ANNALES DE L'I. H. P., SECTION A

CLAUDE HENRY
EDUARDO DE RAFAEL

Relativistic theory of angular correlations in successive two-body decays of unstable particles

Annales de l'I. H. P., section A, tome 2, nº 2 (1965), p. 87-104
<http://www.numdam.org/item?id=AIHPA_1965__2_2_87_0>

Ann. Inst. Henri Poincaré,
Vol. II, n° 2, 1965, p. 87-104.

Section A :
Physique théorique.

Relativistic theory of angular correlations in successive two-body decays of unstable particles

by

Claude HENRY [1] **and Eduardo de RAFAEL**
Institut des Hautes Études Scientifiques
Bures-sur-Yvette (Seine-et-Oise), France.

ABSTRACT. — Relations for the study of angular correlations in successive two-body decay processes are derived from general relativistic invariance principles. The description of arbitrary spin states is discussed in terms of the density matrix formalism. Use is made of multipole expansion techniques derived from the basic transformation laws of physical states and operators.

I. — INTRODUCTION

Evidence for the existence of new particles and resonance states has been accumulating during the last few years. The study of angular correlations in their decay products has been providing information on their spin values and parities, and also on the plausible dynamical mechanisms for the production reaction processes. The theoretical description of these studies has been discussed in the literature in relation to the particular phenomena under consideration [*1*]. Usually, they consist in deriving the relevant kinematical relations in a particular frame of reference. The underlying

(1) Aspirant du Fonds National de la Recherche Scientifique.

principle of these techniques is *Relativistic Invariance in Quantum Mechanics* [2]. In this paper we derive general formulæ for the study of angular correlations in successive two-body decay processes. We show how they follow from the basic transformation laws of physical states and operators. The relations obtained are formulated with respect to a frame-tetrad (see Section II A) which can be chosen arbitrarily. Thus the covariance of our equations is assured and can be shown explicitly at any step of a computation (see for example the derivation of a covariant density matrix for a particle of arbitrary spin in Section II C.) The topics discussed have been summarized at the beginning of each section. We refer to Appendix I for notations and a summary of formulæ necessary for the derivation of several equations in the text.

II. — RELATIVISTIC DESCRIPTION OF PARTICLES WITH ARBITRARY SPIN

Let us call X a certain resonance state or a particle produced in some reaction process

$$a + b \rightarrow \mathrm{X} + c + d + \ldots \tag{1}$$

The sample of X's produced in (1) has a spin distribution which is conveniently characterized by the use of the density matrix formalism [*3*]. In this section we wish to summarize the essential features of a relativistic description of this formalism [*4*].

A. — Frames of reference.

We call *tetrad* a set of four 4-vectors $\{ t, n_{(i)} \}$ $i = 1, 2, 3$, such that (see Appendix I for notations)

$$\left.\begin{array}{lll} t \cdot n_{(i)} = 0\,; & n_{(i)} \cdot n_{(j)} = -\delta_{ij} & \text{for} \quad i, j = 1, 2, 3 \\ t^2 = 1\,; & \varepsilon_{\lambda\mu\nu\rho} t^{\lambda} n^{\mu}_{(1)} n^{\nu}_{(2)} n^{\rho}_{(3)} = 1. & \end{array}\right\} \text{and} \tag{2}$$

In what follows we shall refer all the kinematics to a specific frame-tetrad $\{ t, \vec{n} \}$, i. e., if p denotes the 4-vector energy momentum of a particle, say X in (1), then

$$\mathrm{E} = p . t \qquad \text{and} \qquad \vec{p} = - p . \vec{n} \tag{3}$$

will be the energy and momentum components of p in this frame-tetrad. For example, if t is chosen along p, i. e., $t = p/m$, where m denotes the mass of X, then we are in the rest-frame of X. We shall choose one of the axes $n_{(i)}$, say $n_{(3)} \equiv n$, as the direction of quantization of the spin of the X in its rest-frame. If t is taken along another direction (corresponding to the laboratory system, or to the C. M. system in (1)). we shall choose the quantization axis of the spin of the X along the 4-vector s ($s^2 = -1$, $s.p = 0$) obtained from n by the pure Lorentz transformation which brings t upon $p/m = u$ and leaves invariant the 2-plane orthogonal to t and u,

$$s = \Lambda_{t \to u} n, \tag{4}$$

with [5]

$$\Lambda_{t \to u} = 1 - \frac{(u+t) \otimes (u+t)}{1 + u.t} + 2u \otimes t. \tag{5}$$

Explicitly

$$s = n - \frac{n.u}{1 + u.t}(u + t). \tag{6}$$

B. — One-particle states.

We shall use Dirac's notation $| p, j\lambda \rangle$ for the state of a particle of energy-momentum p, spin j, and magnetic quantum number $\lambda = -j, \ldots, j$, defined as the spin-projection along s (see Eq. (4)). The states $| p, j\lambda \rangle$ are eigenstates of the energy-momentum operator P^μ, corresponding to the eigenvalue p^η,

$$P^\mu \, | p, j\lambda \rangle = p^\mu \, | p, j\lambda \rangle. \tag{7}$$

Let W_p denote the restriction of the polarization operator [6] to the eigenstates of P^μ. Then

$$-\frac{W_p^2}{m^2} \, | p, j\lambda \rangle = j(j+1) \, | p, j\lambda \rangle, \tag{8}$$

and

$$-\frac{W_p}{m} . s \, | p, j\lambda \rangle = \lambda \, | p, j\lambda \rangle. \tag{9}$$

The states $| p, j\lambda \rangle$ are normalized as follows

$$\left. \begin{aligned} \langle p', j'\lambda' \, | p, j\lambda \rangle &= 2\omega_p \delta^{(3)}(\vec{p} - \vec{p}')\delta_{jj'}\delta_{\lambda\lambda'}, \\ \text{with} \qquad \omega_p &= \sqrt{\vec{p}^2 + m^2}. \end{aligned} \right\} \tag{10}$$

The set of vectors $|\, p, j\lambda \rangle$ with $\lambda = -j, \ldots, j$ defines a basis in the representation space of a particle of mass $m \neq 0$ and spin j.

For a given Poincaré transformation [7] (a, Λ), the induced transformation of the vector basis is performed by a unitary operator $U(a, \Lambda)$ according to the relation [8]

$$U(a, \Lambda)\,|\, p, j\lambda \rangle = e^{i\Lambda p \cdot a} D^{(j)}[R(p, \Lambda)]^{\lambda'}{}_{\lambda}\,|\, \Lambda p, j\lambda' \rangle. \tag{11}$$

Here, $R(p, \Lambda)$ is a rotation of the little group of t (l. g. of t) defined thus

$$R(p, \Lambda) = \Lambda^{-1}_{t\to\Lambda p/m} \Lambda \Lambda_{t\to p/m}\,; \tag{12}$$

and $D^{(j)}[R(p, \Lambda)]^{\lambda'}{}_{\lambda}$ the matrix elements of the irreducible representation of the rotation group. From the set of Λ transformations let us consider those $\tilde{\Lambda}$'s belonging to the l. g. of t, i. e., such that $\tilde{\Lambda} t = t$. According to (12) we can identify $R(p, \tilde{\Lambda})$ and $\tilde{\Lambda}$ if and only if

$$\Lambda_{t\to\tilde{\Lambda} p/m} = \tilde{\Lambda} \Lambda_{t\to p/m} \tilde{\Lambda}^{-1}. \tag{13}$$

This means that for a given p, once $\Lambda_{t\to p/m}$ is chosen (for example as in (5)), $\Lambda_{t\to\tilde{\Lambda}' p/m}$, where $\tilde{\Lambda}'$ denotes a specific transformation belonging to the l. g. of t, must be fixed according to (13), i. e.,

$$\Lambda_{t\to\tilde{\Lambda}' p/m} = \tilde{\Lambda}' \Lambda_{t\to p/m} \tilde{\Lambda}'^{-1}.$$

Then

$$\tilde{\Lambda} \Lambda_{t\to\tilde{\Lambda}' p/m} \tilde{\Lambda}^{-1} = \tilde{\Lambda}\tilde{\Lambda}' \Lambda_{t\to p/m} \tilde{\Lambda}'^{-1}\tilde{\Lambda}^{-1} = (\tilde{\Lambda}\tilde{\Lambda}') \Lambda_{t\to p/m} (\tilde{\Lambda}\tilde{\Lambda}')^{-1} = \Lambda_{t\to\tilde{\Lambda}\tilde{\Lambda}' p/m},$$

and hence relation (13) holds for all the vectors $\tilde{\Lambda}' p/m$. Therefore, when restricted to rotations R of the l. g. of t, the transformation law (11) can be simply written

$$U(R)\,|\, p, j\lambda \rangle = D^{(j)}(R)^{\lambda'}{}_{\lambda}\,|\, Rp, j\lambda' \rangle. \tag{14}$$

The Hilbert space of the states consists of the vectors

$$|\, \Phi \rangle = \int \frac{d^3\vec{p}}{2\omega_p} \sum_{\lambda} |\, p, j\lambda \rangle \langle p, j\lambda \,|\, \Phi \rangle, \tag{15}$$

with the scalar product defined, according to (10), thus

$$\langle \psi \,|\, \Phi \rangle = \sum_{\lambda} \int \frac{d^3\vec{p}}{2\omega_p} \langle \psi \,|\, p, j\lambda \rangle \langle p, j\lambda \,|\, \Phi \rangle. \tag{16}$$

The expressions

$$\Phi(p)^{\lambda} = \langle p, j\lambda \,|\, \Phi \rangle, \tag{17}$$

are the $(2j+1)$ spin-components of the wave-function describing a particle of energy-momentum p and spin j. Their transformation law, first derived by Wigner [*9*], reads

$$[U(a, \Lambda)\Phi](p) = e^{ip \cdot a} Q(p, \Lambda)\, \Phi(\Lambda^{-1}p), \tag{18}$$

where $Q(p, \Lambda)$ are the unitary $(2j+1)$ by $(2j+1)$ matrices

$$D^{(j)}\left(\Lambda^{-1}_{t \to p/m} \Lambda \Lambda_{t \to \Lambda^{-1}p/m}\right)$$

of the rotation group.

C. — Spin density matrix.

Consider a sample of X's produced in (1), with fixed energy-momentum p. This is described by an incoherent mixture of $(2j+1)$ orthogonal states $| \Phi_n \rangle$ of pure polarization, each with probability c_n $(0 \leqslant c_n \leqslant 1)$. This mixture can be characterized by the density matrix operator

$$\rho(p) = \sum_{n=-j}^{j} | \Phi_n \rangle c_n \langle \Phi_n |. \tag{19}$$

Let us call $\rho(p)^{\lambda}{}_{\tilde{\lambda}}$ the matrix elements of $\rho(p)$ in the basis defined in the last section. According to (18) their transformation law is (see AI.1)

$$\begin{aligned} \rho'(p)^{\lambda}{}_{\tilde{\lambda}} &= [U(a, \Lambda)\rho U^{-1}(a, \Lambda)](p)^{\lambda}{}_{\tilde{\lambda}} \\ &= (-1)^{2j} D^{(j)}[R(\Lambda^{-1}p, \Lambda)]^{\lambda}{}_{\lambda'} D^{(j)}\Big[R(\Lambda^{-1}p, \Lambda)\Big]^{\tilde{\lambda}'}{}_{\tilde{\lambda}}\, \rho(\Lambda^{-1}p)^{\lambda'}{}_{\tilde{\lambda}'}. \end{aligned} \tag{20}$$

The product of representations appearing in (20) is equivalent to the direct sum of irreducible representations $D^{(L)}[R(\Lambda^{-1}p, \Lambda)]$, with $0 \leqslant L \leqslant 2j$. Therefore, it will be convenient to express $\rho(p)$ as a sum of irreducible tensor operators [*10*] $T^{(L)}_{M}(p)$ $(0 \leqslant L \leqslant 2j, -L \leqslant M \leqslant L)$ i. e., such that

$$U(a, \Lambda) T^{(L)}_{M}(p) U^{-1}(a, \Lambda) = \bar{D}^{(L)}[R(\Lambda^{-1}p, \Lambda)]^{M'}{}_{M} T^{(L)}_{M'}(\Lambda^{-1}p). \tag{21}$$

Using the normalization

$$\operatorname{tr}\left(T^{(L')}_{M'}(p) T^{(L)}_{M}(p)^{*}\right) = \frac{2j+1}{2L+1}\, \delta_{LL'} \delta_{MM'}, \tag{22}$$

the density matrix operator $\rho(p)$ can be expanded thus

$$\rho(p) = \frac{1}{2j+1} \sum_{L=0}^{2j} \sum_{M=-L}^{L} (2L+1)\, \bar{t}^{(L)}_{M}(p) T^{(L)}_{M}(p), \tag{23}$$

where

$$t^{(L)}_{M}(p) = \mathrm{tr}\,(\rho(p)T^{(L)}_{M}(p)), \tag{24}$$

are the so called multipole-parameters of the density matrix. The hermiticity of $\rho(p)$ implies

$$t^{(L)}_{M}(p) = (-1)^{M}\bar{t}^{(L)}_{-M}(p)\,; \tag{25}$$

and tr $\rho(p) = 1$ requires $t^{(0)}_{0}(p) = 1$. The magnitudes of the non zero $t^{(L)}_{M}(p)$ are further restricted by the fact that $\rho(p)$ is positive definite. Thus e. g., tr $\rho^2(p) \leqslant$ tr $\rho(p)$, leads to the condition

$$\sum_{L=0}^{2j}\sum_{M=-L}^{L}(2L+1)\,|\,t^{(L)}_{M}(p)\,|^2 \leqslant 2j+1. \tag{26}$$

In the basis defined by the one-particle states $|\,p, j\lambda\rangle$, the matrix elements $T^{(L)}_{M}(p)^{\lambda}{}_{\tilde{\lambda}}$ are $3-j$ symbol coefficients (see AI.2). This follows from the theorem of Wigner-Eckart

$$\langle p, j\lambda\,|\,T^{(L)}_{M}(p)\,|\,p, j\tilde{\lambda}\rangle = \sqrt{2j+1}\begin{pmatrix}\lambda & L & j\\ j & M & \tilde{\lambda}\end{pmatrix}\langle p, j\,\|\,T^{(L)}(p)\,\|\,p, j\rangle, \tag{27}$$

and the normalization defined in (22). Thus

$$T^{(L)}_{M}(p)^{\lambda}{}_{\tilde{\lambda}} = \frac{\langle p, j\lambda\,|\,T^{(L)}_{M}(p)\,|\,p, j\tilde{\lambda}\rangle}{\langle p, j\,\|\,T^{(L)}(p)\,\|\,p, j\rangle} = \sqrt{2j+1}\begin{pmatrix}\lambda & L & j\\ j & M & \tilde{\lambda}\end{pmatrix}. \tag{28}$$

In this representation, the multipole-parameters are given by

$$t^{(L)}_{M}(p) = (-1)^{L}\sqrt{2j+1}\begin{pmatrix}j & L & \lambda\\ \lambda' & M & j\end{pmatrix}\rho(p)^{\lambda'}{}_{\lambda}. \tag{29}$$

The components $t^{(L)}_{0}(p)$ satisfy the restrictions

$$(t^{(L)}_{0}(p))^2 \leqslant (2j+1)\max_{\lambda\in\{-j,\ldots,j\}}\begin{pmatrix}j & L & j\\ \lambda & 0 & -\lambda\end{pmatrix}^2. \tag{30}$$

The operators $T^{(L)}_{M}(p)$ can be explicitly constructed in the following way. Let us consider the spin operator

$$\vec{S}_p = -\frac{W_p}{m}.\vec{s}, \tag{31}$$

where $\vec{s}$ denotes the three 4-vectors obtained from the space-like components $\vec{n}$ of the frame-tetrad (see Eq. (2)) by the pure Lorentz transformation $\Lambda_{t \to p/m}$ defined in (5),

$$\vec{s} = \Lambda_{t \to p/m} \vec{n}. \tag{32}$$

From the operator $\vec{S}_p$ we form the spherical harmonic $Y^{(L)}_M(\vec{S}_p)$, completely symmetrized with respect to the components of $\vec{S}_p$. Then

$$T^{(L)}_M(p) = (\langle p, j \| Y^{(L)}(\vec{S}_p) \| p, j \rangle)^{-1} Y^{(L)}_M(\vec{S}_p). \tag{33}$$

The expansion of $\rho(p)$ in terms of these operators can be rearranged in a completely covariant way [11] ($\alpha_i = 0, 1, 2, 3$ for $1 \leqslant i \leqslant L$)

$$\rho(p) = \frac{1}{2j+1} + \sum_{L=1}^{2j} \left(-\frac{1}{m}\right)^L W_p^{\alpha_1} \dots W_p^{\alpha_L} \sigma_{\alpha_1 \dots \alpha_L} =$$

$$= \frac{1}{2j+1} - \frac{1}{m} W_p^\alpha \sigma_\alpha + \frac{1}{m^2} W_p^\alpha W_p^\beta \sigma_{\alpha\beta} - \frac{1}{m^3} W_p^\alpha W_p^\beta W_p^\gamma \sigma_{\alpha\beta\gamma} + \dots, \tag{34}$$

where W_p^α denote the components of the polarization 4-vector operator, and [12]

$$\sigma_{\alpha_1 \dots \alpha_L} = \frac{2L+1}{2j+1} \sum_{M=-L}^{L} \bar{t}^{(L)}_M(p) \frac{Y^{(L)}_M(\vec{s}_{\alpha_1}, \dots, \vec{s}_{\alpha_L})}{\langle p, j \| Y^{(L)}(\vec{S}_p) \| p, j \rangle}. \tag{35}$$

The properties of the pseudo-tensors $\sigma_{\alpha_1 \dots \alpha_L}$ follow from their definition:

a) they are completely symmetric and their components are real;
b) they are orthogonal to the energy-momentum 4-vector p,

$$p^{\alpha_i} \sigma_{\alpha_1, \dots, \alpha_i, \dots, \alpha_L} = 0 \quad \text{for} \quad i \in (1, \dots, L)\,; \tag{36}$$

c)
$$\sigma_{\alpha_1, \dots, \alpha_i, \dots, {}^{\alpha_i}, \dots, \alpha_L} = 0 \quad \text{for} \quad i \in (1, \dots, L)\,; \tag{37}$$

d)
$$\sigma_{\alpha_1, \dots, \alpha_L} \sigma^{\alpha_1, \dots, \alpha_L} = C(L, j) \sum_{M=-L}^{L} | t^{(L)}_M |^2, \tag{38}$$

with

$$C(L, j) = \left(\frac{2L+1}{2j+1}\right)^2 \frac{Y^{(L)}_0(\vec{s}_{\alpha_1}, \dots, \vec{s}_{\alpha_L}) Y^{(L)}_0(\vec{s}^{\alpha_1}, \dots, \vec{s}^{\alpha_L})}{(\langle p, j \| Y^{(L)}(\vec{S}_p) \| p, j \rangle)^2}. \tag{39}$$

Relation *d)* gives the degree of polarization $\delta^{(L)}$ 2^L — polar:

$$\delta^{(L)} = [(-1)^L \sigma_{\alpha_1, \dots, \alpha_L} \sigma^{\alpha_1, \dots, \alpha_L}]^{1/2} = \left\{ (-1)^L C(L, j) \sum_{M=-L}^{L} | t^{(L)}_M |^2 \right\}^{1/2}. \tag{40}$$

For L = 1, i. e., the ordinary degree of polarization, we get the following upper-bound

$$\delta^{(1)} = (-\ \sigma_\alpha \sigma^\alpha)^{1/2} \leqslant \frac{3}{2j+1}\,\frac{1}{j+1}\,. \tag{41}$$

Sometimes it is convenient to describe the polarization in terms of the so called Stokes-vector $\vec{\xi}(p)$, such that $0 \leqslant \vec{\xi}(p)^2 \leqslant 1$. According to the upper-bound found for $\delta^{(1)}$, the relation between $\vec{\xi}(p)$ and the multipole-parameters $t^{(1)}_{\mathrm{M}}(p)$ is

$$\xi^{(1)}_{\mathrm{M}}(p) = \sqrt{\frac{j+1}{j}}\, t^{(1)}_{\mathrm{M}}(p). \tag{42}$$

III. — RELATIVISTIC DESCRIPTION OF TWO-BODY DECAYS

In this section we discuss the kinematical properties of the scattering amplitude of the process

$$\mathrm{X} \rightarrow \mathrm{Y} + y, \tag{43}$$

where an unstable particle X of energy-momentum p and spin j decays into a particle Y of energy-momentum p_1 and spin s, and a spinless particle y of energy-momentum p_2. We shall use the shorthand notation

$$\mathrm{S}(p_1, p)^{\mu}{}_{\lambda} = \langle p_1, s\mu; p_2 \mid \mathrm{S} \mid p, j\lambda \rangle, \tag{44}$$

for the scattering-amplitude in (43). Here λ and μ, the magnetic quantum numbers of particles X and Y, are defined as follows

$$-\frac{\mathrm{W}_p}{m}\cdot s \mid p, \lambda \rangle = \lambda \mid p, \lambda \rangle, \quad \text{with} \quad s = \Lambda_{t\rightarrow p/m} n\ ; \tag{45}$$

and

$$-\frac{\mathrm{W}_{p_1}}{m_1}\cdot s_1 \mid p_1, \mu \rangle = \mu \mid p_1, \mu \rangle, \quad \text{with} \quad s_1 = \Lambda_{t\rightarrow p_1/m_1} n. \tag{46}$$

The transformations $\Lambda_{t\rightarrow p/m}$ and $\Lambda_{t\rightarrow p_1/m_1}$ are pure Lorentz transformations defined as in (5). The Lorentz transformation which brings p/m upon p_1/m_1 is defined thus

$$\Lambda_{p/m\rightarrow p_1/m_1} = \Lambda_{t\rightarrow p_1/m_1} \Lambda^{-1}_{t\rightarrow p/m}, \tag{47}$$

i. e., we first bring p/m to rest and then upon p_1/m_1. Note that this transformation is such that

$$s_1 = \Lambda_{p/m \to p_1/m_1} s, \tag{48}$$

i. e., it also transforms the corresponding quantization axis. In what follows we derive the multipole expansion of the scattering amplitude in (43), and the decay angular distribution of Y's.

A. — Multipole expansion of the scattering amplitude.

From (14) and (44), and using (AI.1) we can derive the transformation of the matrix elements of the scattering-amplitude, induced by a rotation R of the l. g. of t,

$$S(p_1, p)^{\mu}{}_{\lambda} = (-1)^{2s} D^{(s)}(R)_{\tilde{\mu}}{}^{\mu} D^{(j)}(R)^{\tilde{\lambda}}{}_{\lambda} S(Rp_1, Rp)^{\tilde{\mu}}{}_{\tilde{\lambda}}. \tag{49}$$

Let us multiply both sides of (49) by $\begin{pmatrix} s & \lambda & m \\ \mu & j & l \end{pmatrix}$ (see AI.2) and take summation over μ and λ. Using (AI.3) we are led to the relation (summation over μ, λ and $\tilde{m}$, $\tilde{\mu}$, $\tilde{\lambda}$)

$$\begin{pmatrix} s & \lambda & m \\ \mu & j & l \end{pmatrix} S(p_1, p)^{\mu}{}_{\lambda} = \overline{D}^{(l)}(R)^{m}{}_{\tilde{m}} \begin{pmatrix} s & \tilde{\lambda} & \tilde{m} \\ \tilde{\mu} & j & l \end{pmatrix} S(Rp_1, Rp)^{\tilde{\mu}}{}_{\tilde{\lambda}}. \tag{50}$$

If we call $S^{m}_{(l)}(p_1, p)$ the linear combination (summation over λ and μ)

$$S^{m}_{(l)}(p_1, p) = \begin{pmatrix} s & \lambda & m \\ \mu & j & l \end{pmatrix} S(p_1, p)^{\mu}{}_{\lambda}, \tag{51}$$

then relation (50) tells us that the functions $S^{m}_{(l)}(p_1, p)$ with $m = -l, \ldots, l$, transform under rotations like the complex conjugate spherical harmonics $\overline{Y}^{m}_{(l)}$. Moreover, the kinematical dependence of each $S^{m}_{(l)}(p_1, p)$ can be expressed in terms of two angular parameters which we choose as the components of the unit 3-vector [*13*]

$$\vec{e}(p, p_1) = \frac{2}{m} [\Delta(m, m_1, m_2)]^{-1/2} (p_1 \wedge p)^{\alpha\beta} p_{\alpha} \vec{s}_{\beta}, \tag{52}$$

where

$$\Delta(m, m_1, m_2) = (m + m_1 + m_2)(m + m_1 - m_2)(m - m_1 + m_2)(m - m_1 - m_2). \quad (53)$$

Consider now the invariant quantities (invariant with respect to the rotations of the l. g. of t)

$$A_l = \sum_{m=-l}^{l} \int d\Omega(\vec{e}) Y_m^{(l)}[\vec{e}(p, p_1)] S_{(l)}^m(p_1, p), \quad (54)$$

where $l = j - s, j - s + 1, \ldots, j + s$ is the orbital angular momentum. This we shall call the l^{th} multipole-amplitude of the process (43). According to this definition and using the inverse relation of (51), we can expand the scattering-amplitude in terms of multipole-amplitudes as follows

$$S(p_1, p)^\mu{}_\lambda = \sum_{l,m} A_l \begin{pmatrix} \mu & j & l \\ s & \lambda & m \end{pmatrix} \overline{Y}_{(l)}^m[\vec{e}(p, p_1)]. \quad (55)$$

B. — Decay angular distribution.

The angular distribution of Y's in the decay (43) is defined thus

$$I(p_1, p) = \left(\frac{1}{2j+1} \sum_l |A_l|^2\right)^{-1} \operatorname{tr} S(p_1, p)\rho(p)S^*(p_1, p), \quad (56)$$

where tr means trace in the polarization space. From (55), (23) and (28), and after some algebraic manipulations where use of (AI.4) and (AI.5) is made, the following expression for $I(p_1, p)$ can be derived

$$I(p_1, p) = (-1)^{j+s} \sqrt{\frac{2j+1}{4\pi}} \sum_{L l \tilde{l}} \sqrt{(2\tilde{l}+1)(2L+1)(2l+1)}$$
$$\begin{Bmatrix} j & L & j \\ \tilde{l} & s & l \end{Bmatrix} \begin{pmatrix} \tilde{l} & 0 & l \\ 0 & L & 0 \end{pmatrix} \frac{\overline{A}_{\tilde{l}} A_l}{\sum_l |A_l|^2} t_M^{(L)}(p) \overline{Y}_{(L)}^M[\vec{e}(p, p_1)], \quad (57)$$

where l and $\tilde{l}$ take the values: $j - s, j - s + 1, \ldots, j + s$; and $0 \leqslant L \leqslant 2j$. If parity is conserved in (43) then $\tilde{l} + l$ must be even and hence only terms with even L will be relevant in (57).

IV. — ANGULAR CORRELATIONS

Consider now the set of reactions

$$a + b \to X + c + d + \dots \qquad (1)$$
$$\hookrightarrow Y + y \qquad (43)$$
$$\hookrightarrow Z + z, \qquad (58)$$

and assume that particle Z has energy-momentum p_3 and spin k, and that z is a spinless particle of energy-momentum p_4. The techniques set down in the last section can here be likewise applied to derive the angular distribution of Z's. However, this requires an explicit knowledge of the density matrix of Y's. In this section we derive the general expression for the angular dependence of the multipole-parameters which characterize the sample of Y's. This provides the necessary information for the study of any angular correlation problem. In particular we derive the angular dependence of the longitudinal polarization of Y's, with respect to a frame-tetrad, and the connection between the averaged multipole-parameters of the sample of Y's and those of X's. Some specific examples are discussed.

A. — Angular dependence of multipole parameters.

The density matrix of Y's produced in the decay (43) is

$$\rho(p_1, p) = \frac{S(p_1, p)\rho(p)S^*(p_1, p)}{\mathrm{tr}\,[S(p_1, p)\rho(p)S^*(p_1, p)]}, \qquad (59)$$

and according to (29) the associated multipole parameters are

$$t^{(L')}_{M'}(p_1, p) = (-1)^{L'}\sqrt{2s+1}\begin{pmatrix} s & L' & \mu \\ \mu' & M' & s \end{pmatrix}\rho(p_1, p)^{\mu'}{}_{\mu}, \qquad (60)$$

with $0 \leqslant L' \leqslant 2s$.

Making use of the multipole expansions for $\rho(p)$ and $S(p_1, p)$, as given in (23) and (28), and (55) respectively, the angular dependence of $t^{(L')}_{M'}(p_1, p)$ reads

$$I(p_1, p)t^{(L')}_{M'}(p_1, p)$$
$$= (-1)^{L'+j+s}\sqrt{(2j+1)(2s+1)} \sum_{l,\tilde{l}=j-s}^{j+s} \sum_{L=0}^{2j} (2L+1)\bar{t}^{(L)}_{M}(p)\frac{\bar{A}_{\tilde{l}}A_l}{\sum_l |A_l|^2}$$
$$\times \begin{pmatrix} s & L' & \mu \\ \mu' & M' & s \end{pmatrix}\begin{pmatrix} \mu' & j & l \\ s & \lambda & m \end{pmatrix}\begin{pmatrix} \lambda & L & j \\ j & M & \tilde{\lambda} \end{pmatrix}\begin{pmatrix} \tilde{\lambda} & s & \tilde{m} \\ j & \mu & \tilde{l} \end{pmatrix}\bar{Y}^{m}_{(l)}[\vec{e}(p, p_1)]Y^{(\tilde{l})}_{\tilde{m}}[\vec{e}(p_1, p)]. \qquad (61)$$

Here, we first use (AI.5) to express the angular dependence in a spherical harmonic $Y^{(\tilde{L})}_{\tilde{M}}[\vec{e}(p, p_1)]$ alone, and next the following relation (see e. g. ref. [*20*])

$$\begin{pmatrix} s & L' & \mu \\ \mu' & M' & s \end{pmatrix}\begin{pmatrix} \mu' & j & l \\ s & \lambda & m \end{pmatrix}\begin{pmatrix} \lambda & L & j \\ j & M & \tilde{\lambda} \end{pmatrix}\begin{pmatrix} \tilde{\lambda} & s & \tilde{l} \\ j & \mu & \tilde{m} \end{pmatrix}\begin{pmatrix} m & \tilde{m} & \tilde{M} \\ l & \tilde{l} & \tilde{L} \end{pmatrix}$$

$$=(-1)^{\tilde{L}+l+j+s}\sum_k(-1)^{2k}(2k+1)\begin{Bmatrix} l & j & k \\ L & s & j \end{Bmatrix}\begin{Bmatrix} s & \tilde{L} & k \\ l & j & \tilde{l} \end{Bmatrix}\begin{Bmatrix} L & \tilde{L} & L' \\ s & s & k \end{Bmatrix}\begin{pmatrix} L & \tilde{M} & L' \\ M & \tilde{L} & M' \end{pmatrix} \tag{62}$$

This expression can be further condensed introducing a $9-j$ symbol (see (AI.6)). We are thus led to the rather compact formula

$$I(p_1, p)t^{(L')}_{M'}(p_1, p)=\sqrt{\frac{2j+1}{4\pi}}\sqrt{2s+1}\sum_{l,\tilde{l}=j-s}^{j+s}\sum_{\tilde{L}=0}^{2j}\sqrt{(2\tilde{L}+1)(2\tilde{l}+1)(2l+1)}$$

$$\times\sum_{L=0}^{2j}(2L+1)\begin{Bmatrix} L & j & j \\ \tilde{L} & \tilde{l} & l \\ L' & s & s \end{Bmatrix}\begin{pmatrix} \tilde{L} & 0 & l \\ 0 & \tilde{l} & 0 \end{pmatrix}\frac{\bar{A}_{\tilde{l}}A_l}{\sum_l |A_l|^2}\begin{pmatrix} L' & \tilde{L} & M \\ M' & \tilde{M} & L \end{pmatrix}t^{(L)}_{M}(p)Y^{\tilde{M}}_{(\tilde{L})}[\vec{e}(p, p_1)]. \tag{63}$$

This relation gives the angular dependence of the multipole-parameters of Y's in terms of the multipole-amplitudes of the decay (43) and the multipole-parameters of the density matrix describing the sample of X's.

B. — Angular dependence of the Stokes-vector.

The Stokes-vector (see (42)) describing the polarization of Y's produced in a given direction $\vec{e}(p, p_1)$, is obtained from the expression given in (63) for $L' = 1$,

$$\xi^{(1)}_{M'}(p_1, p)=\sqrt{\frac{s+1}{s}}\,t^{(1)}_{M'}(p_1, p). \tag{64}$$

As a particular case, the angular dependence of the longitudinal component, $I(p_1, p)\vec{\xi}(p_1, p).\vec{e}(p, p_1)$, with respect to a frame-tetrad $\{ t, \vec{n} \}$ (see equations (2)), can be easily derived. Using (AI.5) we get

$$\mathrm{I}(p_1, p)\vec{\xi}(p_1, p)\cdot\vec{e}(p, p_1) = \sqrt{\frac{2j+1}{4\pi}}\sqrt{\frac{(2s+1)(s+1)}{s}}$$
$$\times \sum_{l,\tilde{l}=j-s}^{j+s} \sum_{\mathrm{L},\tilde{\mathrm{L}}=0}^{2j} (2\tilde{\mathrm{L}}+1)\sqrt{(2\tilde{l}+1)(2l+1)(2\mathrm{L}+1)}$$
$$\begin{Bmatrix} \mathrm{L} & j & j \\ \tilde{\mathrm{L}} & \tilde{l} & l \\ 1 & s & s \end{Bmatrix} \begin{pmatrix} \tilde{\mathrm{L}} & 0 & l \\ 0 & \tilde{l} & 0 \end{pmatrix} \begin{pmatrix} 1 & \tilde{\mathrm{L}} & 0 \\ 0 & 0 & \mathrm{L} \end{pmatrix} \frac{\mathrm{A}_{\tilde{l}}\mathrm{A}_l}{\sum_l |\mathrm{A}_l|^2} t^{(\mathrm{L})}_{\mathrm{M}}(p)\overline{\mathrm{Y}}^{\mathrm{M}}_{(\mathrm{L})}[\vec{e}(p, p_1)]. \quad (65)$$

This expression reduces to a particularly simple form when $s = 1/2$ [*14*]. Then $l, \tilde{l} = j \pm 1/2$. If we call α the asymmetry parameter

$$\alpha = \frac{2Re\mathrm{A}_{j-1/2}\mathrm{A}^*_{j+1/2}}{|\mathrm{A}_{j-1/2}|^2 + |\mathrm{A}_{j+1/2}|^2}, \quad (66)$$

the final result is

$$\mathrm{I}(p_1, p)\vec{\xi}(p_1, p)\cdot\vec{e}(p, p_1) = \sqrt{\frac{2j+1}{4\pi}} \sum_{\mathrm{L}=0}^{2j} \sqrt{2\mathrm{L}+1}\, \frac{1-(-1)^{\mathrm{L}}+\alpha(1+(-1)^{\mathrm{L}})}{2}$$
$$\begin{pmatrix} 1/2 & \mathrm{L} & j \\ j & 0 & 1/2 \end{pmatrix} t^{(\mathrm{L})}_{\mathrm{M}}(p)\overline{\mathrm{Y}}^{\mathrm{M}}_{(\mathrm{L})}[\vec{e}(p, p_1)]. \quad (67)$$

C. — Averages of multipole parameters.

Let us call $\langle t^{(\mathrm{L}')}_{\mathrm{M}'}(p_1, p) \rangle$ the average of the multipole-parameters of Y's over their angular distribution

$$\langle t^{(\mathrm{L}')}_{\mathrm{M}'}(p_1, p) \rangle = \int d\Omega[\vec{e}(p, p_1)]\mathrm{I}(p_1, p) t^{(\mathrm{L}')}_{\mathrm{M}'}(p_1, p). \quad (68)$$

These averages can be easily evaluated using the general expression given in (63). For Y's produced from a sample of X's with fixed energy-momentum p,

$$\langle t^{(\mathrm{L})}_{\mathrm{M}}(p_1, p) \rangle = \sqrt{(2j+1)(2s+1)}$$
$$\times \sum_{l=j-s}^{j+s} (2l+1)(2\mathrm{L}+1) \begin{Bmatrix} \mathrm{L} & j & j \\ 0 & l & l \\ \mathrm{L} & s & s \end{Bmatrix} \begin{pmatrix} l & 0 & 0 \\ 0 & l & 0 \end{pmatrix} \begin{pmatrix} \mathrm{L} & \mathrm{M} & 0 \\ \mathrm{M} & \mathrm{L} & 0 \end{pmatrix} \frac{|\mathrm{A}_l|^2}{\sum_l |\mathrm{A}_l|^2} t^{(\mathrm{L})}_{\mathrm{M}}(p). \quad (69)$$

Using (AI.7) and the explicit values of the $3-j$ symbols appearing in (69), we finally get

$$\langle t^{(L)}_M(p_1, p)\rangle = \sum_l (-1)^{j+s+l+L}\sqrt{(2j+1)(2s+1)}\begin{Bmatrix} s & j & l \\ j & s & L \end{Bmatrix}\frac{|A_l|^2}{\sum_1 |A_l|^2} t^{(L)}_M(p). \tag{70}$$

This relation gives the proportionality coefficient between averages of the multipole-parameters of Y's and the corresponding multipole-parameters of X's. As a particular case of (70), when $L = 1$, we get

$$\langle t^{(1)}_M(p_1, p)\rangle = \sum_l \frac{j(j+1)+s(s+1)-l(l+1)}{2\sqrt{j(j+1)s(s+1)}}\frac{|A_l|^2}{\sum_l |A_l|^2} t^{(1)}_M(p). \tag{71}$$

Let us choose the axis of quantization along the normal to the scattering 3-plane of the reaction (1). Then, if δ_Y and δ_X denote the component along this axis of the Stokes-vector of Y's and X's, from (42) and (71) it follows that

$$\langle \delta_Y \rangle = \frac{1}{s}\sum_{l=j-s}^{j+s} [j(j+1)+s(s+1)-l(l+1)]\frac{|A_l|^2}{\sum_l |A_l|^2}\frac{\delta_X}{2j+2}. \tag{72}$$

The application of the preceding relation in the determination of the spin-parity of strange baryonic resonances has recently been discussed by one of us [*15*]: assume that (43) is a strong decay and particle Y has spin 1/2 (e. g. [*16*] $Y^*(1\,660) \to \Lambda + \pi$; $Y^*_0(1\,815) \to \Sigma^+ + \pi^-$); then it follows from (72) that

$$\langle \delta_Y \rangle = \delta_X \qquad \text{if} \qquad l = j - 1/2 \tag{73}$$

and

$$\langle \delta_Y \rangle = -\frac{j}{j+1}\delta_X \qquad \text{if} \qquad l = j + 1/2. \tag{74}$$

In reference (15) it is discussed how $\langle \delta_Y \rangle$ and δ_X can be obtained from the experimental data. Then for a sufficiently high number of events, relation (73) or (74) leads to the spin-parity determination of the X.

As an application of the restriction given in (30) we would like to comment on the $\pi^+\pi^-$ resonance at 1 250 MeV, called f^0. At present there is no conclusive assignment for its spin value, although recent experimental data [*17*] indicate that the f^0 is an isotopic singlet ($I = 0$) and, therefore, that its spin j is even. The experimental data on the decay angular distribution of the f^0 indicate [18] that $j \geqslant 2$. Let us consider the quantities

$$\gamma^{(L)} = \sqrt{\frac{4\pi}{2L+1}} \langle Y_0^{(L)}(\hat{\pi}) \rangle, \tag{75}$$

$$L \text{ (even)} = 0, 2, \ldots, 2j$$

where $\hat{\pi}$ is a unit vector along the momentum of one of the decay π's in the rest system of the f^0, and $\langle Y_0^{(L)}(\hat{\pi}) \rangle$ the average of the spherical harmonic $Y_0^{(L)}(\hat{\pi})$ over the angular distribution of the f^0 decay. From the restriction stated in (30) it follows that

$$\gamma^{(L)} \leqslant (2j+1) \left| \begin{pmatrix} j & L & j \\ 0 & 0 & 0 \end{pmatrix} \right| \sqrt{\operatorname*{Max}_{\lambda \in \{-j, \ldots, j\}} \begin{pmatrix} j & L & j \\ \lambda & 0 & -\lambda \end{pmatrix}^2}. \tag{76}$$

For example:

$$\text{if } j=2 \text{ and } L=2, \quad \gamma^{(2)} \leqslant 2/7; \tag{77}$$

$$\text{if } j=4 \text{ and } L=2, \quad \gamma^{(2)} \leqslant 2/7; \tag{78}$$

$$\text{if } j=4 \text{ and } L=4, \quad \gamma^{(4)} \leqslant 27/143. \tag{79}$$

These upper-bounds can be used to eliminate some of the possible values of the spin of the f^0.

Further applications of the formalism presented in this paper are at present being studied (see e. g., ref. [19]).

APPENDIX I

In this appendix we define our notations and collect a number of formulæ which are necessary for the derivation of several relations stated in the text. We adopt the definitions and conventions of Wigner's book on Group Theory [20] with two exceptions:

1) complex conjugation is indicated by a bar (e. g., $\bar{t}^{(\mathrm{L})}_{\mathrm{M}}$ is the conjugate of $t^{(\mathrm{L})}_{\mathrm{M}}$);
2) the (Hermitian) adjoint of an operator or a matrix ρ is denoted by ρ^*.

We use the scalar product notation $t.n = g_{\alpha\beta}t^{\alpha}n^{\beta}$ with:

$$g^{00} = -g^{11} = -g^{22} = -g^{33} = 1.$$

$\varepsilon_{\lambda\mu\nu\rho}$ is the totally antisymmetric pseudo tensor of the fourth rank with $\varepsilon_{0123} = 1$. Summation is understood when the same indices appear co-and contravariantly (The same applies to the $3-j$ symbols).

Summary of formulæ

$$\mathrm{D}^{(j)}(\mathrm{R})_{\lambda}{}^{\lambda'} = (-1)^{2j}\bar{\mathrm{D}}^{(j)}(\mathrm{R})^{\lambda}{}_{\lambda'}. \tag{AI.1}$$

The connection between co-and contravariant components of the $3-j$ symbols is:

$$\begin{pmatrix} j_1 & j_2 & m_3 \\ m_1 & m_2 & j_3 \end{pmatrix} = (-1)^{j_3 - m_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}. \tag{AI.2}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}\mathrm{D}^{(j_1)}(\mathrm{R})^{n_1}{}_{m_1}\mathrm{D}^{(j_2)}(\mathrm{R})^{n_2}{}_{m_2} = \mathrm{D}^{(j_3)}(\overset{*}{\mathrm{R}})^{m_3}{}_{n_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{AI.3}$$

$$\begin{pmatrix} \tilde{m} & s & \tilde{\lambda} \\ \tilde{l} & \mu & j \end{pmatrix}\begin{pmatrix} \lambda & \mathrm{L} & j \\ j & \mathrm{M} & \tilde{\lambda} \end{pmatrix}\begin{pmatrix} j & \mu & l \\ \lambda & s & m \end{pmatrix} = \begin{Bmatrix} \tilde{l} & \mathrm{L} & l \\ j & s & j \end{Bmatrix}\begin{pmatrix} \tilde{m} & \mathrm{L} & l \\ \tilde{l} & \mathrm{M} & m \end{pmatrix}. \tag{AI.4}$$

$$\mathrm{Y}^{(l)}_{m}(\vec{e})\mathrm{Y}^{(\tilde{l})}_{\tilde{m}}(\vec{e}) = \sqrt{\frac{(2l+1)(2\tilde{l}+1)(2\mathrm{L}+1)}{4\pi}}\begin{pmatrix} l & \tilde{l} & \mathrm{L} \\ m & \tilde{m} & \mathrm{M} \end{pmatrix}\begin{pmatrix} l & \tilde{l} & \mathrm{L} \\ 0 & 0 & 0 \end{pmatrix}\bar{\mathrm{Y}}^{\mathrm{M}}_{(\mathrm{L})}(\vec{e}). \tag{AI.5}$$

(summation over M and L)

$$\sum_k (-1)^{2k}(2k+1)\begin{Bmatrix} \mathrm{L} & \tilde{\mathrm{L}} & \mathrm{L}' \\ s & s & k \end{Bmatrix}\begin{Bmatrix} j & \tilde{l} & s \\ \tilde{\mathrm{L}} & k & l \end{Bmatrix}\begin{Bmatrix} j & l & s \\ k & \mathrm{L} & j \end{Bmatrix} = \begin{Bmatrix} \mathrm{L} & j & j \\ \tilde{\mathrm{L}} & \tilde{l} & l \\ \mathrm{L}' & s & s \end{Bmatrix}. \tag{AI.6}$$

$$\begin{Bmatrix} \mathrm{L} & j & j \\ 0 & l & l \\ \mathrm{L} & s & s \end{Bmatrix} = \frac{(-1)^{j+s+l+\mathrm{L}}}{\sqrt{(2l+1)(2\mathrm{L}+1)}}\begin{Bmatrix} s & j & l \\ j & s & \mathrm{L} \end{Bmatrix}. \tag{AI.7}$$

REFERENCES AND FOOTNOTES

[1] We give a list of recent works on these subjects, where references to the earlier literature can be found :
N. Byers and S. Fenster, *Phys. Rev. Letters*, t. 11, 1963, p. 52;
M. Ademollo and R. Gato, *Phys. Rev.*, t. 133, 1964, p. B 531;
M. Ademollo, R. Gato and G. Preparata, *Phys. Rev. Letters*, t. 12, 1964, p. 462;
K. Gottfried and J. D. Jackson, *Nuovo Cimento*, t. 33, 1964, p. 309 ; *Phys. Rev. Letters*, t. 8, 1964, p. 144;
J. D. Jackson and H. Pilkuhn, *Nuovo Cimento*, t. 33, 1964, p. 906 and errata.
J. D. Jackson, CERN preprint.
For a survey concerning the non-relativistic treatment of angular correlations in nuclear physics, see S. Devons and L. J. B. Goldfarb, *Handbuch der Physik*, t. 42, 1957, p. 362.

[2] E. P. Wigner, *Ann. Math.*, t. 40, 1939, p. 149. See also A. S. Wightman, *L'invariance dans la mécanique quantique relativiste*, Les Houches, 1960, p. 160-226. H. Joos, *Forstchr. Physik*, t. 10, 1962, p. 65.

[3] See e. g., U. Fano, *Rev. Modern Phys.*, t. 29, 1957, p. 74.

[4] L. Michel, *Suppl. Nuovo Cimento*, t. 14, 1959, p. 95.

[5] See e. g., H. Bacry, Thèse, Université de Marseille, 1963.

[6] If $M_{\nu\rho}$ denotes the total angular momentum tensor operator, the polarization operator is $W^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}M_{\nu\rho}P_{\sigma}$. Its mathematical significance and physical implications are discussed in L. Michel, *Loc. cit.*, ref [4].

[7] We call Poincaré transformations those of the inhomogeneous Lorentz Group $\mathfrak{L}_{+}^{\uparrow}$. In (a, Λ), a denotes the translation element and Λ the homogeneous Lorentz transformation.

[8] See e. g., H. Joos, *Loc. cit.*, ref [2].

[9] See E. P. Wigner, *Loc. cit.*, ref [2].

[10] See e. g., U. Fano and G. Racah, *Irreducible Tensorial Sets*, Academic Press Inc., New York, 1959.

[11] See L. Michel, *Loc. cit.*, ref. [4].

[12] Let us illustrate the construction of $Y_{M}^{(L)}(\vec{s}_{\alpha_1}, \ldots, \vec{s}_{\alpha_L})$ in (35) by an example:

$$Y_{1}^{(2)}(\vec{s}_{\alpha_1}, \vec{s}_{\alpha_2}) = -\frac{1}{2}\sqrt{\frac{15}{4\pi}}\left(s_{\alpha_1}^{(3)}s_{\alpha_2}^{(1)} + s_{\alpha_1}^{(1)}s_{\alpha_2}^{(3)} + is_{\alpha_1}^{(3)}s_{\alpha_2}^{(2)} + is_{\alpha_1}^{(2)}s_{\alpha_2}^{(3)}\right).$$

[13] The expression $(p_1 \wedge p)^{\mu\nu}$ is the antisymmetric tensor $p_1^{\mu}p^{\nu} - p^{\mu}p_1^{\nu}$. Note that in the rest frame of X, i. e., when $t = p/m$, $\vec{e}(p, p_1)$ is the unit 3-vector along the momentum of Y.

[14] See e. g., N. Byers and S. Fenster, *Loc. cit.*, ref. [1].

[15] E. de Rafael, *Nuovo Cimento*, t. 33, 1964, p. 237.

[16] For a recent survey concerning Strange-Particle Resonant States and references to the literature, see R. H. Dalitz, *Ann. Rev. Nuc. Sci.*, t. 13, 1963, p. 339.

[17] N. Gelfand *et al.*, *Phys. Rev. Letters*, t. 12, 1964, p. 567;
G. Benson *et al.*, *Phys. Rev. Letters*, t. 12, 1964, p. 600;
M. Aderholz *et al.*, *Phys. Rev. Letters*, t. 10, 1964, p. 240.
[18] Y. Y. Lee *et al.*, *Phys. Rev. Letters*, t. 12, 1964, p. 342.
[19] E. de Rafael, *Physics Letters*, t. 11, 1964, p. 260.
[20] E. P. Wigner, *Group Theory and its application to the quantum mechanics of atomic spectra*, Academic Press, New York, 1959.

ACKNOWLEDGMENTS

We wish to thank Professor L. Michel for many enlightening discussions and constant guidance in this work. We acknowledge gratefully the kind hospitality of Dr. L. Motchane at the Institut des Hautes Études Scientifiques and the financial support of the Fonds National de la Recherche Scientifique of Belgium for one of us (C. H.) and of the Centre National de la Recherche Scientifique for the other (E. de R.) Furthermore we would like to thank Dr. D. N. Williams for pointing out an error in Appendix II of the preprint version of this paper which concerned the properties of multipole parameters and has now been suppressed.

(Manuscrit reçu le 5 novembre 1964).