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Section A :
Physique théorique.

A maximum entropy principle in general relativity and the stability of fluid spheres (1)

by

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SUMMARY. — The principle that the total entropy be a maximum is applied to gravitational systems of a perfect, relativistic fluid in adiabatic motion. An expression for the entropy density of the fluid is derived as an explicit function of the energy density, and the conservation equations imply that this expression satisfies the proper continuity equation. The entropy principle is shown to yield the equation of hydrostatic equilibrium and a stability criterion equivalent to one derived from dynamics. Application to features of equilibrium models is studied.

SOMMAIRE. — Le principe d'entropie maximum est appliqué au système gravitationnel : fluide relativiste parfait en mouvement adiabatique. On trouve une expression de la densité d'entropie, fonction explicite de la densité d'énergie. Les équations de conservation entraînent que cette fonction satisfait identiquement à une équation de continuité. On déduit du principe d'entropie maximum l'équation d'équilibre hydrostatique, et un critère de stabilité équivalent à celui qui découle de la dynamique. Des applications aux modèles d'équilibre sont étudiées.

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INTRODUCTION

This paper deals chiefly with setting up a maximum entropy principle for gravitationally bound spheres of perfect fluid, treated *via* Einstein's general theory of relativity. We will derive an expression for the total entropy of such a fluid sphere in terms of the pressure, energy density, and gravitational potentials and require that this expression be a *maximum* with respect to small perturbations conserving the total energy of the fluid.

We will show that this requirement implies both the relativistic equation of hydrostatic equilibrium and a criterion for the stability of static configurations of fluid satisfying this equation.

Considerable interest will lie in the stability criterion, and it will be shown that it is equivalent to one previously obtained by S. Chandrasekhar from dynamical considerations. We will also use it to elucidate certain features of actual equilibrium models. In particular, we will consider how unstable configurations are related to the location of maxima and minima of plots of equilibrium mass curves, parametrized as functions of the central pressure or density.

Throughout the paper we consider all motions to be adiabatic, so that we neglect heat flow, but we make no other assumption as to the form of the equation of state, except that it be a function of the usual thermodynamic variables. We will limit our discussion to stability against spherically symmetric perturbations, although the formalism could be extended to include other types.

In general there are two methods of testing the stability of a static configuration of fluid, both of which have been applied to the Newtonian case. We will now briefly discuss these two methods and indicate the connection between them.

The first method is known as the « dynamical method » and has been applied to both Newtonian and relativistic fluid mechanics [*1*, *2*]. It involves assuming that the quantities ψ_j characterizing the state of the fluid system deviate, as functions of positions and time, only very slightly from a set of given static quantities ψ_{0j}. One may then take the time-dependent equations of motion to be linear equations with the small differences $\psi_j - \psi_{0j} = \delta\psi_j$ as dependent variables.

If the motions are assumed to be adiabatic, and hence reversible, the time-dependance can be separated out by assuming a functional dependence on $\vec{r}$, t of the form $\delta\psi_j = \delta\psi_j(\vec{r}) \cos \sigma t$, where σ is a constant. The equations for $\delta\psi_j(\vec{r})$ can then be transformed into a Sturm-Liouville eigenvalue problem, with σ^2 as eigenvalue. See in particular section I of this paper, where we present a sketch of the dynamical method in general relativity.

In general the admissible eigenvalues σ^2 form a discrete, countably infinite set, and stability is tested by ascertaining whether or not all the σ^2's are positive. If not, then some of the σ's are imaginary, and excitation of these modes will result in an exponential time-dependence instead of an oscillatory one. This will carry the configuration so far away from the given static state that the assumed linear approximation will become invalid. A static configuration is then stable with respect to small perturbations only if all the eigenvalues σ^2 associated with it are positive.

The second method is known as the « energy method » and has so far only been applied in Newtonian theory [*3*, *4*, *5*]. It states the following: if the total conserved energy E of a mass of fluid can be written as a sum $E = T + U$, where T is a positive definite kinetic energy and U is a potential energy which does not depend on the hydrodynamic velocities, then a configuration is in stable equilibrium only if U is a *minimum* with respect to infinitesimal variations of the quantities ψ_j which conserve the *total amount of material* (in Newtonian theory, the total mass). That this is necessary for the stability of a static distribution of material is evident, since otherwise any lessening of the potential energy U by a small perturbation would release kinetic energy and provide further impetus to the displacement.

In terms of the calculus of variations, requiring U to be a minimum means that we must have $\delta U = 0$ and $\delta^2 U > 0$, where δU is the first variation and $\delta^2 U$ is the second variation. $\delta U = 0$ implies an Euler equation, which turns out to be simply the hydrostatic equilibrium equation, and $\delta^2 U > 0$ is then a stability criterion.

The connection between the dynamical method and the energy method is made as follows: it turns out [*6*] that a necessary and sufficient condition for $\delta^2 U > 0$ for non-zero variations is that the eigenvalues μ of a particular eigenvalue problem all be positive. This eigenvalue problem is in general not quite the same as the one associated with the dynamical method (as discussed above), but one should be able to show that all the μ's are positive if and only if all the σ^2's are positive. This would show that the two stability criteria are in fact the same. This has been done in the Newtonian case [*5*, *6*].

In general relativity, however, the formulation of the energy method is complicated by the apparent lack of a quantity characterizing the « total amount of material ». In Newtonian mechanics this quantity is simply the total mass of the fluid, but in relativity the fundamental equivalence of mass and energy implies that one could alter the total mass of a body without actually removing or adding any « material ». Many authors have used the concept of a total particle number or so-called « proper mass », but this usage seems foreign to hydrodynamics, which is a theory of continuous fields [7].

We may, however, turn to the thermodynamic aspect of the problem and investigate the requirement that the total entropy of the system be a *maximum* for stability. It is this principle which, as stated above, we will develope in this paper, and with which we propose to replace the minimum potential energy method.

Since we consider only adiabatic motions, our expression for the total entropy S will turn out to be a constant of the motion. We will derive an expression for the entropy density in terms of quantities entering directly into Einstein's field equations, and then verify from the field equations that it satisfies the proper continuity equation. An expression for the conserved total entropy S will then follow. This will give us a non-trivial integral of Einstein's field equations for general adiabatic fluid motions.

We will next show that the statement $\delta S = 0$, taken relative to variations which conserve the total energy, implies the relativistic equation of hydrostatic equilibrium. The additional maximum requirement $\delta^2 S < 0$ will then be shown to lead to the same stability criterion obtained by Chandrasekhar from the dynamical method. In the last section, the maximum entropy principle will then be applied to some other interesting questions.

At this point we might say a few words regarding the assumption that no heat flow occurs. It is of course evident that this postulate is never exactly satisfied in nature, but we may regard it as approximately true over appropriately short intervals of proper time relative to an observer moving with the fluid. In situations involving very strong gravitational fields, such as in a gravitational collapse problem, this approximation would appear to be *a propos*, in as much as proper times inside the fluid would be much slowed down compared to the proper times of external observers. However, we will not here investigate the validity of our assumptions, but will content ourselves with exploring their consequences. It would be interesting to see how the methods of this paper might be extended to include the flow of heat and radiation coupled with the matter fields.

We use the following notation and conventions. The expression for the interval length will be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

with Greek indices taking on the values 0, 1, 2, 3. Latin indices will be understood to assume the values 1, 2, 3. Thus the fluid 4-velocity $u^\mu = dx^\mu/ds$ satisfies $u_\mu u^\mu = 1$.

We choose units so that the velocity of light $c = 1$ and the Newtonian gravitational constant $G = 1$. Thus we will write the field equations as

$$S^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -8\pi T^{\mu\nu}$$

where $S^{\mu\nu}$ is the Einstein tensor, which satisfies the conservation identity $\nabla_\alpha S^{\mu\alpha} = 0$, ∇_α denoting the operation of covariant differentiation, and $T^{\mu\nu}$ is the energy-momentum tensor.

The energy-momentum tensor of the perfect fluid without heat flow is then expressed in these units as

$$T^{\mu\nu} = (p + \varepsilon) u^\mu u^\nu - g^{\mu\nu} p,$$

where ε is the total proper energy density, p is the pressure, and u^μ is the hydrodynamic flow 4-velocity.

1. — THE DYNAMICAL METHOD IN GENERAL RELATIVITY

To bring into focus some of the ideas of the introduction, we here present a sketch of the dynamical method of testing the stability of fluid spheres in general relativity. We have mentioned that we will only consider stability against spherically symmetrical perturbations, and since any static configuration of fluid will exhibit such symmetry, we way conveniently take this as a starting point. It will be useful to employ spherical polar coordinates and to define the radius parameter r by using the metric form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^\nu dt^2 - e^\lambda dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \tag{1-1}$$

ν and λ being real functions of r and t.

Einstein's field equations $S^\mu_\nu = -8\pi T^\mu_\nu$ then give [8], using a prime to denote $\partial/\partial r$, for $\mu = \nu = 1$,

$$\frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) = 8\pi[(p + \varepsilon) u_1 u^1 - p] \tag{1-2}$$

and for $\mu = \nu = 0$,

$$\frac{1}{r^2} + e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) = 8\pi[(p + \varepsilon)u_0 u^0 - p]$$
$$= 8\pi[\varepsilon + (p + \varepsilon)e^{\lambda}(u^1)^2] \tag{1-3}$$

where we have used $u_\mu u^\mu = 1$.

Integration of Eq. (1-3) yields

$$e^{-\lambda} = 1 - \frac{8\pi}{r}\int_0^r [\varepsilon + (p + \varepsilon)e^{\lambda}(u^1)^2]r^2 dr. \tag{1-4}$$

For $\mu = 1$, $\nu = 0$, we obtain

$$- e^{-\lambda}\dot{\lambda}/r = 8\pi(p + \varepsilon)u_0 u^1 \tag{1-5},$$

where the dot indicates $\partial/\partial t$. The equation for $\mu = \nu = 2$ would provide a fourth equation (There are four independent unknowns, which may be taken to be ε, u^1, λ, and ν, p being given as a function of the other variables. See section II). However, it is more convenient to use the $\mu = 1$ component of the conservation equation $\nabla_\alpha T^{\mu\alpha} = 0$. We shall not write it out in full, but rather refer the reader to the literature [9].

We now denote

$$\varepsilon(r, t) - \varepsilon_0(r) = \delta\varepsilon, \quad p(r, t) - p_0(r) = \delta p, \quad \lambda(r, t) - \lambda_0(r) = \delta\lambda, \text{ etc.},$$

where ε_0, p_0, λ_0, ν_0 are solutions of the static equations

$$r\nu_0' = e^{\lambda_0} - 1 + 8\pi e^{\lambda_0} r^2 p_0 \tag{1-2'}$$

$$r\lambda_0' = 1 - e^{\lambda_0} + 8\pi e^{\lambda_0} r^2 \varepsilon_0 \tag{1-3'}$$

or

$$e^{-\lambda_0} = 1 - \frac{8\pi}{r}\int_0^r \varepsilon_0 r^2 dr \tag{1-4'}$$

and of the static part of the conservation equation $\Delta_\alpha T^{1\alpha} = 0$, which reads

$$p_0' = - \nu_0'(p_0 + \varepsilon_0)/2. \tag{1-6}$$

Equation (1-6) is the equation of hydrostatic equilibrium, which we will derive again from the maximum entropy principle in section III. It is, of course, the analogue of the Newtonian equation $dp/dr = - \rho d\varphi/dr$, ρ being the mass density and φ the gravitational potential.

Following Chandrasekhar [1], we assume that $\delta\varepsilon$, δp, $\delta\lambda$, $\delta\nu$, and u^1 are small quantities defined on the interval $0 \leqslant r \leqslant R_0$, where R_0 is the boundary radius of the static distribution; and we write to first order of smallness

$$r\delta\nu' = e^{\lambda_0}\delta\lambda + 8\pi r^2 e^{\lambda_0}(\delta p + p_0\delta\lambda) \tag{1-2''}$$

$$e^{-\lambda_0}\delta\lambda = \frac{8\pi}{r}\int_0^r \delta\varepsilon r^2 dr. \tag{1-4''}$$

Letting $\xi(r, t)$ be the « Lagrangian displacement » of an element of the fluid from an initial position, we have to first order

$$\xi = \int_{t_0}^{t} \frac{dr}{dt} dt \simeq \int_{t_0}^{t} e^{\nu_0/2} \frac{dr}{ds} dt = e^{\nu_0/2} \int_{t_0}^{t} u^1 dt,$$

and integrating Eq. (1-5), we get, with $u_4 \simeq e^{\nu_0/2}$,

$$e^{-\lambda_0}\delta\lambda = -\ 8\pi(p_0 + \varepsilon_0) r\xi. \tag{1-5''}$$

From $\nabla_\alpha T^{1\alpha} = 0$, to first order, one may obtain the relation

$$e^{\lambda_0 - \nu}(p_0 + \varepsilon_0)\ddot{\xi} + \delta p' + \frac{1}{2}(p_0 + \varepsilon_0)\delta\nu' + \frac{1}{2}\nu_0'(\delta p + \delta\varepsilon) = 0.$$

Since this equation is linear in the dependent variables and contains only the second time derivative, we may separate variables by assuming a time-dependence of the form $\cos \sigma t$, σ being a constant, getting

$$\delta p' + \frac{1}{2}(p_0 + \varepsilon_0)\delta\nu' + \frac{1}{2}\nu_0'(\delta p + \delta\varepsilon) = \sigma^2 e^{\lambda_0 - \nu_0}(p_0 + \varepsilon_0)\xi, \tag{1-7}$$

where we now consider $\delta\lambda$, $\delta\nu$, δp, $\delta\varepsilon$, and ξ to be functions of r only.

In the next section we show that for adiabatic motions we may write p in the form $p = p(\varepsilon, x)$, where the variable x indexes the fluid element itself and follows its motion. See also reference [*1*]. Then we have in our case

$$\delta p \simeq \frac{\partial p}{\partial \varepsilon}\delta\varepsilon - \left[\frac{\partial p}{\partial x}\right]_{x=r} \xi$$

and we may use this relation to substitute for δp in Eq. (1-7), and use Eq. (1-2″) (1-4″) and (1-5″) to eliminate other variables, finally writing Eq. (1-7) with ξ as dependant variable

$$\xi'' + Q\xi' + (R + P\sigma^2)\xi = 0,$$

where one can check to see that $P \geqslant 0$. Multiplication by the quantity $q(r) = \exp\left[\int^r Q(r')dr'\right]$ allows us to write the form

$$\frac{d}{dr}[q(r)\xi'] + [t(r) + v(r)\sigma^2]\xi = 0, \tag{1-8}$$

where $q(r)$, $v(r) \geqslant 0$. The boundary conditions on ξ are arranged so that the pressure vanishes to first order on the new boundary radius $R_0 + \xi(R_0)$. This condition reads

$$\delta p(R_0) + p_0'(R_0)\xi(R_0) = 0, \tag{1-9}$$

which the reader may easily write himself in terms of ξ. At $r = 0$ we must have of course $\xi = 0$.

Equations (1-8) and (1-9) constitute a Sturm-Liouville eigenvalue problem The problem is of the singular type since q and v may be shown to tend to 0 at both ends of the interval $0 \leqslant r \leqslant R_0$. However, one may show [*2*] that in spite of the singular character of the problem, there exists an infinite discrete set of eigenvalues $\{\sigma_j^2\}$, of which there is a least member σ_0^2.

It is easily shown [*2*, *10*] that we may solve Eq. (1-8) for σ^2 as

$$\begin{aligned}\sigma^2 &= \int_0^{R_0}[q(\xi')^2 - t(\xi)^2]dr \Big/ \int_0^{R_0} v(\xi)^2 dr \\ &= \mathfrak{D}[\xi]/\mathcal{H}[\xi],\end{aligned}$$

thus defining the functionals $\mathfrak{D}[\xi]$, $\mathcal{H}[\xi]$. The boundary terms vanish because $q(0) = q(R_0) = 0$. According to theorems on the extremum properties of eigenvalues [*2*, *10*] the lowest eigenvalue σ_0^2 is given by

$$\sigma_0^2 = \min_{\varphi} \mathfrak{D}[\varphi]/\mathcal{H}[\varphi],$$

where φ ranges over all functions with bounded, piece-wise continuous φ' which satisfy the boundary conditions for ξ and do not vanish identically. The condition

$$\sigma_0^2 = \min_{\varphi} \mathfrak{D}[\varphi]/\mathcal{H}[\varphi] > 0$$

for stability was first applied to this problem by S. Chandrasekhar [*1*]. Note that since q, $v \geqslant 0$, if $t \leqslant 0$ for all $0 \leqslant r \leqslant R_0$, then $\sigma_0^2 > 0$ would follow immediately, and any such configuration would be dynamically stable.

Having studied the application of the dynamical method in general relativity, we now proceed to develope our replacement of the energy method, the maximum entropy principle.

2. — THE TOTAL ENTROPY

In the introduction we stated the need for expressing the total entropy S of the fluid sphere in terms of variables already entering into the field equations. To accomplish this end, we now construct a relation between the proper entropy density s per unit volume and the pressure p and energy density ε.

First let V be the volume of a very small element of the fluid. Since we consider all motions to be adiabatic, the total entropy $S = sV$ of the element will be constant over such motions, and we may write the thermodynamic identity for the element as $dE + pdV = TdS = 0$; or, since $E = \varepsilon V$,

$$d\varepsilon = -(p + \varepsilon)dV/V.$$

Now since a change in volume is the only means of changing the thermodynamic state of the fluid element, heat flow having been outlawed, it follows that any change in state will be accompanied by a non-zero change in ε, except for elements in which p and ε are both zero simultaneously. Thus, except for such « empty » elements, we may say all the thermodynamic variables may be written as functions of the energy density ε, including of course the pressure. However, the equation of state $p = f(\varepsilon)$ will then in general be a different function of ε for different fluid elements.

Let us now consider the use of a co-moving coordinate system x^μ in which the spatial components u^j ($j = 1, 2, 3$) of the fluid 4-velocity vanish throughout the system for all time. Such coordinates are presumably always possible since they may be defined by simply attaching the spatial reference system to the identifiable points of the fluid.

By virtue of the above arguments we may now write the pressure as a function of ε and of the spatial coordinates x^j of the co-moving frame: $p = p(\varepsilon, x^j)$, the x^j serving to index the different points of the fluid; i. e., the different fluid elements. It must be undertood that the function $p(\varepsilon, x^j)$ is to be derived from purely thermodynamic considerations after specifying the initial pressure and energy density distributions at a time $x^0 = (x^0)_0$.

But since the total entropy of the fluid element $S = sV$ is conserved, we may form $dS = d(sV) = sdV + Vds = 0$, or $-dV/V = ds/s$. Hence, from above,

$$d\varepsilon/[p(\varepsilon, x^j) + \varepsilon] = -dV/V = ds/s.$$

Integration with $x^j =$ constant yields

$$\ln s = \int^{\varepsilon} [p(z, x^j) + z]^{-1} dz + \text{constant},$$

or >

$$s = s_0 \exp \left\{ \int_{\varepsilon_0}^{\varepsilon} [p(z, x^j) + z]^{-1} dz \right\} = s(\varepsilon, x^j), \tag{2-1}$$

where s_0 and ε_0 are functions of x^j such that $s(\varepsilon_0) = s_0$.

We have thus expressed the entropy density s as a definite function of the energy density ε and the spatial coordinates x^j of the co-moving frame x^μ. Let it be emphasized again that this expression is valid only for adiabatic changes of state.

Note that of it is possible to define a conserved particle number for the system such that n is the proper particle number density, the total particle number $N = nV$ of a small fluid element would also be conserved; and likewise, $- dV/V = dn/n$. Hence we would similarly have

$$n = n_0 \exp \left\{ \int_{\varepsilon_0}^{\varepsilon} [p(z, x^j) + z]^{-1} dz \right\} = n(\varepsilon, x^j),$$

and thus n and s would be proportional to the same function of ε for fixed x^j.

As an example let us find an expression for the entropy density of a perfect non-relativistic Boltzmann gas with constant specific heat. We may write the equation of state for an element of such a gas as $pV = NkT$, or $p = nkT$. For adiabatic compression or expansion we further have, with γ as the constant ratio of specific heats per particle $\gamma = c_p/c_V$ [11], $pV^\gamma =$ constant or $p =$ constant n^γ, or $n = n_0(p/\bar{p}_0)^{1/\gamma}$, where n_0 and $\bar{p}_0$ are functions of x^j. Then the energy density is given by

$$\varepsilon = (m_0c^2 + \bar{\varepsilon}_0 + c_V T)n = (m_0c^2 + \bar{\varepsilon}_0)n + c_V p/k,$$

where m_0 is the rest mass per particle and $\bar{\varepsilon}_0$ is an arbitrary constant. Therefore

$$\varepsilon = (m_0c^2 + \bar{\varepsilon}_0)n_0(p/\bar{p}_0)^{1/\gamma} + c_V p/k, \tag{2-2}$$

which definies $p = p(\varepsilon, x^j)$ as an implicit function. Then

$$\int_{\varepsilon_0}^{\varepsilon} [p(\varepsilon, x^j) + \varepsilon]^{-1} d\varepsilon = \int_{p_0}^{p} \frac{(\partial\varepsilon/\partial p)dp}{p + \varepsilon(p, x^j)} = \frac{1}{\gamma} \ln \left(\frac{p}{p_0}\right),$$

where we have used $c_p - c_V = k$. It then follows that

$$s = s_0[p(\varepsilon, x^j)/p_0(x^j)]^{1/\gamma} = f(x^j)n(\varepsilon, x^j).$$

We have thus verified the proportionality of s and n.

The total entropy of the fluid element may be written [11]

$$\mathrm{S} = \mathrm{N}k \ln (1/n) + \mathrm{N}c_{\mathrm{V}} \ln k\mathrm{T} + (k + k\zeta + c_{\mathrm{V}})\mathrm{N},$$

where ζ is the chemical constant of the gas. Hence

$$\begin{aligned} s = \mathrm{S/V} &= -nk \ln n + nc_{\mathrm{V}} \ln (p/n) + (k + k\zeta + c_{\mathrm{V}})n \\ &= n \ln n[(\gamma - 1)c_{\mathrm{V}} - k] + [c_{\mathrm{V}} \ln (p_0 n_0^{-\gamma}) + k(1 + \zeta) + c_{\mathrm{V}}]n \\ &= g(x^j)n = s_0(p/p_0)^{1/\gamma}. \end{aligned}$$

This verifies the expression for s derived above from Eq. (2-1).

We must now show that our expression for s satisfies the proper conservation equation $\nabla_\alpha(su^\alpha) = 0$. Let us write the scalar $\nabla_\alpha(su^\alpha)$ in terms of our co-moving frame in which only u^0 is non-vanishing :

$$\nabla_\alpha(su^\alpha) = u^\alpha\partial_\alpha s + s\nabla_\alpha u^\alpha = u^0\partial_0 s + s\nabla_\alpha u^\alpha,$$

where ∂_μ indicates $\partial/\partial x^\mu$. Now we have from Eq. (2-1) that

$$s = s(\varepsilon, x^j) = s_0 \exp \left\{ \int_{\varepsilon_0}^{\varepsilon} [p(z, x^j) + z]^{-1} \, dz \right\},$$

so that

$$\partial_0 s = \frac{\partial s}{\partial \varepsilon}\partial_0\varepsilon = s\partial_0\varepsilon/(p + \varepsilon),$$

and hence

$$\begin{aligned} \nabla_\alpha(su^\alpha) &= s\left(\frac{u^0\partial_0\varepsilon}{p + \varepsilon} + \nabla_\alpha u^\alpha\right) = s\left(\frac{u^\alpha\partial_\alpha\varepsilon}{p + \varepsilon} + \nabla_\alpha u^\alpha\right) \\ &= \frac{s}{p + \varepsilon}[u^\alpha\partial_\alpha\varepsilon + (p + \varepsilon)\nabla_\alpha u^\alpha]. \end{aligned} \tag{2-3}$$

But this is a tensor equation, and hence is true not only in the co-moving frame but in any system of coordinates whatever. Now the field equations imply that

$$\begin{aligned} \nabla_\alpha \mathrm{T}^{\mu\alpha} &= \nabla_\alpha[(p + \varepsilon)u^\mu u^\alpha - g^{\mu\alpha}p] = 0 \\ &= u^\mu u^\alpha\partial_\alpha(p + \varepsilon) + (p + \varepsilon)(u^\alpha\nabla_\alpha u^\mu + u^\mu\nabla_\alpha u^\alpha) - g^{\mu\alpha}\partial_\alpha p. \end{aligned}$$

Let us further construct the scalar product $u_\mu\nabla_\alpha\mathrm{T}^{\mu\alpha} = 0$, using the relation $u_\mu u^\mu = 1$ and its consequent $u_\mu\nabla_\alpha u^\mu = 0$:

$$\begin{aligned} u_\mu\nabla_\alpha\mathrm{T}^{\mu\alpha} &= u^\alpha\partial_\alpha(p + \varepsilon) + (p + \varepsilon)\nabla_\alpha u^\alpha - u^\alpha\partial_\alpha p \\ &= u^\alpha\partial_\alpha\varepsilon + (p + \varepsilon)\nabla_\alpha u^\alpha = 0. \end{aligned}$$

Comparison with Eq. (2-3) then shows that in fact, if the field equations are satisfied, $\nabla_\alpha(su^\alpha) = 0$.

We may use this relation to show that the total proper entropy of the fluid system is a conserved quantity. Let t be a time-like coordinate and x, y, z be space-like coordinates (not necessarily Galilean at infinity). Let Σ_F be the space-like surface of the mass of fluid, and let Σ_a and Σ_b be the intersections of the time-like surfaces $t = a$ and $t = b$, respectively, with the space-time volume of the fluid. Then if $\tilde{\Sigma}_F$ be the part of Σ_F which lies between Σ_a and Σ_b, the surfaces Σ_a, Σ_b, and Σ_F together form a closed 3-space Σ in space-time.

Now let V denote the space-time volume of fluid enclosed by Σ. Then for an arbitrary vector field A^μ, we may write Green's theorem for this volume and surface as [12]

$$\iiiint_V \nabla_\alpha A^\alpha dv_{(4)} = \iiint_\Sigma A^\alpha n_\alpha dv_{(3)},$$

where n_μ is the outward unit normal to Σ, and $dv_{(3)}$ and $dv_{(4)}$ are generalized 3- and 4-volume elements.

If $f(x^\mu) = 0$ is the equation of Σ, then we have

$$n_\mu = \pm \partial_\mu f / | \partial_\alpha f \partial^\alpha f |^{\frac{1}{2}},$$

with the sign chosen so that n_μ is directed outward from Σ. Inside V we can let $dv_{(4)} = \sqrt{-g}\ dt\,dx\,dy\,dz$ and on Σ_a and Σ_b we can set $dv_{(3)} = \sqrt{-g_{(3)}}\ dx\,dy\,dz$, where $g_{(3)}$ is the determinant of the subtensor g_{ij} $(i, j = 1, 2, 3)$.

Then if we choose $A^\mu = su^\mu$, we will have $\nabla_\alpha A^\alpha = \nabla_\alpha(su^\alpha) = 0$ inside V, and consequently, by Green's theorem,

$$\iiint_\Sigma su^\alpha n_\alpha dv_{(3)} = 0.$$

Now the equation for Σ_a is $f(x^\mu) = t - a = 0$, so that on Σ_a, for $a < b$,

$$n_\mu = - \delta^0_\mu / | g^{\alpha\beta} \delta^0_\alpha \delta^0_\beta |^{\frac{1}{2}} = - \delta^0_\mu / \sqrt{g_{00}}.$$

Similarly, on Σ_a,

$$n_\mu = + \delta^0_\mu / \sqrt{g_{00}}.$$

Thus it follows that, since $g^{00} = g_{(3)}/g$,

$$\iiint_\Sigma su^\alpha n_\alpha dv_{(3)} = \iiint_{\Sigma_b} su^0 \sqrt{-g}\,dx\,dy\,dz$$
$$- \iiint_{\Sigma_a} su^0 \sqrt{-g}\,dx\,dy\,dz + \iiint_{\tilde{\Sigma}_F} su^\alpha n_\alpha dv_{(3)} = 0.$$

Further, we must also satisfy the Lichnerowicz junction conditions [13], from which one may easily deduce that on the surface Σ_F, $u_\alpha n^\alpha = 0$. This can also be shown by expressing u^μ and n_μ in co-moving coordinates, in which the equation for Σ_F will have the form $f(x^j) = 0$.

Thus we obtain

$$\iiint_{\tilde{\Sigma}_F} s u^\alpha n_\alpha dv_{(3)} = 0,$$

and hence

$$\iiint_{\Sigma_b} s u^0 \sqrt{-g}\, dx\, dy\, dz = \iiint_{\Sigma_a} s u^0 \sqrt{-g}\, dx\, dy\, dz$$

for all $t = a, b$. Or,

$$S(t) = \iiint_{\Sigma_t} s u^0 \sqrt{-g}\, dx\, dy\, dz = \text{constant}, \tag{2-4}$$

where Σ_t is the 3-space $t =$ constant. The integral S may be said to represent the total entropy in the same sense that conservation of charge in electrodynamics in the form $\nabla_\alpha(\rho u^\alpha) = 0$, where ρ is the charge density, leads to an exactly similar expression for the total conserved charge.

We now proceed to formulate our variational principle S = maximum with respect to variations which conserve the total energy.

3. — THE FIRST VARIATION

We now require that for hydrostatic equilibrium the total entropy S be an extremum (maximum) for all infinitesimal adiabatic variations of the energy density ε and pressure p which conserve the total energy of the system. As stated, we will restrict ourselves to variations which preserve the spherical symmetry of the fluid, and thus we will be able to use the expressions for the metric form and for the field equations presented in section I.

It should be stated at the outset that in performing the variations we will not consider the variations $\delta g_{\mu\nu}$ as independent from $\delta\varepsilon$ and δp, but will rather let them be given *via* the field equations (1-2″) (1-4″) and (1-5″). We shall *not* use the equations $T_2^2 = -8\pi S_2^2$ and $\nabla_\alpha(T^{1\alpha}) = 0$, since they themselves imply the equation of hydrostatic equilibrium. Further, the variation δp will be linked to $\delta\varepsilon$ by means of the adiabatic equation of state $p = p(\varepsilon, x^j)$, x^j representing, as it were, the point of origin of the fluid element. Thus we will have only one independent variational function.

Let us also note that even though the variations are adiabatic, they will not conserve the total entropy identically, since we will always consider the spatial velocity of the material to be zero.

In order to carry out our variations we will need an expression for the total energy. It is easily shown that for the metric from (1-1), a conserved energy may be written in the forms [14]

$$\begin{aligned} M &= 4\pi\int_0^R T_0^0 r^2 dr = 4\pi\int_0^R [(p+\varepsilon)u_0u^0 - p]r^2 dr \\ &= 4\pi\int_0^R [\varepsilon + (p+\varepsilon)e^{\lambda}(u^1)^2]r^2 dr. \end{aligned}$$

Thus for the static problem we take $u^1 = 0$ and write

$$M = 4\pi\int_0^R \varepsilon r^2 dr. \tag{3-1}$$

Using Eq. (2-4), one may then express the total entropy, after performing elementary integrations, as

$$S = 4\pi\int_0^R s\frac{dt}{ds}e^{(\nu+\lambda)/2}r^2 dr.$$

But with $u^1 = dr/ds = 0$, we have from Eq. (1-1) that $dt/ds = e^{-\nu/2}$. Hence

$$S = 4\pi\int_0^R se^{\lambda/2}r^2 dr. \tag{3-2}$$

However, in taking the variations of M and S, we must allow for the fact that the boundary radius R will also vary. Thus it behooves us to transform the independent variable r so that the condition M = constant automatically fixes the end points of the interval of integration. We do this by transforming to the variable

$$m = m(r) = 4\pi\int_0^r \varepsilon r^2 dr = r[1 - e^{-\lambda(r)}]/2,$$

which represents the quantity of energy inside the radius r.

We denote dr/dm by $\dot{r}$ (we trust there will be no confusion with a time derivative in this context) and find $dm/dr = 4\pi r^2\varepsilon$, or $dr/dm = \dot{r} = (4\pi r^2\varepsilon)^{-1}$, and thus

$$\varepsilon = \varepsilon(r, \dot{r}) = (4\pi r^2\dot{r})^{-1} \tag{3-3}.$$

Also we have

$$\lambda = \lambda(r, m) = -\ln(1 - 2m/r) \tag{3-4}.$$

We may hence express the integrand of S as a function of the independent variable m and the dependent variable r and its derivative $\dot{r}$. The principle $\delta S = 0$ will then determine $r = r(m)$ from the Euler equation, and the condition $M =$ constant will be equivalent to fixing the upper end point of the interval of integration $0 \leqslant m \leqslant M$.

However, careful note must be made of the fact that the dependence of the entropy density $s = s(\varepsilon, x^j)$ on the spatial variables x^j is such that x^j always follows the motion of the fluid element. Thus in reality, we have for the spherically symmetric problem $s = s(\varepsilon, r - \xi)$, where ξ is the displacement of the element of fluid which finds itself at r after having been moved from the point $r - \xi$. ξ is then given in terms of the other variables through the relation (1-5″). Let us denote $x = r - \xi$, so that $s = s(\varepsilon(r,\dot{r}), x)$.

Our next step would be to write the variation of S in terms of the variation $\delta r = \delta r(m)$ of the dependent variable r. However, we must first be able to express ξ in terms of δr. Here $\delta r(m)$ is the infinitesimal change of the radius $r = r(m)$ of a sphere of fixed energy content m, and not the Lagrangian displacement, which we have designated as ξ.

Let us denote the change in energy content $m(r)$ of a sphere of radius r by $\delta m(r) = 4\pi \int_0^r \delta\varepsilon r^2 dr$. Then writing $m_0(r)$ as the equilibrium distribution function around which the variation is taken, we have by definition

$$m = m_0(r) + \delta m(r) = m_0(r - \delta r(m)) \cong m_0(r) - (dm_0/dr)\delta r(m),$$

and hence

$$\begin{aligned}\delta m(r) &= -(dm_0/dr)\delta r(m)\\ &= -4\pi\varepsilon r^2 \delta r\end{aligned}$$

to first order.

Thus, using Eq. (1-4″) and (1-5″), we obtain

$$\delta r = -e^{-\lambda}(8\pi r\varepsilon)^{-1}\delta\lambda = (p + \varepsilon)\xi/\varepsilon. \tag{3-5}$$

We now write

$$\begin{aligned}S &= 4\pi \int_0^R s(\varepsilon, x)e^{\lambda/2}r^2 dr\\ &= 4\pi \int_0^M s(\varepsilon(r, \dot{r}), x)e^{\lambda(r,m)/2}r^2\dot{r}dm\\ &= 4\pi \int_0^M L(\dot{r}, r, x, m)dm.\end{aligned}$$

The first variation of S will thus be expressed as

$$\delta S = 4\pi \int_0^M \left[\frac{\partial L}{\partial \dot{r}} \delta \dot{r} + \frac{\partial L}{\partial r} \delta r + \frac{\partial L}{\partial x} (\delta r - \xi) \right] dm$$
$$= 4\pi \int_0^M \left[L_r - \frac{d}{dm}(L_{\dot{r}}) + L_x\left(1 - \frac{\varepsilon}{p+\varepsilon}\right) \right] \delta r dm + [L_{\dot{r}} \delta r]_0^M.$$

According to the fundamental lemma of the calculus of variations [15], the condition $\delta S = 0$ implies the Euler equation

$$L_r - d(L_{\dot{r}})/dm + L_x p/(p + \varepsilon) = 0, \tag{3-6}$$

and the condition $L_{\dot{r}}]_M = 0$ since in general $\delta r(M) \neq 0$, although of course $\delta r(0) = 0$.

We first investigate the boundary term: We have

$$L_{\dot{r}} = e^{\lambda/2} r^2 \left[\frac{\partial s}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial \dot{r}} \dot{r} + s \right],$$

and using Eq. (3-3), which gives $\partial \varepsilon / \partial \dot{r} = -(4\pi r^2 \dot{r}^2)^{-1} = -\varepsilon / \dot{r}$ one obtains easily $L_{\dot{r}} = e^{\lambda/2} r^2 s p (p + \varepsilon)^{-1}$.

Thus if the energy density does not vanish at the surface, we will have $L_{\dot{r}}]_M = 0$ through the condition that the pressure on the surface vanish. But suppose $\varepsilon = 0$ also. Then we may write $L_{\dot{r}} = e^{\lambda/2} r^2 p \partial s / \partial \varepsilon$, and demand simply that $\partial s / \partial \varepsilon$ remain finite as $\varepsilon \to 0$. This is certainly a very reasonable stipulation, and one can show, for example, that for equations of state of the form $\varepsilon = \alpha p + \beta p^{1/\gamma}$ (see Eq. (2-2)), $\partial s / \partial \varepsilon$ stays finite as $\varepsilon \to 0$ for positive α, β, γ if and only if $\gamma > 1$. However, one may also show that the product $p(\varepsilon) \partial s / \partial \varepsilon$ goes to zero as $\varepsilon \to 0$ for all $\gamma > 0$, so that $L_{\dot{r}}]_M = 0$ still tells us very little about the equation of state.

We now proceed to examine the Euler equation (3-6). Let us write $d(L_{\dot{r}})/dm = (dr/dm) d(L_{\dot{r}})/dr = (4\pi r^2 \varepsilon)^{-1} d(L_{\dot{r}})/dr$ and express Eq. (3-6) as a total differential equation with r as independent variable as before, again using a prime to denote d/dr. We have

$$L_r = e^{\lambda/2} s \left\{ \left[\frac{1}{2} \frac{\partial \lambda}{\partial r} + (p + \varepsilon)^{-1} \frac{\partial \varepsilon}{\partial r} \right] r^2 + 2r \right\} \dot{r},$$

where Eq. (3-3) and (3-4) give

$$\frac{\partial \lambda}{\partial r} = -\frac{2m}{r}\left(1 - \frac{2m}{r}\right)^{-1} = (1 - e^{\lambda})/r$$

and

$$\frac{\partial \varepsilon}{\partial r} = -2(4\pi r^3 \dot{r})^{-1} = -2\varepsilon/r.$$

Further,

$$\begin{aligned} d(\mathrm{L}_r^{\cdot})/dm &= (4\pi r^2\varepsilon)^{-1}d(\mathrm{L}_r^{\cdot})/dr \\ &= (4\pi r^2\varepsilon)^{-1}e^{\lambda/2}\{ s[r^2 p\lambda'(p+\varepsilon)^{-1}/2 + r^2\varepsilon p'(p+\varepsilon)^{-2} \\ &\quad + 2rp(p+\varepsilon)^{-1}] + r^2(\partial s/\partial x)p(p+\varepsilon)^{-1}\}. \end{aligned}$$

But

$$\begin{aligned} \mathrm{L}_x p(p+\varepsilon)^{-1} &= e^{\lambda/2}r^2(\partial s/\partial x)\dot{r}p(p+\varepsilon)^{-1} \\ &= (4\pi r^2\varepsilon)^{-1}e^{\lambda/2}r^2(\partial s/\partial x)p(p+\varepsilon)^{-1}, \end{aligned}$$

so that the terms in $\partial s/\partial x$ will cancel. Then after using Eq. (1-3′) and collecting terms, we may write Eq. (3-6) as

$$rp'(p+\varepsilon)^{-1} + (e^\lambda - 1 + 8\pi e^\lambda r^2 p)/2 = 0,$$

which with the aid of Eq. (1-2′) may be written $p'(p+\varepsilon)^{-1} + \nu'/2 = 0$, which we recognize as Eq. (1-6), the equation of hydrostatic equilibrium.

Thus we have obtained Eq. (1-6) by means of the maximum entropy principle and with the help of the equations $S^0_0 = -8\pi T^0_0$, $S^1_1 = -8\pi T^1_1$, and $S^1_0 = -8\pi T^1_0$. We now proceed to examine the second variation $\delta^2 S$ and require that it be negative.

4. — THE SECOND VARIATION AND EQUIVALENCE WITH THE DYNAMICAL METHOD

Now that we have found a necessary condition (the Euler equation) for S to be an extremum, the principle S = maximum requires us to investigate $\delta^2 S < 0$ for infinitesimal variations about the extremum. We will show that this requirement is equivalent to be criterion developed by Chandrasekhar by means of the dynamical method. It will be shown in fact that if σ_0^2 be the least eigenvalue satisfying Eq. (1-8), then $\sigma_0^2 > 0$ if and only if $\delta^2 S < 0$ for all non-zero infinitesimal variations conserving total energy and satisfying the usual continuity and boundary conditions.

We will use again the independent variable m as developped in the last section to accommodate variations in the boundary radius R and to comply automatically with M = constant by fixing the upper end point of the interval of integration.

The second variation δ^2S is given by [6, 16]

$$\delta^2 S = 4\pi \int_0^M [L_{\dot{r}\dot{r}}(\delta\dot{r})^2 + L_{rr}(\delta r)^2 + L_{xx}(\delta r - \xi)^2 + 2L_{\dot{r}r}\delta\dot{r}\delta r + 2L_{\dot{r}x}\delta\dot{r}(\delta r - \xi) + 2L_{rx}\delta r(\delta r - \xi)]dm.$$

Substitution of δr for ξ *via* Eq. (3-5) then allows *us* to write δ^2S in the form

$$\delta^2 S = 8\pi \int_0^M \Omega(\delta\dot{r}, \delta r)dm,$$

thus defining the quadratic form $\Omega(\delta\dot{r}, \delta r)$.

A rather simple test to be satisfied for S to be a maximum is the so-called *Legendre-Weierstrass* test, which is a necessary condition for $\delta^2S < 0$ with respect to both strong and weak variations. It is a local test and is written [16, 17]

$$L_{\dot{r}\dot{r}}[\dot{r} + \theta(\dot{\rho} - \dot{r}), r, r, m] \leqslant 0,$$

where r and $\dot{r}$ are actual extremal values, $\dot{\rho}$ is any physically realizable value of $\dot{r}$, and θ is any number between zero and one. The equality sign can hold only for isolated points.

But

$$L_{\dot{r}\dot{r}} = -4\pi e^{\lambda/2} r^4 \varepsilon^3 s(p + \varepsilon)^{-2} \partial p/\partial\varepsilon,$$

where the dependence on $\dot{r}$ is through $\varepsilon = \varepsilon(r, \dot{r})$, and hence for positive ε the test is satisfied if $\partial p/\partial\varepsilon > 0$, which certainly holds for any real one-phase fluid. This also shows that S certainly cannot be a minimum.

We now proceed to formulate a more general test (related to the so-called *Jacobi* test [18]) which includes the *Legendre-Weierstrass* test in its formulation and which we will show is equivalent to Chandrasekhar's criterion.

First we demonstrate that a necessary and sufficient condition for $\delta^2S < 0$ is that all the eigenvalues μ of the associated linear problem (abbreviating $\Omega_{\delta r} = \partial\Omega/\partial\delta r$, etc.)

$$\frac{d}{dm}(\Omega_{\delta\dot{r}}) - \Omega_{\delta r} - \mu\delta r = 0 \tag{4-1}$$

must be positive. The boundary conditions to be applied in finding the μ's are given as

$$\Omega_{\delta\dot{r}}]_{m=0} = \Omega_{\delta\dot{r}}]_{m=M} = 0,$$

the latter of which may be shown to be the same as Eq. (1-9), which states that the pressure must vanish to first order on the varying boundary of the sphere.

Now let us multiply Eq. (4-1) by δr and integrate, getting

$$\begin{aligned}\mu\int_0^M(\delta r)^2dm &= \int_0^M[\delta r d(\Omega_{\dot{\delta r}})/dm - \delta r\Omega_{\delta r}]dm \\ &= [\delta r\Omega_{\dot{\delta r}}]_0^M - \int_0^M[\dot{\delta r}\Omega_{\dot{\delta r}} + \delta r\Omega_{\delta r}]dm \\ &= -2\int_0^M\Omega(\dot{\delta r}, \delta r)dm = -\delta^2S/4\pi.\end{aligned}$$

Again exploiting the extremal properties of eigenvalues, one may easily show that for the smallest eigenvalue μ_0,

$$\mu_0 = \min_\varphi\left\{-2\int_0^M\Omega(\dot{\varphi}, \varphi)dm\Big/\int_0^M\varphi^2dm\right\}$$

for all admissible φ not identically zero satisfying the same boundary and continuity conditions as δr. Thus if $\mu_0 > 0$, then $\delta^2S < 0$, and conversely.

We are now in a position to compare our condition $\delta^2S < 0$ with Chandrasekhar's criterion, which as we have said results from linearizing the time-dependent equations of motion. Substituting ξ for δr in Eq. (4-1) with the use of Eq. (3-5), one may, after changing the independent variable back to r and performing very tedious manipulations, obtain an equation of the form

$$\frac{d}{dr}[q(r)\xi'] + [t(r) + u(r)\mu]\xi = 0, \tag{4-2}$$

where $q,\ u \geqslant 0$. Comparison with Eq. (1-8), the eigenvalue equation for σ^2 in Chandrasekhar's criterion, would show that $q(r)$ and $t(r)$ are the *same functions* in both equations, but that in general $u(r) \neq v(r)$. The same boundary conditions of course hold for both equations.

We will now prove that all the eigenvalues μ of Eq. (4-2) are positive if and only if all the σ^2's in Eq. (1-8) are positive. As before, we write for the lowest eigenvalues μ_0 and σ_0^2, using $q(0) = q(\mathrm{R}) = 0$,

$$\mu_0 = \min_\varphi \mathfrak{D}[\varphi]/\mathfrak{K}[\varphi]$$

and

$$\sigma_0^2 = \min_\varphi \mathfrak{D}[\varphi]/\mathfrak{J}[\varphi],$$

the functional $\mathfrak{D}[\varphi]$ being of course the same in both equations. Now one can show [2] that there exists an admissible φ_0 such that

$$\mu_0 = \mathfrak{D}[\varphi_0]/\mathfrak{J}[\varphi_0].$$

Now suppose $\mu_0 \leqslant 0$. Then $\mathcal{D}[\varphi_0] \leqslant 0$, and hence

$$\sigma_0^2 \leqslant \mathcal{D}[\varphi_0]/\mathcal{K}[\varphi_0] \leqslant 0.$$

Therefore $\mu_0 \leqslant 0$ implies $\sigma_0^2 \leqslant 0$, and one may similarly show that $\sigma_0^2 \leqslant 0$ implies $\mu_0 \leqslant 0$. Hence all the μ's are positive if and only if all the σ^2's are positive, and hence $\delta^2 S < 0$ if and only if $\sigma_0^2 > 0$.

This shows the equivalence of the two stability criteria. We now pass on to consider applications of our formalism to certain features of actual models.

5. — APPLICATION TO FEATURES OF ACTUAL MODELS

Actual equilibrium models for a variety of equations of state have been fairly thoroughly investigated, and the stability of some of these models has been tested with Chandrasekhar's criterion [*19*]. Since the results which we will discuss [*19*, *20*] were obtained for equations of state of the form $p = p(\varepsilon)$ with no dependence on a co-moving variable x, we will use $p = p(\varepsilon)$ throughout this section as a simplifying assumption.

One aspect of the results obtained takes the form of curves of total energy (mass) of the models as a function of a parameter such as the central pressure p_c. We write $M = M_0(p_c)$. One may show that only one such function $M_0(p_c)$ is obtainable for a given equation of strate.

A stability analysis for a given $p(\varepsilon)$ then shows which values of p_c are associated with stable configurations and which with unstable ones. One feature that usually appears is that for a given equation of state there is a point p_c^0 such that for $0 \leqslant p_c < p_c^0$ the configurations are stable, while for $p_c > p_c^0$ the configurations are unstable. We call the points p_c^0 « transition points ». Further, the transition points p_0^c are known to be « mass peaks », or maxima of the functions $M_0(p_c)$. However, it appears that not all mass peaks are such transition points. We will prove a plausible sufficient condition for a mass peak to be a point of either neutral or unstable equilibrium. This will apply as well to *minima* on the mass curves. Oppenheimer and Volkoff [*20*] have presented arguments which lead one to the conclusion that *all* maxima and minima must be such stability-instability transition points. Calculations by Misner and Zapolsky [*19*] have shown that this is not the case, and we will explain why Oppenheimer and Volkoff's argument fails.

We now prove a sufficient condition for a mass peak to be a point of neutral or unstable equilibrium. For a model with a given equation of state $p = p(\varepsilon)$, consider a curve $M = M_0(p_c)$ such that $M_0(p_c)$ is continuous and dM_0/dp_c is piece-wise continuous. Suppose also that there exists a point p_c^0 such that for some interval $p_c^a \leqslant p_c < p_c^0$, we have $dM_0/dp_c > 0$; and for an interval $p_c^0 < p_c \leqslant p_c^b$, $dM_0/dp_c < 0$. Then it follows that for some range $M_m \leqslant M \leqslant M_0(p_c^0)$, where $M_m < M_0(p_c^0)$, we may solve $M_0(p_c)$ for two continuous functions of M: $p_c = f^1(M)$ and $p_c = f^2(M)$, so that $M = M_0[f^1(M)] = M_0[f^2(M)]$. Of course, $f^1[M_0(p_c^0)] = f^2[M_0(p_c^0)] = p_c^0$. Thus $f^1(M)$ and $f^2(M)$ are one-to-one continuous mappings of the interval $M_m \leqslant M \leqslant M_0(p_c^0)$ onto $f^1(M_m) \leqslant p_c \leqslant p_c^0$ and $p_c^0 \leqslant p_c \leqslant f^2(M_m)$, respectively, where $f^1(M_m) < p_c^0 < f^2(M_m)$. This set of circumstances then describes the properties of a mass peak.

Our sufficient condition for p_c^0 to be a point of neutral or unstable equilibrium is the following: Suppose that there exists a parametrized set of pressure distribution functions $p = p(M, p_c; m)$, where M is the total energy (mass) associated with the distribution and is given *via* Eq. (3-1), the energy density distribution $\varepsilon(M, p_c; m)$ being given by inverting the equation of state $p = p(\varepsilon)$. p_c is of course the central value of the pressure, and m is the independent variable. Suppose further that $p[M_0(p_c), p_c; m]$ is the actual ***equilibrium*** configuration for central pressure p_c; and suppose that $p(M, p_c; m)$ is defined on the set $M_m \leqslant M \leqslant M_0(p_c^0), f^1(M_m) \leqslant p_c \leqslant f^2(M_m)$, $0 \leqslant m \leqslant M$, and that $\partial p/\partial p_c$ and $\partial^2 p/\partial p_c \partial m = \partial \dot{p}/\partial p_c$ are continuous on this same set. We will have of course $p(M, p_c; 0) = p_c$ and $p(M, p_c; M) = 0$.

We can write the value of the entropy S associated with the set of distributions as a function of M and p_c: $S = S(M, p_c)$. We now *further assume* as part of our sufficient condition that with this set of distributions $\partial^2 S/\partial p_c^2$ is continuous on the rectangle $M_m \leqslant M \leqslant M_0(p_c^0)$, $f^1(M_m) \leqslant p_c \leqslant f^2(M_m)$.

We now prove that the above assumptions imply that $\partial^2 S/\partial p_c^2 = 0$ at the point $M = M_0(p_c^0)$, $p_c = p_c^0$. We will abbreviate $\partial S/\partial p_c = DS$ and $\partial^2 S/\partial p_c^2 = D^2 S$. It is apparent from the variational principle that $DS[M_0(p_c), p_c] = 0$. But for any $\bar{p}_c \neq p_c^0$ such that also

$$f^1(M_m) \leqslant \bar{p}_c \leqslant f^2(M_M),$$

there exist p_c^1 and p_c^2 such that $p_c^1 = f^1[M_0(\bar{p}_c)]$ and $p_c^2 = f^2[M_0(\bar{p}_c)]$, where $p_c^1 \neq p_c^2$, but where of course either p_c^1 or $p_c^2 = \bar{p}_c$. Hence

$$DS[M_0(\bar{p}_c), p_c^1] = DS[M_0(\bar{p}_c), p_c^2] = 0.$$

But since D^2S is continuous, Rolle's theorem implies that there exists $\widehat{p}_c$ such that $p_c^1 < \widehat{p}_c < p_c^2$ and $D^2S[M_0(\bar{p}_c), \widehat{p}_2] = 0$.

Now suppose that $D^2S[M_0(p_c^0), p_c^0] \neq 0$. Then by the continuity hypothesis there exists a neighborhood D of the point $M_0(p_c^0)$, p_c^0 such that if Q is the intersection of D with the domain of S, then $D^2S \neq 0$ on Q. But we may obviously bring $\bar{p}_c$, and therefore $M_0(\bar{p}_c)$ and $\widehat{p}_c$, as close to $M_0(p_c^0), p_c^0$ as we like, and therefore bring the point $M_0(\bar{p}_c)$, $\widehat{p}_c$ inside any such set Q. However this contradicts the conclusions of the above paragraph, and hence

$$D^2S[M_0(p_c^0), p_c^0] = 0.$$

This conclusion may then be used to prove that the point p_c^0 is a point of neutral or unstable equilibrium. From $dm/dr = 4\pi r^2\varepsilon$, we may integrate

$$r = r(M, p_c\,; m) = \left\{ \frac{3}{4\pi} \int_0^m dm'/\varepsilon[p(M, p_c\,; m')] \right\}^{1/3}.$$

And since $r[M_0(p_c^0), p_c^0]$ is an equilibrium distribution and must satisfy the Euler equation (3-6) and the boundary condition $L_{\dot{r}}]_M = 0$, one easily obtains

$$D^2S[M_0(p_c^0), p_c^0] = 8\pi \int_0^M \Omega\left(\frac{\partial \dot{r}}{\partial p_c}, \frac{\partial r}{\partial p_c}\right) dm = \delta^2 S.$$

It can be shown that since $p(M, p_c; M) = 0$, the function $\partial r/\partial p_c$ satisfies the proper boundary conditions to be used as a variational function δr. Thus we may conclude that unless $\partial r/\partial p_c$ vanishes identically for $M = M_0(p_c^0)$ and $p_c = p_c^0$, it will follow that $\delta^2 S = 0$ for the admissible variation $\partial r/\partial p_c$. Hence p_c^0 will be a point of neutral or unstable equilibrium. But

$$\frac{\partial r}{\partial p_c} = -\frac{1}{4\pi} \left\{ \frac{3}{4\pi} \int_0^m dm'/\varepsilon[p(M, p_c\,; m')] \right\}^{-2/3} \int_0^m \frac{dm'}{\varepsilon^2} \frac{d\varepsilon}{dp} \frac{\partial p}{\partial p_c},$$

and hence $\partial r/\partial p_c = 0$ would mean $\partial p/\partial p_c = 0$. However $p(M, p_c; 0) = p_c$, so that for $m = 0$, $\partial p/\partial p_c = 1$. The assumed continuity of $\partial p/\partial p_c$ and $\partial \dot{p}/\partial p_c$ then implies that $\partial r/\partial p_c \neq 0$.

Thus the mass peak is a point of neutral or unstable equilibrium, provided that the given existence and continuity assumptions hold. The argument is virtually the same for *minima* on the mass curve. Thus our assumptions are sufficient conditions for neutral or unstable equilibrium at maxima and minima on the mass curve $M = M_0(p_c)$.

Oppenheimer and Volkoff [20] have argued that if p_c^0 is a mass maximum, and if for $dM_0/dp_c > 0$ the configurations are stable, then configurations must be unstable on the other side of p_c^0, where $dM_0/dp_c < 0$.

Phrased in our terminology, their discussion would run as follows: first assume the same sufficient conditions posed above. Then for any fixed $\overline{M}$ such that $M_m \leqslant \overline{M} \leqslant M_0(p_c^0)$, a maximum for S at both $p_c = f^1(\overline{M})$ and $p_c = f^2(\overline{M})$ would imply that $DS = 0$ and $D^2S < 0$ at both points. Then the continuity of D^2S would imply the existence of a *third* point $\widehat{p_c}$ for the fixed $\overline{M}$ such that $f^1(\overline{M}) < \widehat{p_c} < f^2(\overline{M})$ where also $DS = 0$. This follows. However, the conclusion is then drawn that since $DS = 0$, the point $\overline{M}$, $\widehat{p_c}$ is also an equilibrium point. But this is unwarranted; for $DS = 0$ is a necessary condition for equilibrium, but not a sufficient one. For equilibrium we must have at least for an infinite set of parameters $\{ a_j \}$, the condition $\partial S/\partial a_j = 0$ be fulfilled, where the distributions $p(M, a_1, a_2, \ldots, a_j, \ldots, m)$ represent all possible distributions in some neighborhood of the distribution in question.

Their argument has also been used to show that all mass maxima and minima are points of transition between regions of stability and instability. However, even if Oppenheimer and Volkoff's argument were correct, it could not be extended to show the converse of the above; namely, that if one side of a maximum (or minimum) of a mass curve were a region of instability, then the other side must be a region of stability. For, although stability implies $D^2S < 0$, instability does not imply $D^2S > 0$. $D^2S > 0$ is a sufficient condition for instability, but not a necessary one.

Evidence that all maxima and minima are not necessarily such transition points is shown by recent calculations by Misner and Zapolsky [19]. They calculated equilibrium masses for models consisting of a core having the equation of state $\varepsilon = 3p$ and joining smoothly onto an envelope with equation of state $\varepsilon = \alpha p^{3/5} + 3p/2$, the relation for a non-relativistic degenerate Fermi gas. Their plot of mass *versus* central pressure shows three mass maxima and minima, and their stability analysis shows that only one of these is a transition point. Further, all three points are points of either neutral or unstable equilibrium.

Thus our conclusion about mass maxima and minima appears justified in this case. It would be interesting to see whether or not the sufficient conditions that have been posed could be proved analytically from theorems on the continuity of integrals, etc.

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