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Physique théorique.

Covariant decomposition of symmetric tensors and the gravitational Cauchy problem

by

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ABSTRACT. — We look for a covariant equivalent of the decomposition, in a fixed frame, of a symmetric tensor into its irreductible parts which is essential to the reduction of the gravitational field to canonical Hamiltonian form. This search is motivated by the fact that, if one takes as field variables the Einstein tensor G_{ij}, of the three-space $t =$ const. and the second fundamental form coefficients, the constraint equations are simply the requirements that these two tensors be *covariantly* transverse and that the trace of G_{ij} be fixed. The existence of such a covariant decomposition is demonstrated, and its properties exhibited. The latter enable us to characterize the general solutions of the constraint problem in a relatively explicit, though formal, fashion. These are then used to investigate the problem of positiveness of the field energy. The remaining, time-development, Einstein equations are also recast in terms of the present variables. Possible applications are discussed.

RÉSUMÉ. — Nous cherchons l'analogue covariant de la décomposition, dans un système donné de coordonnées, d'un tenseur symétrique en parties irréductibles qui est essentielle à la réduction du champ gravitationnel à une forme Hamiltonienne canonique. Cette recherche est motivée par le

(*) Ce travail a fait l'objet d'une série d'exposés au séminaire L. de Broglie en 1966-1967. Pour l'ensemble des sujets qui ont fait partie de cette série, consulter les références citées auxquelles l'auteur est associé.

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fait que, si l'on prend comme variables du champ le tenseur d'Einstein G_{ij}, de l'hypersurface t = const. et les coefficients de la seconde forme fondamentale, les équations de contrainte, à part de fixer la trace G, exigent simplement que les deux tenseurs soient transverses de façon covariante. L'existence d'une telle décomposition covariante est démontrée, et ses propriétés sont données. Ces dernières nous permettent de caractériser les solutions générales des équations de contrainte d'une manière relativement explicite, quoique formelle. Ces solutions sont alors employées pour traiter le problème du signe de l'énergie du champ. Les autres équations d'Einstein, qui fixent l'évolution dans le temps, sont aussi formulées en fonction de ces variables. Des applications possibles sont discutées.

I. — INTRODUCTION

An essential tool in the canonical formulation of General Relativity [*1*] is the decomposition (at any instant) of those field variables which constitute the initial value data into their irreducible (spin) components. In terms of these, the gravitational field is exhibited as a Hamiltonian system with two unconstrained degrees of freedom corresponding to a (self-coupled) massless spin-two theory; the remaining components are either gauge (coordinate) variables or dependent variables given (implicitly) as functions of the rest by the constraint equations. The Cauchy data being symmetric tensors in the three-space t = const., the required decomposition was a natural generalization of that of a vector field into curl—and divergence—free components (which plays the corresponding role in the canonical analysis of flat-space electrodynamics). The flat-space decomposition is naturally performed in terms of a Cartesian frame, where the metric is just δ^{ij} and covariant and ordinary derivatives coincide. It corresponds, equivalently, to an algebraic classification of the Fourier components of the vector or tensor with respect to the propagation vector (gradient) at each point.

This non-covariant decomposition (which we abbreviate as N-decomposition) can also be performed in a curved space, simply by decomposing with respect to any fixed (but arbitrary) coordinate system which is asymptotically cartesian, and is just the basis mentioned for the canonical analysis of the gravitational field. Although non-covariant in that an arbitrary coordinate change in the 3-space mixes its various components in a complicated way, the decomposition is quite satisfactory physically, and is capable of yielding in principle the full content of the theory, including those of its properties which are invariant.

Indeed, the usual definition of a Hamiltonian formulation requires that in it, all redundant variables of the theory are eliminated, leaving only a set of conjugate amplitudes with respect to a chosen set of space-time coordinates. For relativity, the elimination of redundant « gauge » variables is identical to making a choice of frame, so that a covariant Hamiltonian form in the usual sense is not possible. A more concrete argument against attempting to introduce a covariant (C-) reduction for the gravitational field (as against matter variables such as the electromagnetic field components) was that the metric tensor itself, whose spatial components are the basic field variables, could not be subject to a decomposition in terms of covariant derivatives or traces. For, being by definition the unit tensor of the space, the metric has vanishing covariant derivatives. Further, while the deviation of the metric from its flat space value (which is an equally good variable) does not have this problem, it is not a general coordinate tensor, and hence is not suitable for a covariant analysis.

Despite the physically satisfactory use of particular coordinate frames (and so of the N-decomposition) in curved space, there are some strong arguments, both of principle and in applications, in favor of the introduction of a C-decomposition. First, such an analysis is manifestly independent of the choice of spatial coordinates. Since it is not always possible to cover a curved space globally with a single coordinate system, as is implicitly required by the non-local nature of the N-decomposition, one would (formally at least) avoid problems raised by the need for several coordinate « patches ». Further, the solution (in principle) of the highly non-linear constraint equations of the theory in terms of a particular set of variables as functionals of the rest may not be well-defined for all values of the latter in any one coordinate frame [1]. Since these equations are known [*1*] to have an iterative perturbation solution, this difficulty would presumably manifest itself by the non-convergence (in a given frame) of such a series for very strong fields, i. e. beyond some ranges of values of the independent variables. In this case (in contrast to theories like electrodynamics with linear constraints), a specific field component in a given frame could not always be regarded as *the* constraint variable. These, and analogous difficulties, may perhaps be viewed physically as points of rigor which are bound to arise in a non-linear theory of a Riemann space when sufficiently pathological

(1) This possibility (which is often met with in implicit functions) was pointed out by L. D. Faddeev (private communication).

(or at least sufficiently intense) field configurations are considered. They are, in any case, not the primary concern of this investigation, which also operates at a lower level of mathematical rigor. Concretely, our motivation lies rather in the extremely simple and symmetric form taken by the Einstein field equations in terms not of the spatial metric g_{ij}, but of the Ricci tensor R_{ij} (or Einstein tensor G_{ij}) of the three-space $t =$ const. and of a set of conjugate momenta π^{ij}. The constraint equations, which operate on this hypersurface, are just a set of simple (three-dimensionally) covariant conditions on the divergences and traces of these two tensors, both of which (unlike the metric) can be C-decomposed. The time-development equations also turn out to be quite simple and symmetric in these variables; the non-linearity of the theory, if not avoided, is at least by-passed in what may be a useful way. To be sure, there are a number of other, more abstract, reasons for attempting to define a C-decomposition. A coordinate-covariant formulation of the dynamics of the gravitational field might have important application to the quantization problem, for example, avoiding the question of (operator) relations among different gauges, and permitting a gauge-covariant statement of the quantization conditions, but still within a generalized Hamiltonian framework. Another possible application, to which we shall return, is to the problem of the positive-definiteness of the gravitational field energy and to similar overall coordinate-invariant functional properties of the field which require explicit characterization of the constraint variables. It is assumed throughout that our spaces are complete, simply connected and asymptotically flat, namely that they describes isolated non-singular physical systems (2). Also we emphasize that our operations are (three-dimensionally) coordinate—*covariant*, in distinction to coordinate—*invariant* one (such as use of local invariants as coordinates) which are not all suitable for physical applications.

The relation of the C- to the N-decomposition is roughly like that between the latter and one performed with respect to a fixed axis. That is, just as the propagation vector or gradient provides a direction at each point natural to the field being analysed (rather than one fixed *a priori*) to classify the different vector or tensor components, the covariant gradient generalizes

(2) The precise rates of asymptotic falling-off of the metric required and details of the N-decomposition for arbitrary asymptotically decaying behavior of vector and tensor fields are to be found in ref. (12). For a rough orientation, the requirements that there exist an asymptotic frame at spatial infinity in which

$$g_{\mu\nu} - \eta_{\mu\nu} \sim O(r^{-1}), \qquad \partial_\alpha g_{\mu\nu} \sim O(r^{-2})$$

in a finite neighborhood of the fixed time are quite sufficient for our purposes.

this notion of a variable natural local axis to curved space. We shall see that there does exist a C-decomposition although its properties are somewhat more complicated, as might be expected, since the simple Fourier description of the N-method is not longer available. This restriction will prevent us from giving in a completely explicit way a general functional solution of the constraints; however, the solution will be reduced to a linear problem in terms of elliptic equations of the Poisson type. It may thus be hoped that this method can yield some useful sights into the properties of the Einstein field.

Before turning to the derivation of the C-decomposition, we shall briefly recall the properties of the N-decomposition on which it is modelled. Applications to the gravitational field will then be given, after expressing the Einstein equations (both constraint and time development) in terms of the proposed variables. In the last section, we discuss a number of open problems for which the present methods may prove of some use.

II. — NON-COVARIANT DECOMPOSITION

We summarize here some basic properties of the usual, non-covariant decomposition, starting with the well-known vector case to fix ideas. Here, as throughout, we deal with vector and tensor fields in the non-compact three-dimensional space $t = \text{const.}$ with positive signature. We assume that the fields vanish at spatial infinity and have no singularities, while the space is complete, simply connected, and asymptotically flat ([2]).

For a vector field $\mathbf{V}(r)$, one may write

$$\mathbf{V} = (\theta^{\mathrm{T}} + \theta^{\mathrm{L}})\mathbf{V} \equiv \mathbf{V}^{\mathrm{T}} + \mathbf{V}^{\mathrm{L}} \tag{1}$$

where the projection operators θ^{T}, θ^{L} are dyadics given in coordinate space by the expressions

$$\theta^{\mathrm{L}} \equiv \partial\, \Delta_0^{-1} \partial \quad , \quad \theta^{\mathrm{T}} \equiv (1 - \theta^{\mathrm{L}}). \tag{2}$$

Here ∂ is the gradient operator and Δ_0^{-1} is the inverse Laplacian operator vanishing at infinity. The non-locality of the decomposition is due to the Δ_0^{-1}, which is an integral operator involving the values of $\mathbf{V}$ everywhere. The vector $\mathbf{V}^{\mathrm{T}}$ is transverse, namely $\mathbf{V}^{\mathrm{T}} \equiv \partial \times \boldsymbol{u}$, while $\mathbf{V}^{\mathrm{L}}$ is curl-free, being the gradient of a scalar field V^{L}:

$$\partial \cdot \mathbf{V}^{\mathrm{T}} \equiv 0 \equiv \partial \times \mathbf{V}^{\mathrm{L}} \tag{3}$$

The decomposition is 1) linear, 2) unique and 3) orthogonal. The linearity is obvious from Eqs (1, 2) while uniqueness follows from the expression for $\mathbf{V}^{\mathrm{L}}$ in terms of $\mathbf{V}$:

$$\partial \cdot \mathbf{V}^{\mathrm{L}} \equiv \Delta_0 \mathrm{V}^{\mathrm{L}} = \partial \cdot \mathbf{V}$$

a Poisson equation for V^{L} in terms of $\partial \cdot \mathbf{V}$. By subtraction, $\mathbf{V}^{\mathrm{T}}$ is also uniquely determined: $\mathbf{V}^{\mathrm{T}} = \mathbf{V} - \mathbf{V}^{\mathrm{L}}[\partial \cdot \mathbf{V}]$ [Alternatively, $\partial \times \mathbf{V}^{\mathrm{T}} \equiv \partial \times (\partial \times \boldsymbol{u}) = \partial \times \mathbf{V}$, which fixes $\boldsymbol{u}$ and hence $\mathbf{V}^{\mathrm{T}}$ uniquely.] The projection operators θ^{T}, θ^{L} may thus be used to produce transverse or longitudinal vectors from an arbitrary field; the former have two components at each point (being orthogonal to the propagation axis) the latter have just one, along the axis, with magnitude $|\partial \mathrm{V}^{\mathrm{L}}|$. The orthogonality property expresses the vanishing of the inner product $\int d^3 r \mathbf{V}^{\mathrm{T}} \cdot \mathbf{W}^{\mathrm{L}}$ for arbitrary $\mathbf{V}^{\mathrm{T}}$, $\mathbf{W}^{\mathrm{L}}$ as the integrand is a divergence.

For a vector field, it is known that a C-decomposition exists, enjoying all the above properties, in spaces where the metric g_{ij} has positive signature. Its details will also be relevant to later discussion. Denoting covariant differentiation by ∇_i, we may write

$$\begin{aligned} \mathrm{V}_i = (\theta^{\mathrm{T}}_{ij} + \theta^{\mathrm{L}}_{ij})\mathrm{V}^j &\equiv [(g_{ij} - \nabla_i \Delta^{-1} \nabla_j) + \nabla_i \Delta^{-1} \nabla_j]\mathrm{V}^j \\ &\equiv \mathrm{V}^{\mathrm{T}}_i + \mathrm{V}^{\mathrm{L}}_i \end{aligned} \tag{4}$$

where Δ^{-1} is now the inverse of the covariant Laplacian (3) $\Delta \equiv \nabla_i \nabla^i$, and all indices are moved by means of the metric g_{ij}, whose Christoffel symbol is the affinity. The metric is a covariant constant and may of course be taken through all covariant differentiations. In particular, co- and contra-variant components of a vector field have the same transverse components: $\mathrm{V}^{\mathrm{T}}_i \equiv g_{ij}(\mathrm{V}^{\mathrm{T}j})$, but (unlike the N-case) we shall see that the vectors $\Delta \mathbf{V}^{\mathrm{T}}$ or $\Delta \mathbf{V}^{\mathrm{L}}$ each have in general, both transverse and longitudinal parts, due to the lack of commutativity of covariant derivatives when acting on a vector. This fact will be relevant to the tensor case. The C-decomposition is clearly still linear and unique, since $\nabla^i(\mathrm{V}^{\mathrm{L}}_i + \mathrm{V}^{\mathrm{T}}_i) \equiv \Delta \mathrm{V}^{\mathrm{L}} = \nabla^i \mathrm{V}_i$ is once more a Poisson equation for the scalar V^{L}, which uniquely determines it also in the curved space. The vanishing of $\nabla^i \mathrm{V}^{\mathrm{T}}_i$ and of

$$\boldsymbol{\nabla} \times \mathbf{V}^{\mathrm{L}} \sim \nabla_i \mathrm{V}^{\mathrm{L}}_j - \nabla_j \mathrm{V}^{\mathrm{L}}_i = (\nabla_{ij} - \nabla_{ji})\mathrm{V}^{\mathrm{L}} \equiv 0, \qquad (\nabla_{ij} \equiv \nabla_i \nabla_j)$$

(3) We are not concerned here with possible global problems of the Δ^{-1} operator in a Riemann space, and assume Δ^{-1} to be well-defined globally.

is also obvious as two covariant derivatives acting on a scalar (of scalar density) commute. Orthogonality remains, with the required insertion of a factor $\sqrt{g}$ in the definition of inner product:

$$\int d^3r\sqrt{g}V_i^T W^{Li} \equiv \int d^3r\sqrt{g}g^{ij}V_i^T W_j^L = 0.$$

All of the above results hold just as well for a vector density (of any weight) since any power of $\sqrt{g}$ can be moved past derivatives (of course, the inner product is always defined such that the integrand as a whole is a scalar density of rank 1). These results for the vector field are physically very reasonable; they assure us, for example, in a frame-independent way, that the electromagnetic field in the presence of gravitation still has all the transversality properties and hence the same degrees of freedom as in flat space ([4]). The whole radiation gauge formalism can thus be taken over intact, with separation of gauge ($\mathbf{A}^L$) and dynamical components ($\mathbf{A}^T$) of the vector potential, and of transverse ($\mathcal{E}^T$) and coulomb ($\mathcal{E}^L$) parts of the electric field. Before turning to the tensor case, we may note that the existence of a C-decomposition for vectors is in a sense, related to the fact that the covariant curl, divergence and gradient operators are identical to the ordinary ones on the appropriate representatives (covariant vector, contravariant vector density and scalar) of the fields considered. There is no such correspondence for symmetric tensors, and we must expect that at least some of the properties of the N-decomposition will be lost there.

The N-decomposition of a symmetric tensor field T_{ij} is somewhat more involved than that of a vector, because in addition to its divergence, one may also take the trace of T_{ij}; we shall merely summarize the detailed derivation given elsewhere [*1*]. One first decomposes T_{ij} into a transverse and longitudinal part according to ([5])

$$T_{ij} = T_{ij}^t + \partial_i T_j + \partial_j T_i \quad , \quad \partial_j T_{ij}^t \equiv 0 \tag{5}$$

where T_{ij}^t has three components, at each point, being a symmetric tensor in the plane orthogonal to the propagation vector.

The vector field T_i constitutes the remaining three of the six components

([4]) This has been demonstrated in detail in ref. (2) where the appropriate electromagnetic field variables for the complete Einstein-Maxwell system are exhibited.

([5]) In the N-decomposition, the effective metric is just δ_{ij}, so that no distinction need be made between upper and lower indices; in particular, summation is over any pair of replated indices irrespective of position. Throughout, latin indices and summations range over 1, 2, 3, greek over 0, 1, 2, 3.

of T_{ij}; it is linearly and uniquely determined as an explicit functional of T_{ij} by a Poisson—like equation and can of course be further N-decomposed into its T_i^T and T_i^L parts. If $\partial_i T_{ij} = 0$, the vector part of T_{ij} vanishes. These properties can easily be deduced from the equation

$$\Delta_0 T_i + T_{j,ij} \equiv \Delta_0 T_i^T + 2\Delta_0 T^L_{,i} = \partial_j T_{ij}.$$

Its divergence defines $\Delta_0 T^L \equiv \partial_i T_i^L$ in terms of $\Delta_0^{-1}\partial^2_{ij} T_{ij}$, while $\mathbf{T}^T$ is then defined from the remaining (transverse) part of $\partial_j T_{ij}$. The transverse tensor T^t_{ij} is next separated into a traceless part and one with trace, both transverse:

$$T^t_{ij} = T^{TT}_{ij} + T^T_{ij} \tag{6a}$$

where

$$\partial_j T^{TT}_{ij} \equiv 0 \equiv T^{TT}_{ii} \quad , \quad \partial_j T^T_{ij} \equiv 0 \quad , \quad T^T_{ii} = T^t_{ii} \tag{6b}$$

the « TT » tensor is thus the traceless part of the symmetric 2×2 matrix T^t_{ij}. It has 2 components. The remaining component of T^t_{ij} is the scalar T^T_{ii} which determines T^T_{ij} by

$$T^T_{ij} \equiv \frac{1}{2}(\delta_{ij}\Delta_0 - \partial^2_{ij})T^t_{ll} \tag{6c}$$

this also exhibits the transversality of T^T_{ij} as well as its linearity and uniqueness. By subtraction, T^{TT}_{ij} is also a unique functional of T_{ij} and we may write, for any symmetric T_{ij} the decomposition

$$T_{ij} = [\theta^{TT} + \theta^T + \theta^L]^{lm}_{ij} T_{lm} \equiv T^{TT}_{ij} + T^T_{ij} + (\partial_j T_i + \partial_i T_j). \tag{7}$$

where the projection operators are simple but slightly lengthy combinations of derivatives and inverse laplacians, which we need not reproduce here. Orthogonality, which states that for any two tensors T_{ij} and S_{ij}, the inner products

$$\int d^3r\, T^{TT}_{ij} S^T_{ij} \quad , \quad \int d^3r\, T^{TT}_{ij}\partial_j S_i \quad , \quad \int d^3r\, T^T_{ij}\partial_j S_i$$

vanish, follows directly from the properties of the components and integration by parts. A slightly different grouping of the components of the decomposition, whose form will be useful later, consists in writing

$$\begin{aligned} T_{ij} &= T^{TT}_{ij} + \left(\partial_j W_i + \partial_i W_j - \frac{2}{3}\delta_{ij}\partial_l W_l\right) + \frac{1}{2}(\delta_{ij}\Delta_0 - \partial^2_{ij})\psi \\ &\equiv T^{TT}_{ij} + \qquad (LW)_{ij} \qquad + \theta_{ij}\psi \end{aligned} \tag{8}$$

where the vector part is now traceless, the scalar part still transverse. It is related to the previous one by $(\triangle_0 T^T + 2\partial_l T^l) = \triangle_0\psi$ and $W_i = T_i + \frac{1}{2}T^L_{,i}$. Despite its traceless form, the W-term is sufficient to determine W_i uniquely and completely in terms of the divergence of T_{ij}, while the trace of T just fixes $\triangle_0\Psi$, so that this form corresponds to a slightly different decomposition of the unit dyadic into three projection operators. It is not quite orthogonal, however, since

$$\int d^3r(LW)_{ij}\theta_{ij}\psi = -\frac{2}{3}\int d^3r(\partial_i W_i)\triangle\psi \neq 0$$

[The orthogonality requirements leads uniquely to the form of Eq. (7).] Both the vector and tensor decompositions exhibit the irreducible « spin » parts of the corresponding fields; thus, the six components of T_{ij} have been separated into a transverse-traceless spin 2 (corresponding to a massless field), a transverse spin one (again massless) and two spin 0 scalars (6) (the traces of the transverse and longitudinal parts). The latter two field may be superposed into different linear combinations, which is just the difference between the forms in Eqs (7) and (8). Owing to the use of a decomposition direction natural to each point (the gradient) rather than one with respect to an axis fixed in space, the different components do not mix under rigid rotations of the (Cartesian) coordinates. Arbitrary coordinate transformations, of course, are not meant to preserve the N-decomposition. If a field T_{ij} has vanishing divergence or trace, it follows that its W- or ψ- part, respectively, vanishes. If both vanish, so do both the W- and ψ-parts. Thus one may construct, from an arbitrary tensor, one which is transverse, by subtracting from it the explicitly known functional representing the longitudinal part, $(LW)_{ij}[\partial_k T_{lk}]$. These simple properties are useful not only in the linearized approximation to the Einstein theory, but also in the full theory expressed in an arbitrary (global) coordinate frame. Indeed, it was in terms of this method that the full Einstein field was exhibited [*1*] as a (non-linear) massless spin-two system with two pairs of transverse-traceless unconstrained conjugate variables together with four constraint variables and four gauge variables. The latter eight quantities were just the remaining (lower spin) components of the two initial variables ($h_{ij} \equiv g_{ij} - \delta_{ij}$ and the second fundamental form coefficients π^{ij}) describing

(6) This is the reason to the usefulness of the decomposition in analyzing field equations for higher-spin fields in terms of their irreducible spin pieces.

the field at any instant [In electrodynamics, there are two three-vectors $\mathbf{A}$, $\mathcal{E}$ which divide into the four transverse dynamical components $\mathbf{A}^T$, $\mathcal{E}^T$, one constraint variable $\mathcal{E}^L$ and the gauge part $\mathbf{A}^L$.]

III. — COVARIANT DECOMPOSITION

We now attempt to define, in a completely covariant fashion, a decomposition with properties as near as possible to those found in the previous Section. One may still write, as we shall see,

$$T_{ij} = T^t_{ij} + (\nabla_j T_i + \nabla_i T_j) \tag{9}$$

or also

$$T_{ij} = T^t_{ij} + \left(\nabla_j W_i + \nabla_i W_j - \frac{2}{3} g_{ij} \nabla_k W^k\right) \tag{10}$$

where T^t_{ij} is divergenceless, $\nabla^j T^t_{ij} \equiv 0$. For then we have, from say Eq (10), the equation

$$\nabla^j (LW)_{ij} = \nabla^j T_{ij} \tag{11}$$

To see that this is an elliptic equation, with a unique solution for $(LW)_{ij}[\nabla^k T_{lk}]$, consider the operator

$$-\int d^3r\sqrt{g}\, W^i \nabla^j (LW)_{ij} = \frac{1}{2}\int d^3r\sqrt{g}(LW)_{ij}(LW)^{ij} \tag{12a}$$

where we have integrated by parts to obtain the equality. This operator is thus positive-definite, and if T_{ij} is transverse ($\nabla^j T_{ij} = 0$), $(LW)_{ij}$ itself must vanish. This is the standard argument which shows the positive nature of $-\Delta\Phi$ in terms of $\frac{1}{2}\int d^3r\,\sqrt{g}(\nabla\Phi)^2 \geqslant 0$, and holds also for the equation

$$\nabla^j(\nabla_i T_j + \nabla_j T_i) = \nabla^j T_{ij} \tag{12b}$$

by use of the inequality

$$\int d^3r\sqrt{g}(\nabla_j T_i + \nabla_i T_j)^2 \geqslant 0.$$

Equations (9), (10) may also be viewed as giving a covariantly transverse tensor, T^t_{ij}, in terms of an arbitrary one, T_{ij}, namely

$$T^i_{ij} = T_{ij} - (LW)_{ij}[\nabla^k T_{lk}] \tag{13a}$$

or

$$T^t_{ij} = T_{ij} - (\nabla_j T_i + \nabla_i T_j)[\nabla^k T_{lk}] \tag{13b}$$

In either form, the subtracted parts are fixed unique linear functionals given, in principle, from Eqs (11) or (12) in terms of $\triangle^{-1}$, $\boldsymbol{\nabla}$, etc. It should be noted, though, that the solution is not as simple to write down explicitly as in the N-case, owing to the complications of non-commutation of covariant derivatives, to which we shall return. However, it gives a useful functional characterization of transverse tensors. The difference between the Eqs (13*a*) and (13*b*) is that the trace of the former yields $T^t = T$ (the covariant trace of a tensor, T^i_i, is denoted by T), whereas in the latter $T^t = T - 2\nabla_i T^i$. In either case, the trace of T^t remains, as it should, arbitrary.

The complication we have been anticipating in the C-decomposition first arises when we attempt to isolate a « TT » part from the pure trace part of T^t_{ij} by writing, as in the N-case,

$$T^t_{ij} = T^{TT}_{ij} + \frac{1}{2}(g_{ij}\triangle - \nabla_{ij})\psi \equiv T^{TT}_{ij} + \theta_{ij}\psi \tag{14}$$

where we require

$$\nabla^j T^{TT}_{ij} \equiv 0 \equiv T^{TT} \tag{15}$$

The trace of Eq. (14) yields, as before, the simple relation

$$\triangle\psi = T^t \tag{16}$$

but on taking the divergence of Eq. (14) we find

$$\nabla^j T^t_{ij} = \frac{1}{2}(\nabla_{ij} - \nabla_{ji})\nabla^j\psi. \tag{17}$$

While it is true that for a scalar (or scalar density) χ, we have $(\nabla_{ji} - \nabla_{ij})\chi = 0$, the Ricci identity tells us that this commutator acting on a vector (or vector density) yields

$$(\nabla_{ji} - \nabla_{ij})V^k = V^l R_l{}^k{}_{ij} \tag{18a}$$

(in particular, then, $(\nabla_{ji} - \nabla_{ij})V^i = R_{ji}V^i$) with corresponding additional terms for the commutator on higher rank tensors. Hence, the division (14) is not consistent with the desired property (15). The Ricci tensor R_{ij} is arbitrary; it need only satisfy the contracted Bianchi identity

$$\nabla^j\left(R_{ij} - \frac{1}{2}g_{ij}R\right) \equiv 0 \tag{19}$$

and, in general relativity, its trace R is also specified in terms of other variables by Eq. (29*b*) below. We also recall that, in three dimensions, R_{ij} is entirely equivalent to the full Riemann tensor R_{ijkl} and the identity (19) is equivalent to the uncontracted identities on R_{ijkl}. In three dimensions, we have the equality

$$R_{ijkl} = g_{il}R_{jk} + g_{jk}R_{il} - g_{ik}R_{jl} - g_{jl}R_{ik} - \frac{1}{2}R(g_{il}g_{jk} - g_{ik}g_{jl}).$$

One might attempt to construct a transverse generalization of $\theta_{ij}\psi$ by adjoining to it the only other available tensor operators with correct dimensions, namely $\frac{1}{2}(\alpha R_{ij} + g_{ij}\beta R)\psi$ (α, β constant). However, it is easy to see that no values of α, β can accomplish this in general. Simplification can only occur in the special cases $R_{ij} = \lambda g_{ij}$ (where λ is necessarily a constant by Eq. (19)) or $R = 0$. Indeed, in the former case (where the α and β terms are equivalent), the choice $\alpha + 3\beta = 1$ yields a transveıse operator, and, as has been shown by Barbance [*3*], thereby leads to a C-decomposition with most of the properties of the N-case. Unfortunately, for isolated systems, which are asymptotically flat, only the possibility $\lambda = 0$ would be useful, but the equivalence of R_{ij} and R_{ijkl} shows that this case is just flat space. We shall note later the reason for the strength of the condition $R_{ij} = \lambda g_{ij}$; however, it is clear already that our problems are caused by the presence of a « preferred » background field R_{ij} which mixes various spin parts. If the latter, instead of being arbitrary, were proportional to the unit tensor g_{ij}, it would become harmless. Similarly, spaces of vanishing scalar curvature, $R = 0$ (and hence also $\nabla^i R_{ij} = 0$) permit us to restore transversality by the choice $\alpha = 1$. However, as we shall see below these are very special cases in general relativity, for which in fact the decomposition itself is less relevant.

Returning to the general decomposition, we saw that if we wish to isolate a transverse-traceless part (or to give the general solution for a tensor satisfying the « TT » requirements) we must give up the piecewise decomposition of T_{ij} into $T^t_{ij} + T^L_{ij}$, followed by that of T^t_{ij} into T^{TT}_{ij} and T^T_{ij}. Let us simply analyze the consequence of assuming the form

$$T_{ij} = T^{TT}_{ij} + (\nabla_j T_i + \nabla_i T_j) + \theta_{ij}\chi \tag{20}$$

or alternately

$$T_{ij} = T^{TT}_{ij} + (\nabla_j W_i + \nabla_i W_j - \frac{2}{3} g_{ij} \nabla^k W_k) + \theta_{ij}\psi \tag{21}$$

with $\nabla^j T^{TT}_{ij} = 0 = g^{ij} T^{TT}_{ij}$. The two forms (20), (21) differ, as before, by a simple algebraic rearrangement, $\Delta\psi \equiv \Delta(\chi + 2T^L)$, $W_i \equiv T_i + \frac{1}{2}\nabla_i T^L$. Consider, for example, Eq. (21). To check its consistency, we must see that there exist unique solutions for W_i and ψ such that the assumed transverse-traceless character of T^{TT}_{ij} is allowed. On taking the trace of Eq. (21), we find as before,

$$\Delta\psi = T \tag{22}$$

which is a unique determination of ψ as $\Delta^{-1}T$. Further, there is no coupling of W_i parts in the trace, by construction. The divergence, on the other hand, now reads:

$$\nabla^j(LW)_{ij} = \nabla^j T_{ij} - \nabla^j \theta_{ij}(\Delta^{-1}T) \tag{23}$$

where we have used relation (22) to eliminate ψ. Once again, the elliptic operator on the left of Eq. (23) insures that this is a unique linear equation for (LW). However, owing to the non-transversality of the θ_{ij} operator, it is not only $\nabla^j T_{ij}$, but also T which determines (LW), and (LW) is non-zero even if $\nabla^j T_{ij}$ vanishes; its source would then be $\nabla^j \theta_{ij}\psi = \frac{1}{2} R_{ij} \nabla^j \psi \neq 0$. Instead, (LW) will vanish only if $T_{ij} - \theta_{ij}(\Delta^{-1}T)$ is transverse, namely when this tensor (being traceless automatically) is itself « TT ». Of course, if T_{ij} itself is already « TT », both ψ and (LW) will vanish, as they must if the decomposition (21) is to be at all meaningful.

Clearly, the existence, as demonstrated, of the form (21) entails that of (20), by simple algebraic substitution; the latter can also be verified directly using the elliptic character of the vector part, $\nabla^j(\nabla_j T_i + \nabla_i T_j)$. The existence of the « TT » decomposition (20) or (21), with the linear elliptic equations (22) and (23) gives once again a complete formal characterization of a general « TT » tensor in terms of the solutions of these equations, namely

$$T^{TT}_{ij} = T_{ij} - (LW)_{ij}[T] - \theta_{ij}(\Delta^{-1}T)$$

For the same reasons as before, we cannot, however, put this in the form $T = [\theta^{TT} + \theta^{T} + \theta^{L}]T$ with explicitly given projection operators. Orthogonality is also disturbed by the non-transverse nature of θ_{ij}; while T^{TT}_{ij} is still orthogonal to any $\theta_{ij}\chi$ or to any $\nabla_j T^{ij}$, the latter two quantities are no longer mutually orthogonal. If the tensor as a whole is transverse, then $\nabla_j T_i + \nabla_i T_j + \theta_{ij}\chi$ is divergenceless, and any tensor which is of the form $\nabla_i Z_j$ is orthogonal to it; in particular, $\nabla_i T_j$ and $\nabla_{ij}\chi$ are in this class; but $g_{ij}\triangle\chi$ is not. Of course, if the tensor is also traceless, $g_{ij}\triangle\chi$ becomes orthogonal to this combination as well, and it then follows that

$$\int d^3r\sqrt{g}[\nabla_j T_i + \nabla_i T_j + \theta_{ij}\chi]^2 = 0,$$

with the conclusion, as before that $T_{ij} = T_{ij}^{TT}$ since the integrand must vanish everywhere.

It was mentioned that a special case in which $\theta_{ij}\chi$ can be made transverse occurs when the scalar curvature R, and with it the divergence $\nabla^j R_{ij}$ vanishes. If one then writes, for the case $R = 0$,

$$T_{ij} = T_{ij}^{TT} + (\nabla_j T_i + \nabla_i T_j) + \left(\theta_{ij} + \frac{1}{2} R_{ij}\right)\chi \tag{24}$$

the conditions become simply

$$\triangle\psi \equiv \triangle(\chi + 2T^L) = T \tag{25a}$$

$$\nabla^j(\nabla_j T_i + \nabla_i T_j) = \nabla^j T_{ij} \tag{25b}$$

which have all the properties of the corresponding N-form, except that writing $(\nabla_j T_i + \nabla_i T_j)(\nabla^l T_{lm}]$ explicitly from (25*b*) is still difficult. The simplification here is due to the fact that the « mixing » between lower spins (1 and 0) found in the general case disappears. The Ricci tensor, being itself « TT » (pure spin 2) can no longer affect the lower spins. Even here, there is some loss of orthogonality, namely that between T_{ij}^{TT} and the $R_{ij}\chi$ part, just because R_{ij} is itself « TT ». The mathematical reason behind the complications of the C-decomposition was linked to the presence of a (non-constant) Ricci tensor and the consequent lack of commutation among covariant derivatives acting on a vector, which mix the lower spin components. A precise statement of this difficulty may be found in a theorem of Avez [*4*] to the effect that for any transverse (longitudinal) vector φ_i the vector $A_{ij}\varphi^j$ is also transverse (longitudinal) if and only if $A_{ij} = \lambda g_{ij}$; in our case the Ricci identity brings in $A_{ij} \equiv R_{ij}$. Also [*4*], the operation $\triangle(\varphi_i)$ respects the transverse or longitudinal character of φ_i if and only if $R_{ij} = \lambda g_{ij}$, and this is the other basic operation in the decomposition. Thus, no « better » C-decomposition seems likely to exist.

IV. — THE EINSTEIN EQUATIONS

The basic interest of the C-decomposition lies in its possible usefulness in analyzing the initial value equations of general relativity, namely in providing a more or less explicit solution of the constraint problem and thereby also a covariant characterization of the dynamical (unconstrained) variables, which determine a system's evolution in time. We first recall the form taken by the Einstein equations in a « 3 + 1 » dimensional form in which spatial and temporal indices are separated. For simplicity, we assume there are no sources, but their inclusion would not alter any of our results. The reader is referred to the original papers [*1*] for details.

The Einstein action,

$$I = \int d^4\chi \mathfrak{R}(g_{\mu\nu}, \Gamma^{\alpha}_{\mu\nu}) \equiv \int d^4\chi \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\Gamma),$$

may be varied independently with respect to the metric and affinity to yield the field equations in first order form. These read $R_{\mu\nu}(\Gamma) = 0$ and $\Gamma^{\alpha}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\}$ and are to be considered on an equal footing. The action may be put into particularly simple form upon eliminating algebraic constraints such as the field equations $\Gamma^{i}_{jk} = \left\{ {i \atop jk} \right\}$, and introducing as variables

$$
\begin{array}{llll}
g_{ij} & , \quad N \equiv (-g^{00})^{-\frac{1}{2}} & , \quad N_i \equiv g_{0i} & \\
\pi_{ij} = -N\sqrt{{}^3g}\,(\Gamma^{0}_{ij} - g_{ij}\Gamma^{0}_{lm}\,{}^3g^{lm}) & , & \pi \equiv {}^3g^{ij}\pi_{ij}. &
\end{array}
\tag{26}
$$

Throughout, a consistent use is made of the 3-space metric g_{ij}, its inverse ${}^3g^{ij}$ and determinant ${}^3g$ to raise and lower indices, define (3-dimensional) covariant derivatives, ∇_k, etc. All operations are thus entirely within the three-space, the quantities N, N_i, g_{ij}, π^{ij} transforming as a scalar, vector and symmetric tensor (density) respectively under coordinate transformations within a $t = \text{const.}$ surface (but only g_{ij}, π_{ij} are invariant to transformations off the surface). In particular, note that the quantity ${}^4g^{ij}$ (the spatial component of the full contravariant metric) is related to ${}^3g^{ij}$ according to ${}^4g^{ij} = {}^3g^{ij} - N^{-2}N^iN^j$ (all such relations are obtained by use of the identity $g_{\mu\alpha}{}^4g^{\alpha\nu} = \delta^{\nu}_{\mu}$ and the definitions (26)—for example, the determinants are related by $\sqrt{-g} = N\sqrt{{}^3g}$). In terms of the set (g_{ij}, π^{ij}) and of (N, N_i) the source-free Einstein action can be evaluated to be

$$I = \int d^4\chi \mathfrak{R} = \int d^4\chi [\pi^{ij}\partial_0 g_{ij} - NR^0(g, \pi) - N_iR^i(g, \pi)] \tag{27a}$$

where

$$R^0 \equiv {}^3\mathfrak{R} - \frac{1}{\sqrt{g}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) \quad , \quad R^i \equiv 2\nabla_j\pi^{ij} \tag{27b}$$

aside from an irrelevant divergence [7]. In this action, it is understood that π^{ij}, g_{ij}, N and N_i are to be varied independently. The remaining components of $\Gamma^{\alpha}_{\mu\nu}$, other than the π^{ij}, are given in terms of the reduced set of variables by simple algebraic or differential relations. The action (27) is characteristic of the general covariance of the theory; the basic property of such actions is exhibited in the « parameterized » form of a simple particle action with n degrees of freedom in terms of an arbitrary parameter τ

$$\mathrm{I} = \int d\tau \left[\sum_{i=1}^{n+1} p^i (dq_i/d\tau) - \mathrm{N}(\tau)\mathrm{R}(p^i, q_i) \right]$$

Arbitrary « coordinate » transformations, $\tau \to \bar{\tau}(\tau)$, leave I invariant. The « Hamiltonian » NR is of the form of a Lagrange multiplier N (such quantities always occur when there is gauge invariance; in electrodynamics, A_0 plays this role) times R, a function only of the variables (p, q) occurring in the kinetic energy [7] $\sum_i p^i (dq_i/d\tau)$. The constraint $\mathrm{R} = 0$, obtained from varying N, enables one to solve for a constraint variable, say p^{n+1} in terms of the remaining variables, thereby reducing the problem to one of n degrees of freedom. The sixteen field equations obtained from Eq. (27) by varying with respect to $(g_{ij}, \pi^{ij}, \mathrm{N}, \mathrm{N}_i)$ may be divided into two sets. The first (from varying π^{ij} and g_{ij}) describes the time evolution of the system:

$$\partial_0 g_{ij} = 2\mathrm{N}(g)^{-\frac{1}{2}}\left(\pi_{ij} - \frac{1}{2} g_{ij}\pi\right) + \nabla_i \mathrm{N}_j + \nabla_j \mathrm{N}_i \tag{28a}$$

$$\begin{aligned} \partial_0 \pi^{ij} = & - \mathrm{N}\sqrt{g}\left(\mathrm{R}^{ij} - \frac{1}{2} g^{ij}\mathrm{R}\right) + \frac{1}{2}\mathrm{N}(g)^{-\frac{1}{2}} g^{ij}\left(\pi_{lm}\pi^{lm} - \frac{1}{2}\pi^2\right) \\ & - 2\mathrm{N}(g)^{-\frac{1}{2}}\left(\pi^{im}\pi_m{}^j - \frac{1}{2}\pi\pi^{ij}\right) + \nabla_m(\pi^{ij}\mathrm{N}_m) \\ & - \pi^{im}\nabla_m \mathrm{N}^j - \pi^{jm}\nabla_m \mathrm{N}^i + \sqrt{g}(\nabla^{ij} - g^{ij}\triangle)\mathrm{N}. \end{aligned} \tag{28b}$$

The second set (from varying the N, N_i) is independent of time derivatives and of the multipliers $(\mathrm{N}, \mathrm{N}_i)$; it represents four constraint conditions to be satisfied by the g_{ij}, π^{ij} at every instant:

$$\nabla_j \pi^{ij} = 0 \tag{29a}$$

[7] The criterion that the R^{μ} be independent of the N_{μ} uniquely defines the types of divergences which may be neglected in such problems. For a complete discussion of this delicate point see ref. (5).

$$(g)\mathrm{R} = \left(\pi_{ij}\pi^{ij} - \frac{1}{2}\,\pi^2\right) \tag{29b}$$

Thus the role of the multipliers is just to yield the constraints upon variation. Equations (28) and (29) are completely equivalent to the Einstein equations $G_{\mu\nu} = 0$ in their usual form. The constraints require that the Cauchy data at every instant be such that π^{ij} is transverse and the trace R of R_{ij} is a given function of the π^{lm}. Consistency between these requirements and the time evolution equations is guaranteed [1] by the four-dimensional Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$.

If we adjoin to Eqs (29) the identity (19), $\nabla^j G_{ij} = 0$, we have the very symmetric transversality conditions

$$\nabla_j \pi^{ij} = 0 = \nabla^j G_{ij} \tag{30a}$$

together with a trace condition

$$G \equiv -2\mathrm{R} = -2({}^3g)^{-1}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\,\pi^2\right) \tag{30b}$$

on G; π remains arbitrary (it is basically a gauge variable, in fact). The general resolution of the constraint equations is, from the usual point of view, a very complicated problem. Condition (30*b*) is a highly non-linear differential equation to be obeyed by the metric g_{ij}. The Bianchi identity, of course, is trivially satisfied by G_{ij} when it is expressed as a function of g_{ij}. The metric may be regarded as a « superpotential » such that, in terms of it, the form

$$G_{ij}(g) \equiv \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ki} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{il}\Gamma^l_{kj} - \frac{1}{2}\,g_{ij}\mathrm{R}(g) \tag{31}$$

is covariantly transverse. To be sure, a full solution of the Einstein equations in any particular case requires going through the complicated and coupled constraint equations to determine allowed sets of initial data, (g_{ij}, π^{ij}) satisfying the constraints, in terms of which the time evolution equations (28) yield the solution for all times [The four N, N_i enter as arbitrary functions, whose specification for all space-time corresponds to a choice of coordinates: the simplest choice, for example, $N = 1$, $N_i = 0$ is the Gaussian normal coordinate frame.] The viewpoint adopted here bypasses the difficulties of the full solution of the problem. We assume the metric g_{ij}, of the space to be fixed, while π^{ij} and R_{ij} (or G_{ij}) are considered as arbitrary tensors in this Riemannian space, subject to Eqs (30). That is, no use is made of the relation (31) at this level. Instead, we shall give the solutions of Eqs (30) by the results of the previous section, which tell us how to obtains tensors $(\bar{\pi}^{ij}, \bar{G}_{ij})$ satisfying Eqs (30), starting from two completely arbitrary tensors (π^{ij}, G_{ij}). These (somewhat implicit) general solutions $(\bar{\pi}^{ij}, \bar{G}_{ij})$ then describe all possible allowed initial data as (π_{ij}, G_{ij}) range over arbitrary values. To complement these results, we will also express the time development equations in terms of $(\partial_0 R_{ij}, \partial_0 \pi^{ij})$ rather than of $(\partial_0 g_{ij}, \partial_0 \pi^{ij})$, so that the problem is formally expressed in terms only of R_{ij}, π^{ij}, and the « background » metric g_{ij}. To find the latter involves finally solving the R equation to get one condition on g_{ij} in terms of our C-decomposition solution. However, the interest in the method lies in the possible usefulness of the simple functional solution of the constraints as it stands, without having to perform the inversion to find g_{ij}. To mention one important application, the constants of the motion of the gravitational field are functionals of the initial values only. Thus, one may investigate some of their functional properties without solving the full problem.

Before turning to these two programs, namely solving Eqs (30) and expressing Eqs (28) in terms of (R_{ij}, π^{ij}), it is useful to analyze the linearized approximation from our point of view. As we shall see, there is a degeneracy in this limit between the two approaches, since the various parts of the N-decomposition for G_{ij}, say, are automatically of the same form as that obtained from the metric expression (31)—the g_{ij} (or, more conveniently, the h_{ij}) are explicitly the « superpotentials »— in terms of which the various components G_{ij}^{TT}, etc. are defined by projection operators. Indeed, this fact underlies the use of the N-decomposition to analyze the constraints (in a given frame) of the full theory in the canonical reduction [1], since their linear part is made the basis of an iteration solution.

From Eq. (31), it follows that the Ricci tensor reads

$$R_{ij}(h_{lm}) = -\,\triangle_0 h_{ij} - h_{kk,ij} + h_{ik,kj} + h_{jk,ki} \tag{32a}$$

so that

$$G_{ij} \equiv R_{ij} - \frac{1}{2}\,\delta_{ij} R_{ll} = R_{ij}(h_{lm}) - \delta_{ij}(h_{kl,kl} - \triangle_0 h_{kk}) \tag{32b}$$

and of course $G_{ij,j} \equiv 0$. If we write next

$$G_{ij} = G_{ij}^{TT} + G_{ij}^{T} + (G_{i,j} + G_{j,i}) \tag{33}$$

we find

$$G_{i,jj} + G_{j,ij} \equiv 0 \;\Rightarrow\; (G_{i,j} + G_{j,i}) = 0 \tag{34a}$$

and using this, that

$$G^{T} = -\frac{1}{2}\,R = h_{kk,ll} - h_{kl,kl} \equiv \triangle_0 h^{T} \tag{34b}$$

while

$$G_{ij}^{TT} = G_{ij} - G_{ij}^{T} = f_{ij}^{TT}(h_{lm}) \equiv - \Delta_0 h_{ij}^{TT} \tag{34c}$$

where $f_{ij}^{TT}(h)$ represents the function obtained by writing $G_{ij}(h)$ from (32) and $G_{ij}^{T}(h)$ from (34*b*). Similarly, R_{ij} has the form

$$\begin{aligned} R_{ij} = - \Delta_0 h_{ij}^{TT} + \frac{1}{2}(\delta_{ij}\Delta_0 + \partial_{ij}^2)h^T &= - \Delta_0 h_{ij}^{TT} + \theta_{ij}^{T} h^T + \partial_{ij}^2 h^T \\ &= R_{ij}^{TT} + R_{ij}^{T} + (R_{i,j} + R_{i,j}) \end{aligned} \tag{35}$$

Conversely, these results could have been obtained without any knowledge of the fact that G_{ij} is the Einstein tensor of the metric g_{ij}, simply by noting that the most general form for a transverse tensor G_{ij} is given by Eq. (32) where h_{ij} is an arbitrary tensor. One could also proceed directly, by decomposing h_{ij} itself,

$$G_{ij}^{TT} \equiv - \Delta_0 h_{ij}^{TT} \quad , \quad G_{ij}^{T} \equiv \Delta_0 h^T \quad , \quad G_{i,j} + G_{j,i} \equiv 0$$

The gauge invariance of G_{ij} and R_{ij} with respect to arbitrary changes of the vector h_i in Eq. (32) would alternately tell us that they can only depend on h_{ij}^{TT} and h^T, which are the gauge-invariant parts of h_{ij}: under the transformation $h_{ij} \to h_{ij} + \xi_{i,j} + \xi_{j,i}$ it is clear that only the $(h_{i,j} + h_{j,i})$ part is affected. Thus, in the linearized limit, the curvature tensor is an *invariant* under the linearized limit of coordinate transformations (namely the above gauge transformations) rather than a *covariant*; in the full theory, of course, the curvature transforms as a tensor, and not as an invariant, and all relarions are then covariant. If the final constraint, $R_{ll} = 0$, is inserted into the above equations, it requires that $\Delta_0 h^T = 0$ and as a result

$$G_{ij} = G_{ij}^{TT} = - \Delta_0 h_{ij}^{TT}. \tag{36}$$

The π constraint is equally trivial to solve; since $\partial_j \pi^{ij} = 0$, it follows that $\partial_j \pi_i + \partial_i \pi_j = 0$ and the general solution $\bar{\pi}^{ij}$ has the form

$$\bar{\pi}^{ij} = \pi^{ijTT} + \pi^{ijT} = [\theta^{TT} + \theta^{T}]_{lm}^{ij} \pi^{lm} \tag{37}$$

where π^{lm} is an arbitrary tensor.

In the presence of sources, the right sides of the constraints $\partial_j \pi^{ij}$ and R no longer vanish (though of course $\partial_j G^{ij} \equiv 0$ still). This is the case either when there are external $T^{\mu\nu}$, or more interestingly, for the full theory when the constraints are put into a form where the linearized part is separated explicitly, the non-linear part being an effective source of the former :

$$\begin{aligned} \partial_j \pi^{ij} &\equiv \Delta_0 \pi^i + \pi^j{}_{,ij} = - \pi^{lm} \Gamma^i_{lm} \\ R^L &\equiv 2(h_{ii,jj} - h_{ij,ij}) \equiv 2\Delta_0 h^T = (R - R^L) - ({}^3g)^{-1}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) \end{aligned} \tag{38}$$

This system may be viewed as a non-linear set to determine (π^i, h^T) as functions of the remaining variables (h^{TT}_{ij}, π^{ijTT}) and ($h_{i,j}$, π^T), which appear (together with their first derivatives) only in quadratic or higher terms; the gauge variables ($h_{i,j}$, π^T) may be chosen to determine the frame, while the « TT » components are the dynamic, unconstrained variables. An iteration solution for (π^i, h^T) as a power series in the « TT » and gauge variables is then obtained. The quadratic terms in this expansion are precisely the momentum and energy densities of the linearized theory in an arbitrary gauge. Their integrals, but not the densities themselves, are gauge invariant, in contrast to the situation in Maxwell theory, where of course $T^{\mu\nu}$ itself is gauge invariant (8). In terms of the iteration procedure, h^T and π^i may be obtained to any order. The Hamiltonian and momentum generator are determined, in this way, in any frame.

One application of the N-decomposition lies in proving that, in the linearized approximation, the kinetic energy

$$T = \int d^3r \left[\pi_{ij}\pi_{ij} - \frac{1}{2}\,\pi_{ii}\pi_{jj} \right] \tag{39}$$

is positive-definite as a consequence of the constraints. This example is useful to analyze, for later comparison with the full theory case. The quantity T is certainly not intrinsically positive: if we write

$$\pi_{ij} \equiv \tilde{\pi}_{ij} + \frac{1}{3}\,\delta_{ij}\pi_{ll} \quad , \quad \text{where} \quad \tilde{\pi}_{ll} \equiv 0 \quad , \quad \text{we find} \quad T = \int d^3r \left[\tilde{\pi}_{ij}\tilde{\pi}_{jj} - \frac{1}{6}\,\pi^2 \right]$$

which can be negative since $\tilde{\pi}_{ij}$ and π_{ll} are independent. On the other hand, we know in general that the energy is gauge-invariant and since only π_{ij} is affected by the gauge $\xi^0(\pi_{ij} \rightarrow \pi_{ij} - \theta_{ij}\xi^0)$ which leaves the rest of the energy unchanged, we expect that the gauge choice $\pi = 0$ can be made to exhibit the sign of T [Under gauge transformations, $h_{ij} \rightarrow h_{ij} + \xi_{i,j} + \xi_{j,i}$ and $\pi_{ij} \rightarrow \pi_{ij} - \theta^T_{ij}\xi^0$.] However, the invariance of T holds only as a consequence of the constraint $\partial_j\pi^{ij} = 0$, and is absent for arbitrary π_{ij}. We must therefore use the transversality of π^{ij} in the expression for T, before finding that $T \geqslant 0$, independent of the gauge. We have, by orthogonality,

$$\begin{aligned} T &= \int d^3r \left[\pi^{TT^2}_{ij} + \frac{1}{2}\,\pi^{T^2} + (\pi_{i,j} + \pi_{j,i})^2 - \frac{1}{2}\,(\pi^T + 2\pi_{i,i})^2 \right] \\ &= \int d^3r [\pi^{TT^2}_{ij} + \{ (\pi_{i,j} + \pi_{j,i})^2 - 2\pi_{i,i}^2 \} - 2\pi^T\pi_{i,i}]. \end{aligned} \tag{40}$$

(8) Some of the difficulties of massless theories with spin > 1 related to the non-invariance of their $T^{\mu\nu}$ may be found in S. Deser, J. and S. Trubatch, *Nuov. Cim.*, t. 39, p. 1159 (1965) and in C. M. Bender and B. M. McCoy, *Phys. Rev.*, t. 148, p. 1375 (1966).

In the last equality, the quantity in curly brackets is positive definite, becoming $2\int d^3r\pi_{i,j}\pi_{i,j}$ upon integration by parts. However, the gauge-dependent term $-2\pi^T\pi_{i,i}$ is totally arbitrary in general, vanishing only by use of the solution of the transversality condition, $\partial_j\pi^{ij}=0$. Thus, use of the constraints in an explicit fashion is already essential in the linear case to prove positiveness. It is also needed for the « potential », part of the energy. The latter is just obtained, from Eq. (38) and integration by parts:

$$\begin{aligned} V = \int d^3r(R - R^L) = \int d^3r\{\delta_{ij}R^Q_{ij} - h_{ij}R^L_{ij}\} \\ = \frac{1}{4}\int d^3r[(\nabla h_{ij})^2 - 2(h_{ij,j})^2 + h_{ij,j}h_{mm,i}] \qquad (41) \\ = \frac{1}{4}\int d^3r[(\nabla h^{TT}_{ij})^2 + \frac{1}{2}(\nabla h^T)^2 - 2h_{i,i}\Delta_0 h^T] \end{aligned}$$

where R^L_{ij}, R^Q_{ij} are the linear and quadratic parts in h_{ij} of R_{ij}. We see once again that the gauge contribution has indefinite sign and must be eliminated by use of the $\Delta_0 h^T = 0$ constraint. From the foregoing, it is clear that the determination of the sign of the full field's energy cannot easily proceed order by order by these linearized methods; a covariant attack on the expression may offer more hope.

V. — SOLUTION OF CONSTRAINTS AND APPLICATIONS

We now return to the solution of the full constraints, Eq. (30). The general solution (9) for π^{ij} may be written in either of two forms. That corresponding to Eq. (13) is

$$\pi^{ijt} \equiv \pi^{ij} - (LW)^{ij}[\nabla_l\pi^{lm}] \quad \text{or} \quad \pi^{ijt} \equiv \pi^{ij} - (\nabla^j\pi^i + \nabla^i\pi^j)[\nabla_l\pi^{lm}] \qquad (42)$$

where the π^{ij} in the last members are arbitrary and the longitudinal parts satisfy Eqs (11) or (12). However, the structure of π^{ij} is made most explicit

(9) The first discussion of the π constraints is given in ref. (17), in terms of their form in a « repère mobile » at any point. It is there shown how, effectively, the appropriate components may be solved in terms of the rest. This *local* approach corresponds to our non-local N- or C-decompositions, the emphasis and applications being rather different.

if we use the complete « TT » decomposition of Eqs (20) or (21), along with the defining equations (22, 23):

$$\pi^{ij} = \pi^{ij\,\mathrm{TT}} + (\mathrm{LW})^{ij} + \theta^{ij}(\triangle^{-1}\pi) \tag{43a}$$

where $(\mathrm{LW})^{ij}$ is determined by the elliptic equation

$$\nabla_j(\mathrm{LW})^{ij} = -\nabla\,\theta^{ij}(\triangle^{-1}\pi) \tag{43b}$$

Similarly, we may write the general solution of G_{ij},

$$\mathrm{G}_{ij} = \mathrm{G}_{ij}^{\mathrm{TT}} + (\mathrm{LU})_{ij} + \theta_{ij}\chi \tag{44}$$

where

$$\triangle\chi = \mathrm{G} = -\frac{2}{g}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) \tag{45a}$$

and

$$\nabla^j(\mathrm{LU})_{ij} = -\nabla^j\theta_{ij}\chi = 2\nabla^j\theta_{ij}\triangle^{-1}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right)(g)^{-1} \tag{45b}$$

It should perhaps be reemphasized that the solution (44-45) for G_{ij} is independent of the fact that it is the Einstein tensor of the metric g_{ij}, which enters implicitly in all the operations. The task of determining g_{ij} as a function of the variables in (44) by equating $\mathrm{G}_{ij}(g_{lm})$ to the solution (44) is not considered here, the idea being to display explicitly as much as possible of the form of the tensors governing the initial value problem.

In the special case for which the right side of Eq. (45*a*) vanishes—as when $\pi^{ij} = 0$ at the instant in question (a moment of time symmetry), χ vanishes, and with it (LU), so that G_{ij} is a « TT » tensor. However, the simplification in the C-decomposition which was noted when R vanishes is of no use to us in this one place where $\mathrm{R} = 0$, for there is now no π^{ij} left to decompose, and the C-decomposition of G cannot make use of the simplification either.

Let us now turn to the analysis of the kinetic energy in the full Einstein theory. It should be noted that, as in the linear approximation, the value of the total energy of an isolated system has been shown [*6*] to be coordinate-invariant with respect to all formations preserving the asymptotic conditions. One might then be tempted to use the formal argument that since one choice of time coordinate is that of the « minimal surface » $\pi = 0$ (if it exists—as we shall discuss later), the energy can be evaluated with this choice, and $\int d^3r(g)^{-\frac{1}{2}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) \rightarrow \int d^3r(g)^{-\frac{1}{2}}\pi^{ij}\pi_{ij} \geqslant 0$. However, aside from questions about the validity of the global $\pi = 0$ condition, there are two other difficulties with this simple argument. First, although the total energy's value is invariant, its functional form is *not* (this being a fundamental difference between generally covariant and Lorentz covariant theories: almost arbitrary changes of the time coordinate are permitted, the corresponding tim translation generator also changing in form). Second, the transformations on π also affect the potential energy, in contrast to the linear gauge transformations. Thus it is also possible that the kinetic energy alone is not positive, but only the total energy ([10]), when the information from the constraint (30*b*) has been fed into the spatial metric dependence of the potential energy. Thus, one cannot really speak of the form (as against the *value*) of the Hamiltonian as a function of the dynamical variables in a frame-independent (in particular, in a time choice-independent) way, since it varies with this choice. The *value* of the energy, on the other hand, is defined from a flux theorem in terms of the asymptotic form of the « T » component of the N-decomposition of the metric, namely from $h^{T} \sim E/4\pi r$ (in units of $16\pi\gamma = 1 = c$) and this coefficient of $1/r$ is *frame-invariant*, if the boundary conditions $h_{\mu\nu} \sim O(1/r)$ at spatial infinity are respected [*6*]. This is as in electrodynamics, where the total charge Q may be read off from the term $\sim \nabla Q/r$ in $\mathbf{E}^{L}$ at infinity, but its *form* and sign for any particular charge distribution is of course to be calculated by taking the volume integral over all space of the charge density $j^{0}(\boldsymbol{r})$.

However, it should not be concluded from the foregoing that nothing can be said about the functional form (and, in particular, the sign) of E in a frame-independent way. For since the energy is exhibited as the monopole moment of the source of h^{T}, we may use the R constraint equation, written as a Poisson equation for h^{T}

$$-\Delta_0 h^{T} = (R - \Delta_0 h^{T}) - \frac{1}{g}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) \tag{46}$$

The function Δh^{T} is just the linearized R, R^{L}, and the integral over all space (monopole moment) of the right side of (46) is just the value of the energy:

$$E = \int d^3r\left[-\delta R + \frac{1}{g}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right)\right] \tag{47}$$

where $\delta R \equiv R - R^{L}$. The form of E may be altered by adding divergences which vanish sufficiently rapidly at infinity, but this does not change

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([10]) Of course, it is also conceivable *a priori* that T can be positive in some cases but the total energy negative! A soluble model to test these possibilities might be offered by a system whose spatial metric is conformally flat at the instant, with arbitrary π^{ij}. The π constraint is relatively simple here, and so is the energy's form [15].

its value. For a discussion of the positiveness problem and of the known cases where $E > 0$, see ref. (15).

The *form* (47) is the same in all frames; what is different is the value of the gauge variables and the choice of canonical unconstrained variables [In this respect, the analogy between E and Q ceases to hold, since $Q = \int d^3r j^0$ may have arbitrary form for different systems, whereas E is fixed in form as the energy of this *particular* field. If, in addition, there are external $T^{\mu\nu}$ sources, these would be the analogs of the j^0 in electrodynamics. In the presence of $T^{\mu\nu}$, the energy can only be expected to be positive, of course if $-\int d^3r T^0_0 \geqslant 0$, as is the case for all the usual matter sources [*15*].

Thus, if E can be shown, in a covariant way, to be positive for isolated systems in asymptotically Minskowskian frames [*15*], the problem is solved. One would also conjecture then that the vanishing of the single number E implies space is flat, so that $E = 0$ is both necessary and sufficient for flatness (as is the case in all situations where E is known to be positive).

While the general evaluation of the « potential », δR is complicated, the « kinetic » energy T is a quadratic functional of π^{ij}:

$$T = \int d^3r (g)^{-\frac{1}{2}} \left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2 \right) \tag{48}$$

and one may evaluate it in terms of the general solution (43) for π^{ij}. We shall see that T is « almost », but not quite, positive-definite. It follows immediately by the orthogonality of π^{TT}_{ij} and $(LW)_{ij}$ or $\theta_{ij}\psi$ that

$$T = \int \frac{d^3r}{\sqrt{g}} \left\{ \pi^{TT^2} + (LW)^2 + (\theta\psi)^2 + 2(LW)_{ij}\theta^{ij}\psi - \frac{1}{2}(\Delta\psi)^2 \right\} \tag{49a}$$

The cross terms may be evaluated, using the constraint equation in the form $\nabla^j[(LW)_{ij} + \theta_{ij}\psi] = 0$, to yield

$$T = \int \frac{d^3r}{\sqrt{g}} \left\{ \pi^{TT^2} + \frac{1}{4}[(\Delta\psi)^2 - \nabla^{ij}\psi\nabla_{ij}\psi] + (LW)^2 \right\} \tag{49b}$$

This expression is not quite positive, since $\int [(\Delta\psi)^2 - (\nabla\psi)^2]$ is zero in engeral only if one may interchange deiivatives, i. e. if R_{ij} is neglected. Indeed, from the inequality [11]

$$\int d^3r\sqrt{g}[a\nabla_{ij}\psi + bg_{ij}\Delta\psi]^2 \geqslant 0 \tag{50a}$$

valid for all values of a, b and arbitrary ψ, we find the maximal statement

$$\int d^3r\sqrt{g}\left[(\nabla\psi)^2 - \frac{1}{3}(\Delta\psi)^2\right] \geqslant 0 \tag{50b}$$

where the equality implies $\nabla_{ij}\psi = \frac{1}{3}g_{ij}\Delta\psi$, which has no non-trivial well-behaved solutions.

In reaching the simple form (49 b), and to reduce the $(LW)^2$ term, one uses the fact that

$$\int d^3r\sqrt{g}(\nabla^i\psi)\nabla^j(LW + \theta\psi)_{ij} = 0 = \int d^3r\sqrt{g}W^i\nabla^j(LW + \theta\psi)_{ij} \tag{51}$$

on integration by parts, the first relation gives $\int d^3r\nabla^{ij}\psi(LW)_{ij}$ in terms of ψ alone, the second gives $\int(LW)^2$ in terms of $\int d^3r\nabla^jW^i\nabla_{ij}\psi$. This yields

$$T = \int\frac{d^3r}{\sqrt{g}}\left[\pi^{TT^2} - \frac{2}{3}\Delta\psi\nabla_lW^l\right]. \tag{52}$$

For positiveness, the second term must be always non-negative, since it is independent of π^{TT}_{ij}; $\Delta\psi$ is just the (arbitrary) trace of π^{ij} and so the scalar

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([11]) This is one of a large class of inequalities which can be obtained from the general form

$$\int d^3r\sqrt{g}\,\{a\nabla_iW_j + bg_{ij}\nabla_lW^l + c\nabla_{ij}\psi + dg_{ij}\Delta\psi\}^2 \geqslant 0.$$

However, these do not, for example, permit us to show $T \geqslant 0$ in general. One may also draw inequalities on integrals of the form $-\int d^3r\sqrt{g}\nabla_i\psi R^{ij}\nabla_j\psi$, since these just represent $\int d^3r\sqrt{g}[(\Delta\psi)^2 - (\nabla_{ij}\psi)^2]$ with similar results for expressions like $\int d^3r\sqrt{g}V_iR^{ij}V_j$; thus, using Eq. (50), we learn that

$$\int d^3r\sqrt{g}\nabla_i\psi R^{ij}\nabla_j\psi \leqslant -\frac{2}{3}\int d^3r\sqrt{g}(\Delta\psi)^2$$

for arbitrary ψ and R^{ij}. These inequalities also do not seem to be very useful in the present context.

function $\nabla_l W^l$ must be orthogonal to this trace. In linear theory, this was ensured by $\partial_l W^l = 0$. Here, $\nabla_l W^l$ is the solution of the equation

$$\nabla^{ij}\left(\nabla_i W_j + \nabla_j W_i - \frac{2}{3} g_{ij} \nabla_l W^l\right) \equiv \frac{4}{3} \Delta(\nabla_l W^l) - 2\nabla^j(W^i R_{ij}) = -\nabla^{ij}\theta_{ij}\psi \tag{53}$$

the first equality holding by virtue of the Ricci identities. We may write

$$\begin{aligned} -\frac{2}{3}\int d^3r\, \Delta\psi \nabla_l W^l &= -\frac{2}{3}\int d^3r \psi\, \Delta(\nabla_l W^l) \\ &= \int d^3r \left\{\frac{1}{4}\left[(\Delta\psi)^2 - (\nabla\psi)^2\right] + \nabla_i\psi R^{ij} W_j\right\} \\ &= \int d^3r \nabla_i\psi R^{ij}(\nabla_j\psi + W_j) \end{aligned} \tag{54}$$

Alternately, the second equality of Eq (51), together with the Ricci identity, yields the relation $\int d^3r \nabla_i\psi R^{ij} W_j = \int d^3r (LW)^2$, for arbitrary ψ and R_{ij}. A number of other equivalent, but also non-manifestly definite, forms may be written. These always involve indefinite terms like $((\Delta\psi)^2 - (\Delta\psi)^2)$ which can be reduced to $\sim \int \nabla_i\psi R^{ij} \nabla_j\psi$ or negative terms like $-(\nabla_l W^l)^2$ on which no useful lower bounds have been found. Naturally an integral of the form $\int U_i R^{ij} U_j$ will always be positive if R^{ij} acts as a positive metric. However, if for example $g_{ij}R^{ij}$ is positive, this already means, by the R-constraint that $\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right)$ is positive locally, in which case T is positive *ab initio*.

Of course, it may be that only detailed solution of the constraints in terms of g_{ij} can settle the sign question in a decisive way, or that further ingenuity would show $T > 0$ in our framework; the latter possibility certainly cannot be ruled out, since we have not been able to express W^i explicitly in terms of ψ. Note, however, the very satisfactory way in which the physical « TT » (spin 2) contribution has been exhibited. It always enters in a positive and decoupled fashion.

It is also interesting in this connection that while $\pi^{ij} = 0$ implies $R = 0$, the converse is not true, even making use of the simplifying features of the π^{ij} C-decomposition when $R = 0$. Thus, in this special case (where $R_{ij} = R_{ij}^{TT}$) we write

$$\pi^{ij} = \pi^{ijTT} + (LW)^{ij} + \left(\theta^{ij} + \frac{1}{2} R^{ij}\right)\psi \tag{55}$$

where $\nabla_j(\theta^{ij} + \frac{1}{2}R^{ij})\psi \equiv 0$, so that from $\nabla_j\pi^{ij} = 0$ we conclude directly that $(LW)^{ij} = 0$. Thus, we know, using tracelessness of $\left(\pi_{ij}^{TT} + \frac{1}{2}R_{ij}^{TT}\psi\right)$, that

$$
\begin{aligned}
&\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) \\
&= \left(\pi^{ijTT} + \frac{1}{2}R^{ij}\psi\right)^2 + \theta^{ij}\psi\theta_{ij}\psi - \left(\pi_{ij}^{TT} + \frac{1}{2}R_{ij}\psi\right)\nabla^{ij}\psi - \frac{1}{2}(\Delta\psi)^2 = 0 \quad (56) \\
&= \left(\pi^{ijTT} + \frac{1}{2}R^{ij}\psi\right)^2 + \frac{1}{4}((\nabla\psi)^2 - (\Delta\psi)^2) + \frac{1}{2}\nabla_i\psi R^{ij}\nabla_j\psi + \text{Div}.
\end{aligned}
$$

In the last equality, we have indicated by Div. two terms which arise on integration by parts in $\left(\pi + \frac{1}{2}R\psi\right)\nabla\psi$.

On the other hand, the Ricci identity yields

$$\int[(\Delta\psi)^2 - (\nabla\psi)^2]d^3r = -\int\nabla_i\psi R^{ij}\nabla_j\,\psi d^3r \tag{57}$$

where neither side is, in general, positive definite, since only

$$\int\left[(\nabla\psi)^2 - \frac{1}{3}(\Delta\psi)^2\right]d^3r \geqslant 0.$$

Thus we know that

$$\int d^3r\left[\left(\pi_{ij}^{TT} + \frac{1}{2}R_{ij}\psi\right)^2 + \frac{1}{4}\nabla_i\psi R^{ij}\nabla_j\psi\right] = 0 \tag{58}$$

From thus we could of course conclude, if $\psi = 0$ (i. e. $\pi = 0$) that $\pi_{ij}^{TT} = 0$ and hence π^{ij} vanishes. But even given $\pi_{ij}^{TT} = 0$, we could not conclude that $\psi = 0$ from Eq. (58). This negative result is in contrast with the flat case, where $R_{ij} = 0$ does imply that $\pi_{ij} = 0$ as well [8].

One other remark in connection with the mathematical properties of our decomposition may be in order. We have been using the operator $\Delta = \nabla^i\nabla_i$ uniformly as « the » Laplacian on scalars, vectors and tensors. This operator, as we noted, does not commute with any covariant derivative (including the divergence) unless $R_{ij} = \lambda g_{ij}$. On the other hand, there is another definition of the Laplacian on vectors and tensors which is frequently used mathematically. This is a generalization of the De Rham definition ([12]), which for vectors has the form

$$(\tilde{\triangle}\mathrm{V})_i \equiv \nabla^j(\nabla_j\mathrm{V}_i - \nabla_i\mathrm{V}_j) + \nabla_i(\nabla_j\mathrm{V}^j) = \triangle\mathrm{V}_i - \nabla_{ji}\mathrm{V}^j + \nabla_{ij}\mathrm{V}^j \\ = \triangle\mathrm{V}_i + \mathrm{R}_{ij}\mathrm{V}^j \tag{59}$$

This operator has the property that

$$\nabla_i(\tilde{\triangle}\mathrm{V})^i = \tilde{\triangle}(\nabla_i\mathrm{V}^i) \equiv \triangle(\nabla_i\mathrm{V}^i), \tag{60}$$

as a consequence of the identity $\nabla^{ij}(\nabla_i\mathrm{V}_j - \nabla_j\mathrm{V}_i) = 0$ with

$$\nabla^{ij}\nabla_i\mathrm{V}_j = \triangle\nabla_l\mathrm{V}^l - \nabla^j(\mathrm{V}^l\mathrm{R}_{lj}).$$

Thus, the divergence and $\tilde{\triangle}$ operations commute when acting on vectors. For second rank tensors, the generalized Laplacian reads

$$(\tilde{\triangle}\mathrm{T})_{ij} \equiv \triangle\mathrm{T}_{ij} - \mathrm{R}_{im}\mathrm{T}^m{}_j - \mathrm{R}_{jm}\mathrm{T}_i{}^m + \mathrm{R}_{iljm}\mathrm{T}^{lm}$$

or, in terms of covariant differentiation,

$$(\tilde{\triangle}\mathrm{T})_{ij} = \triangle\mathrm{T}_{ij} - \nabla_{ki}\mathrm{T}^k{}_j - \nabla_{kj}\mathrm{T}_i^k + \nabla_{jk}\mathrm{T}^k{}_i + \nabla_{ik}\mathrm{T}_j^k \\ = \triangle\mathrm{T}_{ij} + (\nabla_{jk} - \nabla_{kj})\mathrm{T}_i^k + (\nabla_{ik} - \nabla_{ki})\mathrm{T}_j^k$$

Note that $(\tilde{\triangle}\mathrm{T})_{ij}$ is symmetric if T_{ij} is; however, it is no longer true here that $[\tilde{\triangle}, \nabla_i] = 0$ unless [7] $\mathrm{R}_{ij} = \lambda g_{ij}$. While the $\tilde{\triangle}$ operator may be useful for antisymmetric tensors (and perhaps for other applications to symmetric ones) it is not directly relevant for purposes of C-decomposition. It is elliptic, since $-\int \mathrm{V}^i(\triangle\mathrm{V}_i - \triangle_{ji}\mathrm{V}^j) = \frac{1}{2}\int(\nabla_i\mathrm{V}_j - \nabla_j\mathrm{V}_i)^2 \geqslant 0$, is essentially the square of the curl of the vector, while $-\int\mathrm{V}^i\nabla_{ij}\mathrm{V}^j = \int(\nabla_l\mathrm{V}^l)^2 \geqslant 0$ (but it does not follow that for the tensor case, $-\int\mathrm{T}^{ij}(\tilde{\triangle}\mathrm{T})_{ij} \geqslant 0$). However, the antisymmetric nature of the curl is what prevents us from utilizing it to decompose our symmetric tensors. Indeed, the combination we use is precisely the symmetric one, $(\nabla_j\mathrm{V}_i + \nabla_i\mathrm{V}_j)$. Similarly, use of scalar parts of the form $[\tilde{\triangle}(\nabla\nabla\psi)]_{ij} + \alpha g_{ij}\triangle\triangle\psi$ is of no help, because they lose orthogonality to $\mathrm{T}_{ij}^{\mathrm{TT}}$, even if they could be made transverse. We shall see shortly, though, that the combination $(\tilde{\triangle}\pi)^{ij}$ happens to arise naturally in the time development parts of the field equations.

([12]) The details of the general definition of $\tilde{\triangle}$ are given in ref. (7), where the special case $\mathrm{R}_{ij} = \lambda g_{ij}$ is treated extensively.

VI. — REFORMULATION OF THE TIME EVOLUTION EQUATIONS

In view of the simple form taken by the constraints in terms of (π^{ij}, R_{ij}), it is interesting to ask whether the time development equations (28) also simplify when recast in terms of $(\partial_0 R_{ij}, \partial_0 \pi^{ij})$ rather than $(\partial_0 g_{ij}, \partial_0 \pi^{ij})$. As they stand, the $\partial_0 g_{ij}$ equations (28*a*) have the simple $\dot{q} = p$ form (although $\sqrt{g}$, g^{ij}, etc. are in fact non-linear functions of g_{ij}). The $\partial_0 \pi^{ij}$ part, however, is very non-linear in g_{ij}, through its dependence on $R^{ij}(g_{lm})$. Clearly, a direct evaluation of $\partial_0 R^{ij}$ in terms of $\partial_0 g_{lm}$ is very lengthy. Fortunately, though, there is a shortcut, owing to a very useful relation (the Palatini identity) which yields the change δR_{ij} of R_{ij} when the affinity changes by an arbitrary, infinitesimal amount $\delta\Gamma^i_{jk}$. This formula is metric-independent, as R_{ij} itself is :

$$\delta R_{ij} \equiv R_{ij}(\Gamma + \delta\Gamma) - R_{ij}(\Gamma) = \nabla_k \delta\Gamma^k_{ij} - \nabla_j \delta\Gamma^k_{ik} \tag{61}$$

where the covariant derivative is with respect to the unvaried affinity, and $\delta\Gamma^k_{ij}$ is a third-rank tensor, being a difference of two affinities $(\delta\Gamma \equiv (\Gamma + \delta\Gamma) - \Gamma)$. We need next to express $\delta\Gamma$ in terms of an arbitrary change δg_{ij} of the metric when, as is the case for a Riemann space, $\Gamma^i_{jk} = \left\{ {i \atop jk} \right\}$. The varied Γ^i_{kj} is the Christoffel symbol of a varied metric $g_{ij} + \delta g_{ij}$, so $\delta\Gamma$ is just the variation of the Christoffel symbol, which is also simple:

$$\begin{aligned} \delta\left[\frac{1}{2} g^{kl}(g_{li,j} + g_{lj,i} - g_{ij,l})\right] &= \frac{1}{2} g^{kl}[\delta g_{jl,i} + \delta g_{il,j} - \delta g_{ij,l}] \\ &= \frac{1}{2} g^{km}\delta g_{mn} g^{nl}[g_{li,j} + g_{lj,i} - g_{ij,l}] \end{aligned} \tag{62}$$

Here we have used the relation $\delta g^{kl} = - g^{km}\delta g_{mn} g^{nl}$ between reciprocal matrices. By expressing ordinary in terms of covariant derivatives (with respect to g_{ij}) one obtains

$$\delta\Gamma_{i\ j}^{\ k} = \frac{1}{2}(\nabla_i \delta g_j^{\ k} + \nabla_j \delta g_i^{\ k} - \nabla^k \delta g_{ij}) \tag{63}$$

where all indices are moved by weans of the imperturbed metric: $\delta g_i^k \equiv g^{kl}\delta g_{li}$. From Eqs (61), (63) we may write δR_{ij} in several forms [13]:

$$2\delta R_{ij} = - \Delta\delta g_{ij} + \nabla_{ki}\delta g_j^{\ k} + \nabla_{kj}\delta g_i^{\ k} - \nabla_{ij}\delta g^l_{\ l} \tag{64}$$

---

[13] Extensive formulae for such variations, and more generally for δR_{ijkl}, may be found in ref. (7).

or, to bring into evidence the appeaıance of the $\tilde{\Delta}$ operator we use the Ricci identity on the two positive terms of (64) to find

$$\begin{aligned} 2\delta R_{ij} &= -\Delta\delta g_{ij} + R_i^l\delta g_{lj} + R_j^l\delta g_{li} - 2R_i{}^l{}_j{}^k\delta g_{lk} + \nabla_{ik}\delta g_j{}^k + \nabla_{jk}\delta g_i{}^k - \nabla_{ij}\delta g_k^k \\ &= -(\tilde{\Delta}\delta g)_{ij} + \nabla_{ik}\delta g_j{}^k + \nabla_{jk}\delta g_i{}^k - \nabla_{ij}\delta g_k^k \end{aligned} \tag{65}$$

Since $\tilde{\Delta}$ is easily seen to commute with contraction, it follows that $g^{ij}(\tilde{\Delta}\Gamma)_{ij} = \tilde{\Delta}(g^{ij}T_{ij}) \equiv \Delta T$. The last equality uses the property of $\tilde{\Delta}$ that $(\tilde{\Delta}gS)_{ij} = \Delta g^{ij}S = g^{ij}\Delta S$ where Δ is the scalar Laplacian $\nabla_k\nabla^k$ and S is a scalar.

For our purposes, the variation δg_{ij} is that due to time evolution, $\delta g_{ij} \equiv \partial_0 g_{ij}\delta t$, and the corresponding $\delta R_{ij} = \partial_0 R_{ij}\delta t$. Thus, we need only substitute equation (28*a*) for $\partial_0 g_{ij}$ into

$$\partial_0 R_{ij} = \frac{1}{2}[-\Delta\partial_0 g_{ij} + \nabla^k\nabla_i\partial_0 g_{jk} + \nabla^k\nabla_j\partial_0 g_{ik} - \nabla_{ij}g^{lm}\partial_0 g_{lm}] \tag{66}$$

Note that ∂_0 does not commute with covariant differentiation (whereas it does with ordinary derivatives). Since we only wish to exhibit the general structure of this equation, we choose, at $t =$ const., Gaussian coordinates in which $N = 1$, $N_i = 0$ to avoid algebraic complications (Actually the choice $N = 1$ is inessential in that we could use $\delta g_{ij} = N^{-1}\partial_0 g_{ij}\delta t$, and thereby find $N^{-1}\partial_0 R_{ij}\delta t$, as $N^{-1}\partial_0$ is the natural scale for time differentiation). Similarly, the terms in N_i could also be carried, but they are lengthy, and irrelevant for our purpose. A choice of N, N_i at a given instant does not imply a coordinate choice at that instant, but rather specifies how this choice changes off the hypersurface. Thus, we may still meaningfully retain the covariant notation in the three-space.

At an instant in which Gaussian coordinates are used,

$$\frac{1}{2}\partial_0 g_{ij} = \frac{1}{\sqrt{g}}\left(\pi_{ij} - \frac{1}{2}g_{ij}\pi\right)$$

Thus, Eq. (66) becomes

$$\partial_0 R_{ij} = (g)^{-\frac{1}{2}}[-\Delta\pi_{ij} + \theta_{ij}\pi + \nabla_{ki}\pi^k{}_j + \nabla_{kj}\pi^k_i] \tag{67a}$$

or in terms of the $\tilde{\Delta}$ operator of Eq. (65), using the π constraint, we get simply

$$\partial_0 R_{ij} = (g)^{-\frac{1}{2}}[-(\tilde{\Delta}\pi)_{ij} + \theta_{ij}\pi] \tag{67b}$$

The last two terms in Eq. (67 *a*) do not vanish by the constraints because of the order of the differentiations. However, one may use the Ricci identity to commute the latter and obtain

$$\begin{aligned}\partial_0 R_{ij} = & - \Delta\pi_{ij} + \theta_{ij}\pi + 3(\pi_{il}R^l{}_j + \pi_{jl}\pi^l{}_i) + R(g_{ij}\pi - \pi_{ij}) \\ & - 2(g_{ij}\pi_{lm}R^{lm} + \pi R_{ij})\end{aligned} \tag{68}$$

From Eqs (67) and (28 *b*) it is obvious that the metric is orthogonal, at every instant, to the rates of change of the variables:

$$g^{ij}\partial_0 R_{ij} = 0 = g_{ij}\partial_0\pi^{ij} \tag{69}$$

from which it follows that

$$\partial_0 R \equiv - 2\partial_0 G \equiv (\partial_0 g^{ij})R_{ij} + g^{ij}(\partial_0 R_{ij}) = - 2G_{ij}\pi^{ij} \tag{70a}$$

$$\partial_0\pi \equiv (\partial_0 g_{ij})\pi^{ij} + g_{ij}(\partial_0\pi^{ij}) = 2\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) = 2R = - 4G \tag{70b}$$

One may also easily verify the consistency of the Bianchi identities with the time development equations, namely that

$$\partial_0\nabla_j\pi^{ij} \equiv \nabla_j\partial_0\pi^{ij} + \pi^{lm}\partial_0\Gamma^i_{lm} = 0 \tag{71a}$$

and

$$\partial_0\left[gR - \left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right)\right] = 0 \tag{71b}$$

by virtue of Eqs (28*b*), (67).

The equations also take on very simple form when the variables have special values at a given instant. Thus, at an instant of time symmetry, $\pi^{ij} = 0$, we have

$$\partial_0 R_{ij} = 0 \quad , \quad \partial_0\pi^{ij} = - \sqrt{g}R^{ij}$$

and conversely, when the 3-space is instantaneously flat, $R_{ij} = 0$ and

$$\partial_0 R_{ij} = - \Delta\pi_{ij} + \theta_{ij}\pi = - \Delta\pi^{TT}_{ij} \quad , \quad \partial_0\pi^{ij} = - \frac{2}{\sqrt{g}}\left(\pi^i_m\pi^{mj} - \frac{1}{2}\pi\pi^{ij}\right)$$

where we have made use of the flat-space decomposition in the $\partial_0 R_{ij}$ equation. Clearly, if both R_{ij} and π^{ij} vanish at any moment, they continue to do so and space is flat. In fact, if R_{ij} alone vanishes at $t = 0$, space is flat, as the vanishing of π^{ij} can then be demonstrated from the constraint equations ([14]). This is in accord with the proposition that if E vanishes space is flat, since $R_{ij} = 0$ implies the existence of the frame $g_{ij} = \delta_{ij}$, and so that $h^T = 0$. Another interesting example of this proposition is furnished by stationary space-time, treated in the Appendix. One might wish to look for an effective action whose variations with respect to R_{ij} and π^{ij} yield the set of equations of motion (28*a*), (67) or (68). However, care must be taken because we have already used the constraints in some places. For this purpose, it is then best to go back to the original form of the equations, keeping N and N_i (whose variation yields the constraints). We do not perform this algebra here.

The relation of the equations for $(\partial_0 R_{ij}, \partial_0 \pi^{ij})$ to the original set has a parallel in electrodynamics. There, instead of the set $(\mathbf{E}^T, \mathbf{A}^T)$, whose dynamics is given by

$$\partial_0 \mathbf{A}^T = -\mathbf{E}^T \quad , \quad \partial_0 \mathbf{E}^T = -\triangle \mathbf{A}^T \quad , \quad H = \frac{1}{2}\int d^3r[\mathbf{E}^{T2} + (\nabla \times \mathbf{A}^T)^2]$$

one may employ the higher order variables ([15]) $(\triangle \mathbf{A}^T, \mathbf{E}^T)$ for which

$$\partial_0 \mathbf{E}^T = -(\triangle \mathbf{A}^T) \quad , \quad \partial_0(\triangle \mathbf{A}^T) = -\triangle \mathbf{E}^T \quad , \quad H' = \frac{1}{2}\int d^3r[(\triangle \mathbf{A}^T)^2 + (\nabla \times \mathbf{E}^T)^2]$$

Since the time-derivative does not commute with C-decomposition, or with, say, the Laplacian, one cannot simply conclude that $\partial_0 R_{ij}^{TT} = (\partial_0 R_{ij})^{TT}$ for example. Similarly, $-\triangle(\pi_{ij}^{TT}) \neq -(\triangle \pi_{ij})^{TT}$ in general and we cannot decompose the right side of Eq. (67*a*) very simply. One can only write $-\triangle \pi_{ij} + \theta_{ij}\pi = -\triangle(\pi_{ij}^{TT} + (LW)_{ij}) + \frac{1}{2}(\triangle \nabla_{ij} \triangle^{-1} - \nabla_{ij})\pi$ but the last term is not zero since $[\triangle, \nabla_{ij}] \neq 0$. It is therefore difficult to divide the time development equations into a set for the « TT » parts plus a remainder. Thus, Eqs (28*b*), (67) have not yet been reduced into a system involving only the unconstrained variables. To do so would require insertion of the C-decomposition solutions for π^{ij} and R_{ij} on both sides of the motion equations. While this is easily accomplished on the right side, the above mentioned lack of commutation between ∂_0 and the C-decomposition on the left makes it more difficult to obtain a simple result parallel to those of the linearized approximation, or of the N-breakup in a given frame.

([14]) These are examples of criteria for a system to be « flat » which are discussed more generally in ref. (8). The proof given there that, if $R_{ij} = 0$ everywhere, space is flat, can actually be extended to show that $R_{ij}(t=0) = 0$ is sufficient.

([15]) The higher order Hamiltonian and the corresponding one for our (R_{ij}, π^{ij}) system may perhaps be related to the Bel-Robinson tensor; whether such considerations might be useful for the problem of energy sign in not known.

VII. — DISCUSSION

In this section, a summary of results is followed by a number of rather speculative remarks concerning various open problems which may (or may not) be usefully approached from the present point of view. Such remarks are therefore to be considered as a list of unsolved questions rather than as necessarily meaningful suggestions towards their solution.

We have developed, in the foregoing, a formal method for reducing the solution of the highly non-linear constraint equations to a linear problem, at least as far as some of the functional properties are concerned. The explicit solution given is a formal one in the sense that we have by-passed the problem of solving the fourth constraint as a function of the metric, considering instead the Einstein tensor as the basic field variable, and performing the C-decomposition in terms of a metric which is arbitrary at this stage. In this way, a number of the functional properties of the constraints can be exhibited explicitly, namely, that the « spin 1 » parts of the two field variables vanish (covariantly) while one of the « spin 0 » parts has a prescribed value. To be sure, the mixing of spin 1 and 0 parts effected by the Ricci tensor made the explicit expression of the various projection operators a difficult task, but did not affect the linear character of the solution. The eventual problem of formulating the effect of the R constraint on the metric g_{ij} can, as far as these results are concerned, be held off without loss of consistency, since these are valid whether or not the « superpotentials » g_{ij} entering in the determining equations have been constructed so as to satisfy the equation $R(g) = \triangle\psi = \left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right)(g)^{-\frac{1}{2}}$. The time development equations also take on a particularly symmetric form when put in terms of $\partial_0 R_{ij}$ and $\partial_0 \pi^{ij}$, the right hand sides being simple polynomials in R_{ij} and π^{ij} (but also depending implicitly on the metric, of course).

It may perhaps be possible to exploit the $\pi^{ij} - G_{ij}$ symmetry to analyze special solutions. If, for example, π^{ij} is proportional to G_{ij}, say $G_{ij} = \frac{1}{3} C\pi_{ij}/\sqrt{g}$ where C is a constant (with dimensions L^{-1}), then if we now invoke the functional form of $G_{ij}(g_{lm})$, transversality of π_{ij} is automatically guaranteed. The trace condition becomes a simple quadratic relation among the π_{ij}, namely $\frac{\pi}{\sqrt{g}} = - C + [C^2 + 6(g)^{-1}\tilde{\pi}_{ij}\tilde{\pi}^{ij}]^{\frac{1}{2}}$, the plus sign on the root being required by the asymptotic boundary conditions (here

$\tilde{\pi}_{ij} \equiv \pi_{ij} - \frac{1}{3} g_{ij}\pi$). This relation is essentially a coordinate condition, namely a choice of π in terms of the traceless part $\tilde{\pi}_{ij}$. It implies, incidentally that the kinetic energy is positive for $C > 0$. If such a choice is possible initially it may then define an interesting class of solutions (not necessarily preserving this relation in time) which constitute one generalization of a plane wave in linear approximation, where π^{ij} and R_{ij} simply differ in phase (and C is the frequency, essentially).

We have noted that the C-decomposition was as « natural » to a curved space as the N-decomposition is to a flat one, in contrast to a classification with respect to a fixed spatial axis (e. g. the « axial gauge », in which derivatives in the same direction everywhere rather than with respect to the local gradient are used). This not only permitted us to solve the constraints but also insured that the solution (which specifies the vector and scalar parts of the tensor) has general validity, as against the possibility that the N-method's solution might not be valid for all possible field configurations in a given frame.

One of the purposes of this attempt was to investigate the still unsolved problem of whether the gravitational field's energy (which is invariantly defined with respect to arbitrary coordinate transformations vanishing at infinity) is positive-definite. It appears from our results that even the rather simple kinetic part is not (at least manifestly) positive-definite in general. While it may be that only the total energy is positive and that the demonstration requires the full solution of the R constraint in terms of the metric, one should perhaps not expect positiveness for all asymptotically flat spaces. A more modest physical demand is that isolated systems which lie on the same « manifold » of spaces as those of weak excitations have positive energy ([16]). Roughly, such spaces are those which reduce « smoothly » down to an everywhere weak gravitational field as certain parameters in them are appropriately altered. In contrast to these, one could conceive of spaces which are asymptotically flat, but sufficiently « pathological » in the interior not to have any such weak limit ([17]).

([16]) « Weak » gravitational fields are not to be confused with the linear approximation, since the constraints link different powers of the amplitude in a more subtle way. See for example, ref. (9). I am indebted to L. D. Faddeev for the ideas on manifolds of solutions.

([17]) However, such systems, if they existed, would not necessarily look pathological from far away. Indeed, general results on the existence of a Newtonian limit of isolated systems (ref. (12)) imply that one would only observe, asymptotically, a Newtonian force $\sim \nabla(\gamma E/r)$ from these. Of course, if for such system

Actually, it is not even known at present whether all weak field ([16]) solutions have positive energy. It has been shown [9] that time-symmetric ($\pi^{ij} = 0$) weak field solutions are positive. The pure kinetic term in the energy is also positive to second order in the weakness parameter by the linearized argument; however, it does not thereby follow that the total energy is positive, for the potential part now satisfies a different equation than if $\pi^{ij} = 0$ and it would have to be shown that it still yields a positive contribution. Thus weak field energy is still an important open question. It may also be that for spaces on the « manifold », conditions such as the existence of a minimal (or sufficiently minimal) surface may be met, thereby assuring positiveness more directly. It is known, for example, that spaces which are well-behaved for all time admit a $\pi = 0$ surface. Similarly, the curvature of « good » spaces may have special properties which give bounds on the type of integral we have encountered, e. g. $\int d^3r \nabla_i\psi R^{ij} \nabla_j\psi$. Of course, the requirement that a system never develop singularities is a very strong one, and in fact no asymptotically flat solution of the empty field equations with this property has yet been given—this being another basic open problem. One positive consequence of our analysis is worth stressing, however. This is the fact that the physical « TT » amplitudes enter in a positive and separate fashion in the energy expression.

Physically, one would expect that at least some ([18]) spaces which start out « almost flat » at $t = -\infty$ evolve in such a way that they come out at $t = +\infty$ without having acquired a singularity. This « S-matrix » idea would seem to be a minimal requirement for general relativity to obey if it is to be really like other field theories. Ostensibly, the energy, being constant, can be evaluated at a non-singular moment, and does not differentiate betweens systems which will or will not develop singularities in time, but there is certainly a relation between the energy and non-singularity questions. Hopefully, non-singular systems should have positive

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$E < 0$ or if $E = 0$ but the space is not flat, they would have rather unusual pro perties. However, there is nothing in the way E appears in the asymptotic force term to forbid negative values for it. For a discussion, from a quite different point of view, of what determines whether gravitation is attractive or repulsive, see ref. (16).

([18]) The reason one would not expect all weak initial situations to evolve without singularities is that the initial conditions might be such as to lead to a « focussing » of all the excitations in a small region at a later time, which would then undergo collapse because of the high density present there. Indeed, the whole question is essentially whether such condensation at a future time can be avoided with *any* incoming conditions.

energy, whether or not the converse holds (and general theorems on gravitational collapse [*10*] would imply that it does not). One simple analogy concerning the existence of different « branches » corresponding to different signs of energy may be instructive. It refers to a situation with sources, but this fact is immaterial to the argument. Consider a spherical distribution of dust with bare mass (i. e. mass in the absence of gravitation) m_0 and radius ε. It can be shown that ([19]), in constrast to Newtonian theory, where the clothed mass or energy m (including gravitational self-energy effects) is $m = m - \frac{1}{2}\gamma m_0^2 \varepsilon^{-1}$, the full theory predicts that $m = m - \frac{1}{2}\gamma m^2 \varepsilon^{-1}$ namely that the self-energy is that due to the total mass itself and not just to the bare mass. In the solution of this quadratic equation,

$$\gamma m = -\varepsilon \pm (\varepsilon^2 + 2\gamma m_0 \varepsilon)^{\frac{1}{2}},$$

the choice of sign determines whether m is positive or negative. The positive branch must be chosen if one wishes to recover the result $m \to m_0$ in the Newtonian limit (small m_0/ε), namely to have a solution on the correct « physical » branch.

In exploring the properties of general systems and the variation in their energy with small changes in the metric, techniques relating δR_{ij} to δg_{ij} may prove useful, particularly in discussing the « potential » part of the energy, which is hardest to deal with explicitly. With sufficient functional information on the initial data, one may be able, for example, to evaluate the variation $\delta_g E$ of the energy as a function of the metric to see whether for all spaces in which $E \geqslant 0$, $(E + \delta E) \geqslant 0$ as well, for example ([19 a]). In some sense, what is needed is a « semi-covariant » method, which distinguishes between physically acceptable frames preserving the asymptotically minkowskian boundary conditions and more general transformations under which invariance is not really desired. Since transformations preserving the physical boundary conditions are known [*12*] to form a group, this is a reasonable requirement, and might permit a useful decomposition of the

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([19]) The derivation of these results is to be found in ref. (11), the second paper of which gives a general coordinate-invariant discussion of the limiting situation $\varepsilon \to 0$.

([19 a]) Since this was written, D. R. Brill (private communication) has made an interesting attempt of this type to prove $E \geqslant 0$, at least when $\pi = 0$. He tries to show that $\delta_g E$ has an extremum only at the flat space value $R_{ij} = 0$. The weak field energy being positive in this case $\left(\delta_g^2 E > 0\right)$, the energy is everywhere positive if $E[g_{ij}, \pi^{ij}]$ has no othes « stationary point ».

variable h_{ij}, which cannot be C-decomposed, as it is only an « asymptotic » tensor.

To conclude this list of speculations, two rather different topics, connected with quantization, may be mentioned. The first is the problem of elementary fields of spin greater than one. It has been conjectured from general arguments [*13*], [*14*] that such theories cannot be consistent even in the absence of all interactions other than the unavoidable one with gravitation. The specific role played by the curvature in mixing subsidiary and dynamical components of such fields should be investigated ([20]). Finally, there are a number of problems in the quantization of the Einstein field itself which may most usefully be attacked in terms of a three-dimensionally covariant approach ([21]). For example, certain quantum consistency conditions are related to energy-momentum density commutators, and these is turn to commutations among the constraints R^{μ} of Eq. (27*b*). The latter can be obtained from general requirements using functional derivative techniques [*13*, *14*] and these basic commutators may perhaps then be amenable to further analysis in terms of the solutions of constraints.

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([20]) Some preliminary work along these lines has been done, for spin 2, by C. Latrémolière (private communication).

([21]) Four-dimensional quantum formulations of massless fields always have the drawback of an indefinite Hilbert space metric. For this especially complicated field, such methods can easily lead to spurious difficulties.

APPENDIX

We sketch here a proof ([22]), based on the energy and the Cauchy variables, that source—free stationary space is flat. First, it is shown that the energy vanishes, from which flatness will follow. This method is complementary to the classic proof of Lichnerowicz [7], based on a quite different « slicing » of the metric components and field equations.

The lack of time dependence (in appropriate coordinates) of a stationary metric means that π^{ij} becomes, by (28a),

$$\pi_{ij} = \frac{1}{2}\, \mathrm{N}^{-1}(2g_{ij}\nabla_l \mathrm{N}^l - \nabla_i \mathrm{N}_j - \nabla_j \mathrm{N}_i) \tag{A1}$$

where we drop all $\sqrt{g}$ factors since they are irrelevant to the calculation. The condition $\nabla^i\pi \;\; = 0$ may be integrated to give

$$\begin{aligned}\int \mathrm{N}_i \nabla_j \pi^{ij} d^3r = 0 = -\int d^3r \nabla_j \mathrm{N}_i \pi^{ij} = \int d^3r \mathrm{N}^{-1}\left[\frac{1}{4}(\nabla_i \mathrm{N}_j + \nabla_j \mathrm{N}_i)^2 - (\nabla_l \mathrm{N}^l)^2\right] \\ = \int d^3r \mathrm{N}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) = \int d^3r \mathrm{NR}\end{aligned} \tag{A2}$$

where the last two equalities follow, respectively, from the form (A1) and the R constraint. If the above π^{ij} is inserted into the trace of the other equation of motion (28b), we obtain the following equation for $\triangle \mathrm{N}$:

$$\triangle \mathrm{N} = \nabla_j[\mathrm{N}^{-1}\mathrm{N}^i(\nabla_j \mathrm{N}_i + \nabla_i \mathrm{N}_j)] \tag{A3}$$

This is a Poisson equation whose source has vanishing monopole moment (by the boundary conditions that $g_{0i} \sim \mathrm{O}(r^{-1})$, $\partial_i h_{\mu\nu} \sim \mathrm{O}(r^{-2})$). Hence the leading, $1/r$, term in N is in fact absent:

$$\mathrm{N} \sim 1 + \theta(r^{-2}) \sim -g_{00}$$

the last estimate following from the relation $g_{00} = \mathrm{N}_i\mathrm{N}^i - \mathrm{N}^2$. From this, we can show the energy vanishes. For, it follows from ref. (6) that the $1/r$ term of N is equal to that of $-\frac{1}{4}h^{\mathrm{T}}$ in stationary frames (as a particular case). On the other hand, $h^{\mathrm{T}} \sim \mathrm{E}/4\pi r$ in any asymptotically minkowskian frame. Thus, we have evaluated the invariant E to be zero by using a class of frames respecting $\partial_0 g_{\alpha\beta} = 0$. The conclusion that the energy vanishes may also be reached by comparing the curvature here to that of an exterior Schwarzschild solution. There, as is well-known, the leading $(1/r^3)$ terms in R^0_{i0j} and R_{ijkl} are proportional to the mass M (essentially through $\nabla_{ij}\mathrm{N}$ and R_{ij}). We have seen that $\mathrm{N} - 1$ starts as $\mathrm{O}(r^{-2})$ and in reasonable cases (it can actually be derived), we will have $\triangle_{ij}\mathrm{N} \sim \mathrm{O}(r^{-3-\varepsilon})$. On the other hand Eq. (28b) shows R_{ij} to behave as

$$\begin{aligned}\mathrm{R}_{ij} - \mathrm{N}^{-1}\nabla_{ij}\mathrm{N} = (2\mathrm{N}^2)^{-1}(\nabla_m \mathrm{N}_i \nabla^m \mathrm{N}_j - \nabla_i \mathrm{N}_m \nabla_j \mathrm{N}^m) \\ - (2\mathrm{N})^{-1}\nabla_m[\mathrm{N}^{-1}\mathrm{N}^m(\nabla_i \mathrm{N}_j + \nabla_j \mathrm{N}_i)]\end{aligned}$$

([22]) A slightly fuller version of this derivation is given in a note by the author in *C. R. Acad. Sci.*, Paris, t. 264, p. 805 (1967). A condensed account of the C-decomposition was summarized in *C. R. Acad. Sci.*, Paris, t. 264, p. 311 (1967).

which clearly also falls off faster than $1/r^3$ in the non-linear terms on the right. Thus the « Schwarzschild mass » M vanishes, as was already to be expected from the behavior of g_{00} in Eq. (A4).

We can now prove flatness from the vanishing « energy » integral (A2) by making its density positive-definite. This is done by removing the longitudinal part, $N^L_{,i}$ of N_i using a coordinate transformation preserving the stationary character and boundary conditions. By Eq. (A1), this is equivalent to a tranformation setting $\pi = 0$, and thus accounts for the positiveness. From the general class, $t \to t + f(x^i)$, $x^i \to x^i + h^i(x^l)$ we choose $h^i = 0$ and f such that the new N^i, $N'_i \equiv N_i + g_{00} f_{,i}$ is purely transverse,

$$\nabla^i N'_i \equiv \nabla^i N_i + \left(\nabla^i g_{00}\right)\left(\nabla_i f\right) + g_{00} \Delta f = 0$$

which is an elliptic equation for f in terms of N^L, with solutions preserving the asymptotic conditions ($f_{,i} \sim O(r^{-1})$. In the new frame, then, we have

$$\int d^3 r N^{-1} \sqrt{g} (\nabla_i N_j + \nabla_j N_i)^2 = 0$$

and hence, since N must have the same sign for all space (from its definition in terms of $g_{\mu\nu}$ given following (A4)),

$$\nabla_i N_j + \nabla_j N_i = 0 \tag{A5}$$

The equation (A3) now reads $\Delta N = 1$, whose only solution is $N = 1$. Finally, equation (28*b*) for R_{ij} then becomes $R_{ij} = 0$, and the 3-space is flat as well. To establish cartesian coordinates just involves choosing $g_{ij} = \delta_{ij}$, in terms of which (A5) implies that $N_i = 0$, so that $g_{\mu\nu} = \eta_{\mu\nu}$.

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