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Extremal invariant states

by

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ABSTRACT. — A number of results are derived which are pertinent to the description of physical systems by states on C*-algebras invariant under a symmetry group. In particular an integral decomposition of a state into states of lower symmetry is obtained which is relevant to the study of spontaneously broken symmetries which occur in equilibrium statistical mechanics as existence of crystals, ferromagnetic states, etc. A characterization is given of strongly clustering euclidean invariant states, and it is shown that they cannot be decomposed into states of lower symmetry.

1. INTRODUCTION

In recent papers [*1*] [*2*] [*3*] we began the analysis of the structure of invariant states over C*-algebras and the purpose of the present paper is to continue this analysis. The principal physical motivation for this programme is provided by statistical mechanics, both classical and quantum, where certain invariant states present themselves as natural candidates for the description of equilibrium [*4*]. A fuller description of the motivation and the mathematical concepts which we use is given in the references cited above. We proceed immediately to the introduction of various definitions and notations, which will then be used throughout the paper, and which allow us to describe more precisely the aims of the sequel.

We consider a C*-algebra $\mathcal{A}$ with identity, a topological group G with

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identity e, and a representation τ of G as automorphisms of $\mathcal{A}$ i. e. for every $g \in G$ there is an automorphism τ_g of $\mathcal{A}$,

$$A \in \mathcal{A} \rightarrow \tau_g A \in \mathcal{A}.$$

We almost always assume *strong continuity* of the automorphisms i. e.

$$\| \tau_g A - A \| \xrightarrow[g=e]{} 0, \qquad A \in \mathcal{A} \tag{1}$$

where $\| \quad \|$ is the algebraic norm. We denote by $\mathcal{A}^*$ the dual of $\mathcal{A}$ and by E the set of states (positive linear forms of norm 1) over $\mathcal{A}$ so $E \subset \mathcal{A}^*$. If $f \in \mathcal{A}^*$ we define $\tau'_g f$ by

$$\tau'_g f(A) = f(\tau_g^{-1} A) \qquad \text{for all} \quad A \in \mathcal{A}$$

and if $\tau'_g f = f$ for all g we say that f is G-invariant. Denoting by L_G the subspace of $\mathcal{A}$ generated by elements of the form $A - \tau_g A$ and by $L_G^\perp$ the weakly closed subspace of $\mathcal{A}^*$ defined by

$$L_G^\perp = \{ f \in \mathcal{A}^* : A \in L_G \Rightarrow f(A) = 0 \}$$

it is immediately clear that $f \in \mathcal{A}^*$ is G-invariant if, and only if, $f \in L_G^\perp$. Further, the G-invariant states over $\mathcal{A}$ are the elements of the convex (weakly) compact set $E \cap L_G^\perp$. We denote the extremal points of a subset K of $\mathcal{A}^*$ by $\mathcal{E}(K)$.

Now, using a well known method due to Gelfand, Naimark and Segal, it is possible to construct from a state $\rho \in E$ a representation of $\mathcal{A}$ by bounded operators $\pi_\rho(\mathcal{A})$, acting on a Hilbert space $\mathcal{H}_\rho$, with a normalized vector $\Omega_\rho \in \mathcal{H}_\rho$ which is cyclic for $\pi_\rho(\mathcal{A})$ in $\mathcal{H}_\rho$. The explicit connection of these various quantities is given by

$$\rho(A) = (\Omega_\rho, \pi_\rho(A)\Omega_\rho)$$

If $\rho \in E \cap L_G^\perp$ the above construction also yields a unitary representation U_ρ of G acting on $\mathcal{H}_\rho$ which is strongly continuous if (1) is satisfied and such that for all $g \in G$ and $A \in \mathcal{A}$

$$U_\rho(g)\Omega_\rho = \Omega_\rho \qquad \text{and} \qquad U_\rho(g)\pi_\rho(A)U_\rho(g)^{-1} = \pi_\rho(\tau_g(A)).$$

Thus Ω_ρ is a G-invariant vector in $\mathcal{H}_\rho$.

In this paper we will be principally interested in studying extremal G-invariant states i. e. states $\rho \in \mathcal{E}(E \cap L_G^\perp)$, a property that is equivalent to the property that the set of operators $\pi_\rho(\mathcal{A}) \cup U_\rho(G)$ is irreducible on $\mathcal{H}_\rho$. It was demonstrated in [2] [3] that for C*-algebras which are « asymptotically abelian » with respect to G, i. e. algebras with the property

$$\| [A_1, \tau_g A_2] \| \xrightarrow[g=\infty]{} 0, \qquad A_1, A_2 \in \mathcal{A} \quad \text{and} \quad g \in G,$$

and if certain separability conditions are satisfied, a state $\rho \in E \cap L_G^\perp$ can be uniquely expressed as an integral over states $\rho_k \in \mathcal{E}(E \cap L_G^\perp)$. Thus in such cases the study of states $\rho \in E \cap L_G^\perp$ is effectively reduced to a study of the extremal states.

Actually the above mentioned result was presented in [2] [3] for $G = R^n$ but the generalization to a large class of locally compact groups can be obtained by merely making appropriate notational changes in the proofs.

Our first new result, presented in Section 2, is an integral representation of a state $\rho \in \mathcal{E}(E \cap L_G^\perp)$ in terms of states in $\mathcal{E}(E \cap L_H^\perp)$ where H is a closed invariant subgroup of G such that G/H is compact. If the algebra $\mathcal{A}$ has the asymptotically abelian property with respect to H the decomposition is unique. This theorem is of interest in statistical mechanics in connection with the occurrence of spontaneously broken symmetries; it is a generalization of a theorem obtained in [3] for $G = R^n$. In Section 3 we specialize to the case where G is the Euclidean group in n dimensions and discuss strongly clustering euclidean invariant states. In Section 4 we establish properties of the spectrum of the n-dimensional translation groups for various cases of physical interest.

2. DECOMPOSITION THEOREM

In the following decomposition theorem we use the measure theoretic techniques introduced in [2]. As explained in [2] (for fuller details see [5]) a partial order $\succ$ may be introduced among the positive measures on the convex compact set $E \cap L_H^\perp$ such that $\mu_1 \succ \mu_2$ is equivalent to $\mu_1(\varphi) \geq \mu_2(\varphi)$ for all convex continuous functions φ on $E \cap L_H^\perp$ and thus $\mu_1(\psi) = \mu_2(\psi)$ for all continuous linear functions ψ on $E \cap L_H^\perp$. Thus if for $A \in \mathcal{A}$ we define the complex continuous linear function $\widehat{A}$ on $E \cap L_H^\perp$ by

$$\widehat{A}(\sigma) = \sigma(A), \qquad \sigma \in E \cap L_H^\perp$$

then a measure μ_ρ with the property $\mu_\rho \succ \delta_\rho$, where δ_ρ is the unit mass at ρ, provides the decomposition

$$\rho(A) = \widehat{A}(\rho) = \delta_\rho(\widehat{A}) = \mu_\rho(\widehat{A})$$

THEOREM 1 [1]. — *Let* G *be a topological group and* H *a closed invariant subgroup of* G *such that the quotient space* G/H *is compact and has a* G-*invariant measure* $d\dot{g}$ *normalized to* 1 *i. e.*

$$\int_{G/H} d\dot{g} = 1$$

Further, let $\mathcal{A}$ *be a* C*-*algebra with identity,* τ *a strongly continuous representation of* G *as automorphisms of* $\mathcal{A}$, *and take* $\rho \in \mathcal{E}(E \cap L_G^\perp)$. *Then there exists a measure* μ_ρ *concentrated on* $\mathcal{E}(E \cap L_H^\perp)$ *(and thus maximal for the order* $\succ$*) which majorizes the unit mass* δ_ρ *at* ρ. *Furthermore there exists a* $\tilde{\rho} \in \mathcal{E}(E \cap L_H^\perp)$ *such that for every continuous function* φ *on* $E \cap L_H^\perp$

$$\mu_\rho(\varphi) = \int_{G/H} d\dot{g}\ \varphi(\tau'_g\tilde{\rho}) \tag{2}$$

In particular for all $A \in \mathcal{A}$

$$\rho(A) = \int_{G/H} d\dot{g}\ \tilde{\rho}(\tau_g^{-1}A)$$

We remark firstly that the condition that the space G/H should have a G-invariant measure $d\dot{g}$ would automatically be satisfied if G and H were unimodular locally compact groups (see for instance [*6*]). More generally if G is locally compact and Δ_G, Δ_H are the modular functions of G and H respectively then a necessary and sufficient condition for the existence of $d\dot{g}$ is that

$$\Delta_G(h) = \Delta_H(h) \qquad \text{for all} \qquad h \in H$$

PROOF. — We begin by noting that for $\rho \in E \cap L_H^\perp$ fixed, $\hat{A}(\tau'_g\rho)$ is a continuous function on G and is invariant under right translations by H. Therefore it defines a continuous function on G/H. Next let us define the average $\langle \hat{A} \rangle$ of $\hat{A}$ by

$$\langle \hat{A} \rangle(\rho) = \int_{G/H} d\dot{g}\ \hat{A}(\tau'_g\rho)$$

and we now demonstrate that $\langle \hat{A} \rangle$ is a continuous function on $E \cap L_H^\perp$. We denote by $\dot{g}$ the image of $g \in G$ under the canonical mapping p of G onto G/H and write

$$\psi_\rho(\dot{g}) = \hat{A}(\tau'_g\rho)$$

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(1) We are very indebted to J. Ginibre for discussions of this theorem; the proof given here was, up to minor changes, written by him, and is reproduced with his permission.

Now for any $\varepsilon > 0$ there exists an open neighbourhood $N(g_1)$ of $g_1 \in G$ such that

$$\left\| \tau_g^{-1} A - \tau_{g_1}^{-1} A \right\| \leqslant \varepsilon/4 \qquad \text{for all} \quad g \in N(g_1).$$

As p is open the image $\dot{N}(\dot{g}_1) = pN(g_1)$ is an open neighbourhood of $\dot{g}_1$ in G/H and as for any $\dot{g} \in \dot{N}(\dot{g}_1)$ there exists $g \in p^{-1}(\dot{g}) \cap N(g_1)$ we have

$$| \psi_\rho(\dot{g}) - \psi_\rho(\dot{g}_1) | = \widehat{A}(\tau'_g\rho) - \widehat{A}(\tau'_{g_1}\rho) \leqslant \varepsilon/4$$

Now let $(\dot{N}(\dot{g}_i))_{i=1\ldots n}$ be a finite covering of the compact space G/H by such neighbourhoods (i. e. $\dot{N}(\dot{g}_i) = pN(g_i)$). Consider $\rho' \in E \cap L_H^\perp$ such that

$$\left| \widehat{A}\left(\tau'_{g_i}\rho\right) - \widehat{A}\left(\tau'_{g_i}\rho'\right) \right| \leqslant \varepsilon/2 \qquad \text{for} \quad i = 1 \ldots n$$

Since any $\dot{g} \in G/H$ must lie in $\dot{N}(\dot{g}_i)$ for some value of i, we have

$$| \psi_\rho(\dot{g}) - \psi_{\rho'}(\dot{g}) | \leqslant | \psi_\rho(\dot{g}) - \psi_\rho(\dot{g}_i) | + | \psi_\rho(\dot{g}_i) - \psi_{\rho'}(\dot{g}_i) | \\ + | \psi_{\rho'}(\dot{g}) - \psi_{\rho'}(\dot{g}_i) | \leqslant \varepsilon$$

Therefore $\psi_{\rho'}(\dot{g})$ tends uniformly to $\psi_\rho(\dot{g})$ as ρ' tends to ρ, this proves the continuity of $\langle \widehat{A} \rangle$ as a function of ρ.

We next define, for any $\rho \in E \cap L_H^\perp$, the average of ρ by

$$\langle \rho \rangle (A) = \langle \widehat{A} \rangle (\rho)$$

Clearly $\langle \rho \rangle \in E \cap L_G^\perp$ and if $\rho \in E \cap L_G^\perp$ then $\langle \rho \rangle = \rho$. We now consider a fixed $\rho \in \mathcal{E}(E \cap L_G^\perp)$ and define the set K_ρ by

$$K_\rho = \{ \sigma \quad ; \quad \sigma \in E \cap L_H^\perp \quad , \quad \langle \sigma \rangle = \rho \}$$

or, more explicitly,

$$K_\rho = \{ \sigma \quad ; \quad \sigma \in E \cap L_H^\perp \quad , \quad \langle \widehat{A} \rangle (\sigma) = \widehat{A}(\rho) \qquad \text{for all} \quad A \in \mathcal{A} \}$$

For every $A \in \mathcal{A}$ the set

$$\{ \sigma \quad ; \quad \sigma \in E \cap L_H^\perp \quad , \quad \langle \widehat{A} \rangle (\sigma) = \widehat{A}(\rho) \}$$

is closed, because of the continuity of $\langle \widehat{A} \rangle$. Therefore K_ρ is a closed subset of $E \cap L_H^\perp$ and hence compact. On the other hand K_ρ is convex and not empty so it has extremal points. We next show that if $\tilde{\rho} \in \mathcal{E}(K_\rho)$

then $\tilde{\rho} \in \mathcal{E}(E \cap L_H^\perp)$. In fact suppose that $\tilde{\rho} \notin \mathcal{E}(E \cap L_H^\perp)$ then there exist $\tilde{\rho}_1$, $\tilde{\rho}_2 \in E \cap L_H^\perp$ and λ real $(0 < \lambda < 1)$ such that $\tilde{\rho} \neq \tilde{\rho}_1$ and

$$\tilde{\rho} = \lambda\tilde{\rho}_1 + (1 - \lambda)\tilde{\rho}_2, \qquad \tilde{\rho} \in \mathcal{E}(K_\rho).$$

At least one of $\tilde{\rho}_1$, $\tilde{\rho}_2$ cannot lie in K_ρ and we suppose $\tilde{\rho}_1 \notin K_\rho$. Hence $\rho \neq \langle \tilde{\rho}_1 \rangle$. On the other hand

$$\rho = \langle \tilde{\rho} \rangle = \lambda \langle \tilde{\rho}_1 \rangle + (1 - \lambda) \langle \tilde{\rho}_2 \rangle$$

which is in contradiction with the assumption that $\rho \in \mathcal{E}(E \cap L_G^\perp)$. Therefore $\tilde{\rho} \in \mathcal{E}(E \cap L_H^\perp)$.

Now let φ be a continuous function on $E \cap L_H^\perp$ i. e. $\varphi \in \mathcal{C}(E \cap L_H^\perp)$. Then $\varphi(\tau_g' \rho)$ is a continuous function on G and is invariant under right translations by H. We can therefore define

$$\mu_\rho(\varphi) = \int_{G/H} d\dot{g}\ \varphi(\tau_g' \tilde{\rho})$$

With this definition μ_ρ is a positive linear functional on $\mathcal{C}(E \cap L_H^\perp)$ and hence a positive measure. If $\varphi \in \mathcal{C}(E \cap L_H^\perp)$ vanishes on $\mathcal{E}(E \cap L_H^\perp)$ then $\varphi(\tau_g' \tilde{\rho}) = 0$ for all $g \in G$ and $\mu_\rho(\varphi) = 0$. Therefore μ_ρ is concentrated on $\mathcal{E}(E \cap L_H^\perp)$. Finally for any $A \in \mathcal{A}$ we have $\mu_\rho(\widehat{A}) = \langle \widehat{\rho} \rangle (A) = \rho(A)$ which concludes the proof.

Thus we have now established the existence of a measure $\mu_\rho \succ \delta_\rho$ which is not only maximal with respect to the order relation but is also concentrated on the extremal points $\mathcal{E}(E \cap L_H^\perp)$. If $\mathcal{A}$ is asymptotically abelian with respect to H, the maximal measure μ_ρ is unique due to the results of [*2*] [*3*] and hence the decomposition (2) is unique.

We now turn our attention to properties of extremal invariant states for more specific cases of physical interest.

3. EUCLIDEAN INVARIANT STATES

The first result we derive is independent of algebraic structure.

LEMMA (2). — *Let* $U : (a, R) \to U(a, R)$ *be a strongly continuous unitary representation in the Hilbert space* $\mathcal{H}$ *of the euclidean group of* R^ν, $\nu \geq 2$.

(2) This lemma is related to some results obtained in quantum field theory for the Lorentz group, see for instance Borchers [7], lemma 4.

If Φ, $\Psi \in \mathcal{H}$ *and if* P_0 *is the projection on the subspace of* $\mathcal{H}$ *formed by the vectors invariant under* $U(R^\nu, 1)$, *then*

$$\lim_{a\to\infty} (\Psi, U(a, 1)\Phi) = (\Psi, P_0\Phi) \tag{1}$$

PROOF. — For clarity we shall use in this proof the functional notation $dp\mu(p)$ instead of $d\mu(p)$ for a measure μ.

By Stone's theorem, there exists a projection-valued measure E on $\mathbf{R}^\nu$ such that

$$U(a, 1) = \int dpe^{-ipa}E(p) \tag{2}$$

The measure $\mu = (\Psi, E\Phi)$ then satisfies

$$\begin{aligned}
\int dp \mid \mu(R^{-1}p) - \mu(p) \mid &= \int dp \mid (\Psi, [E(R^{-1}p) - E(p)]\Phi) \mid \\
&= \int dp \mid (U(o, R)\Psi, E(p)U(o, R)\Phi) - (\Psi, E(p)\Phi) \mid \\
&\leqslant \int dp \mid ([U(o, R) - 1]\Psi, E(p)U(o, R)\Phi) \mid \\
&+ \int dp \mid (\Psi, E(p)[U(o, R) - 1]\Phi) \mid \\
&\leqslant \| U(o, R)\Psi - \Psi \| . \| \Phi \| + \| U(o, R)\Phi - \Phi \| . \| \Psi \|
\end{aligned} \tag{3}$$

Thus, given $\varepsilon > 0$ there exists a neighbourhood $\mathcal{N}$ of 1 in the orthogonal group in ν dimensions such that if $R \in \mathcal{N}$, then

$$\int dp \mid \mu(R^{-1}p) - \mu(p) \mid < \varepsilon \tag{4}$$

Because the orthogonal group is compact, we may choose $\mathcal{N}$ invariant under inner automorphisms.

We may choose a C^∞ function $\varphi \geqslant 0$ with support in $\mathcal{N}$, invariant under inner automorphisms of its argument and such that $\int \varphi(R)dR = 1$. We have then

$$\int dp \left| \int dR\varphi(R)\mu(R^{-1}p) - \mu(p) \right| < \varepsilon \tag{5}$$

If we can prove that

$$\lim_{a\to\infty} \int dpe^{-ipa} \int dR\varphi(R)\mu(R^{-1}p) = (\Psi, P_0\Phi) \tag{6}$$

then, (1) will result from (5), (6) and

$$(\Psi, U(a, 1)\Phi) = \int dp e^{-ipa}\mu(p) \tag{7}$$

We have

$$\int dp e^{-ipa} \int dR\varphi(R)\mu(R^{-1}p)$$
$$= \int dp\mu(p) \int dR\varphi(R)e^{-i(Rp)a} \tag{8}$$

where

$$\int dR\varphi(R)e^{-i(Rp)a} = f(pa) \tag{9}$$

depends only on the scalar product pa because of the assumed invariance of φ under inner automorphisms of its argument. We may write

$$f(pa) = \int dp' e^{-ip'a} \int dR\varphi(R)\delta(p' - Rp) \tag{10}$$

and for $p \neq 0$ fixed

$$\int dR\varphi(R)\delta(p' - Rp) = \delta(|p'| - |p|)\psi(p') \tag{11}$$

where $\psi \in \mathcal{D}(R^\nu)$. Since the Fourier transform of $\delta(|\,.\,| - |p|)$ is continuous and tends to zero at infinity and the Fourier transform of ψ is absolutely integrable, it follows from (10) that f tends to zero at infinity. Since $f(0) = 1$, we see from (8), (9) that

$$\lim_{a\to\infty} \int dp e^{-ipa} \int dR\varphi(R)\mu(R^{-1}p) = \lim_{a\to\infty} \int dp\mu(p)f(pa)$$
$$= \mu(\{0\}) = (\Psi, P_0\Phi) \tag{12}$$

which proves (6) and therefore the lemma.

Now, with the aid of the above lemma, we can deduce the following theorem concerning the structure of euclidean invariant states.

THEOREM 2. — *Let $\mathcal{A}$ be a C*-algebra with identity,* G *the euclidean group of* R^ν, $\rho \in E \cap L_G^\perp$, τ *a representation of* G *as automorphisms of $\mathcal{A}$ such that the corresponding unitary representation* U_ρ *of* G *in* $\mathcal{H}_\rho$ *is strongly continuous. The following conditions are equivalent.*

1. $\lim_{a\to\infty} \rho(A_1\tau_{(a,1)}A_2) = \rho(A_1)\rho(A_2)$ *for* $A_1, A_2 \in \mathcal{A}$
2. Ω_ρ *is the unique vector in* $\mathcal{H}_\rho$ *invariant under* $U_\rho(R^\nu, 1)$ *and imply the following equivalent conditions*

3. $\rho \in \mathcal{E}(E \cap L_H^\perp)$.
4. $\pi_\rho(\mathcal{A}) \cap U_\rho(H)$ *is irreducible*

for any closed non compact subgroup H *of* G. *Conversely, if* $\mathcal{A}$ *is asymptotically abelian with respect to* R^ν *and* $H \subset R^\nu$, *then* 3. (or 4.) *implies* 1. (or 2.).

The equivalence 1. $\Leftrightarrow$ 2. follows directly from the Lemma and [*3*] and the equivalence 3. $\Leftrightarrow$ 4. is of general nature (see [*2*] and [*3*] for characterizations of extremal invariant states). The irreducibility of $\pi_\rho(\mathcal{A}) \cup U_\rho(H)$ for $H \subset R^\nu$ implies the irreducibility of $\pi_\rho(\mathcal{A}) \cup U_\rho(R^\nu)$ and therefore if $\mathcal{A}$ is asymptotically abelian 4. $\Rightarrow$ 2.

We conclude the proof of the theorem by showing that 1. $\Rightarrow$ 4. Since H is closed non compact we can choose a sequence (a_i, R_i) of elements of H such that $a_i \to \infty$, and since the orthogonal group is compact we may assume (possibly going to a subsequence) that $R_i \to R_0$. If $\Phi \in \mathcal{H}_\rho$ is invariant under $U_\rho(H)$, we have then $U_\rho(a_i, R_i)\Phi = \Phi$ or $U_\rho(o, R_i)\Phi = U_\rho(-a_i, 1)\Phi$. By construction

$$\lim_{i \to \infty} U_\rho(o, R_i)\Phi = U_\rho(o, R)\Phi \ \text{ strongly}$$

On the other hand the assumption 1 and the lemma yield

$$\lim_{i \to \infty} U_\rho(-a_i, 1)\Phi = P_0\Phi \ \text{ weakly}$$

Therefore

$$\Phi = U_\rho(o, R^{-1})P_0\Phi = P_0\Psi$$

and the subspace of vectors $\Phi \in \mathcal{H}_\rho$ invariant under $U_\rho(H)$ reduces to the scalar multiples of Ω_ρ. It follows then by standard arguments that

$$[\pi_\rho(\mathcal{A}) \cup U_\rho(H)]' = [\pi_\rho(\mathcal{A}) \cup U_\rho(R^\nu, 1)]' = \{ \lambda I \}$$

so that 1. implies 4.

The above theorem is similar to theorems given in [*1*], [*2*] and [*3*] for invariant states with $G = R^n$. The major difference between the above theorem and the previous theorems is that the strong cluster property 1. has replaced the weak cluster property which characterizes the states $\rho \in \mathcal{E}(E \cap L_{R^n}^\perp)$. It is interesting to note that as Theorem 2 establishes that a euclidean invariant state ρ with the strong cluster property is such that $\rho \in \mathcal{E}(E \cap L_H^\perp)$ for all non-compact subgroups $H \subset G$ there can be no (non-trivial) decomposition of ρ of the type derived in theorem 1. Thus one might say that the « natural » invariance of such a state is the full euclidean invariance. Conversely, if $\mathcal{A}$ is asymptotically abelian, $\rho \in \mathcal{E}(E \cap L_G^\perp)$

but is not strongly clustering, then there must exist a non-trivial decomposition of ρ into states with a lower invariance.

In [3] extremal invariant states over an asymptotically abelian algebra were studied with $G = R^n$ and a classifications E_I, E_{II} and E_{III} of such states was introduced. In the light of the above discussion this classification can be understood as follows. An E_I-state has « natural » invariance under all translations $G = R^n$; an E_{II}-state can be decomposed uniquely into states with « natural » invariance under a subgroup of translations, $H = R^{n-n_1} \times Z^{n_1}$ with $0 < n_1 \leq n$; an E_{III}-state can be decomposed with respect to many different subgroups of G but no decomposition leads to states with a « natural » invariance. Actually this classification was introduced in [3] through consideration of the spectral properties of the unitary operators $U_\rho(R^n)$ associated with a state $\rho \in \mathcal{E}(E \cap L^\perp_{R^n})$. In the next section we derive further properties of this type.

4. SPECTRUM PROPERTIES

THEOREM 3 *a.* — *Let* $G = R^{\nu_1} \times Z^{\nu_2}$ *and* $\rho \in E \cap L^\perp_G$. *We assume that the representation* U_ρ *is strongly continuous. Let* E *be the projection-valued measure on* $R^{\nu_1} \times T^{\nu_2}$ *such that*

$$U_\rho(g) = \int dE(p) e^{-ip\cdot g}$$

1. *Let* $\mathcal{A}$ *be abelian. Then,* supp E = − supp E.

2. *Let* $\mathcal{A}$ *contain a dense subset* $\tilde{\mathcal{A}}$ *such that,* if $A_1, A_2 \in \tilde{\mathcal{A}}$, *then* $[A_1, \tau_g A_2] = 0$ *for* g *outside of some compact.*

If $S \subset R^\nu$ *has Lebesgue measure zero and* $E(S) \neq 0$, *then* $E(-S) \neq 0$.

3. *Let* $\mathcal{A}$ *be asymptotically abelian and* $S_d = \{ p \in R^\nu : E(\{ p \}) \neq 0$. *Then* $S_d = -S_d$.

4. *Let* $\mathcal{A}$ *be asymptotically abelian and* $\rho \in \mathcal{E}(E \cap L^\perp_G)$, *then* $S_d + S_d \subset S_d$ *and if* $p \in S_d$, *then* $E(\{ p \})$ *is one-dimensional.*

To prove 1. and 2. we use the fact that, if $A \in \mathcal{A}$, the Fourier transform of the measure

$$\mu(p) = (\pi_\rho(A)\Omega_\rho, E(p)\pi_\rho(A)\Omega_\rho) - (\pi_\rho(A)^*\Omega_\rho, E(-p)\pi_\rho(A)^*\Omega_\rho)$$

is $\rho([A^*, \tau_g A])$. In case 1. we have thus $\mu = 0$. In case 2., for $A \in \tilde{\mathcal{A}}$, μ is an analytic function.

To prove 3. we use lemma 1 of [3] which establishes, for a certain filter of functions f_α, that

$$0 = \lim_{\alpha\to\infty} \int dg f_\alpha(g) e^{-ipg} \rho([A^*, \tau_g A])$$
$$= (\pi_\rho(A)\Omega_\rho, E(\{p\})\pi_\rho(A)\Omega_\rho) - (\pi_\rho(A)^*\Omega_\rho, E(\{-p\})\pi_\rho(A)^*\Omega_\rho)$$

Statement 4. has been included for completeness : it was already proved in [3] (Theorem 4).

With slightly stronger continuity assumptions we have

THEOREM 3 *b* (3). — *Let* $G = R^{\nu_1} \times Z^{\nu_2}$ *and* $\rho \in \mathcal{E}(E \cap L_G^\perp)$. *We assume that for all* $A \in \mathcal{A}$, $\lim_{g\to 0} \| \tau_g A - A \| = 0$. *Let* E *be the projection-valued measure on* $R^{\nu_1} \times T^{\nu_2}$ *such that*

$$U_\rho(g) = \int dE(p) e^{-ipg}$$

If $\mathcal{A}$ *is asymptotically abelian, then* supp E + supp E $\subset$ supp E.

If $A \in \mathcal{A}$, let $p \to \widehat{A}(p)$ be the Fourier transform of $g \to \tau_g A$. If $A_\varphi = \int dg \varphi(g)^* \tau_g A$, then supp $\widehat{A}_\varphi \subset$ supp $\widehat{\varphi}$.

Let now $p_1, p_2 \in$ supp E, and $\mathcal{N}_i$ be a neighbourhood of p_i. One may choose A_i of the form A_φ such that supp $\widehat{A}_i \subset \mathcal{N}_i$ and $\pi_\rho(A_i)\Omega_\rho \neq 0$ (because $\pi_\rho(A_i)\Omega_\rho$ is of the form $E(\widehat{\varphi^*})\pi_\rho(A)\Omega_\rho$).

If $A = A_1\tau_h A_2$ then, for every $h \in R^\nu$, supp $\widehat{A} \subset \mathcal{N}_1 + \mathcal{N}_2$ hence

$$E(\mathcal{N}_1 + \mathcal{N}_2)\pi_\rho(A_1\tau_h A_2)\Omega_\rho = \pi_\rho(A_1\tau_h A_2)\Omega_\rho. \tag{1}$$

We have on the other hand from [3] (lemma 1) that $f_\alpha \geq 0$ exists such that

$$\lim_{\alpha\to\infty} \int dh f_\alpha(h) \| \pi_\rho(A_1\tau_h A_2)\Omega_\rho \|^2$$
$$= \lim_{\alpha\to\infty} \int dh f_\alpha(h) \rho(A^*_2 . \tau_{-h}(A^*_1 A_1) . A_2)$$
$$= \lim_{\alpha\to\infty} \int dh f_\alpha(h) \rho(A^*_2 A_2 . \tau_{-h}(A^*_1 A_1))$$
$$= \rho(A^*_1 A_1)\rho(A^*_2 A_2) \neq 0$$

which shows that, for some h, $\pi_\rho(A_1\tau_h A_2)\Omega_\rho \neq 0$.

(1) implies then that $E(\mathcal{N}_1 + \mathcal{N}_2) \neq 0$, hence that $p_1 + p_2 \in$ supp E, which proves the proposition.

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(3) The proof of this lemma is based on a technique used in relativistic field theory. See Wightman [8], p. 30.

The above theorems have not established that supp E is in general symmetric and, indeed, asymmetry can arise but then the following result is valid.

THEOREM 3 *c.* — *Let* $G = R^\nu$, $\rho \in E \cap L_G^\perp$ *and* U_ρ *be strongly continuous. Let* E *be the projection-valued measure on* R^ν *such that*

$$U_\rho(g) = \int dE(p)e^{-ipg}$$

If $E(\{0\})$ *is one-dimensional and* E *is concentrated on* $\{0\} \cup S$, *where* $E(-S) = 0$, *then* $\pi_\rho(\mathcal{A})$ *is irreducible, i. e.* $\rho \in \mathcal{E}(E)$.

A simple proof is given in [*9*], p. 65.

Thus we see that the asymetry of the spectrum implies that ρ is extremal among all states (a pure state), a situation which is typical of quantum field theory, but which would arise only exceptionally in statistical mechanics.

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