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Galois properties of linear differential equations (*)

MARIUS VAN DER PUT ⁽¹⁾

Dedicated to Jean Écalle for the anniversary of his transseries.

ABSTRACT. — The theme of this paper is rationality, field of definition and descent for linear differential equations over some differential field. The results extend and supplement [H-P, Ho-P, P]. The questions and results are mainly of an algebraic nature, with the exception of differential equations over fields of convergent Laurent series.

RÉSUMÉ. — Le thème de ce papier est rationalité, corps de définition et descent pour des équations différentielles linéaires sur un corps différentiel. Les résultats étendent et complètent [H-P, Ho-P, P]. Les questions et les résultats sont pour la plupart de nature algébrique à l'exception des équations différentielles sur des corps de séries de Laurent convergentes.

1. Introduction

Let k be a differential field of characteristic 0. Its field of constants will be denoted by C_k . Let $\overline{C_k}$ denote the algebraic closure of C_k and put $\underline{k} = \overline{C_k}k$. The skew ring of differential operators over k is denoted by $k[\partial]$. A monic differential operator $L \in k[\partial]$ may factor in $\underline{k}[\partial]$ as a product L_1L_2 of monic operators such that L_1, L_2 belong to $k'[\partial]$ for some finite extension k' of k , contained in \underline{k} . The theme of this paper is to determine the fields k' which are involved in these factorizations. The translation of this question in terms of differential modules can be phrased as follows:

Let M denote the differential module over k corresponding to L , i.e., $M = k[\partial]/k[\partial]L$. Put $\underline{M} = \underline{k} \otimes_k M$. A submodule $N \subset \underline{M}$ will be called

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rational over a field k' with $k \subset k' \subset \underline{k}$ if there exists a submodule N' of $k' \otimes_k M$ such that $N = \underline{k} \otimes_{k'} N'$.

A related notion is the following. A differential module A over (say) \underline{k} descends to a field k' (with say, $k \subset k' \subset \underline{k}$) if there exists a differential module B over k' and an isomorphism $A \cong \underline{k} \otimes_{k'} B$.

Let M be a differential module over k and consider a proper submodule N (if any) of \underline{M} . Over which field k' is N rational and for which fields k' does N descend to k' ?

We make some observations. The module \underline{M} is equal to $\underline{k}[\partial]/\underline{k}[\partial]L$. The submodules N of \underline{M} are in 1-1 correspondence with the monic right hand factors $L_2 \in \underline{k}[\partial]$ of L . This correspondence is given by $L_2 \mapsto \underline{k}[\partial]L_2/\underline{k}[\partial]L$. Let $N \subset \underline{M}$ correspond to L_2 . The group $\text{Gal}(\underline{k}/k) = \text{Gal}(\overline{C_k}/C_k)$ acts in an obvious way on \underline{M} and on $\underline{k}[\partial]$. For $\sigma \in \text{Gal}(\underline{k}/k)$ one has that $\sigma(N)$ corresponds to $\sigma(L_2)$. In particular, $\sigma(N) = N$ if and only if $\sigma(L_2) = L_2$. Furthermore, N is rational over k' if and only if $\sigma(N) = N$ for all $\sigma \in \text{Gal}(\underline{k}/k')$. In particular, let H denote the open subgroup of $\text{Gal}(\underline{k}/k)$ consisting of the σ with $\sigma(N) = N$. The fixed field \underline{k}^H is the smallest field of rationality for N . We will call this field *the field of definition of N* . The above shows that the given translation from differential operators to differential modules is correct.

An important notion, introduced in [Ho-P], is that of a *skew differential field F over k* . This will mean here the following: F is a skew field containing k , k lies in the center of F , the dimension of F over k is finite, and F has a differentiation which extends the differentiation of k . A *differential module M over F* is a finite dimensional left vector space over F , provided with an additive map $\partial : M \rightarrow M$ satisfying $\partial(fm) = f\partial(m) + f'm$ for all $f \in F$ and $m \in M$. By restriction of scalars, M is also a differential module over k . Using skew differential fields over k one can produce in this way differential modules over k with rather special properties.

The above questions are studied in [H-P] and in recent preprints [Ho-P] and [P]. Here we review some of the material (from a somewhat different point of view) and provide some new complementary results, in particular for differential fields of convergent Laurent series.

2. Rationality for submodules

The results of this section extend and correct [H-P], Corollary 4.2 part 2. We note that a mistake in this corollary (namely $\frac{n}{2d}$ should be $\frac{n}{d}$) was discovered by Mark van Hoeij.

THEOREM 2.1. — *Let M be a differential module over k of dimension n . Let $N \subset \underline{M}$ be an irreducible submodule and $N \neq 0, \underline{M}$. Suppose that there is no submodule $A \neq N$ with $A \cong N$.*

(a) *The field of definition k' of N satisfies $[k' : k] \leq \frac{n}{\dim N}$.*

(b) *Suppose moreover that M is irreducible. Then $C := \text{End}_{k[\partial]}(M)$ is a finite field extension of C_k . For a suitable C_k -linear embedding $C \subset \overline{C_k} \subset \underline{k}$ one has $k' = Ck$. Let Z denote M , regarded as differential module over k' . Then $\underline{k} \otimes_{k'} Z \cong N$.*

Proof. — (a) The sum of all $\sigma(N)$, with $\sigma \in \text{Gal}(\underline{k}/k)$, is a submodule \tilde{N} that is rational over k . Let H be the open subgroup of finite index r of $\text{Gal}(\underline{k}/k)$ consisting of the elements σ with $\sigma(N) = N$. Then N is rational over the fixed field k' of H and $[k' : k] = r$. Let $\sigma_1, \dots, \sigma_r$ denote representatives of $\text{Gal}(\underline{k}/k)/H$ with $\sigma_1 = 1$. Then \tilde{N} is the sum of the $\sigma_i(N)$ for $i = 1, \dots, r$. This sum is direct since $\sigma_i(N) \not\cong \sigma_j(N)$ for $i \neq j$. Hence $[k' : k] \leq \frac{n}{\dim N}$.

(b) $\underline{M} = \sigma_1(N) \oplus \dots \oplus \sigma_r(N)$ since M is irreducible. $E := \text{End}_{\underline{k}[\partial]}(\underline{M})$ is the product of r copies of $\overline{C_k}$, since the $\sigma_i(N)$ are irreducible and pairwise non-isomorphic. Put $C := \text{End}_{k[\partial]}(M)$. Then $\overline{C_k} \otimes_{C_k} C \cong E$ (by [Hö-P], lemma 2.3). Since M is irreducible one has that C is a commutative field and $[C : C_k] = r$. Let Z denote M , considered as differential module over Ck . Then

$$\underline{M} = \underline{k} \otimes_k Z = (\underline{k} \otimes_k kC) \otimes_{kC} Z.$$

Furthermore, $\underline{k} \otimes_k kC$ is isomorphic to the product of r copies of \underline{k} . Therefore \underline{M} is the direct sum of $\underline{k} \otimes_{kC} Z$ taken over all k -linear embeddings of kC into \underline{k} . For the embedding of kC into \underline{k} , which is induced from the embedding of C into the first factor of the product $\overline{C_k} \times \dots \times \overline{C_k}$, one has $\underline{k} \otimes_{kC} Z \cong \sigma_1(N) = N$. Finally, one easily sees that $C = \overline{C_k}^H$ and thus $Ck = \underline{k}^H$ is the field of definition of N . \square

Examples 2.2. — *Example for Theorem 2.1 part (b).* For $\lambda \in \overline{\mathbf{Q}}$ one defines the 1-dimensional differential module $E(\lambda) := \mathbf{Q}(x)e_\lambda$ over $\overline{\mathbf{Q}}(x)$ by $\partial e_\lambda = \frac{\lambda}{x}e_\lambda$. Let $\lambda_1, \dots, \lambda_r$ be all solutions in $\overline{\mathbf{Q}}$ of some irreducible polynomial over \mathbf{Q} . Suppose that $\lambda_i - \lambda_j \notin \mathbf{Z}$ for $i \neq j$. On the differential module $A := E(\lambda_1) \oplus \dots \oplus E(\lambda_r)$ over $\overline{\mathbf{Q}}(x)$ one defines an action of $\text{Gal} := \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) = \text{Gal}(\overline{\mathbf{Q}}(x)/\mathbf{Q}(x))$ by $\sigma e_{\lambda_i} = e_{\sigma(\lambda_i)}$ and $\sigma(fa) = \sigma(f)\sigma(a)$ for $f \in \overline{\mathbf{Q}}(x)$ and $a \in A$. This action commutes with ∂ . Then $M = A^{\text{Gal}}$ is a differential module over $\mathbf{Q}(x)$ and $\overline{\mathbf{Q}}(x) \otimes_{\mathbf{Q}(x)} M = A$. Thus M and $N := E(\lambda_1)$ is an example for (b). Clearly $\mathbf{Q}(\lambda_1)$ is the field of definition

of N . Put $Z := \mathbf{Q}(\lambda_1)(x)e_{\lambda_1}$. Then $\overline{\mathbf{Q}}(x) \otimes_{\mathbf{Q}(\lambda_1)(x)} Z \cong N$ (for the inclusion $\mathbf{Q}(\lambda_1)(x) \subset \overline{\mathbf{Q}}(x)$). Furthermore, Z , considered as differential module over $\mathbf{Q}(x)$, is isomorphic to M . Indeed, $\overline{\mathbf{Q}}(x) \otimes_{\mathbf{Q}(x)} Z$ is isomorphic to $(\overline{\mathbf{Q}}(x) \otimes_{\mathbf{Q}(x)} \mathbf{Q}(\lambda_1)(x)) \otimes_{\mathbf{Q}(\lambda_1)(x)} Z$. Furthermore, $\overline{\mathbf{Q}}(x) \otimes_{\mathbf{Q}(x)} \mathbf{Q}(\lambda_1)(x)$ is isomorphic a product r copies of $\overline{\mathbf{Q}}(x)$. It follows that $\overline{\mathbf{Q}}(x) \otimes_{\mathbf{Q}(x)} Z$ is isomorphic to A . Also $\overline{\mathbf{Q}}(x) \otimes_{\mathbf{Q}(x)} M \cong A$. By [Ho-P], lemma 2.3, one has $Z \cong M$.

The above theorem can be supplemented with an algorithm in case $k = C_k(x)$. Let $L \in k[\partial]$ be monic, irreducible of degree n . Put $M := k[\partial]/k[\partial]L$. Write $e \in M$ for the image of 1. There are several efficient ways to calculate a basis of $C := \text{End}_{k[\partial]}(M)$ over C_k (see [P-S]). A multiplication table for C can be calculated. e is also a cyclic vector for Z , which is M seen as a Ck -differential module of dimension $d = \frac{n}{[C:C_k]}$. The monic operator $P \in Ck[\partial]$ of degree d with $Pe = 0$ can be calculated by some linear algebra. Then P is a right hand factor of L . The irreducible monic right hand factors in $\underline{k}[\partial]$ of L are the images of P in $\underline{k}[\partial]$ obtained by the C_k -linear embeddings of C into $\overline{C_k} \subset \underline{k}$.

THEOREM 2.3. — *Let M be a differential module over k of dimension n . Let $N \subset \underline{M}$ be an irreducible submodule and $N \neq 0, \underline{M}$. Suppose that there is a submodule $A \neq N$ with $A \cong N$.*

(a) *There exists a submodule $A \cong N$ having a field of definition k' with $[k' : k] \leq \frac{n}{\dim N}$.*

(b) *Suppose moreover that M is irreducible. Then $F^\circ := \text{End}_{k[\partial]}(M)$ is a skew field of dimension $s^2 > 1$ over its center C . Put $r := [C : C_k]$.*

Assume that $F := F^\circ \otimes_{C_k} k$ is again a skew field.

Then F is a skew differential field. Let Z denote M , considered as a differential module over F . For a suitable maximal commutative subfield $G^\circ \supset C$ of F° and a suitable embedding of $G := G^\circ \otimes_{C_k} k$ into \underline{k} one has $\underline{k} \otimes_G Z \cong N$. In particular, $\dim N$ is a multiple of s .

The finite field extensions of minimal degree $k' \supset k$, contained in \underline{k} , such that there exists a submodule $A \subset \underline{M}$, with $A \cong N$ and field of definition k' , have the form Sk where $S \supset C$ is a maximal commutative subfield of F° .

(c) *Suppose that the differential field k has the form $C_k(x)$. Then:*

(c1) *The assumption in (b) is satisfied.*

(c2) *For the case $\dim N = 1$ in (a), the estimate $[k' : k] \leq \frac{n}{2}$ holds.*

Proof. — (a) Let N^+ denote the sum of all submodules of \underline{M} that are isomorphic to N . Let $H \subset \text{Gal}(\underline{k}/k)$ be the open subgroup consisting of the σ with $\sigma(N^+) = N^+$. Let $1 = \sigma_1, \dots, \sigma_r$ denote representatives of $\text{Gal}(\underline{k}/k)/H$. Then $\sigma_1(N^+) \oplus \dots \oplus \sigma_r(N^+)$ is rational over k . Hence N^+ is rational over a field k^+ with $[k^+ : k] \leq \frac{n}{\dim N^+}$. We have to show that N^+ contains an irreducible $A \cong N$ which is rational over an extension $k' \supset k^+$ with $[k' : k^+] \leq \frac{\dim N^+}{\dim N}$.

For notational convenience we suppose that $k^+ = k$ and that $N^+ = \underline{M}$. Since \underline{M} is a sum of irreducible modules it is also a direct sum $N_1 \oplus \dots \oplus N_s$ of irreducible modules, isomorphic to N . Then $\text{End}_{\underline{k}[\partial]}(\underline{M})$ is isomorphic to the algebra $\text{Matr}(s, \overline{C_k})$ of all $s \times s$ -matrices with entries in $\overline{C_k}$. The algebra $B := \text{End}_{\underline{k}[\partial]}(M)$ has the property that the canonical map $\overline{C_k} \otimes_{C_k} B \rightarrow \text{End}_{\underline{k}[\partial]}(\underline{M})$ is an isomorphism (see [Ho-P], lemma 2.3). Hence B is a simple algebra of dimension s^2 over its center C_k . Let C , with $C_k \subset C \subset B$, be a maximal commutative subfield. It is known that $[C : C_k] = s$ and that C is a splitting field for B , i.e., $C \otimes_{C_k} B \cong \text{Matr}(s, C)$. Put $k' = Ck$ and $M' = k' \otimes_k M$. Then $\text{End}_{k'[\partial]}(M') = C \otimes_{C_k} B \cong \text{Matr}(s, C)$. Let $P \in C \otimes_{C_k} B$ satisfy $P^2 = P$ and the rank of P , considered as element of $\text{Matr}(s, C)$, is 1. Then the image $A' = P(M')$ is a submodule such that $\underline{k} \otimes_{k'} A'$ is irreducible and isomorphic to N . This proves (a).

(b) Let N^+ denote the sum of all submodules $A \subset \underline{M}$ that are isomorphic to N . Let $H \subset \text{Gal}(\underline{k}/k)$ denote the stabilizer of N^+ . Then H is an open subgroup of finite index r . One chooses representatives $1 = \sigma_1, \dots, \sigma_r$ of $\text{Gal}(\underline{k}/k)/H$. Since M is irreducible one has $\underline{M} = \bigoplus \sigma_i(N^+)$. Furthermore, N^+ is the direct sum of $s > 1$ copies of N . One concludes that $E := \text{End}_{\underline{k}[\partial]}(\underline{M}) = \prod_{i=1}^r \text{End}_{\underline{k}[\partial]}(\sigma_i(N^+))$. Thus E is the product of r copies of the matrix algebra $\text{Matr}(s, \overline{C_k})$. Put $F^o = \text{End}_{\underline{k}[\partial]}(M)$. From M irreducible and $\overline{C_k} \otimes_{C_k} F^o \cong E$ one concludes that F^o is a skew field of dimension rs^2 over C_k . The center C of F^o has the property that $\overline{C_k} \otimes_{C_k} C$ is the center of E and hence isomorphic to $\overline{C_k}^r$. It follows that $[C : C_k] = r$. The algebra $F := F^o \otimes_{C_k} k$ is given a differentiation by $(f \otimes a)' = f \otimes a'$ for all $f \in F^o$ and $a \in k$.

By assumption F is a skew field of dimension s^2 over its center Ck . The left action of F on M , makes M into a differential module Z over F . By restricting the scalars from F to k , one obtains M again. The dimension of \underline{M} is equal to $rs \cdot \dim N$. Let z be the dimension of M as vector space over F . Then the dimension of M over k is equal to zs^2r . It follows that $\dim N$ is divisible by s .

Let $G^\circ \supset C$ be a maximal commutative subfield of F° and write $G = G^\circ k$. Then G is a maximal commutative subfield of F and Z can be considered as a differential module over G . Now $\underline{M} = \underline{k} \otimes_k Z = (\underline{k} \otimes_k G) \otimes_G Z$. Furthermore, $\underline{k} \otimes_k G$ is a product of rs copies of \underline{k} . This yields a decomposition of \underline{M} as a direct sum of rs submodules. One of them is isomorphic to N .

A finite extension $S \supset C_k$, contained in $\overline{C_k}$, has the property that $Sk \otimes_k M$ contains a submodule D with $\overline{k} \otimes_{Sk} D \cong N$ if and only if $Sk \otimes_k M$ is a direct sum of rs^2 irreducible submodules. This condition is equivalent to $S \otimes_{C_k} F^\circ$ is isomorphic to $\prod_{i=1}^r \text{Matr}(s, S)$. The latter is equivalent to $S \supset C$ and S is a splitting field for F° . The extensions S of minimal degree, having that property, are the maximal commutative subfields of F° containing C . See [Ho-P] for more details.

(c1) follows easily from the observation that $F^\circ \otimes_{C_k} C_k(x) \cong F^\circ \otimes_C C(x)$.

(c2) Let N be a submodule of \underline{M} of dimension 1. With the above notation, one has that $\sigma_1(N^+) \oplus \dots \oplus \sigma_r(N^+)$ has field of definition k . We may suppose that this module is equal to \underline{M} . By (b), we find that M is reducible, since $\dim N = 1$. Now \underline{M} and M are semi-simple and M has a decomposition $M_1 \oplus \dots \oplus M_t$ into irreducible submodules of the same dimension. For some $i \in \{1, \dots, t\}$ the projection of N to \underline{M}_i is injective. We replace now N by its isomorphic image in \underline{M}_i . By (2) (b) we have that \underline{M}_i contains no submodule $A \neq N$ with $A \cong N$. Thus we can apply Theorem 2.1, part (a) and therefore $[k' : k] \leq \frac{n}{t} \leq \frac{n}{2}$. \square

Examples 2.4. — Example for Theorem 2.3 part (a) $k = \mathbf{Q}(s, t)$ with $s^2 + t^2 = -1$ and differentiation given by $s' = 1$ and $t' = -st^{-1}$. The 2-dimensional vector space $M := k^2$ over k is made into a differential module by $\partial(a_1, a_2) = (a_1' - a_2/2t, a_2' + a_1/2t)$. A calculation shows (compare section 2.5 of [Ho-P]) that $\text{End}_{k[\partial]}(M)$ is isomorphic to $\mathbf{H} = \mathbf{Q}1 + \mathbf{Q}i + \mathbf{Q}j + \mathbf{Q}k$, the standard quaternion field over \mathbf{Q} . Then the algebra of the $\overline{\mathbf{Q}}(s, t)[\partial]$ -linear endomorphisms of \underline{M} is isomorphic to the matrix algebra $\text{Matr}(2, \overline{\mathbf{Q}})$. Thus \underline{M} has a 1-dimensional submodule. Moreover, all the 1-dimensional submodules of \underline{M} are isomorphic. For every splitting field C of degree two over \mathbf{Q} , i.e., the fields $\mathbf{Q}(\sqrt{-m})$ where m is the sum of ≤ 3 rational squares, there is a 1-dimensional submodule N of \underline{M} with field of definition C .

One observes that the assumption of Theorem 2.3 (b) is not satisfied. Indeed, k is a splitting field for \mathbf{H} and thus $\mathbf{H} \otimes k \cong \text{Matr}(2, k)$. Moreover, the conclusions of Theorem 2.3 (b) and (c2) are not valid for this example.

Example for Theorem 2.3 part (b) In [Ho-P] many examples are given. They

are constructed as follows. Consider a differential field $k = C_k(x)$ and a skew field F° of dimension $s^2 > 1$ over its center C_k . One makes the skew field $F = F^\circ \otimes_{C_k} k$ into a skew differential field by defining $(f \otimes a)' = f \otimes a'$ for $f \in F^\circ$ and $a \in k$. One considers a suitable differential module M over F of dimension z . Then M considered as differential module over k has the properties: M is irreducible, $\text{End}_{k[\partial]}(M) = F^\circ$, $\text{End}_{\underline{k}[\partial]}(\underline{M}) \cong \text{Matr}(s, \overline{C_k})$, and \underline{M} has an irreducible submodule N of dimension sz .

The simplest example is given by $k = \mathbf{Q}(x)$, $F^\circ = \mathbf{H}$ the standard quaternion field over \mathbf{Q} , $M = Fe$ and ∂ on M is given by $\partial e = (i + jx)e$. The differential operator corresponding to M and the cyclic vector e is

$$L_4 = \partial^4 + (2 + 2x^2)\partial^2 + 4x\partial + (4 + 2x^2 + x^4).$$

L_4 is irreducible as element of $\mathbf{Q}(x)[\partial]$. For every splitting field $C \subset \overline{\mathbf{Q}}$ of \mathbf{H} , there is a decomposition L_4 as product of two monic irreducible operators of degree 2 in $C(x)[\partial]$. For the case $C = \mathbf{Q}(i)$ this decomposition reads

$$(\partial^2 + x^{-1}\partial + (-x^{-2} - ix^{-1} + 1 + x^2)) \cdot (\partial^2 - x^{-1}\partial + (ix^{-1} + 1 + x^2)).$$

Suppose that $k = C_k(x)$ and let $L \in k[\partial]$ denote a monic irreducible operator. We suppose that $M := k[\partial]/k[\partial]L$ satisfies the condition of Theorem 2.3. Let $e \in M$ denote the image of 1. An algorithm for finding factorizations of L is the following. One calculates a basis over C_k of $F^\circ = \text{End}_{k[\partial]}(M)$ and a multiplication table w.r.t. this basis. From these data it is easy to calculate $C \supset C_k$, the center of F° . Let $f \in F^\circ$ be a “generic” element. Then $G^\circ = C(f)$ has degree s over C and thus G° is a maximal commutative subfield of F° , containing C . Now M , as differential module over $G = G^\circ k$, has again e as cyclic vector. Using linear algebra, one determines the monic differential operator of smallest degree $P \in G[\partial]$ with $Pe = 0$. After taking a C_k -linear embedding of G° into $\overline{C_k} \subset \underline{k}$, one obtains a right hand factor of L of minimal degree in $\underline{k}[\partial]$.

3. Descent for differential modules

Consider a Galois extension $K \supset k$ of differential fields such that $K = C_K k$. Let G denote $\text{Gal}(K/k) = \text{Gal}(C_K/C_k)$. For a differential module M over K and any $\sigma \in G$ one defines the twisted differential module ${}^\sigma M$ by: ${}^\sigma M = M$ as additive group, ${}^\sigma M$ has a new scalar multiplication given by $f * m = \sigma^{-1}(f)m$ (for $f \in K$ and $m \in M$) and the derivation ∂ of ${}^\sigma M$ coincides with the ∂ of M . A *necessary condition* for M to descend to k is the existence of an isomorphism $\phi(\sigma) : {}^\sigma M \rightarrow M$ for every $\sigma \in G$. The

descent problem asks whether this condition is sufficient, or more generally for which intermediate fields ℓ (i.e., $k \subset \ell \subset K$) M descends to ℓ .

The results of [Ho-P] and [P] concerning this question can be formulated as:

Suppose that $K = C_K k$, C_K/C_k is a finite Galois extension with group G , A is a differential module over K with $\text{End}_{K[\partial]}(A) = C_K$ and ${}^\sigma A \cong A$ for all $\sigma \in G$. Then:

(1) *If k is a field of formal Laurent series, say, $k = C_k((x))$, then A descends to k .*

(2) *If k is a field of convergent Laurent series, say $k = \mathbf{R}(\{x\})$, $K = \mathbf{C}(\{x\})$, then A will in general not descend to k . The obstruction to descent is determined by the Stokes matrices belonging to M .*

(3) *If k is a function field in one variable, say, $k = C_k(x)$, then there is a (skew) field F° over C_k such that M descends to ℓ if and only if the field of constants of ℓ is a splitting field for F° .*

The descent problem is related to the rationality results for irreducible submodules formulated in Theorem 2.3. This can be formulated as follows.

PROPOSITION 3.1. — *Let N be an irreducible module over \underline{k} . Suppose that ${}^\sigma N \cong N$ for all $\sigma \in \text{Gal}(\underline{k}/k)$.*

(1) *There exists an irreducible module M over k and an embedding $N \subset \underline{M}$.*

(2) *The module M , with property (1), is unique up to isomorphism.*

(3) *N descends to a finite extension k' of k with $k' \subset \underline{k}$ if and only if there exists a submodule $A \subset \underline{M}$ with $A \cong N$ and A has field of definition k' .*

Proof. — There is a finite Galois extension $K \supset k$ contained in \underline{k} such that N descends to K . Thus $N = \underline{k} \otimes_K B$ for some differential module B over K . One regards B as a differential module over k . By [Ho-P], B is a semi-simple k -differential module and moreover, $\underline{k} \otimes_k B$ is the direct sum of $[K : k]$ copies of N . Indeed,

$$\underline{k} \otimes_k B = (\underline{k} \otimes_k K) \otimes_K B \cong \bigoplus_{\sigma \in \text{Gal}(K/k)} {}^\sigma N \cong N \oplus \cdots \oplus N.$$

Then $\overline{C_k} \otimes_{C_k} \text{End}_{k[\partial]}(B) \cong \text{End}_{\underline{k}[\partial]}(N \oplus \cdots \oplus N) \cong \text{Matr}([K : k], \overline{C_k})$. Therefore $\text{End}_{k[\partial]}(B) \cong \text{Matr}(d, F^\circ)$ for some d and some (skew) field F° with

center C_k . Hence B is a direct sum $B_1 \oplus \cdots \oplus B_d$ of isomorphic irreducible differential modules over k and $\text{End}_{k[\partial]}(B_1) = F^o$. Clearly N can be seen as a submodule of $\underline{k} \otimes_k B_1$. Thus $M = B_1$ has the required property (1).

Suppose that $M(1)$ has also property (1). Then $N = \underline{k} \otimes_K B \subset \underline{M}(1) = \underline{k} \otimes_k M(1)$ and $\underline{B} := \underline{k} \otimes_k B \cong N^{[K:k]} \subset \underline{M}(1)^{[K:k]}$. The last module is semi-simple and there is a projection $P : \underline{M}(1)^{[K:k]} \rightarrow \underline{M}(1)^{[K:k]}$, commuting with ∂ and with image \underline{B} . For any $\sigma \in \text{Gal}(\underline{k}/k)$ one can form the conjugate $\sigma P \sigma^{-1}$. This is again a projection commuting with ∂ and with image \underline{B} . Let $\sigma_i P \sigma_i^{-1}$, $i = 1, \dots, s$ denote the distinct conjugates of P . Then $Q := \frac{1}{s} \sum \sigma_i P \sigma_i^{-1}$ is again a projection, commuting with ∂ and with image \underline{P} . Since Q also commutes with the action of $\text{Gal}(\underline{k}/k)$ on $\underline{M}(1)^{[K:k]}$, one has that Q maps $\underline{M}(1)^{[K:k]}$ onto \underline{B} . Now \underline{B} is a direct sum of copies of \underline{M} and therefore $\underline{M} \cong \underline{M}(1)$. This proves (2).

The “if” part of (3) is obvious. Suppose that N descends to k' . We may suppose that $k' \subset K$, where K is the finite Galois extension of (1). Then $N = \underline{k} \otimes_{k'} C$ and $B = K \otimes_{k'} C$ satisfies $N = \underline{k} \otimes_K B$. As in (1), $B = B_1 \oplus \cdots \oplus B_d$ as k -differential module. The inclusion $C \subset B$ induces an inclusion $\underline{k} \otimes_k C \subset \bigoplus_{i=1}^d \underline{k} \otimes_k B_i$. Moreover, $A := \underline{k} \otimes_{k'} C$ can be considered as a submodule of $\underline{k} \otimes_k C$. Hence A is a submodule of $\underline{k} \otimes_k B_i$ for some i . Now $B_i \cong M := B_1$ for all i and thus A can be considered as a submodule of \underline{M} . \square

Proposition 3.1 has a translation in terms of differential operators. Let $L \in \underline{k}[\partial]$ be a monic irreducible differential operator and suppose that for any $\sigma \in \text{Gal}(\underline{k}/k)$, the operator $\sigma(L)$ is equivalent to L (in the sense that the two operators define isomorphic differential modules). Let $N := \underline{k}[\partial]/\underline{k}[\partial]L$ be the corresponding differential module. Let L_1, \dots, L_s denote the conjugates of L for the group $\text{Gal}(\underline{k}/k)$. Then L^+ denotes the least common right multiple of L_1, \dots, L_s in the skew ring $\underline{k}[\partial]$ (i.e., L^+ is the monic generator of the right ideal $\bigcap_{i=1}^s L_i \underline{k}[\partial]$). Then $L^+ \in k[\partial]$, $L^+ = LA$ for some $A \in \underline{k}[\partial]$ and thus $\underline{k}[\partial]/\underline{k}[\partial]L^+$ contains the submodule $\underline{k}[\partial]A/\underline{k}[\partial]LA$ which is isomorphic to N . The operator L^+ can be reducible in $k[\partial]$. But L^+ is semi-simple by construction and for any irreducible right hand factor $R \in k[\partial]$ of L^+ one has $M \cong k[\partial]/k[\partial]R$. Part (3) of 3.1 reads now: L descends to k' if and only if R has a monic irreducible right hand factor in $\underline{k}[\partial]$ with coefficients in k' .

4. Galois structure on the solution space

Closely related to rationality and descent is Galois structure on the solution space of a differential module. Let A be a differential module over \underline{k} . Let $K \supset \underline{k}$ denote a Picard-Vessiot extension for A . The solution space of A is then $V := \ker(\partial, K \otimes A)$. Let G and G^+ denote the groups of the differential automorphisms of K/\underline{k} and K/k . There is an obvious exact sequence

$$1 \rightarrow G \rightarrow G^+ \xrightarrow{pr} \text{Gal}(\overline{C_k}/C_k),$$

where for any $g \in G^+$, the element $pr(g)$ is the restriction of g to $\overline{C_k}$. Interesting questions are: what is the image of pr and does pr have a quasi-section?

Example. — $k = \mathbf{Q}(x)$, $\underline{k} = \overline{\mathbf{Q}}(x)$, $A = \underline{k}e$ with $\partial e = \frac{\lambda}{x}e$ and $\lambda \in \overline{\mathbf{Q}} \setminus \mathbf{Q}$. The image H of pr consists of the $\sigma \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ such that $\sigma(\lambda) = \lambda$ or $\sigma(\lambda) = -\lambda + a$ for a uniquely determined $a \in \mathbf{Z}$. Moreover, there is a section $H \rightarrow G^+$.

In particular, $G^+ \rightarrow \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is surjective if and only if the minimal polynomial of λ over \mathbf{Q} has the form $T^2 + aT + b$ with $a \in \mathbf{Z}$.

Proof. — The Picard-Vessiot field K is $\overline{\mathbf{Q}}(x, t)$ with t transcendental over $\overline{\mathbf{Q}}(x)$ and $t' = \frac{\lambda}{x}t$. Any σ such that either $\sigma(\lambda) = \lambda$ or $\sigma(\lambda) = -\lambda + a$ has an obvious extension to an element $\sigma^+ \in G^+$, given by the formula $\sigma^+t = t$ in the first case and by $\sigma^+t = x^at^{-1}$ in the second case. Moreover $(\sigma\tau)^+ = \sigma^+\tau^+$.

Suppose that σ extends to an $\tau \in G^+$. A calculation shows that the equation $y' = \frac{\mu}{x}y$ with $\mu \in \overline{\mathbf{Q}}$ has a non-zero solution in K if and only if $\mu \in \mathbf{Z}\lambda + \mathbf{Z}$. Now $\tau(t)' = \frac{\sigma(\lambda)}{x}\tau(t)$. Therefore $\sigma(\lambda) \in \mathbf{Z}\lambda + \mathbf{Z}$. Also $\sigma^{-1}(\lambda) \in \mathbf{Z}\lambda + \mathbf{Z}$. Hence $\sigma(\lambda) = \pm\lambda + a$ with $a \in \mathbf{Z}$. Suppose that $\sigma(\lambda) = \lambda + a$, then $\sigma^n(\lambda) = \lambda + na$ for any $n \geq 1$. Hence $a = 0$.

The last assertion is an immediate consequence. □

LEMMA 4.1. — *Let M be a differential module over $k = C_k(x)$ and let K be the Picard-Vessiot field for $\underline{k} \otimes_k M$. Then $pr : G^+ \rightarrow \text{Gal}(\overline{C_k}/C_k)$ is surjective and there exists a section $s : \text{Gal}(\overline{C_k}/C_k) \rightarrow G^+$.*

Proof. — It suffices to prove the existence of a section s . Consider a point $x = a \in C_k$ at which M has no singularity. Then $C_k((x - a)) \otimes M$ is a trivial differential module and so is $\overline{C_k}((x - a)) \otimes M$. Put $V = \ker(\overline{C_k}((x - a)) \otimes M)$. Then V is a full solution space of \underline{M} , i.e.,

the dimension of V over $\overline{C_k}$ is equal to the dimension of M over k . The field, generated over $\underline{k} \subset \overline{C_k}((x-a))$ by the coordinates of all elements of V with respect to a fixed basis of M over k , is a Picard-Vessiot field for \underline{M} and can be identified with K . Moreover, V can be identified with the above solution space inside $K \otimes \underline{M}$. Now $\text{Gal}(\overline{C_k}/C_k)$ acts in an obvious way on $\overline{C_k}((x-a))$. The induced action of $\text{Gal}(\overline{C_k}/C_k)$ on $\overline{C_k}((x-a)) \otimes M$ commutes with ∂ . Hence the space $V \subset \overline{C_k}((x-a)) \otimes M$ is stable under this action. Therefore K is also stable under this action and we obtain the required section $s : \text{Gal}(\overline{C_k}/C_k) \rightarrow G^+$. \square

COROLLARY 4.2. — *Suppose that $k = C_k(x)$. Let B be a differential module over \underline{k} such that ${}^\sigma B$ has the same Picard-Vessiot field as B , for every $\sigma \in \text{Gal}(\overline{C_k}/C_k)$. Then $G^+ \rightarrow \text{Gal}(\overline{C_k}/C_k)$ is surjective and has a section.*

Proof. — B descends to some finite Galois extension k' of k , contained in \underline{k} . Thus $B = \underline{k} \otimes_{k'} C$ for some C . Let M denote C , considered as differential module over k . Then $\underline{k} \otimes_k M$ is isomorphic to the direct sum $\bigoplus {}^\sigma C$, taken over all $\sigma \in \text{Gal}(k'/k)$. By assumption, $\underline{k} \otimes_k M$ and B have the same Picard-Vessiot field. Now the above statement follows from Lemma 4.1. \square

OBSERVATION 4.3. — *Galois structure on the solution space V of a differential module $\underline{M} = \underline{k} \otimes_k M$. Rationality properties for submodules of \underline{M} .*

As in lemma 4.1 we assume that $k = C_k(x)$. We will use the above notations. The natural action of G^+ on $K \otimes_k M$ induces an action of G^+ on the solution space $V = \ker(\partial, K \otimes_k M)$. Moreover, this action of G^+ on $K \otimes_k M$ induces the natural action of $\text{Gal}(\overline{C_k}/C_k)$ on $\underline{k} \otimes_k M = \underline{M}$.

The choice of a section $s : \text{Gal}(\overline{C_k}/C_k) \rightarrow G^+$ provides V with a C_k -linear action of $\text{Gal}(\overline{C_k}/C_k)$. By [S], Proposition 3, p.159, the C_k -vector space

$$V_0 := \{v \in V \mid \sigma(v) = v \text{ for all } \sigma \in \text{Gal}(\overline{C_k}/C_k)\}$$

has the property that the natural map $\overline{C_k} \otimes_{C_k} V_0 \rightarrow V$ is a bijection. We will formulate this property by “ V has a C_k -structure”. For the section s , considered in the proof of Lemma 4.1, one has $V_0 = \ker(\partial, C_k((x-a)) \otimes M)$. The C_k -structure of V depends of course on the choice of s .

There is a 1-1 correspondence between the submodules N of \underline{M} and the G -invariant subspaces W of V . This correspondence is given by $N \mapsto W := \ker(\partial, K \otimes N) \subset V = \ker(\partial, K \otimes \underline{M})$. Let N correspond to W . Let H be

the open subgroup of finite index of $\text{Gal}(\overline{C_k}/C_k)$ that stabilizes N . Then $pr^{-1}H \subset G^+$ is the stabilizer of W and moreover, H is the stabilizer of W for the chosen C_k -structure on V . Let $C' \subset \overline{C_k}$ denote the fixed field of H . Then W is defined over C' and the field of definition of N is $k' = C'(x)$. One concludes that rationality properties of G -invariant subspaces W of V (and the corresponding $N \subset \underline{M}$) do not depend on the choice of the section s .

5. Fields of convergent Laurent series

The field of convergent Laurent series over \mathbf{R} will be denoted by $k = \mathbf{R}(\{x\})$. Then $\underline{k} = \mathbf{C}(\{x\})$ is the field of convergent Laurent series over the complex numbers. The main observation is that there are examples for part (b) of Theorem 2.3 in this situation. This is in contrast with the formal case, i.e., differential equations over the fields $\mathbf{R}((x))$ and $\mathbf{C}((x))$.

We are looking for an example of an irreducible differential module M over $\mathbf{R}(\{x\})$ and an irreducible submodule N of $\underline{M} = \mathbf{C}(\{x\}) \otimes_{\mathbf{R}(\{x\})} M$ with $N \neq 0, \underline{M}$ such that \underline{M} contains a submodule A with $A \neq N$ and $A \cong N$. For the construction of an example we need a skew differential field F of finite dimension over its center $k = \mathbf{R}(\{x\})$. The most obvious example is $F = \mathbf{H} \otimes_{\mathbf{R}} \mathbf{R}(\{x\})$, where \mathbf{H} is the quaternion field over \mathbf{R} . The differentiation on F is given by $(h \otimes r)' = h \otimes r'$ for any $h \in \mathbf{H}$ and $r \in \mathbf{R}(\{x\})$. According to part (b) of Theorem 2.3, M must have the structure of a differential module over the skew differential field F . The cheapest choice that one can make is $M = Fe$ and $\partial e = de$ for a suitable element $d \in F$. This choice works. The explicit example $d = i + x^{-1}j$ is treated in detail in [P] (in the context of descent). One obtains a differential operator of degree 4 in $\mathbf{R}(\{x\})[\delta]$ (where $\delta = x \frac{d}{dx}$), namely

$$L_4 = \delta^4 + 2\delta^3 + (3 + 2x^{-1})\delta^2 + (2 - 2x^{-2})\delta + (2 + 4x^{-2} + x^{-4}),$$

which has the properties:

$L_4 \in \mathbf{R}(\{x\})[\delta]$ is irreducible.

L_4 factors in $\mathbf{C}(\{x\})[\delta]$ (in many ways) as a product of irreducible operators of degree 2. One right hand factor is for instance $L_2 := \delta^2 + \delta + (1 + x^{-2} - i)$. The complex conjugate $\tilde{L}_2 := \delta^2 + \delta + (1 + x^{-2} + i)$ is obviously also a right hand factor of L_4 and is moreover equivalent to L_2 .

L_4 factors in $\mathbf{R}((x))[\delta]$ as a product of irreducible operators of degree 2.

One can construct large classes of differential operators in $\mathbf{R}(\{x\})[\delta]$ with these somewhat bizarre properties.

One way of explaining these examples is that a differential module over, say, $\mathbf{C}(\{x\})$ can be seen as a differential module over $\mathbf{C}((x))$, provided with the additional data of a family of Stokes matrices. The possibilities for these additional data form a finite dimensional affine space over \mathbf{C} (see [P-S]). These Stokes matrices are responsible for the above phenomena. It seems that similar examples can be constructed for non-linear differential equations over $\mathbf{R}(\{x\})$ and their solutions written in terms of transseries.

There are only a few skew fields of finite dimension over their center $\mathbf{R}(\{x\})$. In fact, the Brauer group of this field is $(\mathbf{Z}/2\mathbf{Z})^2$. More room for examples seems to be available for the field of convergent Laurent series over the p -adic numbers, i.e., for the differential field $\mathbf{Q}_p(\{x\})$. There are many skew differential fields F of finite dimension over their center $\mathbf{Q}_p(\{x\})$. Indeed, the Brauer group of the field \mathbf{Q}_p is known to be \mathbf{Q}/\mathbf{Z} . Let F° be a skew field of finite dimension over its center \mathbf{Q}_p . The skew field $F = F^\circ \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\{x\})$ has the same dimension over its center $\mathbf{Q}_p(\{x\})$. Moreover, F becomes a skew differential field by the formula $(f \otimes a)' = f \otimes a'$ for any $f \in F^\circ$ and $a \in \mathbf{Q}_p(\{x\})$.

The calculation of the full Brauer group $\text{Br}(k)$ goes as follows. The field $\underline{k} = \overline{\mathbf{Q}_p} \otimes_{\mathbf{Q}_p} k$ is a C_1 -field and has trivial Brauer group. As a consequence $\text{Br}(k) = H^2(\text{Gal}(\underline{k}/k), \underline{k}^*)$. There is a Galois equivariant isomorphism

$$\overline{\mathbf{Q}_p}^* \times \mathbf{Z} \times \overline{\mathbf{Q}_p} \otimes_{\mathbf{Q}_p} x\mathbf{Q}_p\{x\} \rightarrow \underline{k}^*, \text{ given by } (c, n, f) \mapsto cx^n \exp(f).$$

Hence $\text{Br}(k) \cong \text{Br}(\mathbf{Q}_p) \times H^2(\text{Gal}(\underline{k}/k), \mathbf{Z})$. Furthermore,

$$H^2(\text{Gal}(\underline{k}/k), \mathbf{Z}) \cong H^1(\text{Gal}(\underline{k}/k), \mathbf{Q}/\mathbf{Z}).$$

The last group is isomorphic to the group of the continuous homomorphisms $\widehat{\mathbf{Z}} \rightarrow \mathbf{Q}/\mathbf{Z}$.

In the final part of this paper we consider a differential field $K := k(\{x\})$, where k is a complete valued field containing \mathbf{Q}_p . We want to compare the classification of differential modules over K and over $\widehat{K} := k((x))$, in order to see whether there is indeed room for examples with “strange” descent properties due to skew differential fields with center K . As we will show, there is no multisummation theory for differential equations over K and there are no Stokes matrices. The difference between differential equations over K and the formal theory, i.e., differential equations over $\widehat{K} = k((x))$, lies in the appearance of p -adic Liouville numbers. An element $\lambda \in \mathbf{Z}_p$ is called a p -adic Liouville number if $\liminf_{n \rightarrow \infty} |\lambda - n|^{1/n} = 0$. In other words, λ is a p -adic Liouville number if λ is too well approximated by positive

integers. This terminology and some of the following results were discovered in [C], and rediscovered in [P80]. The basic example which explains the role of p -adic Liouville numbers is the differential equation

$$(\delta - \lambda)y = \frac{1}{1-x}, \text{ where } \delta := x \frac{d}{dx} \text{ and } \lambda \in \mathbf{Q}_p, \lambda \notin \mathbf{Z}.$$

This equation has the unique formal power series solution $\sum_{n \geq 0} \frac{1}{n-\lambda} x^n$. This solution is divergent precisely when λ is a p -adic Liouville number.

First we recall the *classification of differential modules over $\widehat{K} = k(\!(x)\!)$* , where k is any algebraically closed field. Let δ denote the differentiation $x \frac{d}{dx}$. A differential module $M = (M, \delta)$ over \widehat{K} is called *regular singular* if there exists a $k[[x]]$ -lattice $M_0 \subset M$ (i.e., $M_0 = k[[x]]b_1 + \dots + k[[x]]b_m$ for some basis b_1, \dots, b_m of M over \widehat{K}) which is invariant under δ . One can show that for any regular singular differential module M there exists a basis b_1, \dots, b_m such that the matrix of δ w.r.t. this basis has coefficients in k and moreover the difference of (distinct) eigenvalues of this matrix is not an integer. For $q \in x^{-1}k[x^{-1}]$ one defines the differential module $E(q) = \widehat{K}e$ of dimension 1 by $\delta(e) = qe$. For distinct elements $q_1, \dots, q_s \in x^{-1}k[x^{-1}]$ and regular singular differential modules M_1, \dots, M_s one considers the differential module $\oplus_{i=1}^s E(q_i) \otimes M_i$. The differential modules obtained in this way are called “unramified differential modules” over \widehat{K} . Let us call the above decomposition the *eigenvalue decomposition* of the unramified module. The q_1, \dots, q_s will be called the *eigenvalues* of the unramified module. For any integer $n \geq 1$ we put $\widehat{K}_n := k(\!(x^{1/n})\!)$. The classification of differential modules over \widehat{K} can be formulated as follows:

For any differential module M over \widehat{K} there exists a smallest integer $n \geq 1$ such that $\widehat{K}_n \otimes_{\widehat{K}} M$ is an unramified differential module over \widehat{K}_n . Moreover, the eigenvalue decomposition of $\widehat{K}_n \otimes_{\widehat{K}} M$ is unique.

The following result seems to be new. It gives the first step towards a classification of differential modules over $K = k(\{x\})$, where k is an algebraically closed and complete field containing \mathbf{Q}_p . For convenience we only formulate the result for the unramified case.

PROPOSITION 5.1. — *The eigenvalue decomposition. Let M be a differential module over K . Suppose that $\widehat{M} := \widehat{K} \otimes_K M$ is unramified. Let the decomposition of \widehat{M} be $\oplus_{i=1}^s E(q_i) \otimes N_i$ (with the above notations). Then there exists a unique decomposition $M = \oplus_{i=1}^s E(q_i) \widehat{\otimes} M_i$, with all M_i regular singular. Moreover, the canonical map $M \rightarrow \widehat{M}$ yields*

isomorphisms $\widehat{K} \otimes M_i \rightarrow N_i$. In other words, the eigenvalue decomposition of \widehat{M} is already present over K .

This result is completely opposite to the complex case, i.e., for differential modules over the differential field $\mathbf{C}(\{x\})$. Indeed, the divergence of the eigenvalue decomposition in the complex case lies at the center of the theory of multisummation and the Stokes matrices. For the proof of the above proposition one has to examine the steps in the proof of the formal case (see for instance Chapter 3 of [P-S]) and show that no divergence occurs. We will not carry this out here.

In the sequel of this paper, k is a complete valued field containing \mathbf{Q}_p . Again $K = k(\{x\})$. For the study of regular singular differential equations over K , the matrix form is more convenient. We will use the notation $\delta + A$ with $A \in \text{Matr}(m, k\{x\})$, for a regular singular differential equation. Write $A = A_0 + A_1x + A_2x^2 + \dots$ with all $A_i \in \text{Matr}(m, k)$. The eigenvalues of A_0 can be shifted over integers by using a transformation $B(\delta + A)B^{-1}$ with $B \in \text{GL}(m, K)$. We may and will suppose that the eigenvalues of A_0 do not differ by an integer.

PROPOSITION 5.2. — Regular singular modules.

Let $\delta + A$ be a regular singular differential equation over K . There exists a unique $B \in \text{GL}(m, k[[x]])$ with $B_0 = 1_m$ such that $B(\delta + A_0)B^{-1} = \delta + A$. Moreover, B is convergent, i.e., $B \in \text{GL}(m, k\{x\})$, if no difference of the eigenvalues of A_0 is a p -adic Liouville number.

Proof. — Write $A = A_0 + A_1x + \dots$ and $B = B_0 + B_1x + B_2x^2 + \dots$ with $B_0 = 1_m$. The equation $B(\delta + A_0) = (\delta + A)B$ is equivalent to a sequence of matrix equations

$$nB_n + A_0B_n - B_nA_0 + \sum_{i=1}^n A_iB_{n-i} = 0 \text{ for } n \geq 1.$$

The operator $T_n : X \mapsto nX + A_0X - XA_0$ on the vector space of the $m \times m$ -matrices over k has eigenvalues $n + \lambda_i - \lambda_j$ (all i, j), where $\lambda_1, \dots, \lambda_s$ are the distinct eigenvalues of A_0 . By assumption, 0 is not an eigenvalue of any T_n . Therefore the sequence of equations has a unique solution. The assumption that $\lambda_i - \lambda_j$ is not a Liouville number for any pair i, j , implies that the norm of the inverse T_n^{-1} of T_n can be bounded by R^n for some $R > 0$. From this the convergence of B follows. \square

Example. — The transformation B of Proposition 5.2, satisfying

$$B\left(\delta + \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}\right)B^{-1} = \delta + \begin{pmatrix} \lambda & x(1+x)^{-1} \\ 0 & 0 \end{pmatrix},$$

is divergent if and only if $-\lambda$ is a p -adic Liouville number.

The classification of regular singular differential equations over K , is in contrast to the complex case, a rather complicated combinatorial problem. Although there is enough room for skew differential fields, the use of p -adic Liouville numbers does not seem to lead to strange descent problems. More precisely we formulate the following conjecture.

Conjecture. — Let $k \supset \ell$ be a finite Galois extension of complete valued fields containing \mathbf{Q}_p . Write $K = k(\{x\})$ and $L = \ell(\{x\})$. Suppose that the differential module M over K has the property that ${}^\sigma M \cong M$ for every σ in the Galois group of K/L . Then M descends to L .

By Proposition 5.1, it suffices to verify this conjecture for regular differential modules M over K . We will prove the conjecture in the special case that M has dimension 2. One can reduce to the situation where M corresponds to the matrix differential operator $\delta + A = B(\delta + \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix})B^{-1}$. In case B is convergent, the conjecture is trivial. Suppose that B is divergent. There are three cases to investigate. We consider one of them, namely $A = \begin{pmatrix} \lambda & a \\ 0 & 0 \end{pmatrix}$. The other two cases can be treated in a similar way.

Now B has the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and b is a divergent solution of the equation $b' + \lambda b = -a$. In particular $-\lambda$ is a p -adic Liouville number. A computation shows that the group of the automorphisms of $\delta + A$ is k^* . For every σ in the Galois group of k/ℓ there is given a $B(\sigma) \in \text{GL}(2, k\{x\})$ such that

$$B(\sigma)(\delta + \sigma(A))B(\sigma)^{-1} = \delta + A.$$

We normalize $B(\sigma)$ by $B(\sigma)_0 = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$ for some $* \in k^*$. The two equalities

$$B(\sigma\tau)(\delta + \sigma\tau(A))B(\sigma\tau)^{-1} = \delta + A,$$

$$B(\sigma)\sigma(B(\tau))(\delta + \sigma\tau(A))\sigma(B(\tau))^{-1}B(\sigma)^{-1} = \delta + A$$

imply that $B(\sigma\tau) = C(\sigma, \tau)B(\sigma)\sigma(B(\tau))$ for some automorphism $C(\sigma, \tau)$ of $\delta + A$. Thus $C(\sigma, \tau) \in k^*$ and by the above normalization one concludes that $C(\sigma, \tau) = 1$ for all σ, τ . Using [S], Proposition 3, p.159, and using [Ho-P] (part (3) of Definitions 2.2), one concludes that $\delta + A$ descends to the field $L = \ell(\{x\})$.

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