non-negative real number. We can now prove the

THEOREM 5. — *In order that the sequence* $\{ s_l \}$ *be the scattering amplitude* (*as defined by Eq.* (83)) *generated by a potential* V *in* $\mathscr{F}$ *such that*

$$V(y) = 0 \quad \text{for} \quad y > x, \tag{182}$$

it is necessary and sufficient that there exist a positive number ε *and a function* w *in* $\mathscr{B}(\varepsilon)$ *such that*

$$w\left(l + \frac{1}{2}\right) = x \frac{s_l \psi'_{0+l} + \psi'_{0-l}}{s_l \psi_{0+l} + \psi_{0-l}} \tag{183}$$

for $l = 0, 1, \ldots,$ *where*

$$\psi_{0+l} = x^{-\frac{1}{2}} \mathrm{H}^{(1)}_{l+\frac{1}{2}}(x)$$
$$\psi_{0-l} = x^{-\frac{1}{2}} \mathrm{H}^{(2)}_{l+\frac{1}{2}}(x)$$
$$\psi'_{0+l} = (x^{-\frac{1}{2}} \mathrm{H}^{(1)}_{l+\frac{1}{2}}(x))'$$
$$\psi'_{0-l} = (x^{-\frac{1}{2}} \mathrm{H}^{(2)}_{l+\frac{1}{2}}(x))'.$$

Remark. — If w has a pole at $v = l + \dfrac{1}{2}$, where l is a non-negative integer, we mean by Eq. (183) that for this l,

$$s_l \psi_{0+l} + \psi_{0-l} = 0.$$

And conversely.

Proof: The *necessity* of the condition follows from Lemma 13. To see this, replace w by ξ for the given x, and remember the relation between ξ and $\{ s_l \}$ for a potential satisfying (182) (see Section 3).

To show that the condition is *sufficient*, we argue as follows. By Lemma 16, there exist a positive number η and a function w_1 in $\mathscr{A}(\eta)$ such that (w_1, w) is in $\mathscr{P}$. Let $0 < v_1 < \ldots < v_k$ be the zeroes of w_1 in $\operatorname{Re} v > 0$; we know that they coincide with the poles of w (by (P1)); the residues of these poles are negative, by (B4). We define $b_1, \ldots, b_k$ by

$$b_j = -\, x(2v_j)^{-1}(\partial_v \varphi_1(v))^2 \operatorname{Res} w_1(v_j)$$

$(1 \leq j \leq k)$, where φ_1 is defined by

$$\varphi_1(v) = x^{v-\frac{1}{2}} w_1(v);$$

the $b's$ are positive. By Theorem 4, there is a potential V in $\mathscr{F}$, satisfying (182), such that the corresponding regular solution φ satisfies

$$\varphi(x, v) = x^{v-\frac{1}{2}} w_1(v)$$

(for $\operatorname{Re} v \geq 0$, say), and

$$\int_0^x (\varphi(y, v_j))^2 dy = b_j$$

$(1 \leq j \leq k)$. Let ξ be defined by

$$\xi(v) = x\varphi'(x, v)/\varphi(x, v).$$

The scattering amplitude $\{\bar{s}\}$ corresponding to V is then given in terms of ξ by

$$\xi\left(l+\frac{1}{2}\right) = x\,\frac{\bar{s}_l\psi'_{0+l} + \psi'_{0-l}}{\bar{s}_l\psi_{0+l} + \psi_{0-l}}$$

It remains to prove that $\{\bar{s}\} = \{s\}$. We prove it by showing that $\xi = w$. Lemma 13 tells us that ξ is in $\mathscr{B}(\varepsilon_1)$ for some positive ε_1; by Lemma 14, the pair (w_1, ξ) is in $\mathscr{P}$. Using Lemma 15, we find that

$$\xi(v) - w(v) = \sum_{j=1}^{k} c_j(v - v_j)^{-1}$$

where

$$c_j = \text{Res }\xi(v_j) - \text{Res } w(v_j)$$

But Lemma 17 yields

$$\text{Res }\xi(v_j) = -\,2v_jx^{-1}(\partial_v\varphi_1(v_j))^{-2}b_j$$

therefore $c_j = 0$ for $1 \leq j \leq k$. Q. e. d.

Theorem 5 has been formulated in terms of the sequence of the partial waves amplitudes $a_l = \frac{1}{2}\, i(1 - s_l)(l = 0, 1, \ldots)$. It is perhaps possible to formulate it directly in terms of the function

$$F(z) = \sum_{l=0}^{\infty} (2l + 1)a_l P_l(z)$$

whose modulus squared is proportional to the differential scattering cross section in function of the scattering angle θ when we put $z = \cos\theta$. Let us state in this spirit a consequence of Theorem 5: *In order that* F *be generated by a potential* V *in* $\mathscr{F}$ *such that*

$$V(y) = 0$$

for $y > x$, *it is necessary that* F *be an entire function such that*

$$\limsup_{r\to\infty} r^{-\frac{1}{2}} ln\,|\,M(r)\,| \leq x,$$

M *being defined by*

$$M(r) = \sup\,\{\,|\,F(re^{i\varphi})\,|,\ 0 \leq \varphi \leq 2\pi\,\}.$$

To show this, we remark that under our assumptions the scattering amplitude $\{s\}$ generated by V is given by Eq. (183) in terms of a function w in $\mathscr{B}(\varepsilon)$ (some $\varepsilon > 0$). Now, because of (B5),

$$w\left(l + \frac{1}{2}\right) = l + o(1)$$

as l (integer) goes to $+\infty$. Therefore, using well known asymptotic formulæ for the Bessel functions appearing in Eq. (183), we find

$$a_l = \frac{1}{2} i(1 - s_l) = o\left(\left(\frac{1}{2} x\right)^{2l} \left(\Gamma\left(l + \frac{3}{2}\right)\right)^{-2l}\right).$$

From here, it follows that the function G, defined by

$$G(z) = \sum_{l=0}^{\infty} a_l z^l,$$

is an entire function of « growth » $\left(\frac{1}{2}, x\right)$ (cf. Ref. [24], Chap. 2), since

$$\limsup l^2 |a_l|^{1/l} \leq \left(\frac{1}{2} xe\right)^2$$

(cf. Ref. [24], Theorem 2.2.10). Finally, using an idea of Z. Nehari [29], we see that the same is true of F.

4.2 On the Proof of Theorem 4

We notice first that we can limit ourselves to prove Theorem 4 for $x = 1$ (we let the reader convince himself that if Theorem 4 is true for $x = 1$, then it is true for any positive x).

Our problem is then the following: given w and $b_1, \ldots, b_k$ as in Theorem 4, find V such that the solution $\varphi(x, v)$ of

$$-\frac{d}{dx}\left(x^2 \frac{d\varphi}{dx}\right)(x, v) - \frac{1}{4} \varphi(x, v) + q(x)\varphi(x, v) = -v^2\varphi(x, v) \qquad (184)$$

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[29] Z. NEHARI, On the Singularities of Legendre Expansions. *J. Rat. Mech. Anal.* (now: *J. Math. Mech.*), **5**, 1956, 987-992. Similar ideas have been developed independently in: T. KINOSHITA, J.-J. LOEFFEL and A. MARTIN, Upper Bounds for the Scattering Amplitude at High Energy. *Phys. Rev.*, **135**, 1964, B1464-1482.

on $0 < x \leq 1$ with the boundary condition

$$\lim_{x \to 0} x^{-\nu + \frac{1}{2}} \varphi(x, \nu) = 1 \tag{185}$$

satisfies

$$\varphi(1, \nu) = w(\nu)$$

on Re $\nu \geq 0$, and

$$\int_0^1 |\varphi(x, \nu_j)|^2 dx = b_j$$

for $1 \leq j \leq k$. We have put

$$q(x) = x^2(\mathrm{V}(x) - 1).$$

We now perform a change of variable which will make evident the almost complete identity of our problem with the one treated by Agranovich and Marchenko. Associated with the diffeomorphism $x \to -\ln x$ from the open interval $(0, 1)$ onto the open half-line $(0, \infty)$ there is a linear unitary map $f \to g$ from $\mathscr{L}_2(0, 1)$ onto $\mathscr{L}_2(0, \infty)$, defined by

$$g(r) = e^{-\frac{1}{2}r} f(e^{-r})$$

for f in $\mathscr{L}_2(0, 1)$. Define accordingly u by

$$u(r, k) = e^{-\frac{1}{2}r} \varphi(e^{-r}, -ik)$$

$(0 < r < \infty, \text{Im } k > 0)$; define furthermore U by

$$\mathrm{U}(r) = q(e^{-r}).$$

Notice that the condition

$$\int_0^1 x^{-1-2\varepsilon} |q(x)| dx < \infty$$

is equivalent to the condition

$$\int_0^\infty |\mathrm{U}(r)| e^{2\varepsilon r} dr < \infty. \tag{186}$$

It is easy to verify that for φ to be a solution of the system (184) and (185), it is necessary and sufficient that u is a solution of the system

$$-\frac{d^2 u}{dr^2}(r, k) + \mathrm{U}(r) u(r, k) = k^2 u(r, k) \tag{187}$$

and

$$\lim_{r \to \infty} u(r, k) e^{-ikr} = 1. \tag{188}$$

Thus, in the new variable, our problem becomes: given w and $b_1, \ldots, b_k$ as in Theorem 4, find U such that the solution $u(r, k)$ of the system (187) and (188) satisfies

$$u(0, k) = w(-ik) \tag{189}$$

for $\operatorname{Im} k \geq 0$, and

$$\int_0^\infty |u(r, k_j)|^2 dr = b_j \tag{190}$$

for $1 \leq j \leq k(k_j = iv_j$; the $k's$ are the zeroes of $k \to w(-ik) = u(0, k)$ in $\operatorname{Im} k > 0)$.

But Eq. (187) is nothing else than the radial Schrödinger equation for stationary s-waves, with potential U, at an energy equal to k^2. As we know, the physically acceptable solutions of this radial equation are those which vanish at $r=0$. Let iv_0 (v_0 positive) be a zero of $k \to u(0, k)$; then $\mathrm{E} = -v_0^2$ is the energy of a bound state of the potential U, and $u(r, iv_0)$ is the corresponding wave function, normalized by the condition

$$e^{v_0 r} u(r, k) \to 1$$

as $r \to \infty$. Let k be real; denote by ψ a solution of Eq. (187) with $\psi(0) = 0$ (ψ is unique up to a constant factor); since $u(r, k)$ and $u(r, -k)$ form a fundamental system of solutions of (187), we can write

$$\psi(r) = \mathrm{A}(u(r, -k) - \mathrm{S}(k)u(r, k));$$

$\mathrm{S}(k)$, which is independent of the normalization of ψ, gives the s-phase shift at the energy k^2 for the scattering by the potential U. Putting $r = 0$, we get

$$\mathrm{S}(k) = u(0, -k)/u(0, k)$$

or, in terms of w,

$$\mathrm{S}(k) = w(ik)/w(-ik). \tag{191}$$

Now, the problem solved by Agranovich and Marchenko is the following: to give necessary and sufficient conditions for a function S, defined for k real, as well as for two corresponding sets of positive numbers $v_1, \ldots, v_k$ and $b_1, \ldots, b_k$, to be such that these entities correspond to a continuous potential U with

$$\int_0^\infty |\mathrm{U}(r)|\, dr < \infty \tag{192}$$

$$\int_0^\infty r\, |\mathrm{U}(r)|\, dr < \infty \tag{193}$$

according to (187), (188), (189), (190) and (191). These conditions are written down in Ref. ([10]), Chap. V, § 1.

Thus, the problem of necessary and sufficient conditions solved by Lemma 11 and Theorem 4 is almost identical with the problem of Agranovich and Marchenko. The modifications are as follows:

a) instead of considering $S(k)$ for real k, we consider $w(v)$ for $\operatorname{Re} v \geq 0$;

b) instead of considering continuous potentials, we consider locally integrable potentials;

c) instead of imposing on the potentials conditions of the form (192), (193), we impose on them a condition of the form (186).

Modifications *a*) and *c*) simplify the task; *b*) does not complicate it insuperably.

We now go over to sketch a version of the proof of Theorem 4 for $x = 1$ which borrows heavily from the ideas and the results of the book by Agranovich and Marchenko.

α) Let w be the given function in $\mathscr{A}(\varepsilon)$, ε being some positive number. We have

$$w(v) = 1 + g(v),$$

where

$$g(v) = \int_0^1 l(y) y^{v-\frac{1}{2}} dy \tag{194}$$

for some l in $\mathscr{L}(\varepsilon)$. As we know, g is holomorphic on $\operatorname{Re} v > -\varepsilon$, and

$$|vg(v)| < C$$

for $\operatorname{Re} v > -\varepsilon$ (C: some constant). Therefore, the function h, defined by

$$h(v) = 1 - (w(v)/w(-v)) \tag{195}$$

satisfies the following: there are positive numbers $\varepsilon_1 (< \varepsilon)$ and C_1 such that h is holomorphic on the strip $|\operatorname{Re} v| \leq \varepsilon_1$, and

$$\int_{-\infty}^{+\infty} |h(\sigma + i\tau)|^2 d\tau < C_1$$

for σ real with $|\sigma| \leq \varepsilon_1$. Therefore (Paley-Wiener theorem, see Appendix I), there is a function f_s such that

$$\int_0^{\infty} |f_s(y) y^{\sigma}|^2 dy < \infty$$

for σ real with $|\sigma| \leq \varepsilon_1$, and

$$h(v) = \int_0^\infty f_s(y) y^{v-\frac{1}{2}} dy. \tag{196}$$

We have

$$f_s(y) = \mathscr{L}_2\text{-}\lim_{a\to\infty} (2\pi)^{-1} \int_{-a}^{+a} h(i\tau) y^{-i\tau-\frac{1}{2}} d\tau \tag{197}$$

(see Ref. ([10]), Eq. (5.1.1)). The function f_s is real, since

$$(h(i\tau))^* = h(-i\tau)$$

for τ real. It has differentiability properties: to see this, define h_1 by

$$h(v) = g(-v) - g(v) + h_1(v);$$

i. e. by

$$h_1(v) = g(-v)(g(v) - g(-v))/w(-v)$$

Clearly, h_1 is holomorphic on the strip $|\mathrm{Re}\, v| \leq \varepsilon_1$; furthermore, there is a positive number C_2 such that

$$|v^2 h_1(v)| < C_2$$

in the strip, and

$$\int_{-\infty}^{+\infty} |h_1(\sigma + i\tau)|^2 d\tau < C_2$$

$$\int_{-\infty}^{+\infty} |(\sigma + i\tau) h_1(\sigma + i\tau)|^2 d\tau < C_2$$

for σ real with $|\sigma| \leq \varepsilon_1$. Therefore, the function m on $y > 0$ defined by

$$m(y) = (2\pi)^{-1} \int_{-\infty}^{+\infty} h_1(i\tau) y^{-i\tau-\frac{1}{2}} d\tau$$

satisfies

$$\int_0^\infty |m(y) y^\sigma|^2 dy < \infty$$

for σ real, $|\sigma| \leq \varepsilon_1$; it is absolutely continuous:

$$y^{\frac{1}{2}} m(y) = -\int_0^y m_1(u) u^{-\frac{1}{2}} du = \int_y^\infty m_1(u) u^{-\frac{1}{2}} du$$

where m_1, given by

$$m_1(y) = \mathscr{L}_2\text{-}\lim_{a\to\infty} (2\pi)^{-1} \int_{-a}^{+a} i\tau h(i\tau) y^{-i\tau-\frac{1}{2}} d\tau,$$

is such that

$$\int_0^\infty |m_1(y)y^\sigma|^2 dy < \infty$$

for σ real with $|\sigma| \leq \varepsilon_1$. Thus, we find

$$f_s(y) = \begin{cases} -l(y) + m(y) & 0 < y < 1 \\ y^{-1}l(y^{-1}) + m(y) & y > 1, \end{cases}$$

l being as in Eq. (194). Let us state our result:

A. *The real function f_s defined in terms of w by Eqs.* (195) *and* (197), *is absolutely continuous on* $0 < y < 1$; *on this interval,*

$$y^{\frac{1}{2}} f_s(y) = \int_0^y m_2(u)du;$$

where the real-valued measurable function m_2 satisfies

$$\int_0^1 |m_2(y)|\, y^{-\eta_1} dy < \infty$$

for some positive η_1. Furthermore,

$$\int_0^\infty |f_s(y)y^{\eta_1}|^2 dy < \infty.$$

Statement A is the analog of statements I_s and II_s of Ref. ([10]), Chap. V, § 1.

β) We define now f by

$$f(x) = \begin{cases} f_s(x) + \sum_{j=1}^{k} b_j^{-1} x^{\nu_j - \frac{1}{2}} & \text{for} \quad x \leq 1 \\ f_s(x) & \text{for} \quad x > 1 \end{cases} \tag{198}$$

f_s being given in terms of w by Eqs. (195) and (197). The function f is easily seen to satisfy

$$\int_0^\infty dx \int_0^1 dy\, |f(xy)|^2 < \infty \tag{199}$$

From this fact, it follows in particular that the map $g \to \bar{g}$, where

$$\bar{g}(y) = \int_0^x g(u) f(uy) du$$

is for all x with $0 < x \leq 1$ a continuous linear map from $\mathscr{L}_2(0, x)$ into

$\mathscr{L}_2(0, \infty)$; or, if we restrict y to the interval $(0, x)$, it is a continuous linear map Φ_x from $\mathscr{L}_2(0, x)$ into itself. The second important step in the proof of Theorem 4 consists in proving that

B. a) *For x with $0 < x \leq 1$, $g \in \mathscr{L}_2(0, \infty)$ and $g + \Phi_x(g) = 0$ imply $g = 0$.*
b) *The function l of Eq. (194) (which is in $\mathscr{L}(\varepsilon)$, and therefore in $\mathscr{L}_2(0, x)$) satisfies the equation*

$$f(y) + l(y) + \int_0^1 l(u) f(uy) du = 0$$

for $0 < y \leq 1$.

The proof of statement B. *a*) is almost identical to (actually, somewhat simpler than) the proof of Theorem 3.5.1 of Ref. ([10]). We shall not elaborate on it. Statement B. *b*) is immediately verified: one only needs to check that the function h, defined for $y > 0$ by

$$h(y) = f(y) + l(y) + \int_0^1 l(u) f(uy) du$$

is in $\mathscr{L}_2(0, \infty)$, and that its Mellin transform $\bar{h}$, defined by

$$\bar{h}(\tau) = \mathscr{L}_2\text{-} \lim_{a \to +\infty} (2\pi)^{-1} \int_{a^{-1}}^{a} h(y) y^{i\tau - \frac{1}{2}} dy$$

can be extended to an holomorphic function, also denoted by $\bar{h}$, on $\operatorname{Im} \tau \geq 0$, and that there is a positive number M such that

$$\int_{-\infty}^{+\infty} |\bar{h}(\sigma + i\tau)|^2 d\tau < \mathrm{M}$$

for $\sigma \geq 0$; it follows then from Hille-Tamarkin's theorem (Ref. ([14])) that $h(y)$ vanishes for $y < 1$ (As a matter of fact, one finds

$$\bar{h}(\tau) = w(-i\tau) - 1 - iw(-i\tau) \sum_{j=1}^{k} b_j^{-1}(\tau - iv_j)^{-1}).$$

Statement B is a substitute for statements III_s, IV and V in Ref. ([10]).

γ) The function f defined by Eq. (198) has the following property, which is analogous to the property of f_s stated in A:

A′. *The real-valued function f is absolutely continuous on $0 < y \leq 1$; on this interval,*

$$y^{\frac{1}{2}} f(y) = \int_0^y \mathrm{F}(u) u^{-1} du,$$

where the real-valued measurable function F *satisfies*

$$\int_0^1 |F(u)| u^{-1-\eta} du < \infty$$

for some positive number η.

Consider for each x with $0 < x \leq 1$ the following integral equation

$$f(xy) + L(x, y) + \int_0^x L(x, z) f(zy) dz = 0 \tag{200}$$

on the interval $0 < y \leq x$, where the function $y \to L(x, y)$ is supposed to be in $\mathscr{L}_2(0, x)$. Statement B. *a*) and (199) allow us to deduce that L is unique; from B. *b*), we get

$$L(1, y) = l(y). \tag{201}$$

Furthermore, from the fact that f satisfies A′ and B. *a*), one derives the following statement (η as in A′):

Γ. *The solution* L *of* (200) *is real-valued and has the following properties:*

a) $(x, y) \to (xy)^{\frac{1}{2}} L(x, y)$ *is continuous on* $0 \leq y \leq x \leq 1$; *there is a positive number* C *such that*

$$|(xy)^{\frac{1}{2}-\eta} L(x, y)| < C,$$

b) *there is a real-valued measurable function* q *such that*

$$\int_0^1 x^{1-2\eta} |q(x)| dx < \infty$$

and

$$2xL(x, x) = \int_0^x u^{-1} q(u) du \tag{202}$$

c) *the function* φ, *defined by*

$$\varphi(x, \nu) = x^{\nu - \frac{1}{2}} + \int_0^x L(x, y) y^{\nu - \frac{1}{2}} dy \tag{203}$$

on $\{(x, \nu) \,|\, 0 < x \leq 1, \operatorname{Re} \nu > -\eta\}$ *is the regular solution, in the sense of Lemma* 1, *of Eq.* (184), q *being given by* (202).

This result is the analog of those which are derived in Chap. V, § 2, 3 and 4, of Ref. ([10]). The proof of Γ, though rather long and tedious, is relatively straightforward. For the sake of completeness, we give a sketch of one of its versions.

All Banach spaces introduced below are on the field of the reals. The norms pertaining to all of them will be denoted by the same symbol $\|\cdot\|$; the context will make clear which norm we mean in each particular occurrence. The reader should bear in mind that certain entities below will be considered as elements of more than one Banach space. Linear continuous maps from one Banach space to another will be called morphisms. The spaces of morphisms between Banach spaces will be tacitly regarded as Banach spaces equipped with the usual norm.

1. We denote by $\mathcal{M}_1$ the Banach space of the real-valued measurable functions g on I ($\mathrm{I} = \{x \mid 0 \leq x \leq 1\}$) which are such that

$$\|g\| = \int_0^1 y^{-1-\eta}|g(y)|\,dy < \infty;$$

and by $\mathcal{M}_2$ the Banach space of the real-valued continuous functions g on I such that

$$\lim_{y \to 0} y^{-\eta} g(y) = 0$$

equipped with the norm

$$\|g\| = \sup\{y^{-\eta} g(y) \mid y \text{ in I}\}$$

Let g be in $\mathcal{M}_1$; we define the function $\mathrm{J}(g)$ by

$$\mathrm{J}(g)(y) = \int_0^y u^{-1} g(u)\,du.$$

It is easy to see that $\mathrm{J}(g)$ is in $\mathcal{M}_1$, and that

$$\|\mathrm{J}(g)\| \leq \eta^{-1}\|g\|;$$

it is equally easy to see that $\mathrm{J}(g)$ is in $\mathcal{M}_2$, and that

$$\|\mathrm{J}(g)\| \leq \|g\|.$$

2. We denote by $\mathcal{M}'_{10}$ the set of those F in $\mathcal{M}_1$ which are such that

$$\begin{gathered}\text{for each } x \text{ in I}, \int_0^1 y^{-1}|g(y)|^2 dy < \infty \text{ and} \\ g(y) + \int_0^1 \mathrm{J}(\mathrm{F})(xyu) g(u) u^{-1} du = 0 \text{ imply } g = 0.\end{gathered} \tag{204}$$

Let f be a function satisfying A′ and B. *a*). Then there is an F in $\mathcal{M}'_{10}$ such that

$$y^{\frac{1}{2}} f(y) = \mathrm{J}(\mathrm{F})(y).$$

It is easily verified that $\mathcal{M}'_{10}$ is contained in the subset $\mathcal{M}_{10}$ of $\mathcal{M}_1$ consisting of those F which are such that

$$\text{for each } x \text{ in I, } g \text{ in } \mathcal{M}_1 \text{ and} \\ g(y) + \int_0^1 \mathrm{J(F)}(xyu)g(u)u^{-1}du = 0 \text{ imply } g = 0. \tag{205}$$

(For any given x in I, define h by

$$h(y) = \int_0^1 \mathrm{J(F)}(xyu)g(u)u^{-1}du,$$

and verify that $g \in \mathcal{M}_1$ implies $h \in \mathcal{M}_1$ and $h \in \mathcal{M}_2$, whence

$$\int_0^1 y^{-1}\,|\,h(y)\,|^2 dy < \infty$$

follows). We shall prove Γ under the assumption that f is of the form

$$y^{\frac{1}{2}}f(y) = \mathrm{J(F)}(y),$$

where F is in $\mathcal{M}_{10}$; this assumption is slightly less restrictive than the original one.

3. Let $\mathscr{A}$ be a Banach space. We denote by $(\mathscr{A})_\varepsilon(\varepsilon > 0)$ the Banach space of the continuous maps $\Phi\colon \mathrm{I} \to \mathscr{A}$ which are such that

$$\|\,\Phi\,\| = \sup\,\{\,x^{-\varepsilon}\,\|\,\Phi(x)\,\|,\ x \text{ in I}\,\}$$

is finite. We denote by $\mathscr{E}$ the Banach algebra of the morphisms from $\mathcal{M}_1$ into itself. We define a morphism H from $\mathcal{M}_1$ into $(\mathscr{E})_\eta$ in the following way: H(F)(x) (F in $\mathcal{M}_1$, x in I) is the morphism $g \to \mathrm{H(F)}(x)(g)$ in $\mathscr{E}$ given by

$$\mathrm{H(F)}(x)(g)(y) = \int_0^1 \mathrm{J(F)}(xyu)g(u)u^{-1}du$$

(we leave the necessary verifications to the reader). In particular, we have the estimate

$$\|\,\mathrm{H(F)}(x)\,\| \leq \eta^{-1}x^{\eta}\,\|\,\mathrm{F}\,\|. \tag{206}$$

One can see furthermore that for g in $\mathcal{M}_1$, H(F)(x)(g) is in $\mathcal{M}_2$, and that H can also be considered as a morphism from $\mathcal{M}_1$ into $(\mathrm{Mor}\,(\mathcal{M}_1, \mathcal{M}_2))_\eta$ (here, Mor $(\mathcal{M}_1, \mathcal{M}_2)$ is the space of the morphisms from $\mathcal{M}_1$ into $\mathcal{M}_2$). Moreover, one can prove that H(F)(x) is a compact operator of $\mathscr{E}$, for all F in $\mathcal{M}_1$ and all x in I. Consequently, F is in $\mathcal{M}_{10}$ if and only if 1 + H(F) has an inverse in the Banach algebra $(\mathscr{E})_0$. Let g be a real-valued function on I; we define $T_x(g)$ by

$$T_x(g) = g(x, y)$$

for $0 < x \leq 1$, and

$$T_0(g) = 0$$

T_x is easily seen to be a morphism from $\mathscr{M}_1$ into itself, with

$$\| T_x \| = x^{\eta}.$$

It is also a morphism from $\mathscr{M}_2$ into itself; again

$$\| T_x \| = x^{\eta}.$$

One can verify that, for each g in $\mathscr{M}_1$ (resp. in $\mathscr{M}_2$), $x \to T_x(g)$ is in $(\mathscr{M}_1)_\eta$ (resp. in $(\mathscr{M}_2)_\eta$). We define now a map N from $I \times \mathscr{M}_{10}$ into $\mathscr{M}_1$ by

$$N(x, F) = -(1 + H(F))^{-1}(x)(T_x \circ J(F)) \tag{207}$$

That N is continuous follows from what has been said. We find also that $F \to (x \to N(x, F))$ is continuous from $\mathscr{M}_{10}$ into $(\mathscr{M}_1)_\eta$. We can now rewrite Eq. (207) in the following way:

$$T_x \circ J(F) + N(x, F) + H(F)(x)(N(x, F)) = 0. \tag{208}$$

From this last relation, and from what has been said, we deduce that N can also be considered as a continuous map from $I \times \mathscr{M}_{10}$ into $\mathscr{M}_2$, and that $F \to (x \to N(x, F))$ can be considered as a continuous map from $\mathscr{M}_{10}$ into $(\mathscr{M}_2)_\eta$. As a consequence, we see that, for F in $\mathscr{M}_{10}$, $(x, y) \to N(x, F)(y)$ is continuous on $I \times I$, and that we have an estimate of the form

$$| N(x, F)(y) | < C(xy)^{\eta},$$

where C depends only on F. F being in $\mathscr{M}_{10}$, put

$$y^{\frac{1}{2}}f(y) = J(F)(y) \tag{209}$$

on $0 \leq y \leq 1$, and define L on $0 < y \leq x \leq 1$ by

$$(xy)^{\frac{1}{2}}L(x, y) = N(x^2, F)(x^{-1}y); \tag{210}$$

it is easy to deduce from Eq. (208) that L is the unique solution of Eq. (200) which satisfies

$$\int_0^x | L(x, y) |^2 dy < \infty$$

for each x in $0 < x \leq 1$. From here, Γ. *a*) follows easily.

4. As a preparation for the proof of statements Γ. *b)* and Γ. *c)*, we show that there is a set of « smooth » functions in $\mathscr{M}_{10}$ which is dense in $\mathscr{M}_{10}$ (in the topology induced by $\mathscr{M}_1$). We shall say that a map $\mathrm{I} \to \mathscr{A}$ ($\mathscr{A}$: a Banach space) is *smooth* if and only if it has continuous derivatives of all orders and vanishes in some neighborhood of $x = 0$. We shall denote the set of the smooth real-valued functions by $\mathscr{M}_0$. $\mathscr{M}_0$ is known to be dense in $\mathscr{M}_1$. On the other hand, we can show that $\mathscr{M}_{10}$ is a non-empty open subset of $\mathscr{M}_1$: this follows from the fact that $\mathrm{F} = 0$ is in $\mathscr{M}_{10}$, and that, if F_0 is in $\mathscr{M}_{10}$, the open set

$$\mathrm{U} = \{ \mathrm{F} \,|\, \mathrm{F} \in \mathscr{M}_1, \ \| \mathrm{F} - \mathrm{F}_0 \| < \eta \, \| (1 + \mathrm{H}(\mathrm{F}))^{-1} \|^{-1} \}$$

is also in $\mathscr{M}_{10}$ (easy exercise in Banach algebras; use (206)). As a consequence, we see that $\mathscr{M}_{10} \cap \mathscr{M}_0$ is non-empty, and dense in $\mathscr{M}_{10}$. Let now F be in $\mathscr{M}_{10} \cap \mathscr{M}_0$. Then one can show that $(x, y) \to \mathrm{N}(x, \mathrm{F})(y)$ has continuous partial derivatives of all orders, and vanishes near $\Delta = (\{0\} \times \mathrm{I}) \cup (\mathrm{I} \times \{0\})$: just go through the arguments of point 3 above, using the « smoothness » of F.

5. We define a morphism DH from $\mathscr{M}_1$ into $(\mathrm{Mor}\,(\mathscr{M}_2, \mathscr{M}_1))_\eta$ in the following way: $\mathrm{DH}(\mathrm{F})(x)$ (F in $\mathscr{M}_2$, x in I) is the morphism $g \to \mathrm{DH}(\mathrm{F})(x)(g)$ from $\mathscr{M}_2$ into $\mathscr{M}_1$ given by

$$\mathrm{DH}(\mathrm{F})(x)(g)(y) = \int_0^1 \mathrm{F}(xyu)g(u)u^{-1}du$$

(again, we leave the verifications to the reader). DH can also be viewed as a morphism from $\mathscr{M}_1$ into $(\mathrm{Mor}\,(\mathscr{M}_2, \mathscr{M}_2))_\eta$. We define a map DN from $\mathrm{I} \times \mathscr{M}_{10}$ into $\mathscr{M}_1$ by

$$\mathrm{DN}(x, \mathrm{F}) = -(1 + \mathrm{H}(\mathrm{F}))^{-1}(x)(\mathrm{T}_x(\mathrm{F}) + \mathrm{DH}(\mathrm{F})(x)(\mathrm{N}(x, \mathrm{F}))) \qquad (211)$$

The continuity of DN is easily established, as well as the fact that $\mathrm{F} \to (x \to \mathrm{DN}(x, \mathrm{F}))$ is a continuous map from $\mathscr{M}_{10}$ into $(\mathscr{M}_1)_\eta$. From Eq. (211), we obtain

$$\mathrm{DN}(x, \mathrm{F}) + \mathrm{T}_x(\mathrm{F}) = -\mathrm{DH}(\mathrm{F})(x)(\mathrm{N}(x, \mathrm{F})) - \mathrm{H}(\mathrm{F})(x)(\mathrm{DN}(x, \mathrm{F})) \quad (212)$$

According to what was said up to now, the right-hand side is in $\mathscr{M}_2$; consequently, so is the left-hand side. Furthermore, we find that

$$(x, \mathrm{F}) \to \mathrm{DN}(x, \mathrm{F}) + \mathrm{T}_x(\mathrm{F})$$

is continuous from $\mathrm{I} \times \mathscr{M}_{10}$ into $\mathscr{M}_2$, and that

$$\mathrm{F} \to (x \to \mathrm{DN}(x, \mathrm{F}) + \mathrm{T}_x(\mathrm{F}))$$

is continuous from $\mathcal{M}_{10}$ into $(\mathcal{M}_2)_{2\eta}$. Let δ_1 be the morphism

$$g \to \delta_1(g) = g(1)$$

from $\mathcal{M}_2$ into the reals R. Applying δ_1 to $\mathrm{DN}(x, \mathrm{F}) + \mathrm{T}_x(\mathrm{F})$, we see that $\mathrm{F} \to (x \to \mathrm{DN}(x, \mathrm{F})(1) + \mathrm{F}(x))$ is continuous from $\mathcal{M}_{10}$ into $(\mathrm{R})_{2\eta}$. Therefore, putting $\mathrm{DN}_1(\mathrm{F})(x) = \mathrm{DN}(x, \mathrm{F})(1)$, we see that DN_1 is a continuous map from $\mathcal{M}_{10}$ into $\mathcal{M}_1$. Now, a direct computation shows that, for F in $\mathcal{M}_{10} \cap \mathcal{M}_0$,

$$\delta_1(\mathrm{N}(x, \mathrm{F})) = \mathrm{N}(x, \mathrm{F})(1) = \int_0^x v^{-1}\mathrm{DN}_1(\mathrm{F})(v)dv \tag{213}$$

(show first that for F in $\mathcal{M}_{10} \cap \mathcal{M}_0$, the partial derivative with respect to x of $(x, y) \to \mathrm{N}(x, \mathrm{F})(y)$ is given by $(x, y) \to x^{-1}\mathrm{DN}(x, \mathrm{F})(y)$). Using then the facts that δ_1 is continuous on $\mathcal{M}_2$, that $\mathrm{F} \to \mathrm{N}(x, \mathrm{F})$ is continuous from $\mathcal{M}_{10}$ into $\mathcal{M}_2$ (for each x in I), that DN_1 is continuous from $\mathcal{M}_{10}$ into $\mathcal{M}_1$, and that $\mathcal{M}_{10} \cap \mathcal{M}_0$ is dense in $\mathcal{M}_{10}$, we see that the relation (213) is valid for all F in $\mathcal{M}_{10}$ and all x in I. Define now q by

$$q(x) = 4\ \mathrm{DN}_1(\mathrm{F})(x^2) \tag{214}$$

We find

$$\int_0^1 |q(x)|\,x^{-1-2\eta}dx < \infty$$

and

$$2x\mathrm{L}(x, x) = \int_0^x u^{-1}q(u)du,$$

L being as in Eq. (210). Thus, we have obtained Γ. *b*).

6. We notice that $\mathrm{F} \to q$ (given by Eq. (214)) is a continuous map from $\mathcal{M}_{10}$ into the Banach space $\mathcal{M}_3$ of those real valued measurable functions g such that

$$\| g \| = \int_0^1 u^{-1-2\eta}\,|g(u)|\,du < \infty.$$

As in the proof of Lemma 1, let $\mathscr{C}$ be the Banach space of the complex-valued continuous, bounded functions on $\{(x, \nu)\,|\,x \in \mathrm{I},\ \mathrm{Re}\ \nu \geq -\eta\}$. Clearly, $\mathrm{F} \to \psi$, where

$$\psi(x, \nu) = 1 + \int_0^1 \mathrm{N}(x^2, \mathrm{F})(y)y^{\nu-1}dy \tag{215}$$

is continuous from $\mathcal{M}_{10}$ into $\mathcal{C}$ (remember that N is a continuous map from $\mathrm{I} \times \mathcal{M}_1$ into $\mathcal{M}_1$). We claim that ψ is the solution of the integral equation

$$\psi(x, \nu) = 1 + \int_0^1 (2\nu)^{-1}(1 - (u/x)^{2\nu})q(u)\psi(u, \nu)u^{-1}du \tag{216}$$

i. e.

$$\psi = \psi_0 + \mathrm{B}(\psi)$$

with

$$\psi_0(x, \nu) = 1,$$

to use notations similar to those used in the proof of Lemma 1 (see in particular Eq. (15)). It is obviously sufficient to verify Eq. (216) for F in $\mathcal{M}_{10} \cap \mathcal{M}_0$, since $\mathrm{F} \to (q, \psi)$ is continuous from $\mathcal{M}_1$ into $\mathcal{M}_3 \times \mathcal{C}$, and since $(q, \psi) \to \mathrm{B}(\psi)$ is a continuous map from $\mathcal{M}_3 \times \mathcal{C}$ into $\mathcal{C}$. It is easy to check that Eq. (216) is true if and only if φ, defined by Eq. (203) and Eq. (210), is the regular solution, in the sense of Lemma 1, of Eq. (184), q being given by Eq. (214) (verify that $\varphi(x, \nu) = x^{\nu - \frac{1}{2}}\psi(x, \nu)$, and recall the proof of Lemma 1).

Let us now indicate how to prove Eq. (184) when F is in $\mathcal{M}_{10} \cap \mathcal{M}_0$. In this case, $\mathrm{L}(x, y)$ is zero as soon as y is in a certain neighborhood of $y = 0$, and L has continuous partial derivatives of all orders on $0 \leq y \leq x \leq 1$. Furthermore, f, given by Eq. (209), is in $\mathcal{M}_0$. Thus, we can operate freely on Eq. (200), from which we obtain the following relation, using the fact that

$$\partial_x(x^2\partial_x\bar{f})(x, y) = \partial_y(y^2\partial_y\bar{f})(x, y)$$

(we have put $\bar{f}(x, y) = f(xy)$):

$$\partial_x(x^2\partial_x\mathrm{L})(x, y) - q(x)\mathrm{L}(x, y) = \partial_y(y^2\partial_y\mathrm{L})(x, y).$$

From this, and from Eq. (203), we deduce that φ satisfies Eq. (184) with q given by Eq. (214). It is then very easy to verify that φ has the other properties which characterize the regular solution of Eq. (184) in the sense of the remark at the end of Subsection 2.2. Thus, Eq. (216) is verified for a smooth F; therefore, by continuity, Eq. (216) is valid for any F in $\mathcal{M}_{10}$. This is statement Γ. *c*).

This concludes the sketch of the proof of Γ.

Defining now V by

$$q(x) = x^2(\mathrm{V}(x) - 1)$$

(q given by Eq. (202)), we see that φ is the regular solution corresponding to V, in the sense of Lemma 1. We also have, from Eqs. (194), (201) and (203):

$$\varphi(1, \nu) = w(\nu) \tag{217}$$

for all ν with Re $\nu \geq -\eta$.

δ) We have still to verify that φ, as given by Eq. (203), satisfies

$$\int_0^1 (\varphi(x, \nu_j))^2 dx = b_j \tag{218}$$

$(1 \leq j \leq k)$.

We obtain Eq. (218) as a consequence of the following result:

Δ. *Define n and $n_1, \ldots, n_k$ by*

$$n(x, \tau) = \varphi(x, i\tau) - \varphi(x, -i\tau)w(i\tau)/w(-i\tau) \tag{219}$$

$(0 < x \leq 1;\ \tau > 0)$ *and*

$$n_j(x) = b_j^{-\frac{1}{2}}\varphi(x, \nu_j) \tag{220}$$

$(0 < x \leq 1;\ 1 \leq j \leq k)$. *Then, for any g in $\mathscr{L}_2(0, 1)$*

$$\mathrm{U}(g)(\tau) = \mathscr{L}_2\text{-}\lim_{a \to 0} \int_a^1 n(x, \tau)g(x)dx$$

exists in $\mathscr{L}_2(0, \infty)$; we have

$$(2\pi)^{-1} \int_0^\infty |\mathrm{U}(g)(\tau)|^2 d\tau + \sum_{j=1}^k |g_j|^2 = \int_0^1 |g(x)|^2 dx, \tag{221}$$

where g_j is defined by

$$g_j = \int_0^1 n_j(x)g(x)dx \tag{222}$$

$(1 \leq j \leq k)$.

This result is identical, *mutatis mutandis*, with the one which is derived in Chap. IV, § 2 of Ref. ([10]). To obtain Eq. (218) from Δ, we put $g(x) = \varphi(x, \nu_j)$ in it. Applying Green's formula, and the fact that

$$n(1, \tau) = n_j(1) = 0$$

(follows from Eqs. (217), (219) and (220)), we obtain from Eq. (221)

$$b_j^{-1}\left(\int_0^1 (\varphi(x, \nu_j))^2 dx\right)^2 = \int_0^1 (\varphi(x, \nu))^2 dx,$$

i. e. Eq. (218).

ε) Finally, we give a (somewhat indirect) argument to show that V is unique. Let V_1 be another potential such that the corresponding regular solution φ_1 satisfies

$$\varphi_1(1, \nu) = w(\nu)$$

and

$$\int_0^1 (\varphi(x, \nu_j))^2 dx = b_j$$

From Lemmata 11, 13, 15 and 17, it follows, as in the proof of Theorem 5, that the logarithmic x-derivative ξ at $x = 1$ of φ is equal to the logarithmic x-derivative ξ_1 at $x = 1$ of φ_1. Therefore, the corresponding scattering amplitudes $\{ s \}$ and $\{ s^{(1)} \}$ are equal; consequently, by Theorem 3,

$$V(y) = V_1(y)$$

(almost everywhere).

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APPENDIX I

The Mellin transformation

A list of properties of the Mellin transformation, most of which can be easily deduced from well known facts about the Fourier transformation. R: the real line; R_+: the open half line $x > 0$.

1. Let f be any complex-valued function on R. We define $U(f)$, function on R_+, by

$$U(f)(x) = x^{-\frac{1}{2}}f(-\ln x)$$

The inverse U^{-1} of U is given by

$$U^{-1}(f)(x) = e^{-\frac{1}{2}x}f(e^{-x}),$$

f being a function on R_+. It is easy to verify that for $1 \leq p < \infty$, U maps $\mathscr{L}_p(R; dx)$ isometrically and bijectively onto $\mathscr{L}_p(R_+; x^{\frac{1}{2}p-1}dx)$. In particular, U is a unitary isomorphism from $\mathscr{L}_2(R; dx)$ onto $\mathscr{L}_2(R_+; dx)$. It is also an isometric isomorphism from $\mathscr{L}_1(R; dx)$ onto $\mathscr{L}_1(R_+; x^{-\frac{1}{2}}dx)$.

2. It is well known that $\mathscr{L}_1(R; dx)$ is a (commutative) *algebra* under the *convolution product* $(f, g) \rightarrow f*g$ defined by

$$f*g(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)dy$$

(almost everywhere). The *Fourier transformation* F_1, defined for f in $\mathscr{L}_1(R; dx)$ by

$$F_1(f)(\tau) = \int_{-\infty}^{+\infty} e^{-i\tau x}f(x)dx$$

has the following *convolution property:* it is an *injective homomorphism* from the algebra $\mathscr{L}_1(R; dx)$ into the algebra $\mathscr{C}$ of the continuous functions on R which tend to zero at infinity (the product in $\mathscr{C}$ being the pointwise multiplication). We denote $F_1(\mathscr{L}_1(R; dx))$ by $\mathscr{C}_0$.

In the proof of Lemma 13, we use the following

THEOREM (Wiener). — *Let g be in $\mathscr{C}_0$. If the function $\tau \rightarrow 1 + g(\tau)$ has no zeroes, then the function h, defined by*

$$1 + h(\tau) = (1 + g(\tau))^{-1},$$

is in $\mathscr{C}_0$.

Applying U, we find that $\mathscr{L}_1(R_+; x^{-\frac{1}{2}}dx)$ is a (commutative) *algebra* under the *convolution product* $(f, g) \rightarrow f*g$ given by

$$f*g = U(U^{-1}(f)*U^{-1}(g))$$

for f, g in $\mathscr{L}_1(R_+; x^{-\frac{1}{2}}dx)$. We obtain

$$f*g(x) = \int_0^{\infty} f(xu^{-1})g(u)u^{-1}du$$

(almost everywhere).

We call *Mellin transformation* M_1 the isomorphism given by

$$M_1 = F \circ U^{-1}$$

from $\mathscr{L}_1(\mathrm{R}_+; x^{-\frac{1}{2}}dx)$ onto the algebra $\mathscr{C}_0$. We have for f in $\mathscr{L}_1(\mathrm{R}_+; x^{-\frac{1}{2}}dx)$:

$$\mathrm{M}_1(f)(\tau) = \int_0^{\infty} f(x)x^{i\tau-\frac{1}{2}}dx.$$

3. The usual *Fourier-Plancherel transformation* F_2 is defined by

$$\mathrm{F}_2(f)(\tau) = (2\pi)^{-\frac{1}{2}}\mathscr{L}_2\text{-}\lim_{a\to\infty}\int_{-a}^{a} f(x)e^{-i\tau x}dx$$

for f in $\mathscr{L}_2(\mathrm{R}; dx)$. It is an unitary isomorphism from $\mathscr{L}_2(\mathrm{R}; dx)$ onto itself, whose inverse F_2^{-1} is given by

$$\mathrm{F}_2^{-1}(f)(x) = (2\pi)^{-\frac{1}{2}}\mathscr{L}_2\text{-}\lim_{a\to\infty}\int_{-a}^{+a} f(\tau)e^{i\tau x}d\tau$$

for f in $\mathscr{L}_2(\mathrm{R}; dx)$.

We have in this context the following.

THEOREM (Paley-Wiener, Ref. ([13])). — *Let* K, α *and* β *be real numbers with* $\mathrm{K} > 0$, $\alpha < \beta$. *Let g be a function which is holomorphic in the closed strip* $\alpha \leq \mathrm{Im}\, z \leq \beta$, *and which is such that*

$$\int_{-\infty}^{+\infty} |g(\tau + i\sigma)|^2 d\tau < \mathrm{K}$$

for any σ *with* $\alpha \leq \sigma \leq \beta$. *Then there is a measurable function* f *on* R *such that*

$$\int_{-\infty}^{+\infty} |e^{\alpha x}f(x)|^2dx < \infty, \qquad \int_{-\infty}^{+\infty} |e^{\beta x}f(x)|\, dx < \infty$$

and

$$g(\tau + i\sigma) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{+\infty} e^{-i(\sigma+i\tau)x}f(x)dx$$

for all σ *with* $\alpha < \sigma < \beta$.

COROLLARY. — *In order that the function g, holomorphic in* $\mathrm{Im}\, z \geq 0$, *be given for* $\sigma > 0$ *as*

$$g(\tau + i\sigma) = \int_{-\infty}^{0} e^{-i(\tau+i\sigma)x}f(x)dx,$$

where f *is a measurable function with*

$$\int_{-\infty}^{0} |f(x)|^2dx < \infty,$$

it is necessary and sufficient that there exists a positive number K *such that*

$$\int_{-\infty}^{+\infty} |g(\tau + i\sigma)|^2 d\tau < \mathrm{K}$$

for each σ *with* $\sigma \geq 0$.

This theorem is used in the proof of Lemma 3.

One more result.

THEOREM. — *In order that the function* f *in* $\mathscr{L}_2(\mathrm{R}; dx)$ *be an absolutely continuous function with its derivative* f' *in* $\mathscr{L}_2(\mathrm{R}; dx)$, *it is necessary and sufficient that its Fourier-Plancherel transform* $\mathrm{F}_2(f)$ *satisfy*

$$\int_{-\infty}^{+\infty} |g(\tau)|^2 d\tau < \infty, \tag{1.1}$$

where g is defined by

$$g(\tau) = -i\tau F_2(f)(\tau).$$

If condition (I.1) *is satisfied, then*

$$f' = F_2^{-1}(g)$$

and

$$\lim_{|x| \to \infty} f(x) = 0$$

We define the *Mellin transformation* M_2 by

$$M_2 = F_2 \circ U^{-1}$$

M_2 is an unitary map from $\mathscr{L}_2(R_+; dx)$ onto $\mathscr{L}_2(R; dx)$, and we have the pair of formulæ:

$$M_2(f)(\tau) = (2\pi)^{-\frac{1}{2}} \mathscr{L}_2\text{-} \lim_{a \to \infty} \int_{a^{-1}}^{a} f(x) x^{i\tau - \frac{1}{2}} dx$$

(τ in R)

$$M_2^{-1}(f)(x) = (2\pi)^{-\frac{1}{2}} \mathscr{L}_2\text{-} \lim_{a \to \infty} \int_{-a}^{a} f(\tau) x^{-i\tau - \frac{1}{2}} d\tau$$

(x in R_+).

The three theorems given above become.

THEOREM (Paley-Wiener). — *Let* K, α, β *and g be as above. Then there is a measurable function f on* R_+ *such that*

$$\int_0^\infty |x^{-\alpha} f(x)|^2 dx < \infty; \quad \int_0^\infty |x^{-\beta} f(x)|^2 dx < \infty$$

and

$$g(\tau + i\sigma) = (2\pi)^{-\frac{1}{2}} \int_0^\infty f(x) x^{i(\tau + i\sigma) - \frac{1}{2}} dx$$

for all σ *with* $\alpha < \sigma < \beta$.

(Used in the proofs of Lemma 13 and of Lemma 16, as well as in Subsection 4.2, α).

COROLLARY. — *In order that the function g, holomorphic in* Im $z \geq 0$, *be given for* $\sigma > 0$ *as*

$$g(\tau + i\sigma) = \int_1^\infty f(x) x^{i(\tau + i\sigma) - \frac{1}{2}} dx$$

where f is a measurable function with

$$\int_1^\infty |f(x)|^2 dx < \infty,$$

it is necessary and sufficient that there exist a positive number K *such that*

$$\int_{-\infty}^{+\infty} |g(\tau + i\sigma)|^2 d\tau < K$$

for each σ *with* $\sigma \geq 0$.

(Used in the proofs of Lemmata 11, 12 and 13, as well as in Subsection 4.2, β).

THEOREM. — *In order that the function f in $\mathscr{L}_2(\mathrm{R}_+; dx)$ be an absolutely continuous function (derivative f') such that $\bar{f}$, defined by*

$$\bar{f}(x) = xf'(x)$$

is in $\mathscr{L}_2(\mathrm{R}_+; dx)$, it is necessary and sufficient that its Mellin transform $\mathrm{M}_2(f)$ satisfies

$$\int_{-\infty}^{+\infty} |g(\tau)|^2 d\tau < \infty, \tag{I.2}$$

where g is defined by

$$g(\tau) = -i\tau \mathrm{M}_2(f)(\tau).$$

If condition (I.2) *is satisfied, then*

$$\bar{f} + \frac{1}{2} f = \mathrm{M}_2^{-1}(g),$$

and

$$\lim_{x \to 0} x^{\frac{1}{2}} f(x) = \lim_{x \to \infty} x^{\frac{1}{2}} f(x) = 0.$$

(Used in the proof of Lemma 11 and of Lemma 16, as well as in Subsection 4.2, α).

4. In the proof of Lemma 16, we make use of the following result.

LEMMA. — *Let g be a function in $\mathscr{L}_2(\mathrm{R}; dx)$; let ψ be defined on* $\mathrm{Im}\, z < 0$ *by*

$$\psi(z) = \pi^{-1} i \int_{-\infty}^{+\infty} g(\rho)(\rho - z)^{-1} d\rho \tag{I.3}$$

Then, for $\mathrm{Im}\, z < 0$,

$$\psi(z) = 2(2\pi)^{-\frac{1}{2}} \int_0^{\infty} f(x) e^{-izx} dx,$$

where f is the inverse Fourier-Plancherel transform of g:

$$f = \mathrm{F}_2^{-1}(g).$$

Remark. — Both integrals exist by Schwarz' inequality.

Proof: The function ψ is holomorphic on $\mathrm{Im}\, z < 0$. From (I.3), we find that the derivative ψ' of ψ is given by

$$\psi'(z) = \pi^{-1} i \int_{-\infty}^{+\infty} g(\rho)(\rho - z)^{-2} d\rho.$$

Choose $\sigma < 0$, and define h by

$$h(\tau) = \psi'(\tau + i\sigma).$$

We have

$$h(\tau) = \pi^{-1} i \int_{-\infty}^{+\infty} g(\rho)(\tau - \rho + i\sigma)^{-2} d\rho$$

We show first that

$$h(\tau) = -2(2\pi)^{-\frac{1}{2}} i \int_{-\infty}^{+\infty} x e^{\sigma x} f(x) e^{-i\tau x} dx, \tag{I.4}$$

where $f = F_2^{-1}(g)$. This relation follows immediately from the convolution property in No. 2 above if g is in $\mathscr{L}_1(\mathrm{R}; dx)$, since $\tau \to (\tau + i\sigma)^{-2}$ is in $\mathscr{L}_1(\mathrm{R}; dx)$, and since

$$\int_{-\infty}^{+\infty} e^{i\rho x}(\rho + i\sigma)^{-2}d\rho = \begin{cases} -2\pi e^{\sigma x} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x < 0. \end{cases}$$

If g is not in $\mathscr{L}_1(\mathrm{R}; dx)$, then we define the sequences $\{g_n\}$, $\{f_n\}$ and $\{h_n\}$ by

$$g_n(\tau) = \begin{cases} g(\tau) & \text{if} \quad |\tau| < n \\ 0 & \text{if} \quad |\tau| \geq n \end{cases}$$

$$f_n = F_2^{-1}(g_n)$$

$$h_n(\tau) = \pi^{-1}i\int_{-\infty}^{+\infty} g_n(\rho)(\tau - \rho + i\sigma)^{-2}d\rho. \tag{I.5}$$

Since g_n is in $\mathscr{L}_1(\mathrm{R}; dx)$, we have

$$h_n(\tau) = -2(2\pi)^{-\frac{1}{2}}i\int_0^{+\infty} xe^{\sigma x}f_n(x)e^{-i\tau x}dx. \tag{I.6}$$

On the other hand,

$$\mathscr{L}_2\text{-}\lim_{n\to\infty} g_n = g$$

therefore.

a) $\mathscr{L}_2\text{-}\lim_{n\to\infty} f_n = f$, so that for all real τ,

$$\int_0^{\infty} xe^{\sigma x}f_n(x)e^{-i\tau x}dx \to \int_0^{\infty} xe^{\sigma x}f(x)e^{-i\tau x}dx$$

as n tends to ∞.

b) From (I.5), it follows that

$$\lim_{n\to\infty} h_n(\tau) = h(\tau)$$

for all real τ.

Thus, we obtain (I.4) from (I.6). We rewrite (I.4) as

$$\psi'(z) = -2(2\pi)^{-\frac{1}{2}}i\int_0^{\infty} xe^{-izx}f(x)dx$$

(for $\operatorname{Im} z < 0$). Thus, for negative σ and σ_0,

$$\psi(\rho + i\sigma) - \psi(\rho + i\sigma_0) = \int_{\sigma_0}^{\sigma} \psi'(\rho + i\tau)(id\tau) = 2(2\pi)^{-\frac{1}{2}}\int_{\sigma_0}^{\sigma} d\tau\int_0^{+\infty} xe^{-i\rho x}e^{\tau x}f(x)dx,$$

whence

$$\psi(\rho + i\sigma) - \psi(\rho + i\sigma_0) = 2(2\pi)^{-\frac{1}{2}}\int_0^{\infty} (e^{\sigma x} - e^{\sigma_0 x})e^{-i\rho x}f(x)dx,$$

by Fubini's theorem. Let now σ_0 go to $-\infty$. Since

$$|\psi(\rho + i\sigma_0)| \leq \pi^{-1}\|g\|\left(\int_{-\infty}^{+\infty}(\rho^2 + \sigma^2)^{-1}d\rho\right)^{\frac{1}{2}} = (\pi|\sigma_0|)^{-\frac{1}{2}}\|g\| \tag{I.7}$$

(use Schwarz' inequality), the left-hand side of (I.7) tends to $\psi(\rho + i\sigma)$; by Lebesgue's theorem, the right-hand side tends to

$$2(2\pi)^{-\frac{1}{2}} \int_0^\infty e^{-i(\rho+i\sigma)x} f(x) dx.$$

Q. e. d.

From this Lemma, we obtain easily the following.

COROLLARY. — *Let g and ψ be as in above Lemma. Then, for* $\operatorname{Im} z < 0$,

$$\psi(z) = 2(2\pi)^{-\frac{1}{2}} \int_0^1 f(y) y^{iz-\frac{1}{2}} dy,$$

where f is the inverse Mellin transform of g:

$$f = \mathrm{M}_2^{-1}(g).$$

Proof: From the Lemma,

$$\psi(z) = 2(2\pi)^{-\frac{1}{2}} \int_0^\infty \bar{f}(x) e^{-izx} dx,$$

where $\bar{f} = \mathrm{F}_2^{-1}(g)$. But $\mathrm{F}_2^{-1} = \mathrm{U}^{-1} \circ \mathrm{M}_2^{-1}$, so that $\bar{f} = \mathrm{U}^{-1}(f)$. Therefore

$$\psi(z) = 2(2\pi)^{-\frac{1}{2}} \int_0^\infty e^{-\frac{1}{2}x} f(e^{-x}) e^{-izx} dx.$$

Put now $y = e^{-x}$. Q. e. d.

APPENDIX II

To illustrate section 4

The proof of the sufficiency of the condition given in Theorem 5 is a constructive one. We show here on an example how to use it to compute the potential which generates a given scattering amplitude.

1) Given the sequence $\{ s_l \}$ $(l = 0, 1, \ldots)$ and the positive number x, suppose that there are two complex numbers a and b such that

$$x \frac{s_l \psi'_{0+l} + \psi'_{0-l}}{s_l \psi_{0+l} + \psi_{0-l}} = l + a\left(l + \frac{1}{2} - b\right)^{-1}$$

for (non-negative integer) l not equal to $b - \frac{1}{2}$, and

$$s_l \psi_{0+l} + \psi_{0-l} = 0$$

for l equal to $b - \frac{1}{2}$ ($\psi_{0+l}, \ldots$ are defined as in Theorem 5). Let us ask *what conditions a and b have to satisfy in order that $\{ s_l \}$ be the scattering amplitude generated by a potential* V *of the class $\mathscr{F}$ with*

$$\mathrm{V}(y) = 0$$

for $y > x$ (for the somewhat unusual meaning we give to the term « scattering amplitude », see Eqs. (82) and (83); for the definition of $\mathscr{F}$ in terms of $\mathscr{V}$, see Section 3; for the definition of $\mathscr{V}$, see Subsection 2.12). The answer is the following: *It is necessary and sufficient that a and b be real, and that (a, b) be in Δ, where Δ is the set $\{ a \leq 0 \} \cup \{ b < 0, a < b^2 \}$.*

Let us show this. According to Theorem 5 (Eq. (183)), it is necessary and sufficient that there be a function w in $\mathscr{B}(\varepsilon)$ (ε: some positive number) such that

$$w\left(l + \frac{1}{2}\right) = l + a\left(l + \frac{1}{2} - b\right)^{-1}$$

for all non-negative integers l with $l \neq b - \frac{1}{2}$. Now, because of condition (B5) in the definition of $\mathscr{B}(\varepsilon)$, all functions of $\mathscr{B}(\varepsilon)$ are seen to satisfy the hypothesis of Carlson's theorem (Ref. ([24]), Chap. 9). Therefore, w must be given by

$$w(v) = v - \frac{1}{2} + a(v - b)^{-1} \tag{II.1}$$

Because of condition (B1), a and b have to be real. This being so, we have

$$\mathrm{Im}\ w(v) = \mathrm{Im}\ v(1 - a\,|\,v - b\,|^{-2})$$

Thus, we see that (a, b) must be in Δ if condition (B4) has to be satisfied. Conversely, if (a, b) is in Δ, then w is easily seen to be in $\mathscr{B}(\varepsilon)$.

2) Let $\{ s_l \}$ and x be as above. Suppose that (a, b) is in Δ, in order that there exists in $\mathscr{F}$ a potential V generating $\{ s_l \}$ as its scattering amplitude. We know that V is unique in $\mathscr{F}$ (Theorem 3) and that its range does not exceed x. We also know that the function w given

by Eq. (II.1) is equal to ξ, the logarithmic derivative at x of the regular solution φ of Eq. (1), divided by x (compare Eqs. (145) and (147) with Eq. (183), remembering Eq. (83)). *Let us now compute the potential* V.

If a vanishes, i. e. if

$$w(v) = v - \frac{1}{2},$$

then

$$\mathrm{V}(y) = \begin{cases} 1 & y \leq x \\ 0 & y > 0 \end{cases}$$

is the answer. This can be verified as follows. If V is as we say it is, the regular solution φ of Eq. (1) is given by

$$\varphi(y, v) = y^{v-\frac{1}{2}}$$

for $y \leq x$. Its logarithmic derivative at x is thus equal to $x^{-1}w(v)$, as it should.

Let us examine now *the case where* a *is not zero.* We apply here the general procedure. The first step consists in finding a function w_1 in $\mathscr{A}(\eta)$ (some positive η) such that the pair (w_1, w) is in $\mathscr{P}$. By Lemma 16, we know that such a function exists and is unique. We find by inspection ([30]) that w_1 is given by

$$w_1(v) = (v - b)(v + c)^{-1}$$

where we have put

$$c = (b^2 - a)^{\frac{1}{2}} > 0.$$

We know that w_1 is related to the regular solution φ of Eq. (1) in the following way:

$$\varphi(x, v) = x^{v-\frac{1}{2}}w_1(v) \tag{II.2}$$

(see Lemma 14). Furthermore, if b is positive (a is then negative), Lemma 17 yields

$$\int_0^x |\varphi(x, b)|^2 dy = \frac{1}{2}x^{2b}d^{-1}b^{-1}, \tag{II.3}$$

where we have put

$$d = (c + b)(c - b)^{-1}.$$

For the moment, we make the simplifying assumption that x is 1. The second step then consists in computing the function f defined by Eqs. (195) to (198) for $0 < y \leq 1$ (in Eq. (195), we have to replace w by w_1). We find

$$f(y) = 2dcy^{c-\frac{1}{2}}.$$

The third step consists in solving the integral Eq. (200) for the function L. This equation having a separable kernel, this is easily done, and we get

$$\mathrm{L}(x, y) = -2dc(xy)^{c-\frac{1}{2}}(1 + dx^{2c})^{-1}.$$

The fourth and last step consists in computing V for $0 < y \leq 1$, using Eq. (202) and remembering that

$$q(y) = y^2(\mathrm{V}(y) - 1).$$

([30]) In more complicated cases, we would have to construct w_1 according to the procedure used in the proof of Lemma 16.

The answer is

$$V(y) = 1 - 8c^2dy^{2c-2}(1 + dy^{2c})^{-2}.$$

If we go over to the case where x is an arbitrary positive number, we find for $0 < y \leq x$

$$V(y) = 1 - 8c^2dy^{-2}(y/x)^{2c}(1 + d(y/x)^{2c})^{-2}.$$

To conclude, we compute the regular solution φ of Eq. (1) with the help of Eq. (203) in the case where $x = 1$. The result is

$$\varphi(y, v) = y^{v-\frac{1}{2}}(1 - 2dcy^{2c}(c + v)^{-1}(1 + dy^{2c})^{-1})$$

for $0 < y \leq 1$. To get the result for $0 < y \leq x$ in the case where x is an arbitrary positive number, it suffices to replace y by y/x inside the parenthesis. One can verify that these functions indeed satisfy Eq. (1) for the potentials given above.

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