ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

NABIL OURIMI

Extension of correspondences between rigid polynomial domains

Annales de la faculté des sciences de Toulouse 6e série, tome 14, nº 3 (2005), p. 501-514

<http://www.numdam.org/item?id=AFST_2005_6_14_3_501_0>

Annales de la Faculté des Sciences de Toulouse Vol. XIV, n° 3, 2005
pp. 501–514

Extension of correspondences between rigid polynomial domains(*)

NABIL OURIMI (1)

ABSTRACT. — We study the holomorphic extension of proper holomorphic correspondences between rigid polynomial domains in $\mathbb{C}^2$ (or convex in $\mathbb{C}^n$). Moreover, we show that any irreducible correspondence between rigid polynomial domains is a mapping, if the target one is strongly pseudoconvex. This can be viewed as an extension of a result of Bell-Bedford [1] for this class of unbounded domains.

RÉSUMÉ. — Nous étudions le prolongement holomorphe des correspondances holomorphes propres entre domaines polynomiaux rigides de $\mathbb{C}^2$ (ou convexes de $\mathbb{C}^n$). Nous montrons aussi qu'une correspondance irréductible entre de tels domaines est une application, si le domaine d'arrivée est strictement pseudoconvexe. Ceci peut être vu comme une extension d'un résultat de Bell-Bedford [1] pour cette classe de domaines non bornés.

1. Introduction

A domain $D \subset \mathbb{C}^n$ is called rigid polynomial if

$$D = \{(w_o, w_1) \in \mathbb{C} \times \mathbb{C}^{n-1} : r(w_o, w_1) = 2Re(w_o) + P(w_1, \bar{w}_1) < 0\}$$

for some real polynomial $P(w_1) = P(w_1, \bar{w}_1)$. The purpose of this paper is to study the boundary regularity of proper holomorphic correspondences between rigid polynomial domains. The main result is the following

THEOREM 1.1. — *Let $f : D \to D'$ be a proper holomorphic correspondence between nondegenerate rigid polynomial domains in $\mathbb{C}^2$ (or convex in $\mathbb{C}^n$). Then we have the following stratification of the boundary :*

(*) Reçu le 16 juin 2003, accepté le 6 octobre 2004

(1) Faculté des Sciences de Bizerte, Jarzouna (Bizerte), Tunisie.
E-mail: ourimin@yahoo.com

$\partial D = S^h \cup S^\infty$ where S^h is the set of holomorphic extendability of f as a correspondence and

$$S^\infty = \{p \in \partial D : \overline{\lim_{z \longrightarrow p}} \ |f(z)| = \infty\}.$$

Moreover, S^h is an open dense subset of ∂D.

Note that the pseudoconvexity of domains is not required. A similar result was proved earlier by Coupet-Pinchuk [8] in the case of proper holomorphic mappings. For bounded domains in $\mathbb{C}^2$ with real analytic boundary the holomorphic extension of proper holomorphic correspondences was studied by K.Verma [24] under additional condition on the correspondence f^{-1}. It is a generalization of the work of Diederich-Pinchuk [12] who proved the same result at the level of proper holomorphic mappings. For other related results and without mentioning the entire list we refer the reader to [11], [9], [1], [2], [14], [15], [20].

Our second theorem concerns a version of Bedford-Bell's theorem (see theorem 3 in [1]) for this class of rigid polynomial domains.

THEOREM 1.2. — *Let $f : D \to D'$ be a proper holomorphic irreducible correspondence between nondegenerate rigid polynomial domains in $\mathbb{C}^2$ (or convex in $\mathbb{C}^n$).*

(1) If D' is strongly pseudoconvex, then f is a mapping.

(2) If f is a mapping and D is strongly pseudoconvex, then f is a biholomorphism and D' is strongly pseudoconvex.

In particular, if $f : D \to M$ is a proper holomorphic mapping from a strongly pseudoconvex nondegenerate rigid polynomial domain in $\mathbb{C}^2$ onto a complex 2-dimensional manifold, the correspondence $F = f^{-1} \circ f$ is a self-proper holomorphic correspondence of D. Theorem 2 implies that each irreducible component of F is a biholomorphic mapping. Let G be the group given by these components. Then for all $z \in D$, $f^{-1} \circ f(z) = \{g(z),\ g \in G\}$. This proves that f is factored by a finite subgroup of automorphisms. The same result holds if D is convex in $\mathbb{C}^n$ and M is a complex n-dimensional manifold.

2. Background material

Let D and D' be two domains in $\mathbb{C}^n$ and let A be a complex purely n-dimensional subvariety contained in $D \times D'$. We denote by $\pi_1 : A \to D$ and

$\pi_2 : A \to D'$ the natural projections. When π_1 is proper, $(\pi_2 \circ \pi_1^{-1})(z)$ is a non-empty finite subset of D' for any $z \in D$ and one may therefore consider the multi-valued mapping $f = \pi_2 \circ \pi_1^{-1}$. Such a map is called a holomorphic correspondence between D and D'; A is said to be the graph of f. Since π_1 is proper, in particular it is a branched analytic covering. Then, there exist an $n-1$-dimensional analytic subset $V_f \subset \operatorname{graph} f$ and an integer m such that π_1 is an m-sheeted covering map from the set $A \backslash \pi_1^{-1}(\pi_1(V_f))$ onto $D \backslash \pi_1(V_f)$. Hence, $f(z) = \{f^1(z), \ldots, f^m(z)\}$ for all $z \in D \backslash \pi_1(V_f)$ and the f^j's are distinct holomorphic functions in a neighborhood of $z \in D \backslash \pi_1(V_f)$. The integer m is called the multiplicity of f and $\pi_1(V_f)$ is its branch locus. If both π_1 and π_2 are proper then f is a proper holomorphic correspondence. If A is irreducible as an analytic set, then f is called an irreducible holomorphic correspondence. Given a holomorphic correspondence $f : D \to D'$ with graph $A \subset D \times D'$, one can find the system of canonical defining functions

$$\phi_I(z, w) = \sum_{|J| \leqslant m} \phi_{IJ}(z) w^J, \ |I| = m, \ (z, w) \in \mathbb{C}^n \times \mathbb{C}^n \tag{2.1}$$

where $\phi_{IJ}(z) \in \mathcal{O}(D)$ and A is precisely the set of common zeros of the functions $\phi_I(z, w)$ (see [7] for details).

For $(z_o, z_o') \in A$, let A_1 be an irreducible component of A containing (z_o, z_o'). Since (z_o, z_o') is isolated in the fiber above z_o in A, there exist neighborhoods $U \ni z_o$ and $U' \ni z_o$ such that the projection $\pi : A_1 \cap (U \times U') \to U$ is proper (see [7]). We denote by $f' : U \cap D \to U' \cap D'$ the correspondence defined by the analytic subset $A' = A_1 \cap (U \times U')$ which is the local correspondence obtained by isolating certain branches of the correspondences f.

DEFINITION. — *Let $f : D \to D'$ be a holomorphic correspondence between domains in $\mathbb{C}^n$ and z_o be a point in ∂D. Then f extends as a holomorphic correspondence near z_o if there exist a connected neighborhood $U \ni z_o$ in $\mathbb{C}^n$ and a closed complex analytic set $A \subset U \times \mathbb{C}^n$ of pure dimension n, which may possibly be reducible, such that,*

i) $\operatorname{graph} f \cap \{(U \cap D) \times \mathbb{C}^n\} \subset A$

ii) $\pi : A \to U$, the natural projection is proper.

Note that in general the extending correspondence of f may have more branches than the correspondence f. Let $D \subset \mathbb{C}^n$ be a rigid polynomial domain as in theorem 1. We say that D is nondegenerate if its boundary contains no nontrivial complex variety. When P is homogeneous these domains naturally appear as an approximation of domains of finite type and may be considered as their homogeneous models. These ones are useful in studies of many problems for more general domains.

Let $z = (z_o, z_1)$ and $w = (w_o, w_1)$ be points in $\mathbb{C} \times \mathbb{C}^{n-1}$. We define $r(z, \bar{w}) = \dfrac{z_o + \bar{w}_o}{2} + P(z_1, \bar{w}_1)$, the complexification of the function r. We call Segre variety of w associated to D the smooth algebraic hypersurface

$$Q_w = \{z \in \mathbb{C}^n : r(z, \bar{w}) = 0\}.$$

It can be expressed as the graph of a holomorphic function; since we can write Q_w as $Q_w = \{(h_w(z_1), z_1), z_1 \in \mathbb{C}^{n-1}\}$, where $h_w(z_1) = -\bar{w}_o - 2P(z_1, \bar{w}_1)$. Segre varieties have played an important role in the study of the boundary regularity of holomorphic correspondences and mappings when the obstructions are real analytic.

For all $p \in \partial D$ and U a neighborhood of p, we denote by the $S = S(U)$ the set of Segre varieties $\{Q_w,\ w \in U\}$ and λ the so-called Segre map defined by

$$\begin{aligned} \lambda :\ & U \to S \\ & w \mapsto Q_w \end{aligned}$$

Let $I_w := \{z : Q_w = Q_z\}$ be the fiber of λ over Q_w. For any $w \in \partial D$, the set I_w is a complex variety of ∂D (see [9] and [13]). Since the domain D is nondegenerate, then for any $w \in \partial D$, there exists a neighborhood U_w of w, such that $I_w \cap U_w$ is finite. We can also define a structure of complex analytic variety of finite dimension in S such that the map λ is a finite antiholomorphic branched covering. The set I_w contains at most as many points as the sheet number of λ. Note also that the Segre map λ is locally one to one near strictly pseudoconvex points of ∂D. We refer again the reader to the papers of Diederich-Fornaess [9] and Diederich-Webster [13] for more details and more properties of Segre varieties.

Finally, we recall that for $z \in \partial D$, the cluster set $cl_f(z)$ is defined as :

$$cl_f(z) = \{w \in \mathbb{C}^2 \cup \infty : \lim_{j \to \infty} \operatorname{dist}(f(z_j), w) = 0, \text{ for } z_j \in D,\ z_j \to z\}.$$

3. Algebraicity of proper holomorphic correspondences

In this section we shall prove the following theorem which will play a big role in the proof of our main result.

THEOREM 3.1. — *Let D and D' be nondegenerate rigid polynomial domains in $\mathbb{C}^2$. Then, any proper holomorphic correspondence $f : D \to D'$ is algebraic (i.e., its graph is contained in an algebraic set of dimension 2 in $\mathbb{C}^2 \times \mathbb{C}^2$).*

In the end of this section, Theorem 3.1 will be generalized to nondegenerate rigid polynomial convex domains in $\mathbb{C}^n$. A similar result was proved earlier by Berteloot-Sukhov in the case of bounded algebraic domains in $\mathbb{C}^n$ and by Coupet-Pinchuk [8] in the case of proper holomorphic mappings between rigid polynomial nondegenerate domains in $\mathbb{C}^n$ (see also [10] in the case of bounded algebraic domains). For the algebraicity of local holomorphism between real algebraic submanifolds, we refer the reader to [25] with references included.

For the proof of Theorem 3.1, we start by the following proposition.

PROPOSITION 3.2. — *Let $f : D \to D'$ be a proper holomorphic correspondence between nondegenerate pseudoconvex rigid polynomial domains in $\mathbb{C}^2$. Then we have the following stratification of the boundary : $\partial D = S^\infty \cup S^c$ where*

$$S^c = \{z \in \partial D : f \text{ extends continuously in a neighborhood of } z\}$$
$$S^\infty = \{p \in \partial D : \overline{\lim_{z \to p}} |f(z)| = \infty\}.$$

Furthermore, S^c is an open dense subset of ∂D.

Proof. — Let $p \in \partial D$. If the cluster set $cl_f(p)$ does not contain infinity, then according to [5] f extends continuously in a neighborhood p. Thus, we get the desired stratification. To show the density of S^c, suppose that S^∞ has an interior point $q \in \partial D$. According to Bedford-Fornaess [3] (see also [4], lemma 1) there exists a holomorphic function $h(w)$ with the following properties :

(i) $|h(w)| \simeq (|w_1|^{2k} + |w_2|^2)^{\frac{1}{N}}$ as $|w| \to \infty$ for some positive integer N;

(ii) $arg h(w) \in [-\frac{\pi}{4}, \frac{\pi}{4}]$.

Set the function $g(w) = (\alpha h(w) - 1)/(\alpha h(w) + 1)$, where $\alpha > 0$ is small enough. Then $g(w)$ is holomorphic in D and satisfies $|g(w)| < 1$ in D and $g(w) \to 1$ as $|w| \to \infty$. The function $G(z) = \prod_{1 \leqslant j \leqslant m} [g \circ f^j - 1]$ is holomorphic in $D \backslash \sigma$, $\sigma \subset D$ is an analytic set of dimension $\leqslant 1$. Since

$G(z)$ is bounded ($|G(z)| \leqslant 2^m$), it extends as a holomorphic function on D. The function $G(z) \to 0$ as z tends to a boundary point close to q. By the boundary uniqueness theorem (see for instance [7]) we get that one of the branches $f^j \equiv \infty$ on D. This contradiction completes the proof of Proposition 3.2.

Proof of Theorem 3.1. — If D is not pseudoconvex, there exist $p \in \partial D$ and a neighborhood U of p such that all functions in the representation (2.1) of f extend holomorphically to U. Moreover, we can replace p by a nearby point $q \in U \cap \partial D$ so that f splits at q to biholomorphic mappings f^j, $j \in \{1, \dots, m\}$. Now the classical Webster's theorem implies that the f^j extend to algebraic mappings; therefore, f is algebraic by the uniqueness theorem.

Now, assume that D is pseudoconvex, which implies that D' is also pseudoconvex. Since S^∞ is nowhere dense in ∂D, there exists a point $p \in \partial D$ such that f extends continuously in a neighborhood U of p and "splits" in U to holomorphic mappings f^j, $j \in \{1, \dots, m\}$ defined on $D \cap U$, which are continuous up the boundary of D. We denote $\omega(\partial D')$ the set of weakly pseudoconvex points of D'. First we will prove that $f^j(U \cap \partial D)$ is not contained in $\omega(\partial D')$ for all j. Fix $j_o \in \{1, \dots, m\}$ and assume that $f^{j_o}(U \cap \partial D) \subset \omega(\partial D')$. The set $\omega(\partial D')$ has the following local stratification

$$\omega(\partial D') = M_1 \cup \ldots \cup M_k,$$

where $M_1, \dots, M_k$ are smooth varieties with holomorphic dimension zero (see [6]). Set d to be the integer defined by :

$$d = \max\{\dim M_i : M_i \cap f^{j_o}(U \cap \partial D) \neq \emptyset\}$$

Let M_{i_o} be the variety of dimension d and $z_o \in U \cap \partial D$ such that $f^{j_o}(z_o) \in M_{i_o}$. By the continuity of f^{j_o}, there exists a small neighborhood $V \ni z_o$ so that $f^{j_o}(V \cap \partial D) \subset M_{i_o}$. Then, without loss of generality we may assume that z_o is a strongly pseudoconvex point. According to S.Sibony [18], in a neighborhood of $f^{j_o}(z_o)$ the variety M_{i_o} is contained in a smooth strongly pseudoconvex hypersurface S_{i_o}. Then, f^{j_o} is a non-constant CR mapping between strongly pseudoconvex hypersurfaces. According to [17], f^{j_o} is a C^∞ diffeomorphism. Using the reflection principle [16], it follows that f^{j_o} extends as a biholomorphism. This contradicts the fact that $f^{j_o}(U \cap \partial D) \subset \omega(\partial D')$.

Hence, there exists a point $p \in S^c \cap \partial D$ such that $f(p) \subset \partial D' \backslash \omega(\partial D')$. In view of the continuity of f we can assume that p is strongly pseudoconvex (moving slightly p if necessary). Then, the same argument as above proves that the mappings f^j extend holomorphically past p for all $j = 1, \dots, m$. Now again the classical Webster's theorem implies that the f^j extend to algebraic mappings; therefore, f is algebraic by the uniqueness theorem. □

Let $\Gamma = \{z \in \partial D : cl_f(z) \subset \partial D'\}$. Note that $\Gamma = \partial D \backslash S^\infty$ and $\Gamma \neq \emptyset$. Indeed, if D is pseudoconvex (which implies that D' is also pseudoconvex), in view of proposition 1, Γ is an open dense set in ∂D. If D is not pseudoconvex, there exist $z \in \partial D$ and a neighborhood $U \ni z$ such that all functions in the representation (2.1) of f extend holomorphically to U. Thus, $U \cap \partial D \subset \Gamma$ and so $\Gamma \neq \emptyset$.

As a consequence of Theorem 3.1, we have the following corollary.

COROLLARY 3.3. — *Γ is a dense set in ∂D.*

Proof. — Since f is algebraic, its components f^1 and f^2 are also algebraic. Then, there exist two polynomials

$$P_j(z, w_j) = a_j^{m_j}(z) w_j{}^{m_j} + \cdots + a_j^1(z) w_j + a_j^0(z), \quad j = 1, 2$$

where $a_j^{k_j}(.)$ are holomorphic polynomials for all $k_j \in \{0, \dots, m_j\}$ such that for $z \in D$, $P_1(z, f^1(z)) = P_2(z, f^2(z)) = 0$. Without loss of generality we may assume that $a_j^{m_j} \not\equiv 0$ in $\mathbb{C}^2$ for $j = 1, 2$. If $p \in S^\infty$, we have either $a_1^{m_1}(p) = 0$ or $a_2^{m_2}(p) = 0$. So the polynomial function $\tilde{a} = a_1^{m_1}.a_2^{m_2}$ vanishes identically on S^∞. If S^∞ admits an interior point, then by the uniqueness theorem the polynomial $\tilde{a}$ vanishes identically in $\mathbb{C}^2$. This completes the proof of Corollary 3.3. □

Remark 3.4. — The proof of Theorem 3.1 uses the existence of a holomorphic peak function at infinity for D (due to Bedford-Fornaess [3] in $\mathbb{C}^2$). Such a function exists also in the case of unbounded hyperbolic convex domains in $\mathbb{C}^n$. Indeed; if D is an unbounded hyperbolic convex domains in $\mathbb{C}^n$, there are n hyperplanes $H_1, \dots, H_n$ independent over $\mathbb{C}$, such that $\overline{D}$ is on one side of each of these hyperplanes. There exist complex coordinates $\tilde{Z} = (\tilde{z}_1, \dots, \tilde{z}_n)$ such that $H_j = \{\tilde{Z} \in \mathbb{C}^n : Re\tilde{z}_j = 0\}$ and D is contained in the half space $\{\tilde{Z} \in \mathbb{C}^n : Re\tilde{z}_j < 0\}$. The image of infinity by the associated Cayley transform is contained in the zero of the function $\tilde{Z} \to \prod_{1 \leqslant j \leqslant n} (\tilde{z}_j - 1)$. According to theorem 6.1.2 of [22] this is the peak set of holomorphic function. Then, Theorem 3.1 is also proved in the case of nondegenerate rigid polynomial convex domains in $\mathbb{C}^n$. It seems very likely that this theorem holds in $\mathbb{C}^n$ without the assumption of convexity of domains, but the main difficulty is to show in proposition 1 that S^∞ is nowhere dense in the boundary of D.

4. Proof of Theorem 1.1

First of all, we give a proof for domains in $\mathbb{C}^2$. Without loss of generality we may assume that the correspondence f is irreducible. Since f is algebraic, there exists an algebraic set $\mathcal{A} \subset \mathbb{C}^2 \times \mathbb{C}^2$ of dimension 2 such that the graph Γ_f of f is contained in $\mathcal{A}$. We may assume that $\mathcal{A}$ is irreducible; otherwise we consider only the irreducible component of $\mathcal{A}$ containing Γ_f. Let $\pi_1 : \mathcal{A} \to \mathbb{C}^2$ be the coordinate projection to the first component and let $E = \{z \in \mathbb{C}^2 : \dim \pi_1^{-1}(z) \geqslant 1\}$. We denote by $F : \mathbb{C}^2 \backslash E \to \mathbb{C}^2$ the multiple valued map corresponding to $\mathcal{A}$; that is,

$$F(w) = \{w' : (w, w') \in \mathcal{A}\}.$$

We denote by S_F its branch locus (i.e., $z \in S_F$ iff the coordinate projection π_1 is not locally biholomorphic near $\pi_1^{-1}(z)$).

Recall that $\Gamma = \{z \in \partial D : cl_f(z) \subset \partial D'\}$. To prove Theorem 1.1, we need to prove that $\Gamma \cap E = \emptyset$ (i.e., for all $z \in \Gamma$, $\pi_1^{-1}(z)$ is discrete). The proof (Lemma 4.1 and Lemma 4.2) uses the ideas of Shafikov developed in [20] to study the analytic continuation of holomorphic correspondences and equivalence of domains. For the sake of completeness we recall it here.

LEMMA 4.1. — *If $a \in \Gamma$ and $Q_a \not\subset E$, then $a \notin E$*

Proof of Lemma 4.1. — By contradiction, suppose that $a \in E$. Since $Q_a \not\subset E$, there exist a point $b \in Q_a$ and a small neighborhood $U_b \ni b$ such that $U_b \cap E = \emptyset$. We may choose small neighborhoods U_a and U_b such that for any $z \in U_a$, the set $Q_z \cap U_b$ is non-empty and connected. Let $\Sigma = \{z \in U_a : Q_z \cap U_b \subset S_F\}$. Since the boundary contains no nontrivial complex variety and $\dim_{\mathbb{C}} S_F = 2 - 1$, by shrinking U_a if necessary, Σ will be a finite set. Following the ideas in [9] and [12] we define

$$X = \{(w, w') \in (U_a \backslash \Sigma) \times \mathbb{C}^2 : F(Q_w \cap U_b) \subset Q'_{w'}\}.$$

We follow the convention of using the right prime to denote the objects in the target domain. For instance, $Q'_{w'}$ will mean the Segre variety of w' with the respect to the hypersurface $\partial D'$. Note that the choose of U_a and U_b such that for any $z \in U_a$, the set $Q_z \cap U_b$ is non-empty and connected, is to avoid ambiguity in the condition $F(Q_w \cap U_b) \subset Q'_{w'}$; since different components of $Q_w \cap U_b$ could be mapped to different Segre varieties.

We prove the following properties on the set X.

Claim. —

i) X is not empty

ii) X is a complex analytic set

iii) X is closed

iv) $\Sigma \times \mathbb{C}^2$ is a removable singularity for X.

Proof. — • Since $\dim E \leqslant 2 - 1$ and Γ is a dense set in the boundary (see Corollary 3.3), there exists a sequence $\{a_j\} \subset (U_a \cap \Gamma)\backslash(E \cup \Sigma)$ such that $a_j \to a$ as $j \to \infty$. Let $w \in Q_{a_j} \cap U_b$ be an arbitrary point, and let $w' \in F(w)$. It follows from the invariance property of Segre varieties under holomorphic correspondences [24] that $F(Q_w \cap U_b) \subset Q'_{w'}$. But $a_j \in Q_w$, so $a'_j \in Q'_{w'}$ for all $a'_j \in F(a_j)$. Since $w \in Q_{a_j} \cap U_b$ was arbitrary, we conclude that $F(Q_{a_j} \cap U_b) \subset Q'_{a'_j}$. Thus, $(a_j, a'_j) \in \mathcal{A}$ and so $\mathcal{A} \neq \emptyset$.

• Let $(w, w') \in X$. Consider an open simply connected set $\Omega \subset U_b \backslash S_F$ such that $Q_w \cap \Omega \neq \emptyset$. The branches of F are globally defined in Ω. Since $Q_w \cap U_b$ is connected, the inclusion $F(Q_w \cap U_b) \subset Q'_{w'}$ is equivalent to

$$F^j(Q_w \cap \Omega) \subset Q'_{w'}\ j = 1, \ldots, m,$$

where the F^j denote the branches of F in Ω. Let $r'(w, \bar{w})$ be a defining function of D'. The inclusion $F^j(Q_w \cap \Omega) \subset Q'_{w'}\ j = 1, \ldots, m$ can be expressed as

$$r'(F^j(z), \bar{w'}) = 0 \quad \text{for any } z \in Q_w \cap \Omega\ , j = 1, \ldots, m.$$

Since $Q_w = \{(h_w(z_1), z_1), z_1 \in \mathbb{C}\}$ where $h_w(z_1) = -\bar{w}_o - 2P(z_1, \bar{w}_1)$, we obtain

$$r'(F^j(h_w(z_1), z_1)\ , \bar{w'}) = 0, \text{ for any } z_1 \in \mathbb{C}. \tag{4.1}$$

Thus, X is defined by an infinite system of holomorphic equations in $(\bar{w}, \bar{w}')$. By the Noetherian property of the ring of holomorphic functions, we can choose finitely many points $z_1^1, \ldots, z_1^m$ so that (4.1) can be written as a finite system

$$\sum_{|J| \leqslant d} \alpha_J^k(w) {w'}^J = 0.$$

where $k = 1, \ldots, m$, d is the degree of r' in w' and α_J^k holomorphic in w. Thus, X is a complex analytic set in $(U_a \backslash \Sigma) \times \mathbb{C}^2$.

• The set X is closed in $(U_a \backslash \Sigma) \times \mathbb{C}^2$. Indeed, let (w^j, w'^j) a sequence in X that converges to $(w^o, w'^o) \in (U_a \backslash \Sigma) \times \mathbb{C}^2$, as $j \to \infty$. Since $Q_{w^j} \to Q_{w^o}$ and $Q'_{w'^j} \to Q'_{w'^o}$, from the inclusion $F(Q_w^j \cap U_b) \subset Q'^j_{w'}$ we obtain $F(Q_{w^o} \cap U_b) \subset Q'_{w'^o}$, which implies that $(w^o, w'^o) \in X$ and thus X is a closed set.

• Now, let us show that $\Sigma \times \mathbb{C}^2$ is a removable singularity for X. Let $p \in \Sigma$. It follows that $\bar{X} \cap (\{p\} \times \mathbb{C}^2) \subset \{p\} \times \{z' : \ F(Q_p) \cap U_b \subset Q_{z'}\}$. If $w' \in F(Q_p) \subset Q_{z'}$, then $z' \in Q_{w'}$. Since $\dim_{\mathbb{C}} Q_{w'} = 1$, then $\{z' : F(Q_p) \cap U_b \subset Q_{z'}\}$ has dimension at most 2 and $\bar{X} \cap (\Sigma \times \mathbb{C}^2)$ has dimension 4-measure zero. Now, Bishop's theorem can be applied to conclude that $\Sigma \times \mathbb{C}^2$ is a removable singularity for X. $\square$

We continue now with the proof of Lemma 4.1. First of all, notice that for small neighborhoods $U_j \ni a_j$ (a_j as defined in the proof of the claim) we have :

$$X|_{U_j \times \mathbb{C}^2} = \mathcal{A}|_{U_j \times \mathbb{C}^2}. \tag{4.2}$$

We denote again by X the closure of X in $U_a \times \mathbb{C}^2$. Without loss of generality we may assume that X is irreducible, then in view of (4.2) and by the uniqueness theorem (see for instance [7]) we have :

$$X|_{U_a \times \mathbb{C}^2} = \mathcal{A}|_{U_a \times \mathbb{C}^2}.$$

Let $\hat{F}$ be the multiple valued mapping corresponding to $\bar{X}$ (the closure of X in $U_a \times \mathbb{C}^2$). By construction, for any $a' \in \hat{F}(a)$, $\hat{F}(a) = I'_{a'}$. Since $\hat{F}(a) \cap \partial D'$ is not empty, it follows that $\hat{F}(a) \subset \partial D'$ and so $\hat{F}(a)$ is a finite set. Therefore, there exists a bounded part Γ' of $\partial D'$ such that $\hat{F}(a) \subset \Gamma'$. Thus, we can choose U_a such that $\bar{X} \cap (U_a \times \partial U') = \emptyset$, where U' is a bounded open neighborhood of Γ'. Otherwise; there exists a sequence $(z_j, z'_j)_j$ in X such that $(z_j)_j$ converges to a and $(z'_j)_j$ converges to $z'_o \in \partial U'$. This implies that $(a, z'_o) \in \bar{X}$ and $z'_o \notin \Gamma'$. This contradiction completes the proof of Lemma 4.1. $\square$

Now, we can conclude that $\Gamma \cap E = \emptyset$ by using the following lemma whose proof is defered until the end of this section.

LEMMA 4.2. — *There exists a holomorphic change of variables, such that in the new coordinates* $Q_a \not\subset E$.

Conclusion of the proof of Theorem 1.1. — In view of Lemma 4.1 and Lemma 4.2, we conclude that $\Gamma \cap E = \emptyset$. Thus, for $a \in \Gamma$ there exists a small neighborhood U_a of a such that $U_a \cap E = \emptyset$. It follows that for any bounded neighborhood U' of $cl_f(a)$ the set $\mathcal{A} \cap (\{a\} \times U')$ is finite. Hence, we can assume that $\mathcal{A} \cap (\{a\} \times \partial U') = \emptyset$. Now, we can choose U_a so small that $\mathcal{A} \cap (U_a \times \partial U') = \emptyset$; otherwise , there exists a sequence $(z_j, z'_j)_j$ in $\mathcal{A}$ such that $(z_j)_j$ converges to a and $(z'_j)_j$ converges to $z'_o \in \partial U'$. This implies that $(a, z'_o) \in \mathcal{A}$ and $z'_o \in \partial U'$: a contradiction. This shows that $\mathcal{A} \cap (U_a \times U')$ defines a holomorphic correspondence from U_a onto U'.

If we assume that the domains are convex in $\mathbb{C}^n$, then the correspondence is algebraic and so we can repeat the same argument of proof. □

We complete this section with the proof of Lemma 4.2 (the proof is given for domains in $\mathbb{C}^n$).

Proof. — Assume that $Q_a \subset E$. From proposition 4.1 of [19] there exists a point $p \in \Gamma \backslash E$ such that $Q_a \cap Q_p \neq \emptyset$. In view of [20] (Lemma 3.1), there exists a neighborhood V of Q_p such that the germ F of the correspondence defined at p (as in the conclusion of the proof Theorem 1.1) extends holomorphically to $V \backslash (\Lambda_1 \cup \Lambda_2)$, where $\Lambda_1 = \pi({\pi'}^{-1}(H_o))$, $H_o \subset \mathbb{P}^n$ is the hypersurface at infinity, $\pi : \bar{X} \to V$ and $\pi' : \bar{X} \to \mathbb{P}^n$ are the natural projections, $X = \{(w, w') \in (V \backslash \Sigma) \times \mathbb{C}^n : F(Q_w \cap U_b) \subset Q'_{w'}\}$ and $\bar{X}$ is the closure of X in $V \times \mathbb{P}^n$; it is well defined since D' is algebraic. Note that Λ_1 is a complex manifold of dimension at most $n-1$ in $\mathbb{C}^n$ and $\Lambda_2 = \pi\{(w, w') \in \mathcal{A} : \dim \pi^{-1}(w) \geqslant 1\}$ is a complex analytic set of dimension at most $n-2$. By dimension considerations, we may assume that $Q_a \cap V \not\subset \Lambda_2$. We may also assume that $Q_a \cap V \not\subset \Lambda_1$; since we can defined a linear fractional transformation such that H_o is mapped onto another complex hyperplane $H \subset \mathbb{P}^n$. Thus by the holomorphic extension along Q_p, we can find points in Q_a where F extends as a holomorphic correspondence. This implies that in the new coordinates $Q_a \not\subset E$. □

5. Proof of Theorem 1.2

We start this section by the following proposition

PROPOSITION 5.1. — *Let z_o be a point in ∂D and z'_o be a strong pseudoconvexity point in $cl_f(z_o)$. Then for small neighborhoods $U \ni z_o$ and $U' \ni z'_o$ the local correspondence $f' : U \cap D \to U' \cap D'$ obtained by isolating certain branches of f is a mapping which extends as a holomorphic mapping to U.*

Proof. — Since f is algebraic, there exists an algebraic irreducible set $\mathcal{A}$ containing the graph of f. Then for small neighborhoods $U \ni z_o$ and $U' \ni z'_o$, $\mathcal{A} \cap (U \times U')$ defines a holomorphic correspondence that extends the correspondence f'. We denote by F the multi-valued mapping corresponding to $\mathcal{A} \cap (U \times U')$. Let $\lambda : U \to S := \{Q_w : w \in U\}$ and $\lambda' : U' \to S' := \{Q'_{w'} : w' \in U'\}$ be the Segre maps. We denote by E' the critical set of λ' and by S_F the branch locus of F. First, we verify $S_F \subset F^{-1}(E')$. Let $x \in S_F$. Then, there exist sequences $(x_n)_n$ converging to x and (y_n) and (z_n) converging to $x' \in F(x)$ such that $y_n \in F(x_n)$ and $z_n \in F(x_n)$ with $y_n \neq z_n$. In view of corollary 4.2 of [12], there exists a single valued mapping $\varphi : S \to S'$ such that $\lambda' \circ F = \varphi \circ \lambda$. Then we conclude that $\lambda'(y_n) = \lambda'(z_n)$. Hence λ' is not one to one in a neighborhood of x'. This implies that $x' \in E'$ and so $x \in F^{-1}(E')$. Since $U \cap \partial D$ is strongly pseudoconvex, λ' is one to one antiholomorphic mapping and so E is empty. This implies that S_F is also empty. Then (by shrinking U and U' if necessary) $F : U \cap D \to U' \cap D$ is a proper holomorphic mapping which extends as a holomorphic mapping in view of [12] (see also [21]). $\square$

Proof of Theorem 1.2. — (1) Let f be a proper holomorphic correspondence as in Theorem 1.2. Note that the domain D is pseudoconvex; since D' is strongly pseudoconvex. We shall prove that the branch locus S_f of f is empty. By contradiction assume $S_f \neq \emptyset$. The correspondence is algebraic, then S_f and its image $f(S_f)$ are algebraic. Let W' be an irreducible component of $f(S_f)$. First of all, we show that $\overline{W'} \cap \partial D'$ is not empty. Let h be an irreducible polynomial in $\mathbb{C}^2$ such that $W' = \{\xi \in D' : h(\xi) = 0\}$. If W' does not extend across $\partial D'$, the defining function r' of D' will be negative in $\hat{W}' = \{\xi \in \mathbb{C}^2 : h(\xi) = 0\}$. According to [7] (see prop.2, pp.76), there exists an analytic cover $\pi : \hat{W}' \to \mathbb{C}$. Let $g_1, \ldots, g_k$ be the branches of π^{-1} which are locally defined and holomorphic in $\mathbb{C} \backslash \sigma$, with $\sigma \subset \mathbb{C}$ an discrete set. Consider the function $\hat{r}'(w) = sup\{r' \circ g_1(w), \ldots, r' \circ g_k(w)\}$. Since π is an analytic cover, $\hat{r}'$ extends as a subharmonic function in $\mathbb{C}$. Then it is constant; since it is negative. This contradicts the fact that the domain D' is nondegenerate. So, we conclude that the variety $f(S_f)$ extends across the boundary $\partial D'$. Let $z'_o \in \overline{f(S_f)} \cap \partial D'$ and $z_o \in \overline{S_f} \cap \partial D$ such that $z'_o \in cl_f(z_o)$. Proposition 5.1 implies that the local correspondence obtained by isolating certain branches of f is a mapping near z_o. This contradicts the fact that $z_o \in \overline{S_f}$. Thus, $S_f = \emptyset$. The correspondence f is irreducible and the domain D is simply connected (since it is homeomorphic to $\{(z'_o, z') : \mathcal{R}e(z'_o) < 0\}$), therefore f is a mapping.

(2) Let f be a mapping from D onto D'. If D is strongly pseudoconvex, then in view of (1) of Theorem 1.2 the correspondence f^{-1} is a mapping from D' onto D. It follows that f is a biholomorphism. Let $q = (q_o, q_1)$ be a point in $\partial D'$ and let us consider the complex line $L = \{(z_o, z) \in \mathbb{C}^2, z = q_1\}$. The line L intersects D' by the set $\Omega = \{(z_o, z) : Rez_o < -P'(q_1),\ z = q_1\}$ and the restriction F' of f^{-1} to Ω is algebraic. Therefore there exists a finite subset $\mathcal{S}$ of $l = \partial\Omega = \{(z_o, z) \in \mathbb{C}^2 : z = q_1,\ Rez_o = -P'(q_1)\}$ such that F' extends holomorphically to $l \backslash \mathcal{S}$. According to [8] (theorem 2.1), f^{-1} extends holomorphically to $l \backslash \mathcal{S}$. Replacing $f^{-1}(Z)$ by $f^{-1}(Z - Z^1)$ with $Z^1 = (Req_o + P'(q_1),\ 0)$, we get a biholomorphism from $D' \to D$ which extends holomorphically in a neighborhood of q. Applying the same arguments to f, we obtain a local biholomorphism in a neighborhood of q. This shows that q is strongly pseudoconvex. Since q is an arbitrary point in $\partial D'$, we get the required result.

If the domains are convex in $\mathbb{C}^n$ the proof is the same; since f is algebraic. □

I would like to thank the referee for his useful remarks and suggestions on this material.

Bibliography

[1] BEDFORD (E.), BELL (S.). — Boundary behavior of proper holomorphic correspondences, Math. Ann. 272, p. 505-518 (1985).

[2] BEDFORD (E.), BELL (S.). — Boundary continuity of proper holomorphic correspondences, Lecture Notes in Mathematics, 1198, p. 47-64 (1983-84).

[3] BEDFORD (E.), FORNAESS (J.E.). — A construction of peak functions on weakly pseudoconvex domains, Ann. of Math. 107, p. 555-568 (1978).

[4] BEDFORD (E.), PINCHUK (S.). — Domains in $\mathbb{C}^{n+1}$ with non-compact automorphisms group, J. Geom. Anal. 1, p. 165-191 (1991).

[5] BERTELOOT (F.), SUKHOV (A.). — On the continuous extension of holomorphic correspondences, Ann. Scuola Norm. Sup. Pisa XXIV, p. 747-766 (1997).

[6] CATLIN (D.). — Boundary invariants of pseudoconvex domains, Ann. of Math., 120, p. 529-586 (1984).

[7] CHIRKA (E.M.). — Complex Analytic Sets, Kluwer Academic Publishers (1989).

[8] COUPET (B.), PINCHUK (S.). — Holomorphic equivalence problem for weighted homogeneous rigid domains in $\mathbb{C}^{n+1}$, in collection of papers dedicated to B. V. Shabat, Nauka, Moscow, p. 111-126 (1997).

[9] DIEDERICH (K.), FORNAESS (J.E.). — Proper holomorphic mappings between real analytic pseudoconvex domains in $\mathbb{C}^n$ Math. Ann. 282, p. 681-700 (1988).

[10] DIEDERICH (K.), FORNAESS (J.E.). — Applications holomorphes propres entre domaines à bord analytique réel. C. R. Acad. Sci. 307, p. 681-700 (1988).

[11] DIEDERICH (K.), PINCHUK (S.). — Regularity of continuous CR-maps in arbitrary dimension, Michigan Math. J. 51, no.1, p. 111-140 (2003).

[12] DIEDERICH (K.), PINCHUK (S.). — Proper holomorphic maps in dimension 2 extend, Indiana Univ. Math. J. 44, p. 1089-1126 (1995).

[13] DIEDERICH (K.), WEBSTER (S.). — A reflexion principle for degenerate real hypersurfaces, Duke Math. J. 47, p. 835-845 (1980).

[14] PINCHUK (S.). — On the boundary behavior of analytic set and algebroid mappings. Soviet Math. Dokl. Akad. Nauk USSR 268, p. 296-298 (1983).

[15] PINCHUK (S.). — Holomorphic mappings in $\mathbb{C}^n$ and the problem of holomorphic equivalence, Encyl. Math. Sci.9, p. 173-199 (1989).

[16] PINCHUK (S.). — On the analytic continuation of holomorphic mappings, Math. USSR Sb. 27, p. 375-392 (1975).

[17] PINCHUK (S.), TSYGANOV (Sh.). — Smoothness of CR mappings between strongly pseudoconvex hypersurfaces, Math. USSR Izvestga 35, p. 457-467 (1990).

[18] SIBONY (N.). — Une classe de domaines pseudoconvexes, Duke Math. Journal, Vol. 55, No.2, p. 299-319 (1987).

[19] SHAFIKOV (R.). — Analytic continuation of Germs of Holomorphic Mappings, Michigan Math. J. 47. (2000).

[20] SHAFIKOV (R.). — Analytic continuation of holomorphic correspondences and equivalence of domains in $\mathbb{C}^n$, Invent. Math. 152, p. 665-682 (2003).

[21] SHAFIKOV (R.),VERMA (K.). — A local extension theorem for proper holomorphic mappings in $\mathbb{C}^2$, J. Geom. Anal. 13, no.4, p. 697-714 (2003).

[22] RUDIN (W.). — Function theory in polydiscs, Mathematics lecture note in series.

[23] WEBSTER (S.). — On the mapping problem for algebraic real hypersurfaces, Invent. Math. 43, p. 53-68 (1977).

[24] VERMA (K.). — Boundary regularity of correspondences in $\mathbb{C}^2$, Math. Z. 231, p. 253-299 (1999).

[25] ZAITSEV (D.). — Algebraicity of local holomorphisms between real algebraic submanifolds of complex spaces, Acta Math. 183, p. 273-305 (1999).