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Motion of concentration sets in Ginzburg-Landau equations (*)

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Abstract. — We discuss a number of results which relate the parabolic Ginzburg-Landau equation with motion by mean curvature. We describe the various concentration phenomena underlying this analysis.

Résumé. — Nous décrivons quelques résultats qui relient l'équation de Ginzburg-Landau parabolique au mouvement par courbure moyenne. Nous discutons les différents phénomènes de concentration liés à cette analyse.

1. Introduction

The asymptotic analysis for Ginzburg-Landau evolution equations has been broadly investigated in the last decade. The purpose of this paper is to review some results both in the scalar and complex case. In particular we try to emphasize some analogies and differences between the two theories.

Our main focus will be the parabolic Ginzburg-Landau equation

$$(\mathrm{PGL})_\varepsilon \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = -\dfrac{1}{\varepsilon^2}\nabla_u V(u_\varepsilon) & \text{on } \mathbb{R}^N \times (0,+\infty), \\ u_\varepsilon(x,0) = u_\varepsilon^0(x) & \text{for } x \in \mathbb{R}^N , \end{cases}$$

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for functions $u_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^d$, $N \geqslant 1$, $d \geqslant 1$, and V represents a non-convex smooth non-negative potential on $\mathbb{R}^d$. Here $\varepsilon > 0$ denotes a small parameter (a characterisitc length), and we are specially interested in the asymptotic limit $\varepsilon \to 0$.

This equation corresponds to the heat-flow for the Ginzburg-Landau energy

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} e_\varepsilon(u) = \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{2} + \frac{V(u)}{\varepsilon^2} \qquad \text{for } u : \mathbb{R}^N \to \mathbb{R}^d.$$

The set

$$\Sigma = \{y \in \mathbb{R}^d\,,\ V(y) = 0\},$$

which we assume to be non-void, is sometimes called the vacuum manifold in the physical literature and plays an important role in the asymptotic analysis. Indeed, since the potential is non-negative, it achieves its infimum on Σ, and therefore the motion law forces u_ε to take values close to Σ for small ε as time evolves, and in appropriate energy regimes. This however cannot be true uniformly on space-time since the initial data u_ε^0 may not be uniformly close to Σ. We will call defects the points where u_ε is far from Σ. As time evolves these defects will disappear. An important aspect of our discussion will be to show that the defects related to the topology of Σ survive up to a time which is independent of ε, whereas the non-topological ones essentially have a life-span which shrinks with ε. For that reason the topology of Σ will enter directly in the discussion.

The energy $\mathcal{E}_\varepsilon$ has been introduced in the early fifties by Ginzburg and Landau in order to describe phase transitions in condensed matter Physics (more precisely, at low temperature). The nature of the predicted defects (e.g. points, lines, walls) depends crucially on d and Σ (see [36]). Among the many variants of Ginzburg-Landau functionals, there are in particular those including electromagnetic effects, as for instance in superconductivity. Related models have been developed in particle physics (as for example, Yang-Mills-Higgs theory).

In this paper we will focus on the cases $d = 1$ and $d = 2$ (i.e. u real or complex-valued). Moreover we assume that the potential is given by

$$V(u) = \frac{(1 - |u|^2)^2}{4}.$$

Note that in this case

$$\Sigma = \{-1, 1\} \qquad \text{if } d = 1\,, \qquad\qquad \Sigma = S^1 \qquad \text{if } d = 2\,,$$

where S^1 is the unit circle in $\mathbb{R}^2$. In the first case, i.e. $d = 1$, the non-connectedness of Σ yields typically codimension one defects, whereas in the second case, i.e. $d = 2$, Σ is not simply connected and allows for defects of codimension two. In Section 2 we will briefly show that the typical energy needed to observe a topological defect for $d = 1$ is of order ε^{-1}, whereas it is of order $|\log \varepsilon|$ for $d = 2$.

With this choice of potential, $(\mathrm{PGL})_\varepsilon$ writes

$$(\mathrm{PGL})_\varepsilon \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{on } \mathbb{R}^N \times (0, +\infty), \\ u_\varepsilon(x, 0) = u_\varepsilon^0(x) & \text{for } x \in \mathbb{R}^N . \end{cases}$$

It is well known that $(\mathrm{PGL})_\varepsilon$ is well-posed for initial datas in H^1_{loc} with finite Ginzburg-Landau energy $\mathcal{E}_\varepsilon(u_\varepsilon^0)$. Moreover, we have the energy identity

$$\mathcal{E}_\varepsilon(u_\varepsilon(\cdot, T_2)) + \int_{T_1}^{T_2} \int_{\mathbb{R}^N} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 (x, t) dx\, dt = \mathcal{E}_\varepsilon(u_\varepsilon(\cdot, T_1)) \quad \forall 0 \leqslant T_1 \leqslant T_2 . \tag{1.1}$$

We assume that the initial condition u_ε^0 verifies the bound

$$(H_0) \qquad \mathcal{E}_\varepsilon(u_\varepsilon^0) \leqslant M_0 k_\varepsilon ,$$

where M_0 is a fixed positive constant, and the definition of k_ε depends on the dimension d, namely we set

$$k_\varepsilon = \frac{1}{\varepsilon} \quad \text{if } d = 1, \qquad k_\varepsilon = |\log \varepsilon| \quad \text{if } d = 2 .$$

The definition of k_ε in both cases $d = 1$ and $d = 2$ should be related to the energy cost needed for a single defect (we will develop this notion later). Notice that, in view of (1.1), we have

$$\mathcal{E}_\varepsilon(u_\varepsilon(\cdot, T)) \leqslant \mathcal{E}_\varepsilon(u_\varepsilon^0) \leqslant M_0 k_\varepsilon \qquad \text{for all } T \geqslant 0. \tag{1.2}$$

In order to analyze the asymptotic properties of solutions to $(\mathrm{PGL})_\varepsilon$ we consider two kinds of objects.

The first ones describe the topological defects of u_ε: for $d = 1$ it is simply given by the gradient ∇u_ε, whereas for $d = 2$ it is the jacobian $J u_\varepsilon$, defined as the 2-form

$$J u_\varepsilon = d u_\varepsilon^1 \wedge d u_\varepsilon^2 .$$

Although this may not be obvious at first glance, they are bounded in suitable norms independently of ε and therefore do not need any kind of renormalization. It can be shown (see Section 3) that in the asymptotic limit $\varepsilon \to 0$ they concentrate on codimension d rectifiable sets in $\mathbb{R}^N \times \mathbb{R}^+$, called respectively the jump set and the vorticity set. This fact is not related to the equation $(\mathrm{PGL})_\varepsilon$, but due only to the energy bound (1.2) and properties of the functional $\mathcal{E}_\varepsilon$. Passing to subsequences, the limiting object J_* is a bounded vector measure on $\mathbb{R}^N \times \mathbb{R}^+$, as well as its restriction J_*^t on each time slice $\mathbb{R}^N \times \{t\}$. In Section 2 we will discuss in more details the structure of J_*.

The second objects are the renormalized energy densities given by the Radon measures μ_ε, defined on $\mathbb{R}^N \times [0, +\infty)$,

$$\mu_\varepsilon = \frac{e_\varepsilon(u_\varepsilon(x,t))}{k_\varepsilon}\, dx\, dt,$$

and of their time slices μ_ε^t, defined on $\mathbb{R}^N \times \{t\}$,

$$\mu_\varepsilon^t = \frac{e_\varepsilon(u_\varepsilon(x,t))}{k_\varepsilon}\, dx,$$

so that in particular $\mu_\varepsilon = \mu_\varepsilon^t\, dt$. In view of assumption (H_0) and (1.2), μ_ε is a bounded measure, independently of ε. We may therefore assume, up to a subsequence $\varepsilon_n \to 0$, that there exists a Radon measure μ_* defined on $\mathbb{R}^N \times [0, +\infty)$ such that

$$\mu_\varepsilon \rightharpoonup \mu_* \qquad \text{as measures.}$$

In view of the semi-decreasing property of the measures μ_ε^t (see [26, 13]), passing possibly to a further subsequence, we may also assume that

$$\mu_\varepsilon^t \rightharpoonup \mu_*^t \qquad \text{as measures on } \mathbb{R}^N \times \{t\}, \text{ for all } t \geqslant 0.$$

In the asymptotic limit $\varepsilon \to 0$, there is a simple relation between the quantities introduced so far, namely

$$||J_*|| \leqslant C_d|\mu_*|\,, \qquad\qquad ||J_*^t|| \leqslant C_d|\mu_*^t| \quad \text{for any } t > 0\,, \qquad (1.3)$$

where $C_1 = \sqrt{2}/3$ and $C_2 = 1$. Moreover these bounds are sharp. This relation will be discussed in Section 2. The evolution of μ_*^t is easier to analyze than that of J_*^t. Indeed, it is possible to derive directly equations governing the motion of μ_*^t, using $(\mathrm{PGL})_\varepsilon$, whereas this is not clear for J_*^t. The structure of μ_*^t can be summarized as follows.

THEOREM 1.1 (Structure of μ_*^t). — *There exists a subset* Σ_μ *in* $\mathbb{R}^N \times (0,+\infty)$, *such that the following properties hold.*

i) Σ_μ *is closed in* $\mathbb{R}^N \times (0,+\infty)$ *and for any compact subset* $\mathcal{K} \subset \mathbb{R}^N \times (0,+\infty) \setminus \Sigma_\mu$

$$|u_\varepsilon(x,t)| \to 1 \qquad \textit{uniformly on } \mathcal{K} \textit{ as } \varepsilon \to 0.$$

ii) *For any* $t > 0$, $\Sigma_\mu^t \equiv \Sigma_\mu \cap \mathbb{R}^N \times \{t\}$ *verifies*

$$\mathcal{H}^{N-d}(\Sigma_\mu^t) \leqslant KM_0.$$

iii) *For each* $t > 0$, *the measure* μ_*^t *can be exactly decomposed as*

$$\mu_*^t = g(x,t)\mathcal{H}^N + \Theta_*(x,t)\mathcal{H}^{N-d} \llcorner \Sigma_\mu^t. \tag{1.4}$$

iv) *In case* $d = 1$, $g \equiv 0$, *while for* $d = 2$, $g = |\nabla\Phi_*|^2$, *where the function* Φ_* *verifies the linear heat equation on* $\mathbb{R}^N \times (0,+\infty)$.

v) *the function* $\Theta_*(\cdot,t)$ *is bounded, and there exists a positive function* η *defined on* $\mathbb{R}_*^+$ *such that, for almost every* $t > 0$, *the set* Σ_μ^t *is* $(N-d)$*-rectifiable and*

$$\Theta_*(x,t) = \Theta_{N-d}(\mu_*^t, x) = \lim_{r\to 0} \frac{\mu_*^t(B(x,r))}{\omega_{N-d} r^{N-d}} \geqslant \eta(t),$$

for $\mathcal{H}^{N-d}$ *a.e.* $x \in \Sigma_\mu^t$.

In the case $d = 1$, Theorem 1.1 has been proved by Ilmanen [26]; earlier related results have been provided, among others, in [15, 16, 17].

In the case $d = 2$, Theorem 1.1 has been proved in [9] (see also related results in [31, 32, 29, 28]). Many arguments rely on the elliptic theory developed in particular in [5, 49, 10, 43, 33, 34, 6, 30, 12, 8]. Some elements in the proofs will be discussed in Section 3 and 4.

In view of the decomposition (1.4), μ_*^t can be split into two parts. A diffuse part $|\nabla\Phi_*|^2$, and a concentrated part

$$\nu_*^t = \Theta_*(x,t)\mathcal{H}^{N-2} \llcorner \Sigma_\mu^t.$$

An important difference between the scalar and the complex case is that in the scalar case there is no diffuse part (i.e. $g \equiv 0$). The presence of the diffuse term in the complex case is due to the possible oscillating behavior of the phase. This part evolves in time according to the linear heat equation. In other words, in the complex case, the energy has two different modes:

– the linear mode, corresponding to Φ_*;

– the topological mode, corresponding to ν_*.

Concerning J_* we have also, as a consequence of (1.3),

$$supp\, J_* \subset \Sigma_\mu\,, \qquad\qquad supp\, J_*^t \subset \Sigma_\mu^t \quad \text{for any } t > 0\,. \tag{1.5}$$

In some cases the inclusions in (1.5) are strict.

Note also that in the critical dimension $N = d$ the concentration set Σ_μ^t reduces to a finite set, in particular the measures ν_*^t are given by a finite sum of Dirac masses with positive coefficients bounded from above and from below.

The next step is to derive the motion law for the concentration set Σ_μ^t. In the critical dimension $N = d$, it turns out that the points of Σ_μ^t do not move at all in the given time scale. A rescaling of time depending on ε is needed to see the defects move in the singular limit (see Section 3): this is the so-called "slow motion" phenomenon of point defects (see [15, 16, 28, 31]. We will not discuss this here.

If $N > d$, then we will see in Section 4 that the concentration set Σ_μ^t evolves according to motion by mean curvature.

2. Analysis of the topological defects

In this Section we review some results concerning the jumps and vorticity sets. As mentioned, the results here rely only on properties of the Ginzburg-Landau functionals $\mathcal{E}_\varepsilon$ and are completely independent of the equation $(\mathrm{PGL})_\varepsilon$.

2.1. The scalar case

The properties of the Ginzburg-Landau functional $\mathcal{E}_\varepsilon$ in the scalar case $d = 1$ have been extensively investigated in the 80's, in particular by the De Giorgi school (starting with the seminal work by Modica-Mortola [39], and [38]) and also, motivated by physical questions, since the works by Gurtin and Sternberg [24, 47]. To simplify a little the arguments, we will restict the attention to a bounded domain $\Omega \subset \mathbb{R}^N$, say for instance the unit ball B_1, and consider families $\{v_\varepsilon\}_{0<\varepsilon<1}$ of scalar functions defined on Ω, verifying a bound of the form

$$\mathcal{E}_\varepsilon(v_\varepsilon) \leqslant M_0 k_\varepsilon = \frac{M_0}{\varepsilon}, \tag{2.1}$$

where $M_0 > 0$ is independent on ε. Clearly such a bound does not yield any control on the L^2 norm of the gradient. However, estimate (2.1) is sufficient to derive some compactness, in particular for the jump set. More precisely, the following holds.

PROPOSITION 2.1. — *Let $(v_\varepsilon)_{\varepsilon>0}$ a sequence such that*

$$\mathcal{E}_\varepsilon(v_\varepsilon) \leqslant \frac{M_0}{\varepsilon}.$$

Then, for a subsequence $\varepsilon_n \to 0$,

$$v_{\varepsilon_n} \to v_* \qquad \text{in } L^1(\Omega),$$

where $v_*(x) \in \{-1, 1\}$ for a.e. $x \in \Omega$, and $v_* \in BV(\Omega)$.

Sketch of proof. — We have

$$\frac{\varepsilon}{2}\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon}\int_\Omega (1 - |v_\varepsilon|^2)^2 = \varepsilon\mathcal{E}_\varepsilon(v_\varepsilon) \leqslant M_0.$$

Hence, from the inequality $ab \leqslant \frac{1}{2}(a^2 + b^2)$, it holds

$$\int_\Omega |\nabla v||1 - |v_\varepsilon|^2| \leqslant \frac{\varepsilon}{\sqrt{2}}\int_\Omega |\nabla v_\varepsilon|^2 + \frac{\sqrt{2}}{4\varepsilon}\int_\Omega (1 - |v_\varepsilon|^2)^2 \leqslant \sqrt{2}M_0, \tag{2.2}$$

that is

$$\int_\Omega |\nabla\zeta(v_\varepsilon)| \leqslant \sqrt{2}M_0, \tag{2.3}$$

where $\zeta(t) = t - t^3/3$. This yields a uniform bound in $W^{1,1}$ for $\zeta(v_\varepsilon)$, and a subsequence $\zeta(v_{\varepsilon_n})$ converges therefore weakly in $BV(\Omega)$, hence strongly in L^1. So does $v_{\varepsilon_n} = \zeta^{-1}(\zeta(v_{\varepsilon_n}))$. Moreover, since $\zeta(v_*) = \frac{2}{3}v_*$, we have

$$\int_\Omega |\nabla v_*| = \frac{3}{2}\int_\Omega |\nabla\zeta(v_*)| \leqslant \frac{3\sqrt{2}}{2}M_0, \tag{2.4}$$

by (2.3) and lower semicontinuity of total variation. Hence $v_* \in BV(\Omega)$. □

Notice that Proposition 2.1 states that ∇v_{ε_n} converges in $W^{-1,1}$ to $J_* = \nabla v_*$, and that the limiting jump set J_* is a bounded measure.

Remark 2.2. — i) Let us emphasize that condition (2.1) does not imply that the sequence v_ε is bounded in BV. A simple example in dimension one is given by

$$v_\varepsilon(x) = 1 + \varepsilon^{1/2}\sin(\frac{x}{\varepsilon}), \qquad \text{for } -1 \leqslant x \leqslant 1.$$

Clearly the maps v_ε satisfy (2.1) but they are not equibounded in BV.

ii) On the other hand, one may prove that given any sequence v_ε satisfying (2.1) there exists another sequence $\tilde{v}_\varepsilon$ verifying also (2.1) which is equibounded in BV and which is close to the original sequence v_ε in the following sense:

$$||v_\varepsilon - \tilde{v}_\varepsilon||_{L^1(\Omega)} \to 0 \qquad \text{as } \varepsilon \to 0.$$

The main point is to get rid of the possible small oscillations of v_ε on the set where it takes values close to $+1$ and -1. This is achieved by a composition with a suitable projection on $\Sigma = \{-1, 1\}$.

The fact that $v_* \in BV(\Omega)$ and $|v_*| = 1$ a.e. in Ω yields some important properties for the jump set. In order to get some insight for this type of result, let us first consider the one dimensional case, which captures already some of the essential features of the problem.

2.1.1. The case $N = 1$

Let $\Omega = I$ be a bounded interval of $\mathbb{R}$. We have

PROPOSITION 2.3. — *Let $v \in BV(I)$, $|v| = 1$ a.e.. Then v has only a finite number ℓ of jumps $a_1, \dots, a_\ell$, and there exists $\chi \in \{-1, 1\}$ such that*

$$v(x) = \chi \prod_{i=1}^{\ell} \left(\frac{x - a_i}{|x - a_i|} \right). \tag{2.5}$$

Proof. — The result follows immediately from the definition of the BV norm in dimension one: it is the sum of the L^1 norm and the total variation V_I, defined by $V_I(v) = \sup\{\sum |v(x_{i+1}) - v(x_i)|\,,\ \{x_i\} \text{ partition of } I\}$. □

Remark 2.4. — Note that if v_* is given by (2.5), then $J_* = \nabla v_* = 2\chi \sum_{i=1}^{\ell} (-1)^i \delta_{a_i}$, and in particular $||J_*|| = 2\ell$.

Next we show inequality (1.3) in dimension one, that is

. PROPOSITION 2.5. — *i) Let v_* given by (2.5). Then for any sequence $(v_\varepsilon)_{0<\varepsilon<1}$ such that $v_\varepsilon \longrightarrow v_*$ in L^1 as $\varepsilon \longrightarrow 0$, we have*

$$\liminf_{\varepsilon \to 0} \varepsilon \mathcal{E}_\varepsilon(v_\varepsilon) \geqslant \frac{2\sqrt{2}}{3}\ell\,. \tag{2.6}$$

ii) The bound (2.6) is sharp, i.e. there exists a sequence $(u_\varepsilon)_{0<\varepsilon<1}$ such that $u_\varepsilon \longrightarrow v_*$ in L^1, as $\varepsilon \longrightarrow 0$, and

$$\lim_{\varepsilon \to 0} \varepsilon \mathcal{E}_\varepsilon(u_\varepsilon) = \frac{2\sqrt{2}}{3}\ell\,. \tag{2.7}$$

Proof. — i) Going back to the first inequality in (2.2), we have

$$\int_\Omega |\zeta(v_\varepsilon)| \leqslant \sqrt{2}\varepsilon \mathcal{E}_\varepsilon(v_\varepsilon)\,. \tag{2.8}$$

On the other hand, $\zeta(v_\varepsilon) \longrightarrow \zeta(v_*)$ in L^1, and lower semicontinuity of the total variation gives

$$\liminf_{\varepsilon \to 0} \int_\Omega |\nabla \zeta(v_\varepsilon)| \geqslant \int_\Omega |\nabla \zeta(v_*)|\,. \tag{2.9}$$

Since $\zeta(v_*) = \frac{2}{3}v_*$, we have $\int_\Omega |\nabla\zeta(v_*)| = \frac{4}{3}\ell$, and (2.6) follows.

ii) The main idea is to construct an optimal profile (on the whole of $\mathbb{R}$) for the transition from -1 to $+1$. Indeed, consider the problem

$$-\ddot{v} = v(1-v^2)\,, \qquad v(-\infty) = -1\,, \quad v(+\infty) = 1\,. \tag{2.10}$$

Actually, it is elementary to show that the solution is the unique minimizer (up to translations) of $\mathcal{E}_1$ subject to the above boundary conditions. It is explicitely given by the formula $v(x) = \tanh(\frac{x}{\sqrt{2}})$.

Next set

$$u_\varepsilon(x) = \chi \prod_{i=1}^{\ell} v\left(\frac{x-a_i}{\varepsilon}\right)\,. \tag{2.11}$$

A few computations show that $u_\varepsilon \longrightarrow v_*$ in L^1, and

$$\varepsilon \mathcal{E}_\varepsilon(u_\varepsilon) = \frac{\sqrt{2}}{3}\ell + O(\exp(-\frac{K}{\varepsilon}))\,, \tag{2.12}$$

for some constant $K > 0$. □

Remark 2.6. — Multiplying equation (2.10) by $\dot{v}$ we obtain the pointwise equality

$$2\dot{v}^2 = (1-v^2)^2\,. \tag{2.13}$$

This yields the equipartition of energy for u_ε

$$\frac{1}{2}\dot{u}_\varepsilon^2 = \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 + O(\exp(-\frac{K}{\varepsilon}))\,. \tag{2.14}$$

More generally, for any sequence w_ε verifying statement ii), it is elementary to prove equipartition of the energies

$$\int_\Omega \frac{|\nabla w_\varepsilon|^2}{2} = \int_\Omega \frac{(1-|w_\varepsilon|^2)^2}{4\varepsilon^2} + o(1)\,. \tag{2.15}$$

This equality holds also in higher dimensions (see Proposition 2.9).

We would like to draw the attention of the reader that in the scalar case considered here the **exact** form of the optimal profile plays a central role in the analysis. We will see that in the complex case the exact form of the optimal profile does not really enter in the corresponding theory.

Remark 2.7. — In view of (2.12), we see that the interaction between jumps is exponentially weak.

2.1.2. The case $N \geqslant 2$

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$. As in dimension one, the fact that $v_* \in BV(\Omega)$ and $|v_*| = 1$ a.e. in Ω allows to deduce regularity properties for the jump set of v_*, which are best expressed in the language of Geometric Measure Theory. More precisely, we have

PROPOSITION 2.8. — *Let $v_* \in BV(\Omega)$, $|v_*| = 1$ a.e.. There exists a set $E \subset \Omega$ of finite perimeter in Ω, such that $v_* = 2\chi_E - 1$, where χ_E is the characteristic of E. In particular, the jump set of v_* is $(N-1)$-rectifiable, and $2Per_\Omega(E) = \int_\Omega |\nabla v_*| = ||J_*||$.*

Comment. — i) We recall that a set $E \subset \mathbb{R}^N$ is k-rectifiable, for $1 \leqslant k \leqslant N$, if it has locally finite k-dimensional Hausdorff measure $\mathcal{H}^k$, and is contained, up to an $\mathcal{H}^k$-negligible set, in a countable union of k-dimensional surfaces of class $\mathcal{C}^1$. For such sets, the tangent space $\mathrm{Tan}(E,x)$ is well-defined in a measure theoretic sense for $\mathcal{H}^k$ a. e. $x \in E$. An important aspect of rectifiable sets is that they are limits of finite unions of k-dimensional polyhedral sets in a suitable weak norm.

ii) The proof of Proposition 2.8 is far from being elementary, and relies on De Giorgi's theory of finite perimeter sets. More precisely, let $w_* \in BV(\Omega)$ (so that Dw_* is a measure), and $|w_*| = 1$ a.e.. Let $\Omega_*^{\pm} = \{x \in \Omega\,,\ w_*(x) = \pm 1\}$. Then Dw_* is supported on the $(N-1)$-rectifiable set $\partial^*\Omega_*^{\pm}$, the reduced boundary of $\Omega_*^{\pm}$. [For the definition of reduced boundary, see e.g. [45]; the reduced boundary is included in the usual topological boundary. In the smooth case they actually coincide, but in general they may be different].

The N-dimensional analog of Proposition 2.5 is the following

PROPOSITION 2.9. — *i) Let $v_* \in BV(\Omega)$, $|v_*| = 1$ a.e.. Then for any sequence $(v_\varepsilon)_{0<\varepsilon<1}$ such that $v_\varepsilon \to v_*$ in L^1 as $\varepsilon \to 0$, we have*

$$\liminf_{\varepsilon\to 0} \varepsilon\mathcal{E}_\varepsilon(v_\varepsilon) \geqslant \frac{\sqrt{2}}{3}\int_\Omega |\nabla v_*| = \frac{\sqrt{2}}{3}||J_*|| = \frac{\sqrt{2}}{3}Per_\Omega(E)\,. \tag{2.16}$$

ii) The bound (2.16) is sharp, i.e. there exists a sequence $(u_\varepsilon)_{0<\varepsilon<1}$ such that $u_\varepsilon \to v_*$ in L^1, as $\varepsilon \to 0$, and

$$\lim_{\varepsilon\to 0} \varepsilon\mathcal{E}_\varepsilon(u_\varepsilon) = \frac{\sqrt{2}}{3}\int_\Omega |\nabla v_*| = \frac{\sqrt{2}}{3}||J_*|| = \frac{\sqrt{2}}{3}Per_\Omega(E)\,. \tag{2.17}$$

Comment. — The previous proposition is a classical example of Γ-convergence (see [39])

Sketch of the proof. — The proof of i) is identical to the proof of i) in Proposition 2.5.

The easiest way to prove ii) is to use an approximation of E by a set with a polyhedral boundary in Ω. Then the u_ε are constructed using essentially the optimal profile (rescaled at the level ε) in the orthogonal direction to the approximating boundary.

2.2. The complex case

Here we will consider $u : \Omega \to \mathbb{C} \simeq \mathbb{R}^2$, so that $\Sigma = \{y \in \mathbb{C}, V(y) = 0\} = \{y \in \mathbb{C},\ |y| = 1\} = S^1$. A new type of singularity can appear here, due to the fact that $\pi_1(S^1) = \mathbb{Z} \neq 0$. Interesting new cases of topological defects appear therefore for planar Ω, i.e. for $N = 2$ (this is somewhat similar to the one dimensional case for scalar problems).

2.2.1. Vortices

We start the discussion here with a minimization problem which, in a vague sense, corresponds to the selection of optimal profiles. For that purpose, let $\Omega = D^2 = \{z \in \mathbb{C} \simeq \mathbb{R}^2\,,\ |z| \leqslant 1\}$, and consider a regular function

$$g : \partial\Omega = S^1 \to S^1$$

with winding number $d \neq 0$. In contrast to the scalar case, there is of course a large choice of boundary conditions verifying $|g| = 1$. Let us consider next the minimization problem

$$I_\varepsilon = \inf\{\mathcal{E}_\varepsilon(v)\,,\ v \in H^1_g(D^2; \mathbb{C})\}\,.$$

If $d \neq 0$, any minimizer for $\mathcal{E}_\varepsilon$ has to vanish at some points. Moreover, it can be proved that $H^1_g(D^2, S^1) = \emptyset$, and therefore I_ε diverges as $\varepsilon \to 0$. The asymptotic analysis here is of course more involved, since we have PDE's instead of ODE's. It was initiated in [5], where the following was established.

PROPOSITION 2.10. — *Assume $d > 0$, and let u_ε be a minimizer for $\mathcal{E}_\varepsilon$. Then we have*

$$\mathcal{E}_\varepsilon(u_\varepsilon) = \pi d|\log\varepsilon| + O(1)\,, \qquad \frac{1}{4\varepsilon^2}\int_\Omega (1 - |u_\varepsilon|^2)^2 = O(1)\,. \qquad (2.18)$$

Moreover, there exists d points $a_1, \ldots, a_d$ in Ω, and a harmonic function $\varphi : \Omega \to \mathbb{R}$ such that $u_\varepsilon \to u_*$ as $\varepsilon \to 0$ in $W^{1,p}(\Omega)$ for any $p < 2$, and in $C^k_{loc}(\Omega \setminus \{a_1, \ldots, a_d\})$, where

$$u_*(z) = \exp(i\varphi(z)) \prod_{i=1}^{d} \frac{z - a_i}{|z - a_i|}\,.$$

The points a_i are usually called "vortices" (in analogy with the terminology of fluid dynamics). Since φ is harmonic, it is completely determined by the boundary condition and the location of the points a_d. As a matter of fact it can be proved that the configuration $(a_1, \ldots, a_d)$ is not arbitrary, but minimizes a suitable renormalized energy (i.e. independent of ε). Again, the boundary condition enters in an essential way in the definition of this energy.

Remark 2.11. — As the reader might already have noticed, there are strong analogies between the 1-dimensional scalar case and the planar complex case: clearly vortices and jumps play a somewhat similar role. Let us stress however a few differences:

i) the typical energy necessary to the formation of a vortex is of order $|\log\varepsilon|$, whereas for jumps it is ε^{-1};

ii) from (2.18) one sees that there is no energy balance in the complex case, and the diverging part of the energy is concentrated in the gradient term;

iii) in a (vague) sense, jumps do not "interact", whereas vortices do. Their interaction is governed by the renormalized energy.

Another striking difference concerns the way the theory has been developed in both cases. Indeed, PDE techniques have played an important role in the starting development for the complex case, while the emphasis was put first, for the scalar case, on variational methods (e.g. compactness, Γ-convergence...).

Remark 2.12.— Consider the boundary condition $g(z) = z = Id_{S^1}$, which is the simplest possible with non-zero winding number.

It is natural, due to the symmetries in the problem, to seek solutions of the form

$$w_\varepsilon(z) = f_\varepsilon(r)\exp(i\theta) = f_\varepsilon(r)\frac{z}{|z|}$$

where $z = r\exp(i\theta)$ (in polar coordinates), and $f_\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ is smooth and such that

$$f_\varepsilon(0) = 0, \qquad f_\varepsilon(r) = 1 \quad \text{for } r \geqslant \varepsilon, \qquad |f'| \leqslant 2\varepsilon^{-1}, \tag{2.19}$$

a simple computation shows that

$$I_\varepsilon \leqslant \mathcal{E}_\varepsilon(w_\varepsilon) \leqslant \pi|\log\varepsilon| + K\,,$$

which establishes the upper bound for I_ε. Actually, it has been proved that the minimizers u_ε, for small ε, do have radial symmetry [37, 40]. Moreover, as in the scalar case, we may define an optimal profile (although it is not given by an explicit formula). More precisely, there exists a unique function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$\begin{cases} -f'' - \frac{1}{r}f' + \frac{1}{r^2}f = f(1-f^2) & \text{on } [0,+\infty) \\ f(0) = 0 & . \end{cases} \tag{2.20}$$

Then we have

$$u_\varepsilon(z) \simeq f(\frac{|z|}{\varepsilon})\exp(i\theta)$$

and

$$u_\varepsilon(z) \to u_*(z) = \frac{z}{|z|} \qquad \text{as } \varepsilon \to 0, \qquad \text{in } W^{1,p}\,,\; p < 2\,,$$

and in $C^k_{loc}(D^2 \setminus \{0\})$. The map $u_*(z) = z/|z|$ realizes thus the prototypical singularity that can appear in the asymptotics for minimization problems.

2.2.2. The quest of compactness

As in the scalar case, the energy bound $\mathcal{E}_\varepsilon(v_\varepsilon) \leqslant M_0|\log\varepsilon|$ enables to derive some compactness for the sequence $(v_\varepsilon)_{0<\varepsilon<1}$. However the discussion is a little more involved. Indeed, a simple example shows that no general compactness result for reasonable norms can be derived, due to possible divergences in the phase. Take, for instance

$$w_\varepsilon(z) = \exp(i\varphi(z)\sqrt{|\log\varepsilon|})\,,$$

with $\varphi : \Omega \to \mathbb{R}$ a non-constant smooth function. We have $|w_\varepsilon| = 1$, hence

$$\mathcal{E}_\varepsilon(w_\varepsilon) = \frac{1}{2}\int_\Omega |\nabla w_\varepsilon|^2 = \frac{|\log\varepsilon|}{2}\int_\Omega |\nabla\varphi|^2 \leqslant K|\log\varepsilon|\,.$$

On the other hand, $|\nabla w_\varepsilon| = O(|\log\varepsilon|^{1/2})$, so that any norm of the gradient will diverge as $\varepsilon \to 0$. Actually, even for solutions of the stationary Ginzburg-Landau equation, no compactness has to be expected even in L^1 (see [14]).

However, one may split the contribution of the "topological" part from the rest of the phase to assert, in analogy with Remark 2.2, ii), (see [2])

PROPOSITION 2.13. — *Let $M_0 > 0$ and $(v_\varepsilon)_{\varepsilon>0}$, $v_\varepsilon : \Omega \to \mathbb{C}$ such that*

$$\mathcal{E}_\varepsilon(v_\varepsilon) \leqslant M_0|\log\varepsilon|\,.$$

Let $G \subset\subset \Omega$ be a smooth open simply connected set. Then, there exists a subsequence $\varepsilon_n \to 0$, ℓ points $a_1, \dots, a_\ell \in G$, integers $d_1, \dots, d_\ell \neq 0$, with $\sum_1^\ell |d_i| \leqslant K'$, for some constant K' depending only on M_0, and functions $\varphi_{\varepsilon_n} : G \to \mathbb{R}$ such that

$$\int_G |\nabla\varphi_{\varepsilon_n}|^2 \leqslant M_0|\log\varepsilon|$$

and

$$v_{\varepsilon_n} \cdot \exp(-i\varphi_{\varepsilon_n}) \to \prod_{i=1}^{\ell}\left(\frac{z-a_i}{|z-a_i|}\right)^{d_i} \qquad \text{in } H^s(G)\,,\; s < 1.$$

Notice that in the previous example, $\ell = 0$ (i.e. there are no vortices) and taking $\varphi_\varepsilon = \varphi \cdot \sqrt{|\log \varepsilon|}$, one may write, as above,

$$w_{\varepsilon_n} \cdot \exp(-i\varphi_{\varepsilon_n}) \equiv 1\,.$$

Sketch of proof. — The idea is to introduce a regularization of v_ε in order to get rid of possible "small dipoles" (i.e. pairs of vortices having opposite multiplicities and whose distance is say $o(\varepsilon^{1/2})$), and to keep only the "relevant" part of the vorticity of v_ε.

Assume for simplicity that $|v_\varepsilon| \leqslant 2$, and consider a minimizer w_ε of

$$F_\varepsilon(u) = \frac{1}{2}\int_\Omega \frac{|u - v_\varepsilon|^2}{\varepsilon} + \mathcal{E}_\varepsilon(u)\,, \qquad u \in H^1(\Omega;\mathbb{R}^2)\,.$$

Then w_ε verifies the perturbed Ginzburg-Landau equation

$$\frac{w_\varepsilon - v_\varepsilon}{\varepsilon} = \Delta w_\varepsilon + \frac{1}{\varepsilon^2} w_\varepsilon (1 - |w_\varepsilon|^2)\,. \tag{2.21}$$

One can easily show that $\mathcal{E}_\varepsilon(w_\varepsilon) \leqslant \mathcal{E}_\varepsilon(v_\varepsilon) \leqslant M_0|\log\varepsilon|$, and

$$\int_\Omega |w_\varepsilon - v_\varepsilon|^2 \leqslant 2M_0\varepsilon|\log\varepsilon|\,.$$

Performing a change of scale, and denoting

$$\begin{aligned} \tilde\varepsilon &= \varepsilon^{1/2} \\ \tilde w(x) &= w(\tilde\varepsilon x)\,, \end{aligned} \tag{2.22}$$

we are then led to the equation

$$\tilde w_\varepsilon - \tilde v_\varepsilon = \Delta \tilde w_\varepsilon + \frac{1}{\tilde\varepsilon^2}\tilde w_\varepsilon(1 - |\tilde w_\varepsilon|^2)\,, \tag{2.23}$$

and the left hand side in (2.23) is bounded in L^∞. Many techniques developed in the context of the stationary Ginzburg-Landau equation (see [5, 47, 10]) apply to (2.23). In particular, on G, the maps w_ε will have a finite number of vortices, bounded independently of ε. More precisely, for any $1/2 \leqslant \delta < 1$, there exists points $a_1^\varepsilon, \dots, a_\ell^\varepsilon$, integers $d_1^\varepsilon, \dots, d_\ell^\varepsilon$, and a constant $\lambda > 0$ such that $|w_\varepsilon| \geqslant \delta$ on $G \setminus \cup_1^\ell B(a_i^\varepsilon, \lambda\varepsilon)$, and

$$\frac{w_\varepsilon(z)}{|w_\varepsilon(z)|} = \exp(i\varphi_\varepsilon(z)) \prod_{i=1}^{\ell} \left(\frac{z - a_i^\varepsilon}{|z - a_i^\varepsilon|}\right)^{d_i^\varepsilon} \qquad \text{on } G \setminus \cup_{i=1}^\ell B(a_i^\varepsilon, \lambda\varepsilon)\,, \tag{2.24}$$

where $\varphi_\varepsilon : G \to \mathbb{R}$ are suitable functions. Moreover, we have

$$\begin{aligned} ||w_\varepsilon - v_\varepsilon||_{L^2} &\leqslant C\varepsilon^{1/2}|\log\varepsilon|^{1/2} \\ ||\nabla(w_\varepsilon - v_\varepsilon)||_{L^2} &\leqslant C|\log\varepsilon|^{1/2}, \end{aligned} \tag{2.25}$$

so that, for $s < 1$, $||w_\varepsilon - v_\varepsilon||_{H^s} \leqslant C\varepsilon^\alpha$, for some $0 < \alpha < 1$, and after a few simple computations the conclusion follows. □

Comment. — i) Proposition 2.13 shows that the possible lack of compactness is merely due to the phase (which is a real-valued function). On the other hand, the "topological" contribution due to the vortices is essentially compact.

ii) In view of the previous remark, some topological properties of the level sets of $\mathcal{E}_\varepsilon$ can be reduced to the properties of the level sets of the renormalized energy on the space of configurations of vortices (which is finite dimensional). This fact has been used in [2, 44, 51, 11] in order to find solutions to the stationary equation by variational methods (mountain pass, Ljusternik-Schnirelman theory, etc...).

2.2.3. Compactness for Jacobians

A related but conceptually different approach for locating the vorticity for maps v_ε satisfying the bound $\mathcal{E}_\varepsilon(v_\varepsilon) \leqslant M_0|\log\varepsilon|$ has been proposed first in [30] and, independently, in [1].

The main idea here is to look at the Jacobians of v_ε, which allows to characterize its topological part. More precisely, for $v = (v^1, v^2) : \Omega \to \mathbb{R}^2$ a smooth map, its Jacobian Jv is the 2-form defined by

$$Jv = dv^1 \wedge dv^2 = \frac{1}{2}d(v^1dv^2 - v^2dv^1).$$

In two dimensions, it may be identified with a scalar function, namely

$$Jv = \det(\nabla v) = v_x \times v_y,$$

where, for $a, b \in \mathbb{R}^2$, $a \times b = a^1b^2 - a^2b^1$. Note that $v_x \times v_y = 0$ whenever v_x and v_y are colinear. Hence, when $|v| = 1$, we have $Jv \equiv 0$. In particular, oscillations in the phase of v are not "seen" by its Jacobian Jv.

It is then proved that

PROPOSITION 2.14. — *Let $v_\varepsilon : \Omega \to \mathbb{R}^2$ such that $\mathcal{E}_\varepsilon(v_\varepsilon) \leqslant M_0|\log\varepsilon|$. Then there exists a subsequence $\varepsilon_n \to 0$, ℓ points $a_1, \ldots, a_\ell \in \Omega$, and integers*

$d_1, \ldots, d_\ell \neq 0$, with $\sum_1^\ell |d_i| \leqslant K'$, for some constant K' depending only on M_0, such that

$$Jv_{\varepsilon_n} \rightharpoonup J_* = \pi \sum_{i=1}^{\ell} d_i \delta_{a_i} \qquad \text{in } [\mathcal{C}_c^{0,\alpha}(\Omega)]^*\,, \text{ for any } \alpha > 0\,. \tag{2.26}$$

Remark 2.15. — i) Recall that the corresponding result in the one dimensional scalar case would be $\dot{v}_{\varepsilon_n} \rightarrow 2\chi \sum_{i=1}^{\ell} (-1)^i \delta_{a_i}$ (see Remark 2.2).

ii) Proposition 2.8 could also be derived using Proposition 2.13. However, the approaches in [30, 1] are more complete and give also interesting results for higher energy levels than the ones considered here.

2.2.4. Γ-convergence

The following result, stated in [30, 1], has to be compared with Proposition 2.5.

PROPOSITION 2.16. — *i) Let J_* be as in (2.26). Then for any sequence $(v_\varepsilon)_{0<\varepsilon<1}$ such that $Jv_\varepsilon \rightharpoonup J_*$ in $[\mathcal{C}_c^{0,\alpha}(\Omega)]^*$ as $\varepsilon \rightarrow 0$, we have*

$$\liminf_{\varepsilon \rightarrow 0} \frac{\mathcal{E}_\varepsilon(v_\varepsilon)}{|\log \varepsilon|} \geqslant ||J_*|| = \pi \sum_{i=1}^{\ell} |d_i|. \tag{2.27}$$

ii) The bound (2.27) is sharp, i.e. there exists a sequence $(u_\varepsilon)_{0<\varepsilon<1}$ such that $Ju_\varepsilon \rightharpoonup J_*$ in $[\mathcal{C}_c^{0,\alpha}(\Omega)]^*$ as $\varepsilon \rightarrow 0$, and

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{E}_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} = ||J_*|| = \pi \sum_{i=1}^{\ell} |d_i|. \tag{2.28}$$

Proof. — i) If the l.h.s. of (2.27) is equal to infinity there is nothing to prove. Therefore we may assume without loss of generality that $\mathcal{E}_\varepsilon(v_\varepsilon) \leqslant M_0|\log \varepsilon|$. Thus, going back to Proposition 2.13, we have $\mathcal{E}_\varepsilon(w_\varepsilon) \leqslant \mathcal{E}_\varepsilon(v_\varepsilon)$, and $Jw_\varepsilon \rightarrow J_*$ as $\varepsilon \rightarrow 0$ by (2.25), and we may work now on w_ε instead of v_ε. The main advantage is that the vorticity of w_ε is located, in view of (2.24), in a finite number of disjoint balls of size ε, and $|w_\varepsilon| > \delta$ outside the balls, where $1/2 \leqslant \delta < 1$ is fixed. Then, elementary computations (see [5], Chapter 1) show that

$$\mathcal{E}_\varepsilon(w_\varepsilon) \geqslant |\log \varepsilon| \delta^2 \pi \sum_{i=1}^{\ell} |d_i| - K\,, \tag{2.29}$$

for a constant $K > 0$ independent of ε. The conclusion follows by letting $\varepsilon \to 0$ and then $\delta \to 1$.

ii) For $j = 1, \dots, d_i$, let $b^\varepsilon_{i,j} = a_i + |\log\varepsilon|^{-1}\exp(i2\pi j/d_i)$. Consider the map

$$u_\varepsilon(x) = \prod_{i=1}^{\ell}\prod_{j=1}^{d_i} f_\varepsilon(x - b^\varepsilon_{i,j})\frac{x - b^\varepsilon_{i,j}}{|x - b^\varepsilon_{i,j}|}, \tag{2.30}$$

where f_ε is defined as in (2.19). Elementary computations show that the sequence u_ε enjoys the desired properties. □

2.2.5. The case $N \geqslant 3$

Since in dimension two vortices are points, and therefore codimension two defects, one expects, likewise, that in higher dimensions defects for the complex Ginzburg-Landau functional will concentrate on sets of codimension two. The following result, first proved in [30] gives a precise formulation of that.

PROPOSITION 2.17. — *Let $(v_\varepsilon)_{0<\varepsilon<1}$ be a sequence such that $\mathcal{E}_\varepsilon(v_\varepsilon) \leqslant M_0|\log\varepsilon|$. Then, for a subsequence $\varepsilon_n \to 0$,*

$$Jv_{\varepsilon_n} \rightharpoonup J_* \qquad in[\mathcal{C}^{0,\alpha}_c(\Omega)]^*, \tag{2.31}$$

where $\frac{1}{\pi}J_*$ is an $(N-2)$- (integer multiplicity) rectifiable current without boundary.

Comment. — i) We recall some terminology from Geometric Measure Theory. A k-dimensional current on Ω is an element of the dual of the space of smooth k-forms with compact support in Ω. A k-current is called rectifiable if it can be represented by integration over a k-rectifiable set, with an integer valued density function.

ii) The proof of Proposition 2.17 in [30] relies on reduction to the two dimensional case by slicing arguments.

A different proof has been derived independently in [1]: the strategy is to approximate the Jacobian of v_ε by polyhedral currents with uniformly bounded mass, and then apply the classical Federer-Fleming compactness theorem.

The corresponding Γ-convergence result (i.e. the generalization of Proposition 2.16 to higher dimensions) is proved in [1].

To conclude Section 2, we emphasize once more that, for maps v_ε verifying the energy bound

$$\mathcal{E}_\varepsilon(v_\varepsilon) \leqslant M_0 k_\varepsilon ,$$

the topological defects concentrate on $N-d$-dimensional sets with some regularity (i.e. they are rectifiable). In view of inequalities (2.16), (2.27), the concentration set for defects is also a concentration set for the energy (however, for arbitrary maps, energy might concentrate outside J_*).

Finally, we also would like to point out that, even though J_* is rectifiable, its geometrical support might not be closed, so that in particular, the distributional support could be the whole domain.

3. Some properties of $(\mathbf{PGL})_\varepsilon$

In this section we discuss some properties of solutions u_ε to equation $(\mathrm{PGL})_\varepsilon$, which will enter directly in the proof of Theorem 1.1. If not otherwise stated, proofs are provided in [9].

We begin with pointwise estimates for u_ε and its derivatives.

PROPOSITION 3.1. — *Let u_ε be a solution of $(PGL)_\varepsilon$ with $\mathcal{E}_\varepsilon(u_\varepsilon^0) < +\infty$. Then there exists a constant $K > 0$ depending only on N such that, for $t \geqslant \varepsilon^2$ and $x \in \mathbb{R}^N$, we have*

$$|u_\varepsilon(x,t)| \leqslant 3, \qquad |\nabla u_\varepsilon(x,t)| \leqslant \frac{K}{\varepsilon}, \qquad |\frac{\partial u_\varepsilon}{\partial t}(x,t)| \leqslant \frac{K}{\varepsilon^2}, \tag{3.1}$$

where K is independent of the initial data.

The proof relies on the maximum principle and the construction of suitable supersolutions.

Remark 3.2. — Equation $(\mathrm{PGL})_\varepsilon$ has standard scaling properties. If u_ε is a solution to $(\mathrm{PGL})_\varepsilon$, then for $R > 0$ the function

$$v_\varepsilon(x,t) \equiv u_\varepsilon(Rx, R^2 t)$$

is a solution to $(\mathrm{PGL})_{R^{-1}\varepsilon}$. The bounds (3.1) are therefore coherent with this invariance.

As mentioned in the Introduction, the evolution properties of the energy density can be directly inferred from $(\mathrm{PGL})_\varepsilon$. This is presumably well reflected in the results of the next section, which are the starting point in the proof of Theorem 1.1.

3.1. Monotonicity formulas

Let u_ε be a solution to $(\mathrm{PGL})_\varepsilon$ verifying (H_0). For $(x_*, t_*) \in \mathbb{R}^N \times \mathbb{R}^+$ we set $z_* = (x_*, t_*)$.

For $0 < R \leqslant \sqrt{t_*}$ we define the weighted energy

$$E_w(z_*, R) = k_\varepsilon^{-1} \int_{\mathbb{R}^N} e_\varepsilon(u(x, t_* - R^2)) \exp(-\frac{|x - x_*|^2}{4R^2}) dx \tag{3.2}$$

$$= \int_{\mathbb{R}^N} \exp(-\frac{|x - x_*|^2}{4R^2}) d\mu_\varepsilon^{t_* - R^2}.$$

We emphasize the fact that the above integral is computed at the time $t = t_* - R^2$, and **not** at time $t = t_*$, i.e. a shift in time $\delta t = -R^2$ has been introduced. Note also that in (3.2) the weight becomes small outside the ball $B(x_*, R)$. More precisely, the following inequality holds

$$\exp(\frac{1}{4}) \frac{1}{R^{N-2}} E_w(z_*, R) \geqslant \frac{1}{R^{N-2}} \int_{B(x_*, R)} d\mu_\varepsilon^{t_* - R^2}. \tag{3.3}$$

The right-hand side of (3.3) arises naturally in the stationary equation, where its monotonicity properties (with respect to the radius R) play an important role. In our parabolic setting, the time t at which E_w is computed is related to R by $t = t_* - R^2$ and this is consistent with the parabolic scaling (for $\lambda > 0$) $x \to \lambda x$, $t \to \lambda^2 t$, which leaves the linear heat equation invariant, and which we mentioned earlier.

In this context, the following monotonicity formula was derived first by Struwe [48] for the heat-flow of harmonic maps (see also [18, 25]). In a different context Giga and Kohn [23] used related ideas.

PROPOSITION 3.3.— *We have*

$$\frac{d}{dR} \left(\frac{1}{R^{N-2}} E_w(z_*, R) \right) \geqslant 0, \tag{3.4}$$

i.e. in particular the quantity $R^{2-N} E_w(z_*, R)$ is non-decreasing in R. Passing to the limit $\varepsilon_n \to 0$ we have therefore

$$\frac{d}{dR} \left(\frac{1}{R^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|x - x_*|^2}{4R^2}) d\mu_*^{t_* - R^2} \right) \geqslant 0. \tag{3.5}$$

As a consequecence of Proposition 3.3, and in view of (3.3) and (H_0), we have

$$\frac{1}{R^{N-2}} \int_{B(x,R)} d\mu_*^t \leqslant C(t)\,, \tag{3.6}$$

where $C(t)$ is a constant depending only on t. Loosely speaking, estimate (3.6) shows that if concentration of energy does occur, the Hausdorff dimension of the concentration set has to be at least $N-2$. This is consistent with the analysis of Section 2, but the context is completely different. In the complex case ($d=2$), the fact that the dimension is exactly $N-2$ will follow from the monotonicity formula and the Clearing-Out Lemma below. However, for the scalar case ($d=1$) , another monotonicity formula has to be worked out as follows (see [26]).

PROPOSITION 3.4. —

$$\frac{d}{dR}\left(\frac{1}{R^{N-1}} E_w(z_*, R)\right) \geqslant -\frac{1}{R^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|x-x_*|^2}{4R^2}) d\xi_\varepsilon^t\,, \tag{3.7}$$

where

$$\xi_\varepsilon^t = \left(\frac{|\nabla u_\varepsilon(x)|^2}{2} - \frac{V(u_\varepsilon(x))}{\varepsilon^2}\right) dx \tag{3.8}$$

is called the "discrepancy" term.

This inequality is less satisfactory than inequality (3.4), unless one is able to prove that the discrepancy term is negative (or small). Using the maximum principle, Ilmanen proved negativity of the discrepancy term under the condition it is negative at initial time. Soner [46] however proved that the r.h.s. is small after time $t \geqslant \varepsilon$, so that in the limit $\varepsilon \to 0$, we have

$$\frac{d}{dR}\left(\frac{1}{R^{N-1}} \int_{\mathbb{R}^N} \exp(-\frac{|x-x_*|^2}{4R^2}) d\mu_*^{t_*-R^2}\right) \geqslant 0\,. \tag{3.9}$$

Here again, (3.9) shows that concentration of energy can occur only on sets of dimension at least $N-1$. The Clearing-Out Lemma is needed as well to prove that it is exactly $N-1$.

3.2. Clearing-Out

In this section we discuss the various versions of Clearing-Out needed. We start with the following

THEOREM 3.5. — *Let $0 < \varepsilon < 1$, u_ε be a solution of $(PGL)_\varepsilon$ with $\mathcal{E}_\varepsilon(u_\varepsilon^0) < +\infty$, $z_* = (x_*, t_*)$ and $\sigma > 0$ be given. There exists $\eta_1 = \eta_1(\sigma) > 0$ depending only on the dimension N and on σ such that if*

$$\frac{1}{R^{N-d}} \int_{\mathbb{R}^N} \exp(-\frac{|x - x_*|^2}{4R^2}) d\mu_\varepsilon^{t_* - R^2} \leqslant \eta_1, \tag{3.10}$$

then

$$|u_\varepsilon(x_*, t_*)| \geqslant 1 - \sigma. \tag{3.11}$$

Sketch of proof for $d = 1$. — By invariances of the equation, it suffices to consider the case $z_* = (0, 1)$. We apply the monotonicity formula (3.7) at the point $z_* = (0, 1)$ between $R = 1$ and $R = \lambda\varepsilon$. We have

$$\frac{1}{(\lambda\varepsilon)^{N-1}} \int_{\mathbb{R}^N} \exp(-\frac{|x - x_*|^2}{4\lambda^2\varepsilon^2}) d\mu_\varepsilon^{1-\lambda^2\varepsilon^2} \leqslant \int_{\mathbb{R}^N} \exp(-\frac{|x - x_*|^2}{4}) d\mu_\varepsilon^1 \leqslant \eta_1 \tag{3.12}$$

In particular, in view of (3.3), the mean-value of $(1 - |u_\varepsilon|^2)^2$ on $B_{\lambda\varepsilon}$, which is achieved at some point x_0, verifies

$$\left(1 - |u_\varepsilon(x_0, 1 - \lambda^2\varepsilon^2)|^2\right)^2 = \frac{1}{\omega_N \lambda^N \varepsilon^N} \int_{B_{\lambda\varepsilon} \times \{1 - \lambda^2\varepsilon^2\}} (1 - |u_\varepsilon|^2)^2 \leqslant \frac{K\eta_1}{\omega_N \lambda}. \tag{3.13}$$

Combining (3.13) with the pointwise estimates (3.1), we obtain

$$\begin{aligned} \left(1 - |u_\varepsilon(0,1)|^2\right)^2 &\leqslant \left(|u_\varepsilon(0,1)|^2 - |u_\varepsilon(x_0, 1 - \lambda^2\varepsilon^2)|^2\right)^2 \\ &\quad + \left(1 - |u_\varepsilon(x_0, 1 - \lambda^2\varepsilon^2)|^2\right)^2 \leqslant K\left(\lambda + \lambda^2 + \frac{\eta_1}{\lambda}\right). \end{aligned} \tag{3.14}$$

We first choose λ such that $K(\lambda + \lambda^2) \leqslant \sigma^2/2$, then we choose η_1 so that $K\eta_1/\lambda \leqslant \sigma^2/2$. □

The case $d = 2$ is much more involved, and we refer to [9] for a proof. A similar result was obtained earlier for $N = 3$ in [35], and for $N = 4$ in [50]. The corresponding result for the stationary case was developed in a series of papers (see [10, 47, 43, 33, 34, 6, 8]).

The condition in (3.10) involves an integral on the whole of $\mathbb{R}^N$. In some situations, it will be convenient to integrate on finite domains. From this point of view, **assuming** (H_0) we have the following result, in the spirit of Brakke's original Clearing-Out [13], Lemma 6.3, but for jumps and vorticity here, not yet for the energy!

PROPOSITION 3.6. — *Let u_ε be a solution of $(PGL)_\varepsilon$ verifying assumption (H_0) and $\sigma > 0$ be given. Let $x_T \in \mathbb{R}^N$, $T > 0$ and $R \geqslant \sqrt{2\varepsilon}$. There exists a positive continuous function λ defined on $\mathbb{R}_*^+$ such that, if*

$$\check{\eta}(x_T, T, R) \equiv \frac{1}{R^{N-d}} \int_{B(x_T, \lambda(T)R)} d\mu_\varepsilon^T \leqslant \frac{\eta_1(\sigma)}{2}$$

then

$$|u_\varepsilon(x,t)| \geqslant 1 - \sigma \qquad \text{for } t \in [T + T_0, T + T_1] \quad \text{and } x \in B(x_T, \frac{R}{2}).$$

Here T_0 and T_1 are defined by

$$T_0 = \max(2\varepsilon, \left(\frac{2\check{\eta}}{\eta_1(\sigma)}\right)^{\frac{2}{N-2}} R^2), \qquad T_1 = R^2.$$

Theorem 3.5 and Proposition 3.6 have many consequences. Some are of independent interest. For instance, the simplest one is the complete annihilation of the topological defects for $N \geqslant d + 1$.

PROPOSITION 3.7. — *Assume that $N \geqslant 3$. Let u_ε be a solution of $(PGL)_\varepsilon$ verifying assumption (H_0). Then*

$$|u_\varepsilon(x,t)| \geqslant \frac{1}{2} \qquad \text{for any } t \geqslant T_f \equiv \left(\frac{M_0}{\eta_1}\right)^{\frac{2}{N-2}} \text{ and for all } x \in \mathbb{R}^N, \quad (3.15)$$

where $\eta_1 = \eta_1(\frac{1}{2})$.

Remark 3.8. — The result of Proposition 3.7 **does not** hold in the critical dimension $N = d$. As already mentioned, this is related to the so-called "slow motion" phenomenon (see [15, 16, 28, 31]).

3.3. Improved pointwise energy bounds

In this section we analyze the situation where $|u_\varepsilon| \geqslant 1 - \sigma$ on some standard cylindrical domain. Note that such a situation may occur when it is possible to apply Theorem 3.5.

THEOREM 3.9. — *Let $B(x_0, R)$ be a ball in $\mathbb{R}^N$ and $T > 0$, $\Delta T > 0$ be given. Consider the cylinder*

$$\Lambda = B(x_0, R) \times [T, T + \Delta T].$$

There exists a constant $0 < \sigma \leqslant \frac{1}{2}$, and $\beta > 0$ depending only on N, such that if

$$|u_\varepsilon| \geqslant 1 - \sigma \qquad \text{on } \Lambda, \tag{3.16}$$

then

$$e_\varepsilon(u_\varepsilon) = |\nabla \Phi_\varepsilon|^2 + \kappa_\varepsilon \qquad \text{in } \Lambda_{\frac{1}{2}}, \tag{3.17}$$

where the functions Φ_ε and κ_ε are defined on $\Lambda_{\frac{1}{2}}$ and verify

$$\frac{\partial \Phi_\varepsilon}{\partial t} - \Delta \Phi_\varepsilon = 0 \qquad \text{in } \Lambda_{\frac{1}{2}}, \tag{3.18}$$

$$\|\kappa_\varepsilon\|_{L^\infty(\Lambda_{\frac{1}{2}})} \leqslant C(\Lambda)\varepsilon^\beta, \qquad \|\nabla \Phi_\varepsilon\|^2_{L^\infty(\Lambda_{\frac{1}{2}})} \leqslant C(\Lambda) M_0 |\log \varepsilon|. \tag{3.19}$$

More precisely, writing $u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon)$, where $\rho_\varepsilon = |u_\varepsilon|$ and φ_ε is a suitable smooth real-valued function, we have

$$\|\nabla \varphi_\varepsilon - \nabla \Phi_\varepsilon\|_{L^\infty(\Lambda_{\frac{1}{2}})} \leqslant C(\Lambda)\varepsilon^\beta. \tag{3.20}$$

Remark 3.10. — In the scalar case $d = 1$, then obviously $|\nabla \Phi_\varepsilon|^2$ vanishes, so that in particular $e_\varepsilon(u_\varepsilon) = o(k_\varepsilon)$.

We briefly sketch the proof in the case $d = 1$. Set $\theta = 1 - \rho_\varepsilon$. The function θ verifies the equation

$$\frac{\partial \theta}{\partial t} - \Delta \theta + \frac{c(x)\theta}{\varepsilon^2} \leqslant 0 \tag{3.21}$$

where $c(x) = (1 + (\theta(x) - 1)^2)$. Then one proves first using a suitable supersolution for (3.21) (see [9], Lemma 1.1) that

$$1 - |u_\varepsilon| \leqslant C(\Lambda)\varepsilon^2 |\log \varepsilon| \tag{3.22}$$

on a slightly smaller cylinder, so that

$$\frac{(1 - |u_\varepsilon|^2)^2}{4\varepsilon^2} \leqslant C(\Lambda)\varepsilon^2 |\log \varepsilon|^2 .$$

The gradient term can be treated similarly with a few computations.

Combining Theorem 3.5 and Theorem 3.9, we obtain the following immediate consequence.

PROPOSITION 3.11. — *There exist an absolute constant $\eta_2 > 0$ and a positive function λ defined on $\mathbb{R}_*^+$ such that if, for $x \in \mathbb{R}^N$, $t > 0$ and $r > \sqrt{2\varepsilon}$, we have*

$$\int_{B(x,\lambda(t)r)} d\mu_\varepsilon^t \leqslant \eta_2 r^{N-d},$$

then

$$e_\varepsilon(u_\varepsilon) = |\nabla\Phi_\varepsilon|^2 + \kappa_\varepsilon$$

in $\Lambda_{\frac{1}{4}}(x,t,r) \equiv B(x,\frac{r}{4}) \times [t+\frac{15}{16}r^2, t+r^2]$, where Φ_ε and κ_ε are as in Theorem 3.9. In particular, if $d = 1$,

$$e_\varepsilon(u_\varepsilon) = o(k_\varepsilon) \qquad \text{uniformly in } \Lambda_{\frac{1}{4}}(x,t,r),$$

whereas, for $d = 2$,

$$e_\varepsilon(u_\varepsilon) \leqslant C(t,r)k_\varepsilon \qquad \text{on } \Lambda_{\frac{1}{4}}(x,t,r).$$

Remark 3.12. — The constant η_2 is actually defined as $\eta_2 = \eta_1(\sigma)$, where σ is the constant in Theorem 3.9 and η_1 is the function defined in Proposition 3.6.

The previous result allows to deduce, passing to the limit as $\varepsilon_n \to 0$, the following properties of the limiting measure μ_*.

THEOREM 3.13. — *There exists an absolute constant $\eta_2 > 0$, and a positive continuous function λ defined on $\mathbb{R}_*^+$ such that, if for $x \in \mathbb{R}^N$, $t > 0$ and $r > 0$ we have*

$$\mu_*^t(B(x,\lambda(t)r)) < \eta_2 r^{N-d}, \tag{3.23}$$

then for every $s \in [t+\frac{15}{16}r^2, t+r^2]$, μ_*^t,

a) if $d = 1$,

$$\mu_*^s \equiv 0 \qquad \text{on } B(x,\frac{1}{4}r).$$

b) if $d = 2$, then μ_*^s absolutely continuous with respect to the Lebesgue measure on the ball $B(x,\frac{1}{4}r)$. More precisely,

$$\mu_*^s = |\nabla\Phi_*|^2\, dx \qquad \text{on } B(x,\frac{1}{4}R),$$

where Φ_* is smooth and satisfies the heat equation in

$$\Lambda_{\frac{1}{4}} = B(x_0,\frac{1}{4}r) \times [t+\frac{15}{16}r^2, t+r^2].$$

4. Densities and concentration sets

In order to analyse geometric properties of the measures μ_* and μ_*^t, an important concept is that of densities. For a given Radon measure ν on $\mathbb{R}^N$, we have the classical definition

DEFINITION 4.1. — *For $m \in \mathbb{N}$, the m-dimensional lower density of ν at the point x is defined by*

$$\Theta_{*,m}(\nu, x) = \liminf_{r\to 0} \frac{\nu(B(x,r))}{\omega_m r^m},$$

where ω_m denotes the volume of the unit ball B^m. Similarly, the m-dimensional upper density $\Theta_m^*(\nu^t, x)$ is given by

$$\Theta_m^*(\nu, x) = \limsup_{r\to 0} \frac{\nu(B(x,r))}{\omega_m r^m}.$$

When both quantities coincide, ν admits a m-dimensional density $\Theta_m(\nu, x)$ at the point x, defined as the common value.

Since the energy measure is expected to concentrate on (N-d)-dimensional objects, our main efforts will be devoted to the study of the density $\Theta_{*,N-d}(\mu_*^t, \cdot)$. As already mentioned, the monotonicity formula provides upper-bounds for $\Theta_{*,N-d}(\mu_*^t, \cdot)$.

In order to prove that the dimension of the concentration set is **exactly** $N-d$, lower bounds are needed as well. However, there are some conceptual difficulties to attack $\Theta_{*,N-d}(\mu_*^t, \cdot)$ directly (since the equation depends on time). Instead, we will first work on the measure μ_*, and recall the notion of parabolic density, which is natural in view of monotonicity.

DEFINITION 4.2. — *Let ν be a Radon measure on $\mathbb{R}^N \times [0, +\infty)$ such that $\nu = \nu^t\, dt$. For $t > 0$ and $m \in \mathbb{N}$, the parabolic m-dimensional lower density of ν at the point (x,t) is defined by*

$$\Theta_{*,m}^P(\nu, (x,t)) = \liminf_{r\to 0} \frac{1}{r^m} \int_{\mathbb{R}^N} \exp(-\frac{|x-y|^2}{4r^2})\, d\nu^{t-r^2}(y).$$

The parabolic upper density and parabolic density are defined accordingly, and denoted respectively by $\Theta_m^{P,*}$ and Θ_m^P.

Remark 4.3. — Notice that Θ^P is not the classical density, in the spirit of Definition 4.1, for the parabolic metric defined by $d_P((x,t),(x',t')) = \max(|x-x'|, |t-t'|^{\frac{1}{2}})$.

It clearly follows from monotonicity that the limit in Definition 4.2 is decreasing, so that $\Theta^P_{N-2}(\mu^*, (x,t))$ exists everywhere in $\mathbb{R}^N \times (0,+\infty)$. We set

$$\Sigma_\mu = \left\{(x,t) \in \mathbb{R}^N \times (0,+\infty) \text{ s.t. } \Theta^P_{N-d}(\mu_*, (x,t)) > 0\right\},$$

and for $t > 0$, $\Sigma^t_\mu = \Sigma_\mu \cap \left(\mathbb{R}^N \times \{t\}\right)$. The parabolic density is related to the $\Theta_{*,N-d}$ by

$$\Theta^P_{N-d}(\mu_*, (x,t)) \geqslant K\Theta_{*,N-d}(\mu^t_*, x),$$

so that in particular

$$\Theta_{*,N-d}(\mu^t_*, x) \equiv 0 \qquad \text{on } \mathbb{R}^N \setminus \Sigma^t_\mu. \tag{4.1}$$

4.1. First properties of Σ_μ

As in Brakke's works ([13]), the main tool in the study of geometric properties of Σ_μ is the following Clearing-Out Lemma.

THEOREM 4.4. — *There exists a positive continuous function η_3 defined on $\mathbb{R}^+_*$, such that for any $(x,t) \in \mathbb{R}^N \times (0,+\infty)$ and any $0 < r < \sqrt{t}$, if*

$$\Xi_\mu((x,t),r) \equiv \frac{1}{r^{N-d}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4r^2}\right) d\mu^{t-r^2}_*(y) \leqslant \eta_3(t-r^2)$$

then

$$(x,t) \notin \Sigma_\mu.$$

Theorem 4.4 is a direct consequence of Theorem 3.13. An immediate corollary is

COROLLARY 4.5. — *For any $(x,t) \in \Sigma_\mu$, we have*

$$\Theta^P_{N-d}(\mu_*, (x,t)) \geqslant \eta_3(t).$$

At this stage, we are in position to derive the following, without invoking any further property of the equation $(\mathrm{PGL})_\varepsilon$.

PROPOSITION 4.6. —

i) The set Σ_μ is closed in $\mathbb{R}^N \times (0,+\infty)$.

ii) For any $t > 0$,

$$\mathcal{H}^{N-d}(\Sigma^t_\mu) \leqslant KM_0 < +\infty.$$

iii) For any $t > 0$, *the measure* μ_*^t *can be decomposed as*

$$\mu_*^t = g(x,t)\mathcal{H}^N + \Theta_*(x,t)\mathcal{H}^{N-d} \llcorner \Sigma_\mu^t,$$

where g *is some smooth function defined on* $\mathbb{R}^N \times (0,+\infty) \setminus \Sigma_\mu$ *and* Θ_* *verifies the bound* $\Theta_*(x,t) \leqslant KM_0 t^{\frac{2-N}{2}}$.

Comment. — a) The function Θ_* in decomposition *iii*) is the Radon-Nikodym derivative of $\mu_*^t \llcorner \Sigma_\mu^t$ with respect to $\mathcal{H}^{N-d}$.

b) Concerning g, it can be locally defined as $|\nabla \Phi_*|^2$ for some smooth Φ_* verifying the heat equation. It is possible to show that the function Φ_* is actually defined globally, and verifies the heat equation on $\mathbb{R}^N \times (0,+\infty)$. This requires some further properties of $(\mathrm{PGL})_\varepsilon$ which we are not going to discuss here (see Theorem 3 in [9]).

In order to show that the Hausdorff dimension of Σ_μ^t is exactly $N-d$ it was sufficient to bound the parabolic density away from zero. To deduce further regularity properties of Σ_μ^t it is crucial to derive lower bounds for the density $\Theta_{*,N-d}$ itself. We have

PROPOSITION 4.7. — *For almost every* $t > 0$,

$$\Theta_{*,N-d}(\mu_*^t, x) = \Theta^*_{N-d}(\mu_*^t, x) \geqslant K\eta_3(t),$$

for $\mathcal{H}^{N-2}$ *almost every* $x \in \Sigma_\mu^t$. *Consequently, for almost every* $t > 0$ *the set* Σ_μ^t *is (N-2)-rectifiable.*

The proof of Proposition 4.7 is not immediate and involves several ingredients. In particular, one has to consider concentration sets for the limit of the measures $k_\varepsilon^{-1}|\partial_t u_\varepsilon|^2 dxdt$, which are uniformly bounded. It can be shown that these sets have small Hausdorff dimension.

Once the existence of the density is established, it follows from the celebrated regularity theorem of Preiss [42] that Σ_μ^t is $(N-d)$-rectifiable for a.e. $t > 0$.

The next discussion will be devoted to the evolution law of the concentrated part ν_*^t of the measure μ_*^t. We recall first some classical facts concerning mean curvature flows.

5. Mean curvature flows

5.1. The classical notion

Let Σ be a smooth compact manifold of dimension k, and $\gamma_0 : \Sigma \to \mathbb{R}^N$ $(N \geqslant k)$ a smooth embedding, so that $\Sigma^0 = \gamma_0(\Sigma)$ is a smooth k-dimensional submanifold of $\mathbb{R}^N$. The mean curvature vector at the point x of Σ^0 is the vector of the orthogonal space $(T_x\Sigma^0)^\perp$ given by

$$\vec{H}_{\Sigma^0}(x) = -\sum_{\alpha=1}^{N-k}\left(\sum_{j=1}^{k}(\tau_j \cdot \frac{\partial \nu^\alpha}{\partial \tau_j})\nu^\alpha\right) = -\sum_{\alpha=1}^{N-k}(\operatorname{div}_{T_x\Sigma^0}\nu^\alpha)\,\nu^\alpha, \tag{5.1}$$

where $(\tau_1, ..., \tau_k)$ is an orthonormal moving frame on $T_x\Sigma^0$, $(\nu^1, ..., \nu^{N-k})$ is an orthonormal moving frame on $(T_x\Sigma^0)^\perp$, and $\operatorname{div}_{T_x\Sigma^0}$ denotes the tangential divergence at the point x. The integral formulation of (5.1) is given by

$$\int_{\Sigma^0} \operatorname{div}_{T_x\Sigma^0}\vec{X}\, d\mathcal{H}^k = -\int_{\Sigma^0} \vec{H}_{\Sigma^0} \cdot \vec{X}\, d\mathcal{H}^k, \tag{5.2}$$

for all $\vec{X} \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$. The vectors $\vec{H}_{\Sigma^0}(\cdot)$ are uniquely determined by (5.2), and in particular the definition in (5.1) does not depend on the choice of orthonormal frames.

Next, we introduce a time dependence, and consider a smooth family $\{\gamma_t\}_{t\in I}$ of smooth embeddings of Σ in $\mathbb{R}^N$, where I denotes some open interval containing 0. We set $\Sigma^t = \gamma_t(\Sigma)$. If χ is a smooth compactly supported function on $\mathbb{R}^N$, a standard computation shows that

$$\frac{d}{dt}\int_{\Sigma^t} \chi(x)\, d\mathcal{H}^k = \int_{\Sigma^t}\Big(-\chi(x)\vec{H}_{\Sigma^t}(x) + P(\nabla\chi(x))\Big)\cdot \vec{Y}(x)\, d\mathcal{H}^k, \tag{5.3}$$

where $\vec{Y}(x) = \frac{d}{ds}\gamma_s(\gamma_t^{-1}(x))$ is the velocity vector at the point x, and P denotes the orthogonal projection on $(T_x\Sigma^t)^\perp$.

The family $(\Sigma^t)_{t\in I}$ is moved by mean curvature in the classical sense if and only if

$$\frac{d}{dt}\gamma_t(m) = \vec{H}_{\Sigma^t}(\gamma_t(m)), \qquad \text{for all } m \in \Sigma \text{ and } t \in I. \tag{5.4}$$

In particular, if $(\Sigma^t)_{t\in I}$ is moved by mean curvature, (5.3) becomes

$$\frac{d}{dt}\int_{\Sigma^t} \chi(x)\, d\mathcal{H}^k = -\int_{\Sigma^t} \chi(x)|\vec{H}_{\Sigma^t}(x)|^2\, d\mathcal{H}^k + \int_{\Sigma^t} \nabla\chi(x)\cdot \vec{H}_{\Sigma^t}(x)\, d\mathcal{H}^k, \tag{5.5}$$

and actually (5.5) is equivalent to (5.4) if χ is taken arbitrary. Notice that the last term in the r.h.s of (5.5) corresponds to a transport term, whereas the first term represents a shrinking of the area. Actually, if $\chi \equiv 1$ in a neighborhood of Σ^t, then

$$\frac{d}{dt}\mathcal{H}^k(\Sigma^t) = -\int_{\Sigma^t} |\vec{H}_{\Sigma^t}(x)|^2 \, d\mathcal{H}^k.$$

In particular, the mean curvature flow is the **gradient flow** for the **area functional**. Finally, existence of a classical solution of (5.4) for small times can be established, but singularities develop in finite time.

5.2. Brakke flows

In the attempt to extend (5.4) or (5.5) to a larger class of (less regular) objects, and in particular to extend the flow past singularities, Brakke [13] relaxed equality in (5.5), and considered instead sub-solutions, i.e. verifying the **inequality**

$$\frac{d}{dt}\int_{\Sigma^t} \chi(x)\, d\mathcal{H}^k \leqslant -\int_{\Sigma^t} \chi(x) |\vec{H}_{\Sigma^t}(x)|^2 \, d\mathcal{H}^k + \int_{\Sigma^t} \nabla\chi(x) \cdot \vec{H}_{\Sigma^t}(x)\, d\mathcal{H}^k, \tag{5.6}$$

for all non-negative $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^N)$. Following Brakke [13], we are thus going to extend (5.6) to less regular objects.

Recall that a Radon measure ν on $\mathbb{R}^N$ is said to be k-rectifiable if there exists a k-rectifiable set Σ, and a density function $\Theta \in L^1_{\text{loc}}(\mathcal{H}^k \llcorner \Sigma)$ such that $\nu = \Theta(\cdot)\, \mathcal{H}^k \llcorner \Sigma$. Since Σ is rectifiable, for $\mathcal{H}^k$-a.e. $x \in \Sigma$, there exist a unique tangent space $T_x\Sigma$. The distributional first variation of ν is the vector-valued distribution $\delta\nu$ defined by

$$\delta\nu(\vec{X}) = \int_\Sigma \mathrm{div}_{T_x\Sigma}\vec{X}\, d\nu \qquad \text{for all } \vec{X} \in \mathcal{C}^\infty_c(\mathbb{R}^N, \mathbb{R}^N). \tag{5.7}$$

In case $|\delta\nu|$ is a measure, absolutely continuous with respect to ν, we say that ν has a first variation and we may write

$$\delta\nu = \vec{H}\nu,$$

where $\vec{H}$ is the Radon-Nikodym derivative of $\delta\nu$ with respect to ν. In this case, formula (5.7) becomes

$$\int_\Sigma \mathrm{div}_{T_x\Sigma}\vec{X}\, d\nu = -\int_\Sigma \vec{H}\cdot\vec{X}\, d\nu. \tag{5.8}$$

Remark 5.1. — Notice that in the smooth case, this notion coincides with the definition (5.1), in view of (5.2). Notice also that the component of $\vec{H}$ which is orthogonal to $T_x\Sigma$ is independent of the density Θ. However, if Θ is non constant, then $\vec{H}$ may have a tangential part.

We are now in position to give the precise definition of a Brakke flow. Let $(\nu_t)_{t\geqslant 0}$ be a family of Radon measures on $\mathbb{R}^N$. For $\chi \in \mathcal{C}_c^2(\mathbb{R}^N, \mathbb{R}^+)$, we define

$$\bar{D}_t\nu_0^t(\chi) = \limsup_{t\to t_0} \frac{\nu^t(\chi) - \nu^{t_0}(\chi)}{t - t_0}.$$

If $\nu^t \llcorner \{\chi > 0\}$ is a k-rectifiable measure which has a first variation verifying $\chi|\vec{H}|^2 \in L^1(\nu^t)$, then we set

$$\mathcal{B}(\nu^t, \chi) = -\int \chi|\vec{H}|^2\, d\nu^t + \int \nabla\chi \cdot P(\vec{H})\, d\nu^t,$$

[here P denotes $\mathcal{H}^k$-a.e. the orthogonal projection onto the tangent space to ν^t].

Otherwise, we set

$$\mathcal{B}(\nu^t, \chi) = -\infty.$$

DEFINITION 5.2 (Brakke flow). — *Let $(\nu_t)_{t\geqslant 0}$ be a family of Radon measures on $\mathbb{R}^N$. We say that $(\nu_t)_{t\geqslant 0}$ is a k-dimensional Brakke flow if and only if*

$$\bar{D}_t\nu^t(\chi) \leqslant \mathcal{B}(\nu^t, \chi), \tag{5.9}$$

for every $\chi \in \mathcal{C}_c^2(\mathbb{R}^N, \mathbb{R}^+)$ and for all $t \geqslant 0$.

The motion by mean curvature in the sense of Brakke has many interesting properties, in particular the fact that the area functional decreases along the flow, as expected from the classical motion. Moreover, it allows to handle a large class of objects. However, an important and essential flaw of Brakke's definition is that there is never uniqueness (unless $\mu_*^0 = 0$). Indeed, if $(\mu^t)_{t\geqslant 0}$ is a Brakke flow, so is also $(g(t)\mu^t)_{t\geqslant 0}$, where g is an arbitrary non increasing function on $\mathbb{R}^+$. In particular, the trivial solution given by $\nu^0 = \mu^0$ and $\nu^t \equiv 0$ for $t > 0$ is not excluded a priori. We will call this last solution the instantaneously vanishing solution.

Although non uniqueness is presumably an intrinsic property of mean curvature flows when singularities appear, a major part of non uniqueness in Brakke's formulation is therefore non intrinsic, and allows as shown for weird solutions. A stronger notion of solution will be discussed in Section 7.

6. Relating PGL$_\varepsilon$ to mean curvature flow

We are now able to describe the evolution law for the concentrated part ν_*^t of the measure μ_*^t.

THEOREM 6.1. — *The family $(\nu_*^t)_{t>0}$ is a mean curvature flow in the sense of Brakke.*

Remark 6.2. — Theorem 6.1 asserts that the linear and the topological mode of the energy do not interact.

For $d = 1$, Theorem 6.1 has been proved by Ilmanen in [26]. In case $d = 2$ the proof given in [9] follows a similar strategy, and relies both on the measure theoretic analysis of Ambrosio and Soner [3] and on the analysis of the structure of μ_* given in Theorem 1.1.

The starting point of the analysis is the formal analogy of equality (5.5), namely

$$\frac{d}{dt}\int_{\Sigma^t}\chi(x)\,d\mathcal{H}^k = -\int_{\Sigma^t}\chi(x)|\vec{H}_{\Sigma^t}(x)|^2\,d\mathcal{H}^k + \int_{\Sigma^t}\nabla\chi(x)\cdot\vec{H}_{\Sigma^t}(x)\,d\mathcal{H}^k,$$

with the classical relation describing the evolution of localized energies

$$\frac{d}{dt}\int_{\mathbb{R}^N}\chi(x)\,d\mu_\varepsilon^t = -\int_{\mathbb{R}^N\times\{t\}}\chi(x)\frac{|\partial_t u_\varepsilon|^2}{k_\varepsilon}\,dx + \int_{\mathbb{R}^N\times\{t\}}\nabla\chi(x)\cdot\frac{-\partial_t u_\varepsilon \nabla u_\varepsilon}{k_\varepsilon}\,dx. \tag{6.1}$$

The comparison of the two formulas suggests, at least formally, that in the limit

$$\omega_\varepsilon^t \equiv \frac{|\partial_t u_\varepsilon|^2}{k_\varepsilon}\,dx \to |\vec{H}|^2\,d\mu_*^t, \tag{6.2}$$

and

$$\sigma_\varepsilon^t \equiv \frac{-\partial_t u_\varepsilon \nabla u_\varepsilon}{k_\varepsilon}\,dx \to \vec{H}\,d\mu_*^t. \tag{6.3}$$

Actually, this is a little over optimistic for two reasons. First we have to deal also with the diffuse part of the energy. Second, since (6.2) involves the quadratic term $|\vec{H}|^2$, only lower semi-continuity can be expected at first sight.

Consider first the measure $\sigma_\varepsilon = \sigma_\varepsilon^t\,dt$ defined on $\mathbb{R}^N \times [0,+\infty)$. It is easy to show that σ_ε is uniformly bounded, so that passing possibly to a subsequence $\varepsilon_n \to 0$, we may assume $\sigma_\varepsilon \rightharpoonup \sigma_*$. Moreover, σ_* is absolutely continuous with respect to μ_*. Therefore, we may write

$$\sigma_* = \vec{\mathfrak{h}}\,\mu_*^t\,dt,$$

where $\vec{\mathfrak{h}} \in L^2(\mathbb{R}^N \times [0,T], \mu_*^t\, dt)$. By Theorem 1.1 and the semi-decreasing property, the measure σ_* decomposes as $\sigma_* = \sigma_*^t dt$, where for a.e. $t \geqslant 0$,

$$\sigma_*^t = -\partial_t \Phi_* \cdot \nabla \Phi_*\, dx + \vec{\mathfrak{h}}\, \nu_*^t.$$

The next step will be to identify the restriction of $\vec{\mathfrak{h}}$ on Σ_μ^t with the mean curvature defined by (5.8). For that purpose, we recall a classical formula involving the stress-energy tensor. Let $\vec{X} \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$. We have, for every $t \geqslant 0$,

$$\frac{1}{k_\varepsilon} \int_{\mathbb{R}^N \times \{t\}} \left(e_\varepsilon(u_\varepsilon)\delta_{ij} - \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j} \right) \frac{\partial X^i}{\partial x_j}\, dx = - \int_{\mathbb{R}^N \times \{t\}} \vec{X} \cdot \sigma_\varepsilon^t. \tag{6.4}$$

Formula (6.4) is already very close to (5.8), in particular the right hand side. In order to handle the diffuse energy, as well as to interpret the l.h.s as a tangential divergence, we need to analyse the weak limit of the stress-energy tensor

$$\alpha_\varepsilon^t = \left(Id - \frac{\nabla u_\varepsilon \otimes \nabla u_\varepsilon}{e_\varepsilon(u_\varepsilon)} \right) d\mu_\varepsilon.$$

Clearly, $|\alpha_\varepsilon^t| \leqslant KN\mu_\varepsilon^t$, and we may assume that $\alpha_\varepsilon^t \rightharpoonup \alpha_*^t \equiv A\cdot\mu_*^t$, where A is an $N \times N$ symmetric matrix. Since the symmetric matrix $\nabla u_\varepsilon \otimes \nabla u_\varepsilon$ is non-negative, we have $A \leqslant Id$. On the other hand, $Tr(e_\varepsilon(u_\varepsilon)\, Id - \nabla u_\varepsilon \otimes \nabla u_\varepsilon) = (N-2)e_\varepsilon(u_\varepsilon) + 2\varepsilon^{-2}V(u_\varepsilon)$. Therefore, since the trace is a linear operation, passing to the limit we obtain

$$Tr(A) = (N-2) + 2\frac{dV_*}{d\mu_*}, \tag{6.5}$$

where the measure V_* is the limit (up possibly to a further subsequence) of $V(u_\varepsilon)/(\varepsilon^2 k_\varepsilon)$. Going to the limit in (6.4), and using the decomposition in Theorem 1.1, we obtain for a.e. $t \geqslant 0$,

$$\begin{aligned} \int_{\mathbb{R}^N} A^{ij} \frac{\partial X^i}{\partial x_j}\, d\nu_*^t + \int_{\mathbb{R}^N} \left(\frac{|\nabla \Phi_*|^2}{2}\delta_{ij} - \frac{\partial \Phi_*}{\partial x_i}\frac{\partial \Phi_*}{\partial x_j} \right) \frac{\partial X^i}{\partial x_j}\, dx \\ = - \int_{\mathbb{R}^N} \vec{X} \cdot \vec{\mathfrak{h}}\, d\nu_*^t - \int_{\mathbb{R}^N} \vec{X} \cdot \nabla \Phi_* \partial_t \Phi_*\, dx. \end{aligned} \tag{6.6}$$

On the other hand, Φ_* verifies the heat equation, so that

$$\int_{\mathbb{R}^N} \left(\frac{|\nabla \Phi_*|^2}{2}\delta_{ij} - \frac{\partial \Phi_*}{\partial x_i}\frac{\partial \Phi_*}{\partial x_j} \right) \frac{\partial X^i}{\partial x_j}\, dx = - \int_{\mathbb{R}^N} \vec{X} \cdot \nabla \Phi_* \partial_t \Phi_*\, dx. \tag{6.7}$$

Combining (6.6) and (6.7) we have therefore proved

LEMMA 6.3. — *For a.e. $t \geqslant 0$, and for every $\vec{X} \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} A^{ij} \frac{\partial X^i}{\partial x_j} \, d\nu_*^t = - \int_{\mathbb{R}^N} \vec{X} \cdot \vec{\mathfrak{h}} \, d\nu_*^t. \tag{6.8}$$

Comparing (6.8) with (5.8), in order to identify $\vec{\mathfrak{h}}$ with the mean curvature of ν^t, we merely have to prove that the matrix A corresponds to the orthogonal projection P onto the tangent space $T_x\Sigma_\mu^t$. By a blow-up argument (see [3]), we deduce

LEMMA 6.4. — *For a.e. $t \geqslant 0$,*

$$A(x) \left[\int_{T_x\Sigma_\mu^t} \nabla\chi(y) \, d\mathcal{H}^{N-d}(y) \right] = 0 \qquad \text{for } \mathcal{H}^{N-d}\text{-a.e. } x \in \Sigma_\mu^t, \tag{6.9}$$

and for all $\chi \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$. In particular, $(T_x\Sigma_\mu^t)^\perp \subseteq \operatorname{Ker} A(x)$.

To conclude, one argues differently in case $d = 1$ and $d = 2$. The simplest case is actually $d = 2$ (see [3]). Indeed, a little elementary linear algebra, combining the fact that $A \leqslant Id$ and $Tr(A) \geqslant N - 2$ by (6.5), implies immediately that A is the orthogonal projection onto the tangent space $T_x\Sigma_\mu^t$, for a.e. $t > 0$.

For $d = 1$ the above argument has to be adapted as follows. Since the discrepancy term ξ_ε^t in (3.8) is negative (by [26]), we have in the limit $\varepsilon \to 0$

$$\frac{dV_*}{d\mu_*} \geqslant \frac{1}{2}. \tag{6.10}$$

Therefore, $Tr(A) \geqslant N - 1$ by (6.5), and one argues similarly.

In both cases, this proves that for a.e. $t \geqslant 0$, ν_*^t has a first variation and $\delta\nu_*^t = \vec{\mathfrak{h}}\,\nu_*^t$, i.e. $\vec{\mathfrak{h}}$ is the mean curvature of ν_*^t.

Remark 6.5. — i) For $d = 1$, using (6.5), we deduce that $\frac{dV_*}{d\mu_*} = \frac{1}{2}$, i.e. the energy balance in the limit. For $d = 2$, we deduce similarly $\frac{dV_*}{d\mu_*} = 0$, i.e. in the limit there is only kinetic energy.

ii) Let $(\tau_1, \dots, \tau_N)$ be an orthonormal frame such that $T_x\Sigma_\mu^t$ is spanned by $(\tau_{d+1}, \dots, \tau_N)$. In view of the expression of the stress-energy tensor in these coordinates, we infer that the energy concentrates in the τ_1 direction for $d = 1$, and in the (τ_1, τ_2) plane for $d = 2$, (i.e. $(T_x\Sigma_\mu^t)^\perp$) and uniformly with respect to the direction.

We next turn to the quadratic term ω_ε^t, and try to convince the reader that for a.e. $t \geqslant 0$,

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N\times\{t\}} \chi \frac{|\partial_t u_\varepsilon|^2}{k_\varepsilon} \geqslant \int_{\mathbb{R}^N\times\{t\}} \chi\, |\vec{\mathfrak{h}}|^2 d\nu_*^t + \int_{\mathbb{R}^N\times\{t\}} \chi\, |\partial_t \Phi_*|^2\, dx.$$

It is tempting to write on Σ_μ^t

$$\frac{|\partial_t u_\varepsilon|^2}{k_\varepsilon} \geqslant \frac{|\partial_t u_\varepsilon \nabla u_\varepsilon|^2}{k_\varepsilon |\nabla u_\varepsilon|^2} \geqslant \frac{1}{2}\frac{|\partial_t u_\varepsilon \nabla u_\varepsilon|^2}{e_\varepsilon(u_\varepsilon)}\, \mu_\varepsilon^t \gtrsim \frac{1}{2}|\vec{H}|^2 \mu_*^t.$$

These formal (but essentially correct) inequalities do not allow to conclude, in view of the factor $\frac{1}{2}$. Fortunately, the last inequality is far from being optimal. Indeed, weak convergence does not imply convergence of the squared quantities!

In the scalar case, the balance between the kinetic and potential terms $|\nabla u_\varepsilon|^2 \simeq e_\varepsilon(u_\varepsilon)$ ([26], Section 8.1) restores the "missing" factor $\frac{1}{2}$.

In the complex case, the missing factor $\frac{1}{2}$ is restored in [3] for a different reason, essentially related to Remark 6.5.

7. Enhanced motion

The analysis of $(\mathrm{PGL})_\varepsilon$, running from Section 3 to 6, was based only on energy estimates, and topology never entered directly in the discussion. In particular, we have been able to deduce the motion law for the energy concentration set in Brakke's weak formulation. This obviously tells us also something about the evolution of J_* since $\mathrm{supp} J_*^t \subset \Sigma_\mu^t$. In general it is difficult however to tell something more about J_* without any additional assumption.

In this section we will show that if the energy of initial data is essentially due to the topological part and concentrates on J_*^0 in the sense of (2.17), (2.28), then a stronger notion of evolution can be obtained: in particular instantaneous vanishing will be excluded. Although the improvement concerns again the energy μ_*^t, one may expect to deduce also better informations for J_*^t.

7.1. Instantaneous vanishing for $(\mathrm{PGL})_\varepsilon$

For flows arising from limits of $(\mathrm{PGL})_\varepsilon$ instantaneous vanishing may occur in at leat three distinct cases:

Low density. We present examples for $d = 1$ and $d = 2$.

i) $d = 1$. First consider a non-negative smooth real function defined on $\mathbb{R}_*^+$ with compact support and consider, on $\mathbb{R}^2$, the function $u_\varepsilon^0(r,\theta) = 1 - \eta f(\varepsilon^{-1}(r-1))$. The energy concentrates on the circle S^1 with density μ_ε^0 proportional to η^2.

ii) $d = 2$. Here we work in dimension 3. In the plane (x_1, x_3), consider a standard dipole of two vortices on the x_1-axis, away from the origin and separated by a length ε^η (where $0 < \eta < 1$ is fixed), so that the energy in the plane is of order $\pi\eta|\log\varepsilon|$. More precisely, let

$$w_\varepsilon(z) = f_\varepsilon(z - b_\varepsilon^+)\frac{z - b_\varepsilon^+}{|z - b_\varepsilon^+|} \cdot f_\varepsilon(z - b_\varepsilon^-)\left(\frac{z - b_\varepsilon^-}{|z - b_\varepsilon^-|}\right)^{-1}, \tag{7.1}$$

where $z = (x_1, x_3)$, $b_\varepsilon^\pm = 1 \pm \varepsilon^\eta$. Rotate the dipole along the x_3 axis so that $e_\varepsilon(u_\varepsilon^0)$ concentrates on a circle with a 1-density proportional to η.

In both cases, if η is chosen sufficiently small, then $\mu_*^t \equiv 0$ for $t > 0$ by the Clearing-Out Lemma.

Hidden mean curvature. Consider in the (x_1, x_2) plane the standard circle S^1. Approximate it, weakly in the sense of measures, by a collection $\mathcal{B}_i$ of small circles centered on S^1 and of radii $\sim \frac{1}{i}$. By Theorem 7.4 below, for each $i \in \mathbb{N}_*$ there exist initial data $(u_\varepsilon^{0,i})$ such that the limiting measures $\mu_*^{t,i}$ evolves according to the classical motion of the small circles, whose lifetime is of the order of i^{-2}. By a diagonal argument, it is therefore possible to construct a sequence u_ε^0 such that $\mu_*^0 = S^1$ but $\mu_*^t \equiv 0$ for $t > 0$.

Concentrated gradients of phase or modulus. We discuss only the concentration of phase gradients for the case $d = 2$. Consider an initial data of the form $u_\varepsilon^0 = \exp(i\varphi_\varepsilon^0\sqrt{|\log\varepsilon|})$, where $|\nabla\varphi_\varepsilon^0|^2$ is bounded in L^1 and concentrates on a (N-2)-dimensional set Σ_0. Also in this case we have $\mu_*^t \equiv 0$ for $t > 0$.

Remark 7.1. — The first and the last case are related to the properties of $(\mathrm{PGL})_\varepsilon$ described in previous sections, whereas the second is intrinsically related to motion by mean curvature.

The three cases have a common feature: the defect set of the initial data u_ε^0 converges to zero as ε tends to 0, at least in the sense of distributions.

7.2. Ilmanen enhanced motion

In order to avoid instantaneous vanishing and weird solutions, Ilmanen [27] introduced the notion of enhanced (mean curvature) motion, which we recall now.

Let $\mathcal{M}_0$ be a $(N-d)$-rectifiable current in $\mathbb{R}^N$, without boundary. Assume for simplicity that $\mathcal{M}_0$ has bounded support and is of finite mass. Let $\mathcal{M}$ be a $(N-d+1)$-rectifiable current in $\mathbb{R}^N \times [0,+\infty)$, and $\{\mu^t\}_{t\geqslant 0}$ a family of non-negative Radon measures on $\mathbb{R}^N$.

DEFNITION 7.2 (Enhanced motion). — *The pair $\{\mathcal{M}, \{\mu^t\}_{t\geqslant 0}\}$ is called an enhanced motion with initial condition $\mathcal{M}_0$ if and only if*

i) $\partial\mathcal{M} = \mathcal{M}_0$.

ii) $\mu^0 = \alpha_d|\mathcal{M}_0|$.

iii) The measure defined on $\mathbb{R}^+$ by $\mathcal{T}(B) = |\mathcal{M}|(\mathbb{R}^N \times B)$, for any Borel set B, is absolutely continuous with respect to the Lebesgue measure.

iv) For all $t \geqslant 0$,

$$\mu^t \geqslant \alpha_d|\mathcal{M}_t|,$$

where $\mathcal{M}_t$ denotes the slice of $\mathcal{M}$ at time t.

v) $\{\mu^t\}_{t\geqslant 0}$ is a Brakke flow.

Here $\alpha_1 = 2\sqrt{2}/3$ and $\alpha_2 = \pi$.

Remark 7.3. — Notice in particular that conditions *i)* and *iii)* are closer to what one actually would normally expect from a Cauchy problem. In Ilmanen's terminology, $\mathcal{M}$ is called the under-current, and provides, in view of *iv)*, a lower bound, which rules out sudden shrinking.

In [27], Ilmanen established the existence of an enhanced motion, for any initial data as above (actually in any codimension). Moreover, in the smooth case, there is uniqueness for an enhanced motion (before singularities appear) and it coincides with the classical notion.

The next result provides an alternative construction in codimension 1 and 2 using the asymptotics for $(\mathrm{PGL})_\varepsilon$. We first introduce some additional notation.

For a map $w : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^d$, set $t \equiv x_0$ and define

$$\mathcal{J}w = \nabla_{x,t} w \quad \text{if } d = 1, \qquad \mathcal{J}w = \sum_{0 \leqslant i < j \leqslant N} \partial_i w \times \partial_j w \, dx_i \wedge dx_j \quad \text{if } d = 2.$$

Next consider the solution u_ε of $(\mathrm{PGL})_\varepsilon$ with an initial datum u_ε^0 verifying (H_0).

In view of the energy identity, the space-time Ginzburg-Landau energy is bounded in $\mathbb{R}^N \times [0, T]$, for every $T > 0$ by

$$\int_{\mathbb{R}^N \times [0,T]} \frac{|\nabla_{x,t} u_\varepsilon|^2}{2} + \frac{V(u_\varepsilon)}{\varepsilon^2} \leqslant M_0(T+1)|k_\varepsilon|.$$

From Proposition 2.9 and Proposition 2.16, it follows that

$$\mathcal{J}u_\varepsilon \rightharpoonup \mathcal{J}_* \qquad \text{in } [\mathcal{C}_c^{0,1}(\mathbb{R}^N \times \mathbb{R}^+)]^*.$$

Moreover, $\alpha_d^{-1}\mathcal{J}_*$ is a $(N-d+1)$-rectifiable current, $\mathcal{J}_*^t = J_*^t$ for any $t \geqslant 0$, and $\partial\mathcal{J} = J_*^0$.

THEOREM 7.4. — *Let $\mathcal{M}_0$ be any given $(N-d)$-rectifiable current without boundary, having bounded support and finite mass. Let $(u_\varepsilon^0)_{\varepsilon>0}$ be a sequence such that $\nabla u_\varepsilon \rightharpoonup \alpha_1\mathcal{M}_0$ in case $d = 1$, or such that $Ju_\varepsilon \rightharpoonup \alpha_2\mathcal{M}_0$ in case $d = 2$, and such that (in both cases)*

$$C_d ||J_*|| = \lim_{\varepsilon \to 0} k_\varepsilon^{-1} \mathcal{E}_\varepsilon(u_\varepsilon).$$

Let u_ε be the solution to $(PGL)_\varepsilon$ with initial data u_ε^0 and set $\mathcal{M} = \alpha_d^{-1}\mathcal{J}_*$. Then $\mathcal{M}$ verifies

$$i)\ \partial\mathcal{M} = \mathcal{M}_0, \qquad ii)\ \mu_*^0 = \alpha_d|\mathcal{M}_0|,$$

and the pair $(\mathcal{M}, \mu_*^t)$ is an enhanced motion in the sense of Ilmanen.

Theorem 7.4 has been proved in [26] for the case $d = 1$, and in [9] for the case $d = 2$.

At this stage, the only point in the above result which requires some clarification is the absolute continuity property of $\mathcal{M}$, as stated in Definition 7.2. In fact, in the context of Theorem 7.4, one is able to show a $\mathcal{C}^{0,1/2}$ continuity with respect to the time interval.

Let us briefly sketch the proof in the case $d = 1$. In view of the energy bounds

$$\int_t^{t+\Delta t} \int_{\mathbb{R}^N} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \leqslant \frac{M_0}{\varepsilon} \quad \text{and} \quad \int_t^{t+\Delta t} \int_{\mathbb{R}^N} (1 - |u_\varepsilon|^2)^2 \leqslant \varepsilon M_0 \Delta t,$$

we deduce $\int_t^{t+\Delta t}\int_{\mathbb{R}^N} |\partial_t\zeta(u_\varepsilon)| \leqslant M_0(\Delta t)^{\frac{1}{2}}$, hence

$$
\begin{aligned}
&|\mathcal{M}|\left(\mathbb{R}^N \times [t, t+\Delta t]\right)\\
&\qquad = \frac{1}{2}\int_t^{t+\Delta t}\int_{\mathbb{R}^N} |\nabla_{x,t}u_*| = \frac{3}{2\sqrt{2}}\int_t^{t+\Delta t}\int_{\mathbb{R}^N} |\nabla_{x,t}\zeta(u_*)|\\
&\qquad \leqslant \liminf_{\varepsilon\to 0} \frac{3}{2\sqrt{2}}\int_t^{t+\Delta t}\int_{\mathbb{R}^N} |\nabla_{x,t}\zeta(u_\varepsilon)| \qquad (7.2)\\
&\qquad \leqslant \liminf_{\varepsilon\to 0} \frac{3}{2\sqrt{2}}\int_t^{t+\Delta t}\int_{\mathbb{R}^N} |\nabla\zeta(u_\varepsilon)| + |\partial_t\zeta(u_\varepsilon)|\\
&\qquad \leqslant \frac{3}{2\sqrt{2}}M_0\left(\Delta t + (\Delta t)^{\frac{1}{2}}\right),
\end{aligned}
$$

which yields the desired result. In the case $d = 2$ one argues along the same lines optimizing the Jacobian estimate (2.27) with respect to space and time variables.

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