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**An example of nonlinear  $q$ -difference equation**

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## An example of nonlinear $q$ -difference equation (\*)

FRÉDÉRIC MENOUS <sup>(1)</sup>

**ABSTRACT.** — We study the formal solutions of the non linear  $q$ -difference equation

$$x\sigma_q f - f = b(f, x)$$

where  $\sigma_q f(x) = f(qx)$ , with a real number  $q > 1$  and  $b(f, x)$  belongs to  $\mathbb{C}\{f, x\}$  with the conditions  $b(0, 0) = 0$  and  $(\partial_f b)(0, 0) = 0$ . We prove that a solution of this equation can be conjugated to the solution  $ue_q(x) = uq^{-\log_q x (\log_q x - 1)/2}$  ( $\sigma_q u = u$ ) of the associated homogeneous equation, with the help of a formal substitution automorphism  $\Theta \in \mathbb{C}[[x, e_q, u, \partial_u]]$ .

Following the methods developed by Jean Ecalle, we first express this conjugating operator  $\Theta$  as a mould-comould expansion. The mould  $W^\bullet$  can be computed and each of its components is a formal series in  $x$ .

When  $b(0, x) = 0$ , these components happen to be convergent and we prove that the conjugating operator is also convergent in a well-adapted topology.

In the generic case, the components of the mould are no more convergent. Nevertheless, these components are  $q$ -multisummable. This is not sufficient to define a good resummation process for the conjugating operator. Some unsolved problems call for new results in the  $q$ -resummation theory. Besides this, it also seem that the arborification of moulds yields some simplifications of the encountered problems.

**RÉSUMÉ.** — On étudie les solutions formelles de l'équation aux  $q$ -différences non-linéaire

$$x\sigma_q f - f = b(f, x)$$

où  $\sigma_q f(x) = f(qx)$ , avec un nombre réel  $q > 1$  et  $b(f, x)$  est dans  $\mathbb{C}\{f, x\}$  avec les conditions  $b(0, 0) = 0$  et  $(\partial_f b)(0, 0) = 0$ . On prouve que les solutions de cette équation peuvent être conjuguées aux solutions  $ue_q(x) = uq^{-\log_q x (\log_q x - 1)/2}$  ( $\sigma_q u = u$ ) de l'équation homogène associée, grâce à un automorphisme de substitution formel  $\Theta \in \mathbb{C}[[x, e_q, u, \partial_u]]$ .

En suivant les méthodes développées par Jean Ecalle, on exprime tout d'abord cet opérateur conjuguant  $\Theta$  comme une série moule-comoule. Le

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moule  $W^\bullet$  peut être calculé et chacune de ses composantes est une série formelle en  $x$ .

Lorsque  $b(0, x) = 0$ , ces composantes sont convergentes et on prouve que l'opérateur conjuguant est aussi convergent pour une topologie adaptée.

Dans le cas générique, les composantes du moule ne sont plus convergentes. Néanmoins, ces composantes sont  $q$ -multisommables. Cette propriété n'est pas suffisante pour définir un procédé de resommation adapté à l'opérateur conjuguant. Ces problèmes semblent nécessiter de nouveaux développements en théorie de la  $q$ -resommation. Parallèlement, il semble que le procédé d'arborification des moules permet de simplifier les problèmes rencontrés.

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## 1. Introduction

Let  $q > 1$ . There has been many recent developments in the theory of linear  $q$ -difference equations. In [2] C. Zhang introduces the notion of  $q$ -Gevrey asymptotic expansions of order 1 and of  $q$ -summable formal series. In [3], C. Zhang and F. Marotte define the notion of  $q$ -multisummable formal series: they develop a  $q$ -analog of the usual accelerosummation or multisummability. With the help of this new resummation theory, they prove that the formal solutions of a linear  $q$ -difference equation are  $q$ -multisummable. There also has been very recent results on the analytic classification of linear  $q$ -difference equations.

Our aim is to try to develop analogous results for non linear  $q$ -difference equations, with the help of the methods developed by J. Ecalle in [1]. We focus here on the most simple case.

In section 2, we introduce the non linear  $q$ -difference equation

$$x\sigma_q f - f = b(f, x)$$

where  $\sigma_q f(x) = f(qx)$ , with  $q > 1$  and  $b(f, x) \in \mathbb{C}\{f, x\}$  with the conditions  $b(0, 0) = 0$  and  $(\partial_f b)(0, 0) = 0$ . We try to express a solution of this equation as a conjugate to the solution  $ue_q(x) = uq^{-\log_q x(\log_q x - 1)/2}$  ( $\sigma_q u = u$ ) of the associated homogeneous equation, with the help of a formal substitution automorphism  $\Theta \in \mathbb{C}[[x, e_q, u, \partial_u]]$ . Following the methods developed by

Jean Ecalle, we first express this conjugating operator  $\Theta$  as a mould-comould expansion, with the help of a mould  $W^\bullet$ .

In section 3, we give a recursive definition of the mould  $W^\bullet$  and prove that it is symmetrel. It implies that the operator  $\Theta$  is a formal substitution automorphism.

In section 4, we solve a family of elementary linear  $q$ -difference equations. These solutions allow us, in section 5, to give a formula for the mould  $W^\bullet$ .

We study in section 6 the *convergent case*: when  $b(0, x) = 0$ , the components of the mould  $W^\bullet$  happen to be convergent and we prove that the conjugating operator is also convergent in a well-adapted topology.

In the generic case, the components of  $W^\bullet$  are no more convergent. We recall in section 7 the definitions related to  $q$ -summability and prove in section 8 that the components of  $W^\bullet$  are  $q$ -multisummable. This is not sufficient to associate a sum to the divergent operator  $\Theta$  and we give in subsection 8.2 a list of problem that remain unsolved.

As this seem to bring some simplifications for the previous unsolved problems, we introduce in section 9 the notion of *arborification* of a symmetrel mould. We give an expression for the arborescent mould  $W^{\bullet<}$  associated to  $W^\bullet$  and translate our unsolved problems in terms of  $W^{\bullet<}$ .

Our unsolved problem are expressed in the conclusion (section 10) in terms of questions about the algebraic structure of  $q$ -multisummable formal series which are solution of a linear  $q$ -difference equation.

## 2. An example of nonlinear $q$ -difference equation

The aim of this paper is to study the formal solutions of the following equation:

$$x\sigma_q f - f = b(f, x) \tag{2.1}$$

where  $\sigma_q f(x) = f(qx)$ , with  $q > 1$  and  $b(f, x) \in \mathbb{C}\{f, x\}$  with the following conditions:

$$b(0, 0) = 0 \text{ and } (\partial_f b)(0, 0) = 0$$

The solution of such an equation is well-known in the case where  $b = 0$

### 2.1. The homogeneous equation

Let

$$(E_0) \quad x\sigma_q f - f = 0 \tag{2.2}$$

The solutions of this equation are the “ $q$ -exponential” functions, by analogy with the usual exponential function, which is a solution of the differential Euler equation.

One can check that the function defined by  $e_q(x) = q^{-\frac{1}{2} \log_q x (\log_q x - 1)}$  (on the Riemann surface  $\mathbb{C}^\bullet$ ) is a solution of equation  $(E_0)$ . Any solution of  $(E_0)$  can then be written  $e_q(x)u(x)$  where  $u(qx) = u(x)$ . Such a function  $u$  is called a  $q$ -constant.

Starting with a solution  $ue_q$  of the homogeneous equation, we try to find a formal solution of equation 2.1 which is formally conjugate to  $ue_q$ . Following the ideas developed in [1], we try to find a conjugating operator

$$\Theta \in \mathbb{C}[[x, e_q, u, \partial_u]] \quad (2.3)$$

where  $\Theta$  is a formal substitution automorphism such that  $\Theta.(ue_q)$  is a formal solution of equation 2.1.

## 2.2. Necessary conditions on $\Theta$

### 2.2.1. Notations

Let us consider once again  $b(f, x) \in \mathbb{C}\{f, x\}$  with  $b(0, 0) = 0$  and  $(\partial_f b)(0, 0) = 0$ . This function can be written:

$$b(f, x) = f \sum_{n \geq -1} b_n(x) f^n = f \sum_{n \geq -1} \sum_{\sigma \geq 0} b_{n,\sigma} x^\sigma f^n \quad (b_{-1,0} = b_{0,0} = 0) \quad (2.4)$$

If we consider the set

$$\begin{aligned} H_0 &= \left\{ \eta = \begin{pmatrix} n \\ \sigma \end{pmatrix}, \quad n \geq -1, \quad \sigma \geq 0 \right\} / \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\ &= H_{-1} \cup H_{\geq 0} \end{aligned} \quad (2.5)$$

where

$$H_{-1} = \{ \eta \in H; n = -1 \} \quad H_{\geq 0} = \{ \eta \in H; n \geq 0 \}$$

then, for a given variable  $u$  (which should be considered as a  $q$ -constant), one can define the following operators.

$$\forall \eta \in H_0 \quad ; \quad \mathbb{B}_\eta = b_\eta u^{n+1} \partial_u \quad (2.6)$$

where  $b_{n,\sigma} = b_\eta$ . These operators are useful in the study of nonlinear differential equations but we will need here some slightly different operators.

DEFINITION 2.1. — *Let*

$$H = \{\eta_1 + \dots + \eta_s; s \geq 1; \eta_i \in H_0\} \quad (2.7)$$

For  $\eta \in H$ , the operator  $\mathbb{D}_\eta$  is defined by:

$$\mathbb{D}_\eta = \sum_{s \geq 1} \sum_{\substack{\eta_1 + \dots + \eta_s = \eta \\ \eta_i \in H_0}} \frac{1}{s!} b_{\eta_1} \dots b_{\eta_s} u^{n_1 + \dots + n_s + s} \partial_u^s \quad (2.8)$$

It is important to notice that, for any  $\eta = \begin{pmatrix} n \\ \sigma \end{pmatrix} \in H$ , the operator  $\mathbb{D}_\eta$  is a finite sum which degree in  $\partial_u$  is at most  $n + 2\sigma$ . In section 6, these operators will be studied in details. It is easy to see that:

$$\forall \eta \in H_0 \quad ; \quad \mathbb{D}_\eta . u = \mathbb{B}_\eta . u \quad (2.9)$$

and,

$$\begin{aligned} \mathbb{B} . u &= \sum_{\eta \in H_0} x^\sigma \mathbb{B}_\eta . u &= b(u, x) \\ \mathbb{D} . u &= \left( \text{Id} + \sum_{\eta \in H} x^\sigma \mathbb{D}_\eta \right) . u &= u + b(u, x) \end{aligned} \quad (2.10)$$

$\mathbb{D}$  is a “convergent” substitution automorphism:

$$\forall f, g \in u\mathbb{C}\{u\} \quad ; \quad \mathbb{D}(fg) = (\mathbb{D}f)(\mathbb{D}g) \quad (2.11)$$

and the notion of convergence will be developed in section 6.

In fact,  $\mathbb{D}$  is actually defined by,

$$\forall \varphi \in \mathbb{C}\{u, x\}, \quad (\mathbb{D} . \varphi)(u, x) = \varphi(u + b(u, x), x)$$

and this is indeed the Taylor expansion of  $\varphi(u + b(u, x), x)$  at the point  $(u, x)$  that allows us to define the components  $\mathbb{D}_\eta$ . This definition of  $\mathbb{D}$  ensures that it is a “convergent” substitution automorphism.

In order to define the conjugating operator  $\Theta$ , we introduce some notations:

DEFINITION 2.2. — *Let  $\mathbf{H} = \cup_{s \geq 1} H^s$  be the set of sequences of elements of  $H$ . If  $\boldsymbol{\eta} = \eta_1, \dots, \eta_s \in \mathbf{H}$ , then*

$$l(\boldsymbol{\eta}) = s \quad ; \quad \boldsymbol{\eta} = \begin{pmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{pmatrix} \quad ; \quad \|\mathbf{n}\| = n_1 + \dots + n_s \quad (2.12)$$

Moreover  $\eta = \eta_2, \dots, \eta_s$ ,  $\eta_{in} = \eta_1$  and, finally,

$$\mathbb{D}_\eta = \mathbb{D}_{\eta_s} \dots \mathbb{D}_{\eta_1} \quad (2.13)$$

If we refer to [1], the set  $\{\mathbb{D}_\eta\}_{\eta \in H}$  defines a cosymmetrel comould. Let us go back now to the definition of  $\Theta$ .

### 2.2.2. The automorphism $\Theta$

As the function  $ue_q$  is a solution of

$$x\sigma_q f - f = 0$$

we must define a substitution automorphism  $\Theta$  on  $\mathbb{C}[[e_q, x, u, \partial_u]]$  such that the function

$$\Theta(ue_q) = \Theta(u)e_q$$

is a solution of

$$(E) \quad x\sigma_q f - f = b(f, x)$$

To do so, we will look for an operator  $\Theta$ :

$$\Theta = 1 + \sum_{\eta \in H} \mathcal{W}^\eta \mathbb{D}_\eta \quad (2.14)$$

where  $\mathcal{W}^\eta \in \mathbb{C}[[e_q, x]]$ . This set of monomials defines a mould  $\mathcal{W}^\bullet$  and, as  $\Theta$  must be a substitution automorphism, this mould has to be symmetrel (see [1]). Let us investigate what should be the definition of the mould  $\mathcal{W}^\bullet$ .

## 3. First steps on the mould $\mathcal{W}^\bullet$

### 3.1. Recursive functional equation for $\mathcal{W}^\bullet$

Suppose that

$$f(u, x) = \Theta(ue_q) = \Theta(u)e_q \quad (3.1)$$

is a solution of

$$(E) \quad x\sigma_q f - f = b(f, x)$$

where  $\Theta$  is a substitution automorphism. Then,

$$f(u, x) = ue_q(x) + \sum_{\eta \in H} \mathcal{W}^\eta \mathbb{D}_\eta(ue_q(x)) = ue_q(x) + \sum_{\eta \in H} e_q(x) \mathcal{W}^\eta \mathbb{D}_\eta(u) \quad (3.2)$$



First,

$$\begin{aligned}
 x\sigma_q f(u, x) &= xu\sigma_q(e_q) + \sum_{\eta \in \mathbf{H}} x\sigma_q(e_q(x))\sigma_q(\mathcal{W}^\eta)\mathbb{D}_\eta(u) \\
 &= ue_q(x) + \sum_{\eta \in \mathbf{H}} e_q(x)\sigma_q(\mathcal{W}^\eta)\mathbb{D}_\eta(u)
 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
 b(f(u, x), x) &= f(u, x) \sum_{n \geq -1} \sum_{\sigma \geq 0} b_{n, \sigma} x^\sigma f^n(u, x) \\
 &= \Theta(u)e_q(x) \sum_{n \geq -1} \sum_{\sigma \geq 0} b_{n, \sigma} x^\sigma (\Theta(u)e_q(x))^n \\
 &= \Theta \left( ue_q(x) \sum_{n \geq -1} \sum_{\sigma \geq 0} b_{n, \sigma} x^\sigma (ue_q(x))^n \right) \\
 &= \Theta \left( \sum_{n \geq -1} \sum_{\sigma \geq 0} b_{n, \sigma} x^\sigma (ue_q(x))^{n+1} \right) \\
 &= \Theta \left( \sum_{\eta \in H_0} x^\sigma e_q^{n+1}(x) \mathbb{B}_\eta(u) \right) \\
 &= \Theta \left( \sum_{\eta \in H} x^\sigma e_q^{n+1}(x) \mathbb{D}_\eta(u) \right) \\
 b(f(u, x), x) &= \sum_{\eta \in \mathbf{H}} x^{\sigma_{\text{in}}} e_q^{\eta_{\text{in}}}(x) \mathcal{W}^\eta \mathbb{D}_\eta(ue_q(x))
 \end{aligned} \tag{3.4}$$

Plugging these expressions in the equation, we get:

**THEOREM 3.1.** — *The monomials  $(\mathcal{W}^\eta)_{\eta \in \mathbf{H}}$  must satisfy the following recursive functional equation:*

$$\forall \eta = \eta_1, \dots, \eta_s \in \mathbf{H}, \quad (\sigma_q - 1)\mathcal{W}^{\eta_1, \dots, \eta_s}(x) = x^{\sigma_1} e_q^{\eta_1}(x) \mathcal{W}^{\eta_2, \dots, \eta_s}(x) \tag{3.5}$$

with  $\mathcal{W}^\emptyset = 1$ .

Assuming that the mould  $\mathcal{W}^\bullet$  is completely determined by these equations, we can already check that  $\Theta$  is a substitution automorphism, that is to say that the mould  $\mathcal{W}^\bullet$  is symmetrel.

### 3.2. Symmetrelity of $\mathcal{W}^\bullet$

We prove here that, if well-defined, the mould  $\mathcal{W}^\bullet$  is symmetrel.

An example of nonlinear  $q$ -difference equation

Let  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) = \begin{pmatrix} n_1 & \dots & n_s \\ \sigma_1 & \dots & \sigma_s \end{pmatrix}$  and  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_t) = \begin{pmatrix} m_1 & \dots & m_t \\ \rho_1 & \dots & \rho_t \end{pmatrix}$  two sequences of  $\mathbf{H} \cup \{\emptyset\}$ . The symmetrelity relations will be proven by induction on  $l = l(\boldsymbol{\eta}) + l(\boldsymbol{\mu})$ .

For  $l = 0$ , it is obvious that

$$\mathcal{W}^{\emptyset} \mathcal{W}^{\emptyset} = 1 = \mathcal{W}^{\emptyset}$$

For  $l = 1$ ,

$$\mathcal{W}^{\boldsymbol{\eta}} \mathcal{W}^{\emptyset} = \mathcal{W}^{\emptyset} \mathcal{W}^{\boldsymbol{\eta}} = \mathcal{W}^{\boldsymbol{\eta}}$$

Suppose that the symmetrelity relations are satisfied for a given  $l \geq 1$ . If  $\boldsymbol{\eta} \in \mathbf{H}$  with  $l(\boldsymbol{\eta}) = l + 1$ , then

$$\mathcal{W}^{\boldsymbol{\eta}} \mathcal{W}^{\emptyset} = \mathcal{W}^{\emptyset} \mathcal{W}^{\boldsymbol{\eta}} = \mathcal{W}^{\boldsymbol{\eta}}$$

If  $(\boldsymbol{\eta}, \boldsymbol{\mu}) \in \mathbf{H}^2$ , with  $l(\boldsymbol{\eta}) + l(\boldsymbol{\mu}) = l + 1$ , then,

$$\begin{aligned} S_{\boldsymbol{\eta}, \boldsymbol{\mu}} &= \sum_{\boldsymbol{\lambda} \in \text{ctsh}(\boldsymbol{\eta}, \boldsymbol{\mu})} \mathcal{W}^{\boldsymbol{\lambda}} \\ &= \sum_{\boldsymbol{\lambda} \in \text{ctsh}(\boldsymbol{\eta}, \boldsymbol{\mu})} \mathcal{W}^{\boldsymbol{\eta}_1, \boldsymbol{\lambda}} + \sum_{\boldsymbol{\lambda} \in \text{ctsh}(\boldsymbol{\eta}, \boldsymbol{\mu})} \mathcal{W}^{\boldsymbol{\mu}_1, \boldsymbol{\lambda}} + \sum_{\boldsymbol{\lambda} \in \text{ctsh}(\boldsymbol{\eta}, \boldsymbol{\mu})} \mathcal{W}^{\boldsymbol{\eta}_1 + \boldsymbol{\mu}_1, \boldsymbol{\lambda}} \end{aligned}$$

thus

$$\begin{aligned} (\sigma_q - 1)S_{\boldsymbol{\eta}, \boldsymbol{\mu}} &= \sum_{\boldsymbol{\lambda} \in \text{ctsh}(\boldsymbol{\eta}, \boldsymbol{\mu})} (\sigma_q - 1)\mathcal{W}^{\boldsymbol{\eta}_1, \boldsymbol{\lambda}} + \sum_{\boldsymbol{\lambda} \in \text{ctsh}(\boldsymbol{\eta}, \boldsymbol{\mu})} (\sigma_q - 1)\mathcal{W}^{\boldsymbol{\mu}_1, \boldsymbol{\lambda}} \\ &\quad + \sum_{\boldsymbol{\lambda} \in \text{ctsh}(\boldsymbol{\eta}, \boldsymbol{\mu})} (\sigma_q - 1)\mathcal{W}^{\boldsymbol{\eta}_1 + \boldsymbol{\mu}_1, \boldsymbol{\lambda}} \\ &= \sum_{\boldsymbol{\lambda} \in \text{ctsh}(\boldsymbol{\eta}, \boldsymbol{\mu})} x^{\sigma_1} e_q^{\eta_1} \mathcal{W}^{\boldsymbol{\lambda}} + \sum_{\boldsymbol{\lambda} \in \text{ctsh}(\boldsymbol{\eta}, \boldsymbol{\mu})} x^{\rho_1} e_q^{m_1} \mathcal{W}^{\boldsymbol{\lambda}} \\ &\quad + \sum_{\boldsymbol{\lambda} \in \text{ctsh}(\boldsymbol{\eta}, \boldsymbol{\mu})} x^{\sigma_1 + \rho_1} e_q^{\eta_1 + m_1} \mathcal{W}^{\boldsymbol{\lambda}} \\ &= x^{\sigma_1} e_q^{\eta_1} \mathcal{W}^{\boldsymbol{\eta}} \mathcal{W}^{\boldsymbol{\mu}} + x^{\rho_1} e_q^{m_1} \mathcal{W}^{\boldsymbol{\eta}} \mathcal{W}^{\boldsymbol{\mu}} \\ &\quad + x^{\sigma_1} x^{\rho_1} e_q^{\eta_1} e_q^{m_1} \mathcal{W}^{\boldsymbol{\eta}} \mathcal{W}^{\boldsymbol{\mu}} \\ &= ((\sigma_q - 1)\mathcal{W}^{\boldsymbol{\eta}}) \mathcal{W}^{\boldsymbol{\mu}} + \mathcal{W}^{\boldsymbol{\eta}} ((\sigma_q - 1)\mathcal{W}^{\boldsymbol{\mu}}) \\ &\quad + ((\sigma_q - 1)\mathcal{W}^{\boldsymbol{\eta}}) ((\sigma_q - 1)\mathcal{W}^{\boldsymbol{\mu}}) \\ &= (\sigma_q - 1) (\mathcal{W}^{\boldsymbol{\eta}} \mathcal{W}^{\boldsymbol{\mu}}) \end{aligned}$$

If we can define the inverse of  $(\sigma_q - 1)$ , then we get the symmetrelity relation at order  $l + 1$ . This ends the proof of the following theorem.

**THEOREM 3.2.** — *If the mould  $\mathcal{W}^\bullet$  is completely defined by the recursive equations 3.5, then this mould is symmetrel and  $\Theta$  is a substitution automorphism.*

It remains to prove that the mould  $\mathcal{W}^\bullet$  is uniquely defined by the equations 3.5. This is the goal of sections 4 and 5. But we can notice first that we can factorize some  $q$ -exponentials in the monomials of  $\mathcal{W}^\bullet$ .

### 3.3. The mould $W^\bullet$

Let us define a new mould  $W^\bullet$ .

**DEFINITION 3.3.** — *For  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}$ . The monomial  $W^\boldsymbol{\eta}$  is defined by*

$$W^{\eta_1, \dots, \eta_s} = e_q^{n_1 + \dots + n_s} W^{\eta_1, \dots, \eta_s} = e_q^{\|\boldsymbol{n}\|} W^{\eta_1, \dots, \eta_s} \quad (3.6)$$

It is clear that  $W^\emptyset = 1$  and

$$\begin{aligned} (\sigma_q - 1)W^{\eta_1, \dots, \eta_s} &= \sigma_q(e_q^{\|\boldsymbol{n}\|})\sigma_q W^{\eta_1, \dots, \eta_s} - e_q^{\|\boldsymbol{n}\|} W^{\eta_1, \dots, \eta_s} \\ &= x^{-\|\boldsymbol{n}\|} e_q^{\|\boldsymbol{n}\|} \sigma_q W^{\eta_1, \dots, \eta_s} - e_q^{\|\boldsymbol{n}\|} W^{\eta_1, \dots, \eta_s} \\ &= x^{\sigma_1} e_q^{\|\boldsymbol{n}\|} W^{\eta_2, \dots, \eta_s} \end{aligned} \quad (3.7)$$

thus

$$S_{\|\boldsymbol{n}\|}(W^{\eta_1, \dots, \eta_s}) = \left(x^{-\|\boldsymbol{n}\|} \sigma_q - 1\right) W^{\eta_1, \dots, \eta_s} = x^{\sigma_1} W^{\eta_2, \dots, \eta_s}$$

We can then conclude that

**THEOREM 3.4.** — *If the mould  $W^\bullet$  is uniquely defined by  $W^\emptyset = 1$  and:*

$$\forall (\eta_1, \dots, \eta_s) \in \mathbf{H}, \quad S_{\|\boldsymbol{n}\|}(W^{\eta_1, \dots, \eta_s}) = \begin{aligned} &(x^{-\|\boldsymbol{n}\|} \sigma_q - 1) W^{\eta_2, \dots, \eta_s} \\ &= x^{\sigma_1} W^{\eta_2, \dots, \eta_s} \end{aligned} \quad (3.8)$$

*then this mould is also symmetrel and the mould  $\mathcal{W}^\bullet$  is uniquely defined. Moreover*

$$\Theta = 1 + \sum_{\boldsymbol{\eta} \in \mathbf{H}} \mathcal{W}^\boldsymbol{\eta} \mathbb{D}_\boldsymbol{\eta} = 1 + \sum_{\boldsymbol{\eta} \in \mathbf{H}} e_q^{\|\boldsymbol{n}\|} W^\boldsymbol{\eta} \mathbb{D}_\boldsymbol{\eta} \quad (3.9)$$

*is a substitution automorphism that conjugate the function  $ue_q$  to a formal solution of*

$$(x\sigma_q - 1)f = b(f, x)$$

In the two following sections we prove that the mould  $W^\bullet$  is uniquely determined by the equations 3.8 and that:

$$\forall \eta \in \mathbf{H}, \quad W^\eta \in \mathbb{C}[[x]] \quad (3.10)$$

To do so we first study, for  $\eta \in H$ , the solutions of

$$(E_\eta) : S_n(\tilde{f}) = x^\sigma \quad (3.11)$$

#### 4. Resolution of $(E_\eta) : S_n(\tilde{f}) = x^\sigma$

Let  $\tilde{f}(x) = \sum_{n \geq 0} f_n x^n$ . For  $\eta = \begin{pmatrix} n \\ \sigma \end{pmatrix} \in H$ , we are looking for the formal solutions of the equation  $(E_\eta)$ :

$$S_n(\tilde{f}) = x^\sigma \quad (4.1)$$

which defines the monomial  $W^\eta$ .

##### 4.1. Case $\eta \in H/H_0$

In this case  $\eta = \begin{pmatrix} n \\ \sigma \end{pmatrix}$  is such that  $n < 0$  and  $\sigma \geq -n$ . If  $m = -n$ , the equation  $(E_\eta)$  becomes:

$$x^m \sigma_q \tilde{f} - \tilde{f} = x^\sigma$$

Thus

$$\begin{aligned} \sum_{k \geq 0} f_k q^k x^{k+m} - \sum_{k \geq 0} f_k x^k &= x^\sigma \\ \sum_{k \geq m} f_{k-m} q^{k-m} x^k - \sum_{k \geq 0} f_k x^k &= x^\sigma \\ \sum_{k \geq \sigma+m} f_{k-m} q^{k-m} x^k - \sum_{k \geq \sigma} f_k x^k &= x^\sigma \end{aligned}$$

If  $\mathcal{V}$  is the valuation of  $\tilde{f}$ , then  $\mathcal{V}(\tilde{f}) = \sigma$ ,  $f_\sigma = -1$  and, for  $k > \sigma$ , if  $k - \sigma \not\equiv 0 \pmod{m}$ , then  $f_k = 0$ . Otherwise, if  $k = \alpha m + \sigma$  ( $\alpha > 0$ ):

$$f_{\sigma+\alpha m} = f_{\sigma+(\alpha-1)m} q^{\sigma+(\alpha-1)m}$$

thus

$$f_{\sigma+\alpha m} = -q^{\alpha\sigma} q^{m\alpha(\alpha-1)/2}$$

It means that

$$\begin{aligned}\tilde{f}(x) &= - \sum_{\alpha \geq 0} x^{\sigma+\alpha n} q^{\alpha \sigma} q^{m\alpha(\alpha-1)/2} \\ &= - \sum_{k \leq 0} x^{\sigma+kn} q^{-k\sigma} q^{-nk(k+1)/2} \\ &= - \sum_{k \leq 0} x^{\sigma+kn} q^{-k\sigma} [k+1]_q^n\end{aligned}$$

where  $[k]_q = q^{-k(k-1)/2}$ .

#### 4.2. Case $n = 0$

We have  $S_0(\tilde{f}) = (\sigma_q - 1)\tilde{f}$  and if  $\begin{pmatrix} 0 \\ \sigma \end{pmatrix} \in H$ , then  $\sigma \geq 1$ . It implies that:

$$\sum_{n \geq 0} (q^n - 1) f_n x^n = x^\sigma$$

thus

$$\tilde{f}(x) = \frac{x^\sigma}{q^\sigma - 1} = \sum_{k \geq 1} x^\sigma q^{-k\sigma} = \sum_{k \geq 1} x^{\sigma+kn} q^{-k\sigma} [k+1]_q^n$$

#### 4.3. Case $n \geq 1$

We have  $S_n(\tilde{f}) = (x^{-n}\sigma_q - 1)\tilde{f}$  and

$$\begin{aligned}\sigma_q(\tilde{f}) - x^n \tilde{f} &= x^{n+\sigma} \\ \sum_{k \geq 0} f_k q^k x^k - \sum_{k \geq 0} f_k x^{k+n} &= x^{n+\sigma} \\ \sum_{k \geq 0} f_k q^k x^k - \sum_{k \geq n} f_{k-n} x^k &= x^{n+\sigma} \\ \sum_{k \geq \sigma+n} f_k q^k x^k - \sum_{k \geq \sigma+2n} f_{k-n} x^k &= x^{n+\sigma} \\ \sum_{\alpha \geq 1} f_{\sigma+\alpha n} q^{\sigma+\alpha n} x^{\sigma+\alpha n} - \sum_{\alpha \geq 2} f_{\sigma+(\alpha-1)n} x^{\sigma+\alpha n} &= x^{n+\sigma}\end{aligned}$$

So  $\mathcal{V}(\tilde{f}) = \sigma + n$ ,  $f_{\sigma+n} = \frac{1}{q^{\sigma+n}}$  and, for  $\alpha > 1$ ,

$$f_{\sigma+\alpha n} q^{\sigma+\alpha n} = f_{\sigma+(\alpha-1)n}$$

thus

$$f_{\sigma+\alpha n} = q^{-\sigma\alpha} q^{-n\alpha(\alpha+1)/2}$$

and finally

$$\tilde{f}(x) = \sum_{k \geq 1} x^{\sigma + kn} q^{-\sigma k} [k + 1]_q^n$$

#### 4.4. Conclusion

In every case:

**THEOREM 4.1.** — *Let  $\eta = \begin{pmatrix} n \\ \sigma \end{pmatrix} \in H$ . The equation  $(E_\eta)$ :*

$$S_n(\tilde{f}) = x^\sigma \tag{4.2}$$

*has a unique formal solution:*

$$W^\eta(x) = \varepsilon_n \sum_{k \in \mathbb{Z}^n} x^{\sigma + kn} q^{-\sigma k} [k + 1]_q^n \tag{4.3}$$

*with  $\varepsilon_n = 1$  (resp.  $\varepsilon_n = -1$ ) if  $n \geq 0$  (resp.  $n < 0$ ),  $\mathbb{Z}^n = \mathbb{Z}^{+*}$  (resp.  $\mathbb{Z}^-$ ) if  $n \geq 0$  (resp.  $n < 0$ ) and  $[k + 1]_q = q^{-k(k+1)/2}$ .*

Using this theorem, one can uniquely define the mould  $W^\bullet$  and we even have a closed formula.

#### 5. A formula for the mould $W^\bullet$

We recall that our goal is to define and study the monomials  $W^\eta$  where

$$\eta \in \mathbf{H} = \{ \eta = (\eta_1, \dots, \eta_s) \ ; \ \forall 1 \leq i < s, \ \eta_i \in H \ ; \ s \geq 1 \}$$

**THEOREM 5.1.** — *If  $\eta = (\eta_1, \dots, \eta_s) = \begin{pmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{pmatrix} = \begin{pmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{pmatrix} \in \mathbf{H}$ , then the monomial  $W^\eta$  is a formal series with the following expression:*

$$W^\eta(x) = \varepsilon_{\mathbf{n}} \sum_{(k_1, \dots, k_s) \in \mathbb{Z}^{\mathbf{n}}} x^{\|\boldsymbol{\sigma}\| + \check{k}_1 n_1 + \dots + \check{k}_s n_s} q^{-\check{k}_1 \sigma_1 - \dots - \check{k}_s \sigma_s} [\check{k}_1 + 1]_q^{n_1} \dots [\check{k}_s + 1]_q^{n_s} \tag{5.1}$$

*with the notations*

- $\varepsilon_{\mathbf{n}} = \prod_{1 \leq i \leq s} \varepsilon_{\hat{n}_i}$  with  $\hat{n}_i = n_i + \dots + n_s$ .
- $\mathbb{Z}^{\mathbf{n}} = \mathbb{Z}^{\hat{n}_1} \times \dots \times \mathbb{Z}^{\hat{n}_s}$ .
- $\check{k}_i = k_1 + \dots + k_i$ .

This result is proven by induction on  $s$ . For  $s = 1$ , the result is simply the Theorem 4.1. Suppose  $s \geq 2$  and  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) = \begin{pmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{pmatrix} \in \mathbf{H}$ .

Then  $\backslash\boldsymbol{\eta} = (\eta_2, \dots, \eta_s) = \begin{pmatrix} n_2, \dots, n_s \\ \sigma_{12}, \dots, \sigma_s \end{pmatrix} \in \mathbf{H}$  and

$$S_{\|\mathbf{n}\|}(W^\boldsymbol{\eta}) = S_{\hat{n}_1}(W^\boldsymbol{\eta}) = x^{\sigma_1} W^{\backslash\boldsymbol{\eta}}$$

thus

$$W^\boldsymbol{\eta} = S_{\hat{n}_1}^{-1}(x^{\sigma_1} W^{\backslash\boldsymbol{\eta}})$$

but

$$W^{\backslash\boldsymbol{\eta}}(x) = \varepsilon_{\backslash\mathbf{n}} \sum_{(k_2, \dots, k_s) \in \mathbb{Z}^{\backslash\mathbf{n}}} x^{\|\boldsymbol{\sigma}\| + \check{k}_2 n_2 + \dots + \check{k}_s n_s} q^{-\check{k}_2 \sigma_2 - \dots - \check{k}_s \sigma_s} [\check{k}_2 + 1]_q^{n_2} \dots [\check{k}_s + 1]_q^{n_s}$$

If  $\check{\mathbf{k}} = (\check{k}_2, \dots, \check{k}_s)$  and  $\left[ \begin{smallmatrix} \backslash\mathbf{n} \\ \check{\mathbf{k}} \end{smallmatrix} \right]_q = [\check{k}_2 + 1]_q^{n_2} \dots [\check{k}_s + 1]_q^{n_s}$  (with  $\check{k}_i = k_2 + \dots + k_i$ ), then

$$W^{\backslash\boldsymbol{\eta}}(x) = \varepsilon_{\backslash\mathbf{n}} \sum_{(k_2, \dots, k_s) \in \mathbb{Z}^{\backslash\mathbf{n}}} x^{\|\boldsymbol{\sigma}\| + \langle \check{\mathbf{k}} | \backslash\mathbf{n} \rangle} q^{-\langle \check{\mathbf{k}} | \boldsymbol{\sigma} \rangle} \left[ \begin{smallmatrix} \backslash\mathbf{n} \\ \check{\mathbf{k}} \end{smallmatrix} \right]_q$$

and

$$\begin{aligned} W^\boldsymbol{\eta}(x) &= \varepsilon_{\mathbf{n}} \sum_{(k_2, \dots, k_s) \in \mathbb{Z}^{\backslash\mathbf{n}}} S_{\hat{n}_1}^{-1} \left( x^{\|\boldsymbol{\sigma}\| + \langle \check{\mathbf{k}} | \backslash\mathbf{n} \rangle} \right) q^{-\langle \check{\mathbf{k}} | \boldsymbol{\sigma} \rangle} \left[ \begin{smallmatrix} \backslash\mathbf{n} \\ \check{\mathbf{k}} \end{smallmatrix} \right]_q \\ &= \varepsilon_{\mathbf{n}} \sum_{(k_2, \dots, k_s) \in \mathbb{Z}^{\backslash\mathbf{n}}} \varepsilon_{\hat{n}_1} \sum_{k_1 \in \mathbb{Z}^{\hat{n}_1}} \left( x^{\|\boldsymbol{\sigma}\| + \langle \check{\mathbf{k}} | \backslash\mathbf{n} \rangle + k_1 \hat{n}_1} q^{-(\|\boldsymbol{\sigma}\| + \langle \check{\mathbf{k}} | \backslash\mathbf{n} \rangle) k_1} \right. \\ &\quad \left. \times [k_1 + 1]_q^{\hat{n}_1} q^{-\langle \check{\mathbf{k}} | \boldsymbol{\sigma} \rangle} \left[ \begin{smallmatrix} \backslash\mathbf{n} \\ \check{\mathbf{k}} \end{smallmatrix} \right]_q \right) \\ &= \varepsilon_{\mathbf{n}} \sum_{(k_1, \dots, k_s) \in \mathbb{Z}^{\mathbf{n}}} x^{\|\boldsymbol{\sigma}\| + \langle \check{\mathbf{k}} | \mathbf{n} \rangle} q^{-\langle \check{\mathbf{k}} | \boldsymbol{\sigma} \rangle} q^{-\langle \check{\mathbf{k}} | \backslash\mathbf{n} \rangle k_1} [k_1 + 1]_q^{\hat{n}_1} \left[ \begin{smallmatrix} \backslash\mathbf{n} \\ \check{\mathbf{k}} \end{smallmatrix} \right]_q \end{aligned}$$

where  $\check{\mathbf{k}} = (\check{k}_1, \dots, \check{k}_s)$  and  $\check{k}_i = k_1 + \dots + k_i$ . But

$$\begin{aligned} q^{-\langle \check{\mathbf{k}} | \backslash\mathbf{n} \rangle k_1} [k_1 + 1]_q^{\hat{n}_1} \left[ \begin{smallmatrix} \backslash\mathbf{n} \\ \check{\mathbf{k}} \end{smallmatrix} \right]_q &= [k_1 + 1]_q^{n_1} \times \\ &\quad \prod_{i=2}^s q^{-\check{k}_i n_i k_1} [k_1 + 1]_q^{n_i} [\check{k}_i + 1]_q^{n_i} \\ &= [k_1 + 1]_q^{n_1} \times \\ &\quad \prod_{i=2}^s \left( q^{-k_1(k_1+1)/2} q^{-\check{k}_i k_1} q^{-\check{k}_i(\check{k}_i+1)/2} \right)^{n_i} \\ &= [k_1 + 1]_q^{n_1} \prod_{i=2}^s [\check{k}_i + k_1 + 1]_q^{n_i} \end{aligned}$$

and it ends the proof.

This theorem calls for some remarks:

- From the formal point of view, we have completely defined a formal substitution automorphism  $\Theta$  such that  $\Theta(ue_q)$  is a formal solution of

$$(x\sigma_q - 1)f = b(f, x)$$

- Generically some monomials in the mould  $W^\bullet$  will be divergent. This call for the definition of a resummation process, in order to get some convergent solutions of the above equation.
- If  $\eta = (\eta_1, \dots, \eta_s) \in \mathbf{H}$  is such that, for  $1 \leq i \leq s$ ,  $\hat{n}_i = n_i + \dots + n_s$  is non-negative, then the monomial  $W^\eta$  is a convergent power series.
- This happens automatically if  $b(0, x) = 0$ . In this “convergent” case, there is no need for resummation and we shall prove in the following section that  $\Theta$  itself is a convergent operator.

### 6. The convergent case $b(0, x) = 0$

We consider once again the equation

$$(x\sigma_q - 1)f = b(f, x)$$

but now  $b(0, f) = 0$  and  $\frac{\partial b}{\partial f}(0, 0) = 0$ . In our previous results, we can restrict ourselves to

$$H_0 = \left\{ \eta = \begin{pmatrix} n \\ \sigma \end{pmatrix}, \quad n \geq 0, \quad \sigma \geq 0 \right\} / \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

and then

$$H = \{ \eta_1 + \dots + \eta_s \quad ; \quad s \geq 1, \quad (\eta_1, \dots, \eta_s) \in H_0^s \} = H_0$$

Once again  $\mathbf{H} = \cup_{s \geq 1} H^s$ . We first prove that the monomials  $W^\bullet$  and  $\mathcal{W}^\bullet$  are convergent and we give some estimates in the neighborhood of  $(0, 0)$ . We give then some bounds for the semi-norms associated to differential operators  $\{\mathbb{D}_\eta \quad \eta \in \mathbf{H}\}$ , so that we can prove the convergence of  $\Theta$  in the “convergent case”. Let us first begin by giving some definitions, as well as some combinatorial results on the structure of  $\mathbf{H}$ .



### 6.1. Seminorms

In order to study the convergence of such operators as  $\Theta$ , we need to define some seminorms on differential operators in  $\partial_u$ , acting on  $\mathbb{C}\{x, u\}$ . Let  $\varphi \in \mathbb{C}\{x, u\}$ , and  $P$  an operator of  $\mathbb{C}\{x, u\}$  into itself. We have the seminorms

$$\|\varphi\|_U = \sup_{u \in U} |\varphi(u)| \quad (6.1)$$

$$\|P\|_{U,V} = \sup_{\varphi: \|\varphi\|_U \leq 1} \|P.\varphi\|_V \quad (6.2)$$

Where  $U$  and  $V$  are two  $(x, u)$ -neighborhood of  $(0, 0)$ . A series  $\sum P_n$  is normally convergent if  $\sum \|P_n\|_{U,V}$  converge for at least a pair  $U, V$ . We should now only consider the following neighborhoods:

$$\forall \alpha > 0, \quad U_\alpha = \{(x, u) \in \mathbb{C}^2 \ ; \ |x| < \alpha, \ |u| < \alpha\} \quad (6.3)$$

As we are going to deal with sums on  $\mathbf{H}$ , we first need some enumerative properties on this set.

### 6.2. Combinatorial properties

For  $\eta \in H$ , we define its weight  $N(\eta) = n + \sigma$ . It is clear that

$$\forall \eta \in H, \quad N(\eta) \geq 1$$

This weight can be extended to  $\mathbf{H}$ :

$$\forall \boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}, \quad N(\boldsymbol{\eta}) = N(\eta_1) + \dots + N(\eta_s) \quad (6.4)$$

DEFINITION 6.1. — *Let*

- For  $N \geq 1$ ,  $q(N) = \#\{\eta \in H \ ; \ N(\eta) = N\}$ .
- For  $\eta \in H$ , and  $s \geq 1$ ,

$$P(\eta, s) = \#\{(\eta_1, \dots, \eta_s) \in H^s; \eta_1 + \dots + \eta_s = \eta\}$$

- For  $\eta \in H$ ,  $P(\eta) = \sum_{s \geq 1} P(\eta, s)$ .

One can see immediately that

LEMMA 6.2. — *We have the following bounds:*

$$\begin{aligned} \forall N \geq 1, \quad q(N) &\leq (N+1) \leq 2^N \\ \forall \eta \in H, s \geq 1, \quad P(\eta, s) &\leq 2^{N(\eta)} C_{N(\eta)-1}^{s-1} \\ \forall \eta \in H \quad P(\eta) &\leq 4^{N(\eta)} \end{aligned} \tag{6.5}$$

and it is clear that, for  $\eta \in H$ , if  $s > N(\eta)$  then  $P(\eta, s) = 0$ .

Let us go back to the study of the moulds  $W^\bullet$  and  $\mathcal{W}^\bullet$ .

### 6.3. Estimates for the moulds $W^\bullet$ and $\mathcal{W}^\bullet$

In the convergent case, every monomial of  $W^\bullet$  is convergent. We give here some estimates for  $W^\bullet$  and  $\mathcal{W}^\bullet$ . These bounds are valid for  $|x| < \alpha < 1$  but we must keep in mind that, if for  $W^\bullet$ , we can restrict ourselves to complex numbers  $x$ , for  $\mathcal{W}^\bullet$ , because of the  $q$ -exponential factor, we will consider  $x$  as an element of  $\tilde{\mathbb{C}}$ , the Riemann surface of the logarithm.

LEMMA 6.3. — *For  $\alpha < 1$  and  $C > 0$ , if  $V_\alpha = \{x \in \mathbb{C}; |x| \leq \alpha\}$  and  $V_{\alpha, C} = \{x \in \tilde{\mathbb{C}}; |x| \leq \alpha \text{ and } |\arg_q x| \leq C\}$ , then, for  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) = \begin{pmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{pmatrix} = \begin{pmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{pmatrix} \in \mathbf{H}$ ,*

$$\|W^\boldsymbol{\eta}\|_{V_\alpha} = \sup_{V_\alpha} |W^\boldsymbol{\eta}(x)| \leq \left(\frac{q}{q-1}\right)^s \alpha^{N(\boldsymbol{\eta})} q^{-\sum_{i=1}^s iN(\eta_i)} \tag{6.6}$$

and

$$\|\mathcal{W}^\boldsymbol{\eta}\|_{V_{\alpha, C}} = \sup_{V_{\alpha, C}} |\mathcal{W}^\boldsymbol{\eta}(x)| \leq \left(\frac{q}{q-1}\right)^s \alpha^{N(\boldsymbol{\eta})} \left(e_q(\alpha) q^{C^2/2}\right)^{\|\mathbf{n}\|} q^{-\sum_{i=1}^s iN(\eta_i)} \tag{6.7}$$

In the convergent case, if  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) = \begin{pmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{pmatrix} = \begin{pmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{pmatrix} \in \mathbf{H}$ , then the monomial  $W^\boldsymbol{\eta}$  is a convergent power series with the following expression:

$$W^\boldsymbol{\eta}(x) = \sum_{(k_1, \dots, k_s) \in \mathbb{N}^s} x^{\|\boldsymbol{\sigma}\| + \check{k}_1 n_1 + \dots + \check{k}_s n_s} q^{-\check{k}_1 \sigma_1 - \dots - \check{k}_s \sigma_s} [\check{k}_1 + 1]_q^{n_1} \dots [\check{k}_s + 1]_q^{n_s} \tag{6.8}$$

where  $[k+1]_q = q^{-k(k+1)/2}$ . If  $k \geq 1$  then  $[k+1]_q \leq q^{-k}$  thus,

$$\begin{aligned} |W^\eta(x)| &\leq \sum_{(k_1, \dots, k_s) \in \mathbb{N}^*} |x|^{\|\sigma\| + \check{k}_1 n_1 + \dots + \check{k}_s n_s} q^{-\check{k}_1 \sigma_1 - \dots - \check{k}_s \sigma_s} [\check{k}_1 + 1]_q^{n_1} \dots [\check{k}_s + 1]_q^{n_s} \\ &\leq \sum_{(k_1, \dots, k_s) \in \mathbb{N}^*} |x|^{\|\sigma\| + \check{k}_1 n_1 + \dots + \check{k}_s n_s} q^{-\check{k}_1 (\sigma_1 + n_1) - \dots - \check{k}_s (\sigma_s + n_s)} \end{aligned}$$

If  $|x| \leq \alpha < 1$  ( $x \in \mathbb{C}$ ) then

$$\begin{aligned} |W^\eta(x)| &\leq \alpha^{N(\eta)} \sum_{(k_1, \dots, k_s) \in \mathbb{N}^*} q^{-\check{k}_1 (\sigma_1 + n_1) - \dots - \check{k}_s (\sigma_s + n_s)} \\ &\leq \alpha^{N(\eta)} \sum_{1 \leq l_1 < l_2 < \dots < l_s} q^{-l_1 N(\eta_1) - \dots - l_s N(\eta_s)} \\ &\leq \alpha^{N(\eta)} \sum_{1 \leq l_1 < l_2 < \dots < l_{s-1}} q^{-l_1 N(\eta_1) - \dots - l_{s-1} N(\eta_{s-1})} \sum_{l_s = l_{s-1} + 1}^{\infty} q^{-l_s N(\eta_s)} \\ &\leq q^{-N(\eta_s)} \frac{q^{N(\eta_s)}}{q^{N(\eta_s)} - 1} \alpha^{N(\eta)} \times \\ &\quad \sum_{1 \leq l_1 < l_2 < \dots < l_{s-1}} q^{-l_1 N(\eta_1) - \dots - l_{s-1} N(\eta_{s-1} + \eta_s)} \\ &\leq \frac{q}{q-1} \alpha^{N(\eta)} q^{-N(\eta_s)} \times \\ &\quad \sum_{1 \leq l_1 < l_2 < \dots < l_{s-1}} q^{-l_1 N(\eta_1) - \dots - l_{s-1} N(\eta_{s-1} + \eta_s)} \\ &\leq \left( \frac{q}{q-1} \right)^s \alpha^{N(\eta)} q^{-\sum_{i=1}^s i N(\eta_i)} \end{aligned}$$

Note also that

$$\begin{aligned} e_q(x) &= q^{-\log_q x (\log_q x - 1)/2} \\ &= q^{-(\log_q |x| + i \arg_q x) (\log_q |x| + i \arg_q x - 1)/2} \\ &= q^{-\log_q |x| (\log_q |x| - 1)/2} q^{\arg_q^2 x / 2} q^{-i \arg_q x (2 \log_q |x| - 1)/2} \end{aligned}$$

So, if  $x \in V_{\alpha, C} = \{x \in \tilde{\mathbb{C}}; |x| \leq \alpha \text{ and } |\arg_q x| \leq C\}$  with  $\alpha < 1$  then

$$|e_q(x)| \leq e_q(\alpha) q^{C^2/2}$$

and this ends the proof of the lemma.

To prove the convergence of  $\Theta$ , it remains to find some bounds on the operators  $\mathbb{D}_\eta$  ( $\eta \in \mathbf{H}$ ).

### 6.4. Bounds for differential operators

There exist  $B > 0$  such that

$$\forall \eta \in H, \quad |b_\eta| \leq |b_{n,\sigma}| \leq B^{n+\sigma} = B^{N(\eta)} \quad (6.9)$$

We remind that  $U_\varepsilon = \{(x, u) \in \mathbb{C}^2; |x| < \varepsilon \text{ and } |u| < \varepsilon\}$ .

LEMMA 6.4. — *Let  $0 < \alpha < \beta$  and  $\varphi$  a holomorphic function in the neighborhood of  $U_\beta$ , then, for any  $\eta = \binom{n}{\sigma} \in H$ ,*

$$\|x^\sigma \mathbb{D}_\eta \cdot \varphi\|_{U_\alpha} \leq \left( \frac{2B\alpha\beta}{\beta - \alpha} \right)^{N(\eta)} \|\varphi\|_{U_\beta} \quad (6.10)$$

thus,

$$\|x^\sigma \mathbb{D}_\eta\|_{U_\alpha, U_\beta} \leq \left( \frac{2B\alpha\beta}{\beta - \alpha} \right)^{N(\eta)} \quad (6.11)$$

We have

$$\begin{aligned} \|x^\sigma \mathbb{D}_\eta \cdot \varphi\|_{U_\alpha} &= \left\| \sum_{\substack{1 \leq s \leq N(\eta) \\ \eta_1 + \dots + \eta_s = \eta \\ \eta_i \in H_0}} x^\sigma \frac{1}{s!} b_{\eta_1} \dots b_{\eta_s} u^{n_1 + \dots + n_s + s} \partial_u^s \cdot \varphi \right\|_{U_\alpha} \\ &\leq (B\alpha)^{N(\eta)} \sum_{s=1}^{N(\eta)} \frac{\alpha^s}{s!} \sum_{\substack{\eta_1 + \dots + \eta_s = \eta \\ \eta_i \in H_0}} \|\partial_u^s \cdot \varphi\|_{U_\alpha} \end{aligned}$$

and, using the Cauchy estimates,

$$\|\partial_u^s \cdot \varphi\|_{U_\alpha} \leq \frac{s!}{(\beta - \alpha)^s} \|\varphi\|_{U_\beta}$$

thus

$$\begin{aligned}
 \|x^\sigma \mathbb{D}_\eta \cdot \varphi\|_{U_\alpha} &\leq (B\alpha)^{N(\eta)} \sum_{s=1}^{N(\eta)} \sum_{\substack{\eta_1 + \dots + \eta_s = \eta \\ \eta_i \in H_0}} \frac{\alpha^s}{(\beta - \alpha)^s} \|\varphi\|_{U_\beta} \\
 &\leq (B\alpha)^{N(\eta)} \sum_{s=1}^{N(\eta)} P(\eta, s) \frac{\alpha^s}{(\beta - \alpha)^s} \|\varphi\|_{U_\beta} \\
 &\leq (B\alpha)^{N(\eta)} \sum_{s=1}^{N(\eta)} 2^{N(\eta)} C_{N(\eta)-1}^{s-1} \frac{\alpha^s}{(\beta - \alpha)^s} \|\varphi\|_{U_\beta} \\
 &\leq \frac{\alpha}{\beta - \alpha} (2B\alpha)^{N(\eta)} \left( \frac{\beta}{\beta - \alpha} \right)^{N(\eta)-1} \|\varphi\|_{U_\beta} \\
 &\leq \left( \frac{2B\alpha\beta}{\beta - \alpha} \right)^{N(\eta)} \|\varphi\|_{U_\beta}
 \end{aligned}$$

This ends the proof of the lemma and we can deduce from this that

LEMMA 6.5. — For  $0 < \alpha < \beta$  and  $\varphi$  a holomorphic function in the neighborhood of  $U_\beta$ , then, for any  $\eta = (\eta_1, \dots, \eta_s) = \begin{pmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{pmatrix} = \begin{pmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{pmatrix} \in \mathbf{H}$ ,

$$\|x^{\|\boldsymbol{\sigma}\|} \mathbb{D}_\eta \cdot \varphi\|_{U_\alpha} \leq s^{N(\eta)} \left( \frac{2B\alpha\beta}{\beta - \alpha} \right)^{N(\eta)} \|\varphi\|_{U_\beta} \quad (6.12)$$

First, consider  $\alpha = \alpha_s < \alpha_{s-1} < \dots < \alpha_1 < \alpha_0 = \beta$ , then

$$\begin{aligned}
 \|x^{\|\boldsymbol{\sigma}\|} \mathbb{D}_\eta \cdot \varphi\|_{U_\alpha} &= \|x^{\sigma_s} \mathbb{D}_{\eta_s} \dots x^{\sigma_1} \mathbb{D}_{\eta_1} \cdot \varphi\|_{U_{\alpha_s}} \\
 &\leq \|x^{\sigma_s} \mathbb{D}_{\eta_s}\|_{U_{\alpha_s}, U_{\alpha_{s-1}}} \dots \|x^{\sigma_2} \mathbb{D}_{\eta_2}\|_{U_{\alpha_2}, U_{\alpha_1}} \\
 &\quad \times \|x^{\sigma_1} \mathbb{D}_{\eta_1}\|_{U_{\alpha_1}, U_\beta} \|\varphi\|_{U_\beta} \\
 &\leq \prod_{i=1}^s \left( \frac{2B\alpha_i \alpha_{i-1}}{\alpha_{i-1} - \alpha_i} \right)^{N_i} \|\varphi\|_{U_{\alpha_0}}
 \end{aligned}$$

where  $N_i = N(\eta_i)$ . We have to estimate

$$T_{\alpha_0}^{\substack{\alpha_1, \dots, \alpha_s \\ N_1, \dots, N_s}} = \prod_{i=1}^s \left( \frac{2B\alpha_i \alpha_{i-1}}{\alpha_{i-1} - \alpha_i} \right)^{N_i}$$

If

$$\alpha_1 = \frac{(N_1 + N_2)\alpha_2\alpha_0}{N_1\alpha_0 + N_2\alpha_2} = f(\alpha_0, \alpha_2, N_1, N_2)$$

then

$$\alpha_0 - \alpha_1 = \frac{N_1\alpha_0^2 - N_1\alpha_2\alpha_0}{N_1\alpha_0 + N_2\alpha_2} = \frac{N_1\alpha_0(\alpha_0 - \alpha_2)}{N_1\alpha_0 + N_2\alpha_2} > 0$$

and

$$\alpha_2 - \alpha_1 = \frac{-N_2\alpha_2(\alpha_0 - \alpha_2)}{N_1\alpha_0 + N_2\alpha_2} < 0$$

thus  $\alpha_2 < \alpha_1 = f(\alpha_0, \alpha_2, N_1, N_2) < \alpha_0$  and

$$\left(\frac{2B\alpha_1\alpha_0}{\alpha_0 - \alpha_1}\right)^{N_1} = \left(\frac{2B(N_1 + N_2)\alpha_2\alpha_0^2}{N_1\alpha_0(\alpha_0 - \alpha_2)}\right)^{N_1} = \left(\frac{2B(N_1 + N_2)\alpha_2\alpha_0}{N_1(\alpha_0 - \alpha_2)}\right)^{N_1}$$

and

$$\left(\frac{2B\alpha_2\alpha_1}{\alpha_1 - \alpha_2}\right)^{N_2} = \left(\frac{2B(N_1 + N_2)\alpha_2^2\alpha_0}{N_2\alpha_2(\alpha_0 - \alpha_2)}\right)^{N_2} = \left(\frac{2B(N_1 + N_2)\alpha_2\alpha_0}{N_2\alpha_2(\alpha_0 - \alpha_2)}\right)^{N_2}$$

thus, for this choice of  $\alpha_1$ ,

$$\begin{aligned} T_{\alpha_0}^{\alpha_1, \dots, \alpha_s} &= \prod_{i=1}^s \left(\frac{2B\alpha_i\alpha_{i-1}}{\alpha_{i-1} - \alpha_i}\right)^{N_i} \\ &= \frac{(N_1 + N_2)^{N_1 + N_2}}{N_1^{N_1} N_2^{N_2}} T_{\alpha_0}^{\alpha_2, \dots, \alpha_s} \end{aligned}$$

Using recursively this formula, for  $s \geq 2$ , we can find ( $\alpha_0 = \beta$  and  $\alpha_s = \alpha$ )

$$\begin{aligned} \alpha_{s-1} &= f(\alpha_0, \alpha_s, N_1 + \dots + N_{s-1}, N_s) \\ \alpha_{s-2} &= f(\alpha_0, \alpha_{s-1}, N_1 + \dots + N_{s-2}, N_{s-1}) \\ &\vdots \\ \alpha_k &= f(\alpha_0, \alpha_{k+1}, N_1 + \dots + N_k, N_{k+1}) \\ \alpha_1 &= f(\alpha_0, \alpha_2, N_1, N_2) \end{aligned}$$

such that  $\alpha < \alpha_{s-1} < \dots < \alpha_1 < \beta$  and

$$T_{\alpha_0}^{\alpha_1, \dots, \alpha_s} = \frac{(N_1 + \dots + N_s)^{N_1 + \dots + N_s}}{N_1^{N_1} \dots N_s^{N_s}} \left(\frac{2B\alpha\beta}{\beta - \alpha}\right)^{N_1 + \dots + N_s}$$

and finally

$$\|x\|^{\|\sigma\|} \mathbb{D}_{\eta} \cdot \varphi \|_{U_\alpha} \leq \frac{N^N}{N_1^{N_1} \dots N_s^{N_s}} \left(\frac{2B\alpha\beta}{\beta - \alpha}\right)^N \|\varphi\|_{U_\beta} \quad (6.13)$$

where  $N_i = N(\eta_i)$  and  $N = N(\eta)$ . Using now that

$$\sup_{N_1 + \dots + N_s = N} \frac{N^N}{N_1^{N_1} \dots N_s^{N_s}} = s^N \quad (6.14)$$

we get the attempted result.

### 6.5. Conclusion

Putting together these different results, we get

THEOREM 6.6. — *Let us consider the equation*

$$(x\sigma_q - 1)f = b(f, x) \quad (6.15)$$

with  $b(0, x) = 0$  and  $\frac{\partial b}{\partial f}(0, 0) = 0$ . For any  $C > 0$ , there exists  $0 < \alpha < 1$  such that the formal solution:

$$f(x, ue_q(x)) = \Theta.(ue_q(x)) = ue_q(x) + \sum_{\boldsymbol{\eta} \in \mathbf{H}} e_q^{\|\mathbf{n}\|+1}(x) W^{\boldsymbol{\eta}}(x) \mathbb{D}_{\boldsymbol{\eta}}.u$$

is indeed a normally convergent series in

$$U_{\alpha, C} = \{(x, u) \in \tilde{\mathbb{C}} \times \mathbb{C} \ ; \ |x| < \alpha, |\arg_q x| < C, \ |u| < \alpha\}.$$

It is clear that

$$\|ue_q(x)\|_{U_{\alpha, C}} \leq \alpha e_q(\alpha) q^{C^2/2}$$

For  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) = \begin{pmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{pmatrix} = \begin{pmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{pmatrix} \in \mathbf{H}$ , we get

$$\|e_q^{\|\mathbf{n}\|+1}(x)\|_{U_{\alpha, C}} \leq (e_q(\alpha) q^{C^2/2})^{\|\mathbf{n}\|+1}$$

and

$$|W^{\boldsymbol{\eta}}(x) \mathbb{D}_{\boldsymbol{\eta}}.u| \leq \left(\frac{q}{q-1}\right)^s \alpha^{\|\mathbf{n}\|} q^{-\sum_{i=1}^s iN(\eta_i)} |x|^{\|\boldsymbol{\sigma}\|} |\mathbb{D}_{\boldsymbol{\eta}}.u|$$

thus, if  $M^{\boldsymbol{\eta}} = \|e_q^{\|\mathbf{n}\|+1}(x) W^{\boldsymbol{\eta}}(x) \mathbb{D}_{\boldsymbol{\eta}}.u\|_{U_{\alpha, C}}$ ,

$$M^{\boldsymbol{\eta}} \leq (e_q(\alpha) q^{C^2/2})^{\|\mathbf{n}\|+1} \left(\frac{q}{q-1}\right)^s \alpha^{\|\mathbf{n}\|} q^{-\sum_{i=1}^s iN(\eta_i)} \left(\frac{2sB\alpha\beta}{\beta-\alpha}\right)^{N(\boldsymbol{\eta})} \|u\|_{U_{\beta}}$$

We can choose  $\beta = 1$  so, if  $\alpha$  is such that  $\alpha < 1$  and  $e_q(\alpha) q^{C^2/2} \leq 1$ ,

$$\begin{aligned} \|e_q^{\|\mathbf{n}\|+1}(x) W^{\boldsymbol{\eta}}(x) \mathbb{D}_{\boldsymbol{\eta}}.u\|_{U_{\alpha, C}} &\leq \left(\frac{q}{q-1}\right)^s q^{-\sum_{i=1}^s iN(\eta_i)} s^{N(\boldsymbol{\eta})} \left(\frac{2B\alpha}{1-\alpha}\right)^{N(\boldsymbol{\eta})} \\ &\leq \left(\frac{q}{q-1} \frac{2B\alpha}{1-\alpha}\right)^{N(\boldsymbol{\eta})} q^{-sN(\boldsymbol{\eta})} s^{N(\boldsymbol{\eta})} \end{aligned}$$

For  $\varepsilon > 0$ , there exists  $\alpha$  is sufficiently small such that

$$\frac{q}{q-1} \frac{2B\alpha}{1-\alpha} < \varepsilon$$

and as

$$\forall s \geq 1, \quad sq^{-s} \leq \frac{1}{e \log q}$$

then, if

$$M_{\alpha,C} = \|ue_q(x)\|_{U_{\alpha,C}} + \sum_{\eta \in \mathbf{H}} \|e_q^{\|\mathbf{n}\|+1}(x)W^\eta(x)\mathbb{D}_{\eta \cdot u}\|_{U_{\alpha,C}}$$

we get

$$\begin{aligned} M_{\alpha,C} &\leq 1 + \sum_{\eta \in \mathbf{H}} \varepsilon^{N(\eta)} q^{-sN(\eta)} s^{N(\eta)} \\ &\leq 1 + \sum_{\eta \in \mathbf{H}} \left( \frac{\varepsilon}{e \log q} \right)^{N(\eta)} \\ &\leq 1 + \sum_{N \geq 1} \sum_{\eta \in H; N(\eta)=N} P(\eta) \left( \frac{\varepsilon}{e \log q} \right)^N \\ &\leq 1 + \sum_{N \geq 1} q(N) 4^N \left( \frac{\varepsilon}{e \log q} \right)^N \\ &\leq 1 + \sum_{N \geq 1} \left( \frac{8\varepsilon}{e \log q} \right)^N \end{aligned}$$

It is clear now that we have the right result by considering  $\varepsilon$  sufficiently small.

We have established the convergence of the formal solution in the convergent case. There is no hope for such a simple result in the general case, as some of the monomials of  $W^\bullet$  are divergent series. The only hope for such monomial is to find a resummation process, which, as in the classical framework of differential equations, yields some sectorial sums having these divergent series as their asymptotic expansion. The usual Borel–Laplace transforms cannot work here as because the classical Borel transform does not yield a convergent function. Nevertheless, there has been some recent results by C. Zhang (see [2],[3]) on a  $q$ -analog to the Borel–Laplace transform.



## 7. Notions of $q$ -summability

We just give some definitions and properties here. For more details, see [3].

### 7.1. Definitions

Here,  $k$  and  $s$  are positive real numbers such that  $ks = 1$ . Let  $f(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]$ . Let

$$\hat{\mathcal{B}}_s f(\zeta) = \sum_{n \geq 0} a_n q^{-sn(n-1)/2} \zeta^n \quad (7.1)$$

be the *formal  $q$ -Borel transform of order  $s$  (or of level  $k$ )* of  $f$ . The inverse of  $\hat{\mathcal{B}}_s$  is the *formal  $q$ -Laplace transform of order  $s$*   $\hat{\mathcal{L}}_s$  and the set of  $q$ -Gevrey series of order  $s$  is

$$\mathbb{C}[[x]]_s = \hat{\mathcal{L}}_s \cdot \mathbb{C}\{\zeta\} \quad (7.2)$$

Once again  $\tilde{\mathbb{C}}$  is the Riemann surface of the logarithm. If  $\log$  is the principal determination of the logarithm then, of course,

$$\log_q x = \frac{\log x}{\log q} = \log_q |x| + i \arg_q x.$$

For  $\theta \in \mathbb{R}$ ,

$$d_\theta = \{x \in \tilde{\mathbb{C}}; \arg x = \theta\} \quad (7.3)$$

is the direction of argument  $\theta$ . A sectorial neighborhood of  $d_\theta$  is a subset of  $\tilde{\mathbb{C}}$ :

$$S_{\theta, \varepsilon} = \{x \in \tilde{\mathbb{C}}; |\arg x - \theta| < \varepsilon\} \quad (7.4)$$

An analytic germ is an analytic function defined in a disc:

$$\tilde{D}(0, R) = \{x \in \tilde{\mathbb{C}}; |x| < R\} \quad (7.5)$$

and  $\tilde{\mathbb{O}}$  is the set of such germs.

An analytic function  $\varphi$  defined on a radially unbounded open set  $V$  has a  *$q$ -exponential growth of order  $k$  and of finite type* if there exists  $\mu \in \mathbb{R}$  such that

$$\varphi(\zeta) = O(\zeta^\mu q^{\frac{k}{2} \log_q^2 |\zeta|}) \quad (\zeta \text{ large, } \zeta \in V) \quad (7.6)$$

We note  $\tilde{\mathbb{H}}_s^\theta$  the set of functions  $\varphi \in \tilde{\mathbb{O}}$  that can be analytically continued in a sectorial neighborhood of  $d_\theta$  such that

1.  $\varphi$  has an asymptotic development in the sense of Poincaré at “ $0 \in \tilde{\mathbb{C}}$ ” in  $V$ ;

2.  $\varphi$  has a  $q$ -exponential growth of order  $k$  and of finite type in  $V$ .

Then

$$\mathbb{H}_s^\theta = \widetilde{\mathbb{H}}_s^\theta \cap \mathbb{C}\{\zeta\} \quad \text{and} \quad \mathbb{E}_s = \bigcap_{\theta \in [0, 2\pi]} \mathbb{H}_s^\theta \quad (7.7)$$

DEFINITION 7.1. — Let  $f \in \mathbb{C}[[x]]_s$  and  $\theta \in \mathbb{R}$ .

(i)  $f$  is  $q$ -summable of order  $s$  in the direction  $d_\theta$  if  $\hat{\mathcal{B}}_s f \in \mathbb{H}_s^\theta$ ; we note  $f \in \mathbb{C}\{x\}_s^\theta$ .

(ii)  $d_\theta$  is a singular direction of order  $s$  for  $f$  if  $\hat{\mathcal{B}}_s f \notin \mathbb{H}_s^\theta$ ;  $\theta \in \text{DS}(f)$ .

(iii)  $f$  is  $q$ -summable of order  $s$  ( $f \in \mathbb{C}\{x\}_s$ ) if  $\text{DS}(f) \cap [0, 2\pi]$  is finite.

For example

$$f(x) = \sum_{n \geq 0} q^{-n(n-1)/2} x^n$$

is  $q$ -summable of order 1 and  $\text{DS}(f) = \{2k\pi \ ; \ k \in \mathbb{Z}\}$  as:

$$\tilde{\mathcal{B}}_1 f(\zeta) = \frac{1}{1-\zeta}$$

If  $f \in \mathbb{C}\{x\}_s$ , we can define the *analytic  $q$ -Laplace transform of  $\hat{f}_\theta = \mathcal{S}^\theta \tilde{\mathcal{B}}_s f$  in the direction  $d_\theta$*  ( $\theta \notin \text{DS}(f)$ ) where  $\mathcal{S}^\theta$  is the the operator of analytic continuation along  $d_\theta$ :

$$f_\theta(x) = \mathcal{L}_s^\theta \hat{f}_\theta(x) = \frac{q^{-1/(8k)} \sqrt{k}}{\sqrt{2\pi \log q}} \int_{d_\theta} q^{-\frac{k}{2}(\log_q \frac{x}{\zeta})(\log_q \frac{x}{\zeta} - \frac{1}{k})} \hat{f}_\theta(\zeta) \frac{d\zeta}{\zeta} \quad (7.8)$$

and  $f_\theta \in \tilde{\mathcal{O}}$  is the  $q$ -sum of order  $s$  of  $f$  in the direction  $d_\theta$ .

There exists also a convolution product on  $\mathbb{C}[[x]]_s$ : Let  $f(x) = \sum a_n x^n$  and  $g(x) = \sum b_n x^n$  two elements of  $\mathbb{C}[[x]]_s$  and  $\hat{f}$  and  $\hat{g}$  be their respective formal  $q$ -Borel transform, then

$$\begin{aligned} \hat{f} *_s \hat{g}(\zeta) &= \tilde{\mathcal{B}}_s(fg) \\ &= \sum_{n \geq 0} a_n q^{-sn(n-1)/2} \zeta^n \hat{g}(\zeta q^{-sn}) \\ &= \sum_{n \geq 0} b_n q^{-sn(n-1)/2} \zeta^n \hat{f}(\zeta q^{-sn}) \end{aligned} \quad (7.9)$$

There is a first difference with the classical Borel-Laplace transform: If  $f$  and  $g$  are two element of  $\mathbb{C}\{x\}_s^\theta$  then  $fg \in \mathbb{C}\{x\}_{s/2}^\theta$  thus  $fg$  is not  $q$ -summable of order  $s$  (see [3]). To circumvent this difficulty, we need to

define  $q$ -multisummability (see [3]). Let  $\Omega^+$  be the set of strictly increasing sequences of elements of  $\mathbb{R}^{+\ast}$ .

DEFINITION 7.2. — If  $\vec{s} = (s_1, \dots, s_r) \in \Omega^+$ ,  $\theta \in \mathbb{R}$  and  $f \in \mathbb{C}[[x]]$ ,

1.  $f$  is  $q$ -multisummable of order  $\vec{s}$  in the direction  $d_\theta$  ( $f \in \mathbb{C}\{x\}_{\vec{s}}^\theta$ ) if:

- $f \in \mathbb{C}[[x]]_{s_r}$ ;
- $\tilde{\mathcal{B}}_{s_r} f \in \mathbb{H}_{s_r - s_{r-1}}^\theta$ ;
- $\mathcal{L}_{s_{r-j}+1-s_{r-j}}^\theta \cdots \mathcal{L}_{s_r-s_{r-1}}^\theta \mathcal{S}^\theta \tilde{\mathcal{B}}_{s_r} f \in \tilde{\mathbb{H}}_{s_{r-j}-s_{r-j-1}}^\theta$  for  $1 \leq j \leq r-2$ ;
- $\mathcal{L}_{s_2-s_1}^\theta \cdots \mathcal{L}_{s_r-s_{r-1}}^\theta \mathcal{S}^\theta \tilde{\mathcal{B}}_{s_r} f \in \tilde{\mathbb{H}}_{s_1}^\theta$ .

2. If  $f \in \mathbb{C}\{x\}_{\vec{s}}^\theta$  the  $q$ -sum of  $f$  in the direction  $d_\theta$  is:

$$\mathcal{S}_{\vec{s}}^\theta f = \mathcal{L}_{s_1}^\theta \mathcal{L}_{s_2-s_1}^\theta \cdots \mathcal{L}_{s_r-s_{r-1}}^\theta \mathcal{S}^\theta \tilde{\mathcal{B}}_{s_r} f \quad (7.10)$$

3. The direction  $d_\theta$  is singular for  $f$  ( $\theta \in \text{DS}(f)$ ) if  $f \notin \mathbb{C}\{x\}_{\vec{s}}^\theta$ .

4.  $f$  is  $q$ -multisummable of order  $\vec{s}$  if  $\text{DS}(f) \cap [0, 2\pi]$  is finite. We note then  $f \in \mathbb{C}\{x\}_{\vec{s}}$ .

For details, see [3].

## 7.2. Applications

We just give two results that are enclosed in [3].

PROPOSITION 7.3. — Let  $\theta \in \mathbb{R}$  and  $s > 0$ .

$$\forall f, g \in \mathbb{C}\{x\}_s^\theta, \quad f.g \in \mathbb{C}\{x\}_{(s/2, s)}^\theta \quad \text{and} \quad \mathcal{S}_{(s/2, s)}^\theta(f.g) = \mathcal{S}_s^\theta f . \mathcal{S}_s^\theta g \quad (7.11)$$

Let  $\Delta$  be a  $q$ -difference operator

$$\Delta = a_0(x) + \dots + a_m(x)\sigma_q^m \quad (7.12)$$

where  $m \geq 1$ ,  $a_0(x)a_m(x) \neq 0$  and  $a_j(x) \in \mathbb{C}\{x\}$ . Let  $\text{val}(a_j)$  be the valuation of  $a_j$ . We note  $\text{PN}(\Delta)$  the Newton polygon of  $\Delta$ : it the convex hull of the ascending half-axis starting from the points  $(j, \text{val}(a_j))$  ( $0 \leq j \leq m$ ). One can suppose that the slopes of  $\text{PN}(\Delta)$  are integers.

**THEOREM 7.4** (*Zhang and Marotte*). — *If  $f \in \mathbb{C}[[x]]$  is such that  $\Delta f \in \mathbb{C}\{x\}$ , then  $f \in \mathbb{C}\{x\}_{\vec{s}}$  where the elements of  $\vec{s} \in \Omega^+$  are the inverses of the positive slopes of  $\text{PN}(\Delta)$ . In particular, if  $\text{PN}(\Delta)$  has no positive slope, then  $f \in \mathbb{C}\{x\} = \mathbb{C}\{x\}_\emptyset$ .*

This last result should be useful to study the  $q$ -multisummability of the monomials of  $W^\bullet$ .

## 8. $q$ -multisummability for $W^\bullet$

### 8.1. First result

**THEOREM 8.1.** — *For  $\eta \in \mathbf{H}$ , there exists  $\vec{s}^\eta$  such that*

$$W^\eta \in \mathbb{C}\{x\}_{\vec{s}^\eta} \tag{8.1}$$

This result becomes obvious by noticing that there exists  $\Delta_\eta \in \mathbb{C}\{x\}[\sigma_q]$  and  $g_\eta \in \mathbb{C}\{x\}$  such that

$$\Delta_\eta W^\eta = g_\eta \tag{8.2}$$

For  $\eta \in \mathbf{H}$ , we note  $\text{DS}(\eta) = \text{DS}(W^\eta)$ . Unfortunately, this result is not sufficient to define a global  $q$ -sum for the operator  $\Theta$ . We give now a list of problems that remain unsolved.

### 8.2. Open problems

#### 8.2.1. Problem 1: Orders of $q$ -multisummability

For  $\eta \in \mathbf{H}$ , we would like to have a complete description of  $\vec{s}^\eta$ . If  $\eta = \begin{pmatrix} n_1 & , \dots , & n_s \\ \sigma_1 & , \dots , & \sigma_s \end{pmatrix}$ , the first computations for  $s = 1, 2, 3$  indicate that the elements of  $\vec{s}^\eta$  are the numbers

$$-\frac{1}{\check{n}_i} = -\frac{1}{n_1 + \dots + n_i} \quad \text{where } 1 \leq i \leq s \quad \text{and } \check{n}_i < 0$$

and it should not be so difficult to prove this.

### 8.2.2. Problem 2: Singular directions

For  $\eta \in \mathbf{H}$ , it does not seem easy to find the set of singular directions  $\text{DS}(\eta)$ . One can note that if  $n > 0$  and  $\sigma \geq n$  then  $\eta = \begin{pmatrix} -n \\ \sigma \end{pmatrix} \in \mathbf{H}$  and  $\Delta_\eta = x^n \sigma_q - 1$ ,  $g_\eta(x) = x^\sigma$  thus

$$W^\eta \in \mathbb{C}\{x\}_{\frac{1}{n}}$$

and  $\text{DS}(\eta) = \{\frac{2i\pi k}{n}, k \in \mathbb{Z}\}$  (see [3]). So, for  $\eta \in \mathbf{H}$ ,  $\text{DS}(\eta)$  is discrete but we would like to define a global  $q$ -sum  $\Theta_\theta$  for  $\Theta$  by

$$\Theta_\theta = 1 + \sum_{\eta \in \mathbf{H}} e_q^{\|\eta\|} (\mathcal{S}_{s^\theta}^\theta W^\eta) \mathbb{D}_\eta \quad (8.3)$$

It means that we must have

$$\theta \notin \text{DS}(\Theta) = \bigcup_{\eta \in \mathbf{H}} \text{DS}(\eta)$$

but this set is no longer discrete. This problem seems unavoidable and is one of the motivation for introducing *arborification* (see section below).

### 8.2.3. Problem 3: Symmetry relations for $W^\bullet$

Let us just forget the previous problem and suppose that we can define a global  $q$ -sum  $\Theta_\theta$  for  $\Theta$  by

$$\Theta_\theta = 1 + \sum_{\eta \in \mathbf{H}} e_q^{\|\eta\|} (\mathcal{S}_{s^\theta}^\theta W^\eta) \mathbb{D}_\eta \quad (8.4)$$

The operator  $\Theta$  was a substitution automorphism because the mould  $W^\bullet$  is symmetrel:

If  $(\eta, \mu) \in \mathbf{H}^2$ , then,

$$W^\eta W^\mu = \sum_{\lambda \in \text{ctsh}(\eta, \mu)} W^\lambda$$

But  $\Theta_\theta$  must also be a substitution automorphism: For

$$\theta \notin \text{DS}(\eta) \cup \text{DS}(\mu) \quad \bigcup_{\lambda \in \text{ctsh}(\eta, \mu)} \text{DS}(\lambda)$$

we must check that:

$$\mathcal{S}_{s^\theta}^\theta W^\eta \cdot \mathcal{S}_{s^\theta}^\theta W^\mu = \sum_{\lambda \in \text{ctsh}(\eta, \mu)} \mathcal{S}_{s^\theta}^\theta W^\lambda \quad (8.5)$$

and this problem is non-trivial. To prove such identities, we would need to prove that there exists  $\bar{s}^{(\eta, \mu)} \in \Omega^+$  such that

$$W^\eta W^\mu \in \mathbb{C}\{x\}_{\bar{s}^{(\eta, \mu)}}^\theta \quad \text{and} \quad \mathcal{S}_{\bar{s}^{(\eta, \mu)}}^\theta(W^\eta W^\mu) = \mathcal{S}_{\bar{s}^\eta}^\theta W^\eta \cdot \mathcal{S}_{\bar{s}^\mu}^\theta W^\mu$$

Note that such a sequence  $\bar{s}^{(\eta, \mu)}$  exists as one can prove that  $W^\eta W^\mu$  is also a solution of a  $q$ -difference equation (see [3]).

#### 8.2.4. Problem 4: Estimates and convergence of $\Theta_\theta$

Let us now forget the two previous problems. It means that we could define a symmetrel mould  $W_\theta^\bullet$  for some  $\theta \in \mathbb{R}$  by:

$$\forall \eta \in \mathbf{H}, \quad W_\theta^\eta = \mathcal{S}_{\bar{s}^\eta}^\theta W^\eta$$

It remains to prove that the substitution automorphism

$$\Theta_\theta = 1 + \sum_{\eta \in \mathbf{H}} e_q^{\|\eta\|} W_\theta^\eta \mathbb{D}_\eta$$

is a convergent operator. On one hand, there should not be any difficulty to get similar results to those obtained in Lemmas 6.4 and 6.5. On the other hand, our guess is that we do not have such good estimates for  $W_\theta^\bullet$  as in Lemma 6.3. This would mean that  $\Theta_\theta$  is no longer normally convergent and this motivates once again the introduction of *Arborification*.

Let us now define the arborification and show what these problems become.

## 9. Arborification for the mould $W^\bullet$

### 9.1. Reminder on the contracting arborification

We follow the definitions of J. Ecalle [1]. Let us consider an additive semigroup  $H$ . The set  $\mathbf{H}$  is the set of sequences on  $H$ , where a sequence is a totally ordered sequence of elements of  $H$ , with possible repetitions.

An *arborescent sequence* on  $H$  is a sequence  $\eta^< = (\eta_1, \dots, \eta_s)^< \in \mathbf{H}^<$  of elements of  $H$  with an arborescent order on the indices  $\{1, \dots, s\}$ : each  $i \in \{1, \dots, s\}$  possess at most one predecessor  $i_-$ . We note  $\eta^< = \eta'^< \oplus \eta''^<$  the disjoint union of  $\eta'^<$  and  $\eta''^<$ , the partial orders of  $\eta'^<$  and  $\eta''^<$  being preserved and the elements  $\eta'^<$  are not comparable with those of  $\eta''^<$ .  $\emptyset$  is the empty sequence. A sequence  $\eta^<$  is irreducible if it is not a disjoint union

of smaller nontrivial sequences; that is to say that it has exactly one least element.

We remind here that a mould  $A^\bullet = \{A^\eta\}$  on  $H$  with values in a commutative algebra is a family of elements  $A^\eta$  indexed by the sequences  $\eta \in \mathbf{H}$  of  $H$ . For example,  $W^\bullet$  is a mould on  $H$  with values in  $\mathbb{C}[[x]]$ . Moreover, this mould is symmetrel:  $W^\emptyset = 1$  and, for any pair  $(\eta', \eta'')$ , we get

$$W^{\eta'} W^{\eta''} = \sum \text{ctsh} \left( \begin{array}{c} \eta', \eta'' \\ \eta \end{array} \right) W^\eta \quad (9.1)$$

Where  $\text{ctsh} \left( \begin{array}{c} \eta', \eta'' \\ \eta \end{array} \right)$  is the number of ways to get  $\eta$  by contracting shuffling of  $\eta'$  and  $\eta''$ .

We also remind that an arborescent mould  $A^{\bullet<} = \{A^{\eta^{<}}\}$  on  $H$  with values in a commutative algebra is a family of elements  $A^{\eta^{<}}$  indexed by the arborescent sequences  $\eta^{<} \in \mathbf{H}^{<}$  of  $H$ . Such an arborescent  $A^{\bullet<}$  is separative if:

$$A^\emptyset = 1 \quad \text{and} \quad \forall \eta^{<}, \eta''^{<}, \quad A^{\eta^{<} \oplus \eta''^{<}} = A^{\eta^{<}} A^{\eta''^{<}} \quad (9.2)$$

We get such arborescent separative moulds by *contracting arborification* of symmetrel moulds. This operation is defined as follows.

Let  $\eta^{<} = (\eta_1, \dots, \eta_s)^{<}$  be an arborescent sequence and  $\eta' = (\eta'_1, \dots, \eta'_{s'})$  a totally ordered sequence. Let  $\text{cont} \left( \begin{array}{c} \eta^{<} \\ \eta' \end{array} \right)$  be the number of monotonic contractions of  $\eta^{<}$  on  $\eta'$ , that is to say the number of surjections  $\sigma$  from  $\{1, \dots, s\}$  into  $\{1, \dots, s'\}$  such that:

$$(i_1 < i_2 \text{ in } \eta^{<}) \implies (\sigma(i_1) < \sigma(i_2) \text{ in } \eta') \quad (9.3)$$

$$\forall j \in \{1, \dots, s'\} \quad ; \quad \eta'_j = \sum_{\sigma(i)=j} \eta_i \quad (9.4)$$

The relation

$$A^{\eta^{<}} = \sum \text{cont} \left( \begin{array}{c} \eta^{<} \\ \eta' \end{array} \right) A^{\eta'} \quad (9.5)$$

defines a homomorphism from the algebra of moulds into the algebra of arborescent moulds. Moreover, the contracting arborification of a symmetrel mould is separative. One can also notice that, if  $\eta$  is a totally ordered sequence and  $\eta^{<}$  is that arborescent sequence with the same order (total), then  $A^{\eta^{<}} = A^\eta$ .

Let us now focus on the mould  $W^\bullet$ .

## 9.2. The arborescent mould $W^{\bullet\prec}$

We will first change some notations. Let  $\boldsymbol{\eta}^{\prec} = (\eta_1, \dots, \eta_s)^{\prec}$  be an arborescent sequence of length  $s$  and of sum  $\|\boldsymbol{\eta}^{\prec}\| = \eta_1 + \dots + \eta_s$ . We redefine the partial sums:

$$\begin{aligned}\check{n}_i &= \sum_{j \leq i} n_j \\ \hat{n}_i &= \sum_{j \geq i} n_j\end{aligned}\tag{9.6}$$

where the orders  $\leq$  and  $\geq$  are now relative to the partial order on  $\{1, \dots, s\}$ . We have the following theorem:

**THEOREM 9.1.** — *If  $\boldsymbol{\eta}^{\prec} = (\eta_1, \dots, \eta_s)^{\prec} = \begin{pmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{pmatrix}^{\prec} = \begin{pmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{pmatrix}^{\prec} \in \mathbf{H}^{\prec}$ , then the monomial  $W^{\boldsymbol{\eta}^{\prec}}$  is a formal series and:*

$$W^{\boldsymbol{\eta}^{\prec}}(x) = \varepsilon_{\mathbf{n}^{\prec}} \sum_{(k_1, \dots, k_s)^{\prec} \in \mathbb{Z}^{\mathbf{n}^{\prec}}} x^{\|\boldsymbol{\sigma}\| + \check{k}_1 n_1 + \dots + \check{k}_s n_s} q^{-\check{k}_1 \sigma_1 - \dots - \check{k}_s \sigma_s} [\check{k}_1 + 1]_q^{n_1} \dots [\check{k}_s + 1]_q^{n_s}\tag{9.7}$$

with the following rules

- The sequences  $\mathbf{n}^{\prec}$  and  $(k_1, \dots, k_s)^{\prec}$  inherit the partial order of  $\boldsymbol{\eta}^{\prec}$ .
- $\varepsilon_{\mathbf{n}} = \prod_{1 \leq i \leq s} \varepsilon_{\hat{n}_i}$  with  $\hat{n}_i = \sum_{j \geq i} n_j$  and  $\varepsilon_n = 1$  (resp.  $\varepsilon_n = -1$ ) if  $n \geq 0$  (resp.  $n < 0$ ).
- $\mathbb{Z}^{\mathbf{n}^{\prec}} = (\mathbb{Z}^{\hat{n}_1} \times \dots \times \mathbb{Z}^{\hat{n}_s})^{\prec}$  and  $\mathbb{Z}^n = \mathbb{Z}^{+*}$  (resp.  $\mathbb{Z}^-$ ) if  $n \geq 0$  (resp.  $n < 0$ ).
- $\check{k}_i = \sum_{j \leq i} k_j$ .

The proof is similar to the proof of Theorem 5.1.

We must now define a dual operation, the *coarborification*, on the comould  $\{\mathbb{D}_{\boldsymbol{\eta}}\}_{\boldsymbol{\eta} \in \mathbf{H}}$ , which is such that

$$\begin{aligned}\Theta &= 1 + \sum_{\boldsymbol{\eta}} \mathcal{W}^{\boldsymbol{\eta}} \mathbb{D}_{\boldsymbol{\eta}} &= 1 + \sum_{\boldsymbol{\eta}} e_q^{\|\mathbf{n}\|} \mathcal{W}^{\boldsymbol{\eta}} \mathbb{D}_{\boldsymbol{\eta}} \\ &= 1 + \sum_{\boldsymbol{\eta}^{\prec}} \mathcal{W}^{\boldsymbol{\eta}^{\prec}} \mathbb{D}_{\boldsymbol{\eta}^{\prec}} &= 1 + \sum_{\boldsymbol{\eta}^{\prec}} e_q^{\|\mathbf{n}^{\prec}\|} \mathcal{W}^{\boldsymbol{\eta}^{\prec}} \mathbb{D}_{\boldsymbol{\eta}^{\prec}}\end{aligned}\tag{9.8}$$

## 9.3. Definition of the coarborification

**THEOREM 9.2.** — *There exists a unique arborescent comould  $\mathbb{D}_{\bullet\prec}$  with the three following properties:*



(i)  $\mathbb{D}_{\bullet}^{\leftarrow}$  is coseparative:  $\mathbb{D}_{\emptyset} = 1$  and

$$\text{col}(\mathbb{D}_{\eta^{\leftarrow}}) = \sum \mathbb{D}_{\eta'^{\leftarrow}} \otimes \mathbb{D}_{\eta''^{\leftarrow}} \quad (\eta'^{\leftarrow} \oplus \eta''^{\leftarrow} = \eta^{\leftarrow}) \quad (9.9)$$

with a sum extended to the arborescent sequences  $\eta'^{\leftarrow}, \eta''^{\leftarrow}$  (even the empty sequences) which disjoint union is  $\eta^{\leftarrow}$ .

(ii) If  $\deg(\eta^{\leftarrow}) = d$ ,  $\mathbb{D}_{\eta^{\leftarrow}}$  is a differential operator of degree  $d$  in  $\partial_u$ : if the sequence  $\eta^{\leftarrow}$  has exactly  $d$  minimal elements and thus:

$$\eta^{\leftarrow} = \eta^{1\leftarrow} \oplus \dots \oplus \eta^{d\leftarrow} \quad (\text{with } \eta^{i\leftarrow} \text{ irreducible and } \neq \emptyset) \quad (9.10)$$

the operator  $\mathbb{D}_{\eta^{\leftarrow}}$  can be written:

$$\mathbb{D}_{\eta^{\leftarrow}} = b(u)\partial_u^d \quad (9.11)$$

(iii) If  $\eta^{\leftarrow} = \eta_1 \cdot \eta^{*\leftarrow}$  ( $\eta^{\leftarrow}$  has a least element  $\eta_1$  followed by an arborescent sequence  $\eta^{*\leftarrow}$ ) we get:

$$\mathbb{D}_{\eta^{\leftarrow}} \cdot u = \mathbb{D}_{\eta^{*\leftarrow}} \cdot \mathbb{D}_{\eta_1} \cdot u \quad (9.12)$$

Moreover, as  $\mathbb{D}_{\bullet}$  is cosympetrel

$$\mathbb{D}_{\eta'} = \sum \text{cont} \left( \begin{array}{c} \eta^{\leftarrow} \\ \eta' \end{array} \right) \mathbb{D}_{\eta^{\leftarrow}} \quad (9.13)$$

These results were proven by Jean Ecalle (see [1]). Note that  $\mathbb{D}_{\emptyset} = 1$  and

$$\mathbb{D}_{\eta_1^{\leftarrow}} = (\mathbb{D}_{\eta_1} \cdot u)\partial_u = \begin{cases} \mathbb{B}_{\eta_1} & \text{if } \eta_1 \in H_0 \\ 0 & \text{otherwise} \end{cases} \quad (9.14)$$

and if the length of  $\eta^{\leftarrow}$  is greater than two:

- Either  $\eta^{\leftarrow}$  is irreducible:  $\eta^{\leftarrow} = \eta_1 \cdot \eta^{*\leftarrow}$  and of degree  $d = 1$ . Thus:

$$\mathbb{D}_{\eta^{\leftarrow}} = (\mathbb{D}_{\eta^{*\leftarrow}} \cdot \mathbb{D}_{\eta_1} \cdot u)\partial_u \quad (9.15)$$

- Either  $\eta^{\leftarrow}$  is reducible of degree  $d \geq 2$  and  $\eta^{\leftarrow} = \eta^{1\leftarrow} \oplus \dots \oplus \eta^{d\leftarrow}$  (with  $\eta^{i\leftarrow}$  irreducible and  $\neq \emptyset$ ), in this case:

$$\mathbb{D}_{\eta^{\leftarrow}} = \frac{1}{d_1! \dots d_s!} (\mathbb{D}_{\eta^{1\leftarrow}} \cdot u) \dots (\mathbb{D}_{\eta^{d\leftarrow}} \cdot u)\partial_u^d \quad (9.16)$$

where  $d_1, \dots, d_s$  are the numbers of identical arborescent sequences  $\eta^{j\leftarrow}$  in the decomposition into irreducible sequences, of course  $\sum d_i = d$ .

One can also notice that if a sequence  $\eta^<$  is irreducible of sum  $\|\eta^<\| \notin H_0$ , then  $\mathbb{D}_{\eta^<} = 0$ . This property remains valid if  $\eta^<$  has at least a monotonic partition  $\eta^{1<}, \dots, \eta^{s<}$  with an irreducible part  $\eta^{i<}$  such that  $\|\eta^{i<}\| \notin H_0$ . It means that we can restrict ourselves to the arborescent sequences  $(\eta_1, \dots, \eta_s)^<$  such that, for  $1 \leq i \leq s$ ,  $\hat{\eta}_i = \sum_{j>i} \eta_j \in H_0$ . We note  $\mathbf{H}_0^<$  this set of sequences. For details, see [1]. We can finally write

$$\begin{aligned} \Theta &= 1 + \sum_{\eta} e_q^{\|\eta\|} W^\eta \mathbf{D}_\eta \\ &= 1 + \sum_{\eta^< \in \mathbf{H}^<} e_q^{\|\eta^<\|} W^{\eta^<} \mathbf{D}_{\eta^<} \\ &= 1 + \sum_{\eta^< \in \mathbf{H}_0^<} e_q^{\|\eta^<\|} W^{\eta^<} \mathbf{D}_{\eta^<} \end{aligned} \tag{9.17}$$

We end this section with a discussion on the interest of using the arborification - coarborification to solve the problems described in section 8.2.

#### 9.4. Open problems

As in section 8.1, we get

**THEOREM 9.3.** — *For  $\eta^< \in \mathbf{H}_0^<$ , there exists  $\bar{s}^{\eta^<}$  such that*

$$W^{\eta^<} \in \mathbf{C}\{x\}_{\bar{s}^{\eta^<}} \tag{9.18}$$

To prove this it is sufficient to notice that, once again, there exists  $\Delta_{\eta^<} \in \mathbf{C}\{x\}[\sigma_q]$  and  $g_{\eta^<} \in \mathbf{C}\{x\}$  such that

$$\Delta_{\eta^<} W^{\eta^<} = g_{\eta^<} \tag{9.19}$$

For  $\eta^< \in \mathbf{H}_0^<$ , we note  $\text{DS}(\eta^<) = \text{DS}(W^{\eta^<})$ .

##### 9.4.1. Problem 1: Orders of $q$ -multisummability

For  $\eta^< \in \mathbf{H}_0^<$ , we would like to describe  $s^{\eta^<}$ . This doesn't seem as simple as in the case of totally ordered sequences. If

$$\eta^< = \eta^{1<} \oplus \dots \oplus \eta^{d<} \quad (\text{with } \eta^{i<} \text{ irreducible and } \neq \emptyset)$$

The set  $s^{\eta^<}$  contains the elements of  $s^{1<}, \dots, s^{d<}$  but, as

$$W^{\eta^<} = W^{\eta^{1<}} \dots W^{\eta^{d<}}$$

the multiplication of the different factors yields new elements in  $s^{\eta^<}$ . Nevertheless, we can assume that if  $\eta^< \in \mathbf{H}_0^<$  is of length  $r$  and for  $1 \leq i \leq r$ ,  $\tilde{n}_i \geq 0$  (with respect to the partial order inherited by  $\mathbf{n}^<$ ), then  $W^{\eta^<}$  is a convergent power series. Otherwise, the greatest element in  $s^{\eta^<}$  is always 1. This means that, to compute the  $q$ -sum, one always starts by a Borel transform of order 1.

### 9.4.2. Problem 2: Singular directions

For  $\eta^< \in \mathbf{H}_0^<$ , if  $W^{\eta^<}$  is not a convergent power series, it seems that, after a first Borel transform of order 1, we always get a meromorphic function whose poles are  $\{q^n ; n \geq 0\}$ ,

$$\{2k\pi; k \in \mathbb{Z}\} \in \text{DS}(\eta^<) \tag{9.20}$$

But it seems that, if  $\theta \neq 0[2\pi]$ , we can perform the successive accelerations (to get the  $q$ -sum), and the accelerated functions do not have any singular direction. This would mean that

$$\{2k\pi; k \in \mathbb{Z}\} = \text{DS}(\eta^<) \tag{9.21}$$

This result needs to be proven but, if it is true, there is no problem of singular directions as

$$\text{DS}(\Theta) = \bigcup_{\eta^< \in \mathbf{H}_0^<} \text{DS}(\eta^<) = \{2k\pi; k \in \mathbb{Z}\} \tag{9.22}$$

This means that, if  $\theta \neq 0[2\pi]$ , we can define a  $q$ -sum for  $\Theta$ :

$$\Theta_\theta = 1 + \sum_{\eta^< \in \mathbf{H}_0^<} e_q^{\|\mathbf{n}^<\|} \left( S_{s^{\eta^<}}^\theta W^{\eta^<} \right) \mathbb{D}_{\eta^<} \tag{9.23}$$

### 9.4.3. Problem 3: Symmetry relations for $W^{\bullet^<}$

The operator

$$\begin{aligned} \Theta &= 1 + \sum_{\eta} e_q^{\|\mathbf{n}\|} W^\eta \mathbb{D}_\eta \\ &= 1 + \sum_{\eta^< \in H^<} e_q^{\|\mathbf{n}^<\|} \|W^{\eta^<}\| \mathbb{D}_{\eta^<} \\ &= 1 + \sum_{\eta^< \in H_0^<} e_q^{\|\mathbf{n}^<\|} \|W^{\eta^<}\| \mathbb{D}_{\eta^<} \end{aligned} \tag{9.24}$$

is a substitution automorphism because the arborescent mould  $W^{\bullet<}$  is *separative*: If  $\eta^{<} = \eta^{1<} \oplus \dots \oplus \eta^{d<}$  then  $W\eta^{<} = W\eta^{1<} \dots W\eta^{d<}$ . If we want  $\Theta_\theta$  to be a substitution automorphism, we must check that, if  $\eta^{<} = \eta^{1<} \oplus \dots \oplus \eta^{d<}$  then, for  $\theta \neq 0[2\pi]$ ,

$$S_{s\eta^{<}}^\theta W\eta^{<} = \left( S_{s\eta^{1<}}^\theta W\eta^{1<} \right) \dots \left( S_{s\eta^{d<}}^\theta W\eta^{d<} \right) \quad (9.25)$$

but this property is not obvious.

#### 9.4.4. Problem 4: Estimates and convergence

Suppose that we could define a separative arborescent mould  $W_\theta^{\bullet<}$  for  $\theta \neq 0[2\pi]$  by:

$$\forall \eta^{<} \in \mathbf{H}_0^{<}, \quad W_\theta^{\eta^{<}} = S_{s\eta^{<}}^\theta W\eta^{<}$$

It remains to prove that the substitution automorphism

$$\Theta_\theta = 1 + \sum_{\eta^{<} \in \mathbf{H}_0^{<}} e_q^{\|\eta^{<}\|} \|W_\theta^{\eta^{<}}\| \mathbb{D}_{\eta^{<}}$$

is a convergent operator. Our guess is that we do not have such good estimates for  $W_\theta^{\bullet<}$  as in Lemma 6.3. But, due to the coarborification, the norms associated to the operators  $\mathbb{D}_{\eta^{<}}$  must become much more smaller than in the Lemma 6.5. This phenomenon did already appear in numerous examples (see [1]) and it should reestablish the convergence of  $\Theta_\theta$ . The problem that remains to solve is to get precise estimates on the behavior of  $W_\theta^{\bullet<}$  in the neighborhood of 0.

## 10. Conclusion

There are still problems to be solved to get a complete understanding of non-linear  $q$ -difference equations. This calls essentially for new theoretical results on  $q$ -multisummability of formal series that are solutions of a linear  $q$ -difference equation.

The different problems can be summarized as follows. The set  $\mathcal{A} \in \mathbb{C}[[x]]$  of formal power series that are solution of a linear  $q$ -difference equation is an algebra. If  $f_1$  and  $f_2$  are two elements of  $\mathcal{A}$ , it is not difficult to see (algorithmic-ally) that  $f_1 + f_2$  and  $f_1 \cdot f_2$  are solutions of some linear  $q$ -difference equation. Because of the results of Zhang and Marotte (see [3]), each element of  $\mathcal{A}$  is  $q$ -multisummable. The results we need can be

formulated as follows: Let  $\mathcal{A}^\theta = \{f \in \mathcal{A} \ ; \ \theta \notin \text{DS}(f)\}$ . If  $f \in \mathcal{A}^\theta$ ,  $f$  is  $q$ -multisummable and we note  $\mathcal{S}^\theta f \in \tilde{\mathcal{O}}$  its  $q$ -sum. This defines an application from  $\mathcal{A}^\theta$  into  $\tilde{\mathcal{O}}$  and the fundamental questions we need to answer to are:

1. Is  $\mathcal{A}^\theta$  an algebra ?
2. Is  $\mathcal{S}^\theta$  an homomorphism from  $\mathcal{A}^\theta$  into  $\tilde{\mathcal{O}}$  (this would solve many of our problems) ?
3. Can we get precise estimates on  $\mathcal{S}^\theta f \in \tilde{\mathcal{O}}$  in a neighborhood of 0, knowing precisely the linear  $q$ -difference equation having  $f$  as a solution (This would help in problem 4) ?

We should be optimistic on these problems.

Besides these questions, there should not be any difficulties to generalize our methods to systems:

$$\left\{ \begin{array}{l} (x\sigma_q - \lambda_1)f_1 = b_1(x, f_1, \dots, f_n) \\ \vdots \\ \vdots \\ (x\sigma_q - \lambda_n)f_n = b_n(x, f_1, \dots, f_n) \end{array} \right. \quad (10.1)$$

with, for  $1 \leq i \leq n$ ,  $b_i(0, 0, \dots, 0) = \frac{\partial}{\partial f_i} b_i(0, 0, \dots, 0) = 0$ . The only new phenomenon should be one similar to the one that appears in the study of vector fields and diffeomorphisms (see [1]): There should be some problem of resonance and small divisors related to the multipliers  $\lambda_i$  and we will certainly have to add some Diophantine condition on these multipliers. On the same way, we should be able to apply our methods to systems with several levels:

$$\left\{ \begin{array}{l} (x^{k_1}\sigma_q - \lambda_1)f_1 = b_1(x, f_1, \dots, f_n) \\ \vdots \\ \vdots \\ (x^{k_n}\sigma_q - \lambda_n)f_n = b_n(x, f_1, \dots, f_n) \end{array} \right. \quad (10.2)$$

with  $(k_1, \dots, k_n) \in \mathbb{Z}^n$ . We should study such systems in the future.

We didn't write anything about the analytic classification of such systems of  $q$ -difference equations. There is a lot of work to be done here. Following the ideas of Jean Ecalle (see [1]), there should be a way to define  $q$ -alien derivations related to the definitions of the  $q$ -summability. Only a few steps have been done on the way to define such operators but they seem

to exist and, as in the classical case of differential equations, these operators should lead us to find a *Bridge equation* that encodes the analytic invariants associated to such systems: this would yield a complete description of the analytic classification and of the Stokes phenomenon.

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