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A new criterion for knots with free periods [(*)]

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ABSTRACT. — Let $p \geqslant 2$ and $q \neq 0$ an integer. A knot K in the three-sphere is said to be a (p, q)-lens knot if and only if it covers a link in the lens space $L(p, q)$. In this paper, we use the second coefficient of the HOMFLY polynomial to provide a necessary condition for a knot to be a (p, q)-lens knot. As an application, it is shown that this criterion rules out the possibility of being $(5, 1)$-lens for 80 among the 84 knots with less than 9 crossings.

RÉSUMÉ. — Soient $p \geqslant 2$ et $q \neq 0$ deux entiers. Un nœud K de la sphère S^3 est dit (p, q)-lenticulaire s'il est invariant par l'action lenticulaire $\varphi_{p,q}$. Dans ce travail, nous étudions le comportement du polynôme de HOMFLY des nœuds lenticulaires. Nous démontrons que la symétrie lenticulaire est reflétée d'une façon très nette par le second coefficient du polynôme de HOMFLY. Comme application, nous démontrons que 80 parmi les 84 nœuds ayant un nombre de croisements inférieur ou égal à 9, ne sont pas $(5, 1)$-lenticulaires.

1. Introduction

This paper is concerned with the question of whether the symmetry of knots and links in the three-sphere is reflected on the quantum invariants. The symmetry we consider in the present paper is the free periodicity. A link L in S^3 is said to be p-freely periodic ($p \geqslant 2$ an integer) if and only if L is fixed by an orientation preserving action of the finite cyclic group $G = \mathbb{Z}/p\mathbb{Z}$ on the three-sphere without fixed points. It has been conjectured

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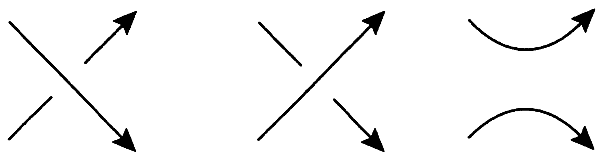
since many decades that such an action is topologically conjugate to an orthogonal action. Consequently, we are going to limit our interest to links which arise as covers of links in the lens space $L(p,q)$. Such a link will be called here a (p,q)-lens link.
The two variable HOMFLY (called also, skein and HOMFLYPT) polynomial is an invariant of ambiant isotopy of oriented links, which generalizes both the Alexander and the Jones polynomials, and can be defined by the following :

$$\begin{array}{ll} (i) & P_{\bigcirc}(v,z) = 1 \\ (ii) & v^{-1}P_{L_+}(v,z) - vP_{L_-}(v,z) = zP_{L_0}(v,z), \end{array}$$

where $\bigcirc$ is the trivial knot, L_+, L_- and L_0 are three oriented links which are identical except near one crossing where they look like in the following figure:



L_+ L_- L_0

Figure 1

It is well known that the HOMFLY polynomial [9] takes its values in the ring $\mathbb{Z}[v^{\pm}, z^{\pm}]$. However, if L is a knot then we have $P_L(v,z) = \sum_{i\geqslant 0} P_{2i,L}(v)z^{2i}$ where $P_{2i,L}$ are elements of $\mathbb{Z}[v^{\pm 2}]$.
Knots with free periods were first studied by Hartly [5] who, motivated by a question of R. Fox, used the Alexander polynomial to provide a criterion for a knot to be freely periodic. The first criteria for periodicity of links using the HOMFLY polynomial is due to Przytycki [10]. In [4], we used the first term of the HOMFLY polynomial to find a necessary condition for a knot to be p-freely periodic, for p prime. This criterion was applied successfully to rule out the possibility of being freely periodic for certain knots. The aim of this paper is to extend this criterion to the second coefficient of the HOMFLY polynomial. Thus we shall prove that similar conditions hold for the polynomial $P_{2,K}(v)$. The proof of our main result is based on the three crucial facts:

- The combinatorial description of lens knots we provided in [1].
- The techniques developed by Traczyk [14] and Yokota [16] in the case of periodic knots and adapted to freely periodic knots in [4].

- The formula for the second term of the HOMFLY polynomial introduced recently by Kanenobu and Miyazawa [8].

An outline of the present paper is as follows. In section 2 we introduce our main results. In section 2, basic properties of freely periodic knots will be summarized. Some properties of the HOMFLY polynomial, needed in the rest of the paper, are given in section 4. In section 5, we shall prove Theorem 2.1. In the last paragraph, our criterion is applied, in the case $p = 5$, to knots with less than 9 crossings.

2. Results and applications

Let p be a prime and $\mathbb{F}_p$ be the cyclic finite field of p elements. Throughout the rest of our paper we denote by $P_{2,K}(v)_p$ the second term of the HOMFLY polynomial considered with coefficients reduced modulo p. If m and n are two integers then $T(n,m)$ denotes the torus link of type (n,m). Recall here that the number of components of $T(n,m)$ is equal to $\gcd(n,m)$. In particular $T(n,m)$ is a knot if and only if n and m are coprime.

THEOREM 2.1. — *Let $p > 3$ be a prime, $q = \pm 1$ and K a (p,q)-lens knot. Then $P_{2,K}(v)_p \in \Gamma_{p,q}$, where $\Gamma_{p,q}$ is the $\mathbb{F}_p[v^{\pm 2p}]$-module generated by $P_{2,T(\alpha,\alpha q \pm p)}(v)_p$ for all $1 \leqslant \alpha \leqslant p-1$.*

This result is more significant for small values of p. Indeed, for such values the generators of the module $\Gamma_{p,q}$ are easily computed using the formula given by V. Jones [7] for the HOMFLY polynomial of torus knots. This fact is illustrated by the following corollary:

COROLLARY 2.2. — *Let $q = \pm 1$ and K be a $(5,q)$-lens knot. Then $P_{2,K}(v)_5 \in \mathbb{F}_5[v^{\pm 10}]$-module generated by v^{q8}.*

Proof of Corollary 2.2. — According to Theorem 2.1, the generators of $\Gamma_{5,1}$ are given by $P_{2,T(\alpha,\alpha\pm 5)}(v)_5$ for $1 \leqslant \alpha \leqslant 4$. We use the formula given in section 4 to compute the HOMFLY polynomial of torus knots. These generators are given by the list below :

$$\begin{array}{ll}
P_{2,T(1,6)}(v)_5 = 1, & P_{2,T(1,-4)}(v)_5 = 1, \\
P_{2,T(2,7)}(v)_5 = 10v^6 - 4v^8, & P_{2,T(2,-3)}(v)_5 = v^{-2}, \\
P_{2,T(3,8)}(v)_5 = 105v^{14} - 21v^8 - 105v^6, & P_{2,T(3,-2)}(v)_5 = v^{-2}, \\
P_{2,T(4,9)}(v)_5 = 770v^{24} - 1210v^{26} - 70v^{30} + 56v^{28}, & P_{2,T(4,-1)}(v)_5 = 1.
\end{array}$$

A similar computation can be easily made in the case $q = -1$.

Remark 2.3. — In the case $p = 7$, the module $\Gamma_{7,1}$ is generated by the two elements: $2v^6 + 3v^8$ and $6v^8 + 4v^{10}$. Consequently the module $\Gamma_{7,-1}$ is generated by $2v^{-6} + 3v^{-8}$ and $6v^{-8} + 4v^{-10}$.

Application. — Corollary 2.2 provides a criterion for a knot of the three-sphere to be fixed by the lens transformation $\varphi_{5,\pm 1}$. Hence, given a knot K, if the polynomial $P_{2,K}(v)_5$ does not satisfy the condition given by Corollary 2.2 then K is not a $(5, \pm 1)$-lens knot. Let us illustrate this by considering the knot $K = 8_{13}$. According to the table in [9] we have

$$P_{2,8_{13}}(v)_5 = v^{-2} - 1 - 2v^2 + v^4.$$

As $P_{2,8_{13}}(v)_5$ is not in the $\mathbb{F}_5[v^{\pm 10}]$-module generated by v^8. Then $K = 8_{10}$ is not a $(5, 1)$-lens knot. It is worth mentioning that the criterion we introduced in [4] using the first coefficient of the HOMFLY polynomial does not decide in the case of the knot $K = 8_{13}$. Thus, Corollary 2.2 is not a consequence of the results we introduced in [4]. More applications are given in the last section of this paper.

3. Freely periodic links

Symmetry of knots and links is a vast subject that has fascinated researchers since the early age of knot theory. Problems as chirality and invertibility have motivated classical knot theory for a long time. Roughly, a knot in S^3 is said to be symmetric if and only if K is fixed by an action of a finite cyclic group on S^3. According to the set of fixed points of the action, we can distinguish many kinds of symmetry. In this section we focus on the case where the action has no fixed points. We define freely periodic knots then we review some basic properties of this family of knots and links.

DEFINITION 3.1. — *Let $p \geqslant 2$ be an integer. A link L in S^3 is said to be p-freely periodic if and only if there exists an orientation preserving diffeomorphism $h : S^3 \longrightarrow S^3$ such that:*

1) h^i has no fixed points for all $1 \leqslant i \leqslant p - 1$,

2) $h^p = Id_{S^3}$,

3) $h(L) = L$.

Example 3.2. — Let L be the torus knot $T(2, 5)$. It is well known that L can be seen as the intersection between an appropriate three-sphere and the complex surface defined by :

$$\Sigma = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C}; z_1^2 + z_2^5 = 0\}.$$

Let us consider the diffeomorphism :

$$\begin{array}{rccc} h: & S^3 & \longrightarrow & S^3 \\ & (z_1, z_2) & \longmapsto & (e^{\frac{2i\pi}{3}} z_1, e^{\frac{2i\pi}{3}} z_2). \end{array}$$

Obviously h satisfies conditions 1 and 2 of Definition 3.1. Moreover, one may easily check that $h(L) = L$. Thus L is a freely periodic knot with period 3.

Remark 3.3. — Let $p \geqslant 2$ and q an integer such that $\gcd(p, q) = 1$. Consider $\varphi_{p,q}$ the diffeomorphism given by:

$$\begin{array}{rccc} \varphi_{p,q}: & S^3 & \longrightarrow & S^3 \\ & (z_1, z_2) & \longmapsto & (e^{\frac{2i\pi}{p}} z_1, e^{\frac{2iq\pi}{p}} z_2). \end{array}$$

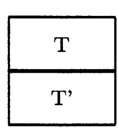
It is easy to see that $\varphi_{p,q}$ is an orientation preserving diffeomorphism of order p and that $\varphi_{p,q}$ has no fixed point. Moreover, we have a p-fold cyclic covering $(\pi_{p,q}, S^3, L(p, q))$.

DEFINITION 3.4. — *Let $p \geqslant 2$ and q an integer such that $\gcd(p, q) = 1$. A link L of S^3 is said to be a (p, q)-lens link if and only if L is mapped onto itself by $\varphi_{p,q}$.*

It is worth mentioning that lens links are the only examples we know of freely periodic links. More precisely we have the following conjecture proved for $p = 2$ and 3.

CONJECTURE 3.5 [13]. — *Let p be a prime and $h : S^3 \longrightarrow S^3$ an orientation preserving diffeomorphim of order p such that for all $1 \leqslant i \leqslant p - 1$, h^i has no fixed points. Then there exists an integer q such that h is topologically conjugate to $\varphi_{p,q}$.*

Let $n \geqslant 1$ be an integer. An n-tangle T is a submanifold of dimension one in $\mathbb{R}^2 \times I$ such that the boundary of T is made up of $2n$ points $\{A_1, \ldots, A_n\} \times \{0, 1\}$. If T and T' are two n-tangles we define the product TT' by putting T over T' as follows:



As in the case of braids we define the closure of T and we denote by $\widehat{T}$ the link obtained from T by joining $A_i \times 1$ to $A_i \times 0$ by a simple arc without adding any crossing. Throughout the rest of this paper $\mathcal{B}_n$ denotes the n-string braid group. It is well known that this group has the following presentation:

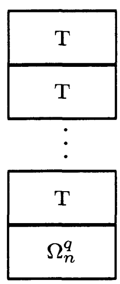
$$B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geqslant 2$$

$$\text{and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \forall 1 \leqslant i \leqslant n-2\rangle.$$

For $n > 2$, the group $\mathcal{B}_n$ is not abelian. Its center is known to be generated by the element $\Omega_n = (\sigma_1\sigma_2 ... \sigma_{n-1})^n$. The following theorem gives a combinatorial description of lens links.

THEOREM 3.6. [1].— *A link K of S^3 is a (p,q)-lens link if and only if there exists an integer $n \neq 0$ and an n-tangle T such that:*

$$K = \widehat{T^p(\sigma_1\sigma_2 ... \sigma_{n-1})^{nq}}.$$



Remark 3.7. — Let n and m be two integers. The torus link $T(n,m)$ is the closure of the braid $(\sigma_1\sigma_2 ... \sigma_{n-1})^m$. Using elementary techniques we can prove that $T(n,m)$ is a (p,q)-lens link if and only if p divides $m - nq$.

4. The HOMFLY polynomial

The discovery of the Jones polynomial [7] led to a significant progress in knot theory. The Jones polynomial was followed by a family of invariants of knots and three-manifolds called the quantum invariants. Among this family of invariants the HOMFLY polynomial which is an invariant of ambiant isotopy of oriented links. This invariant is a two-variable Laurent polynomial which can be seen as a generalization of the Jones and the Alexander polynomial. This section is to review some properties of this polynomial needed in the sequel. At the beginning let us fix some notations.

Let $L = l_1 \cup l_2 \cup \ldots \cup l_n$ be an n-component link of the three-sphere. Throughout the rest of this paper $\lambda_{i,j}$ denotes the linking number of the two components l_i and l_j and λ denotes the total linking number of the link L. It is well known that the HOMFLY polynomial takes values in the ring $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$. Moreover, we can write $P_L(v,z) = \sum_{i \geqslant 0} P_{1-n+2i,L}(v) z^{1-n+2i}$ where $P_{1-n+2i,L} \in \mathbb{Z}[v^{\pm 2}]$ if n is odd and $P_{1-n+2i,L} \in \mathbb{Z}[v^{\pm 1}]$ if n is even.

PROPOSITION 4.1 [9]. — *Let $L = l_1 \cup l_2 \cup \ldots \cup l_n$ be an n-component link then:*

$$P_{1-n,L}(v) = v^{2\lambda}(v^{-1} - v)^{n-1} \prod_{i=1}^{n} P_{0,l_i}(v).$$

Motivated by this proposition, Kanenobu and Miyazawa [8] introduced a similar formula for the polynomial $P_{3-n,L}$.

THEOREM 4.2 [8]. — *Let $n \geqslant 3$ be an integer and $L = l_1 \cup l_2 \cup \ldots \cup l_n$ an n-component link then:*

$$\begin{aligned} P_{3-n,L}(v) = \quad & v^{2\lambda}(v^{-1} - v)^{n-2} \sum_{i<j} (v^{-2\lambda_{i,j}} P_{1,L_{i,j}}(v) \prod_{k \neq i,j} P_{0,L_k}(v)) \\ & -(n-2) v^{2\lambda}(v^{-1} - v)^{n-1} \sum_{i=1}^{n} (P_{2,l_i}(v) \prod_{j \neq i} P_{0,l_j}(v)), \end{aligned}$$

where $L_{i,j}$ denotes the 2-component link $l_i \cup l_j$.

The HOMFLY polynomial of torus knots was computed by V. Jones. To introduce the Jones formula, we find it more convenient to use the polynomial $X_L(q,t)$. This is a version of the HOMFLY polynomial related to $P_L(v,z)$ by the variable changes: $z = q^{1/2} - q^{-1/2}$ and $v = (tq)^{1/2}$. Let $k \geqslant 1$ be an integer, we define: $[k]! = (1-q)(1-q^2)...(1-q^k)$ and $[\bar{k}] = \dfrac{1-q^k}{1-q}$.

THEOREM 4.3 [7]. — *For the torus knot $T(n,m)$ we have:*

$$X_{T(n,m)}(q,t) = \frac{t^{(n-1)(m-1)/2}}{[\bar{n}](1-tq)} \sum_{\substack{\gamma+\beta+1=n \\ \gamma \geqslant 0, \beta \geqslant 0}} (-1)^{\beta} \frac{q^{\frac{2\beta m + \gamma(\gamma+1)}{2}}}{[\gamma]![\beta]!} \prod_{i=-\gamma}^{\beta} (q^i - \lambda q).$$

5. Proof of Theorem 2.1

Most of the techniques used in this section were first developed by Traczyk [14] to study the HOMFLY polynomial of periodic knots (in some sens this class of knots corresponds to the $(p, 0)$-lens knots). In this section we aim to adapt these techniques to the case of freely periodic knots. This will be done in two steps. In the first one, we prove that P_D belongs to $\Gamma'_{p,q}$, where $\Gamma'_{p,q}$ is the $\mathbb{F}_p[v^{\pm 2p}]$-module generated by the polynomials of torus knots $T(n, nq+p)$. The second step explains how to extract a finite set of generators for $\Gamma'_{p,q}$. Let us fix some notations. By T_+, T_- and T_0, we denote three tangles which are identical except near one crossing where they look like in figure 1. By D_+ (respectively D_-, D_0) we denote a diagram of the (p, q)-lens link $D_+ = \widehat{T_+^p \Omega_n^q}$ (respectively $D_- = \widehat{T_-^p \Omega_n^q}$ and $D_0 = \widehat{T_0^p \Omega_n^q})$). It is worth mentioning that if D_+ is a knot then D_- is also a knot. However, D_0 is a link with 2 or $p+1$ components. In the case D_0 has $p+1$ components $D_1 \cup D_2 \cup \ldots \cup D_{p+1}$, then one component (say D_1) is invariant by $\varphi_{p,q}$, the others are cyclically permuted by $\varphi_{p,q}$. We shall prove by induction on the number of crossings of D, that $P_{2,D} \in \Gamma'_{p,q}$. Let D be a (p, q)-lens diagram. Assume that for all (p, q)-lens diagram D' with less crossings than D we have $P_{2,D'} \in \Gamma_{p,q}$. In [4], the following lemma was proved:

Lemma 5.1. — *Let p be a prime. The following congruence holds modulo p:*

$$v^{-p} P_{D_+}(v, z) - v^p P_{D_-}(v, z) \equiv z^p P_{D_0}(v, z).$$

Proposition 5.2. — *Let $p \geqslant 5$ be a prime.*

i) If D_0 has two components then:

$$v^{-p} P_{2,D_+}(v)_p - v^p P_{2,D_-}(v)_p = 0$$

ii) If D_0 has $p+1$ components then:

$$v^{-p} P_{2,D_+}(v)_p - v^p P_{2,D_-}(v)_p = v^{2\lambda}(v^{-1} - v)^p P_{2,D_1}(v)(P_{0,D_2}(v))^p.$$

Proof. — According to Lemma 5.1 we have the following congruence modulo p:

$$v^{-p} P_{2,D_+}(v)_p - v^p P_{2,D_-}(v)_p = P_{3-(p+1),D_0}(v).$$

Obviously, if D_0 has two components then $P_{3-(p+1),D_0}$ is zero. Assume now that D_0 has $p+1$ components D_1,D_2,...,D_{p+1}. Let we denote by $D_{i,j}$ the two-component link $D_i \cup D_j$ and define G and H as follows:

$$G(v) = \quad v^{2\lambda}(v^{-1}-v)^{p-1}\sum_{i<j}(v^{-2\lambda_{i,j}}P_{1,D_{i,j}}(v)\prod_{k\neq i,j}P_{0,D_k}(v)),$$

$$H(v) = \quad -(p-1)v^{2\lambda}(v^{-1}-v)^{p}\sum_{i=1}^{p+1}(P_{2,D_i}(v)\prod_{j\neq i}P_{0,D_j}(v)).$$

It can be easily seen from Theorem 4.2 that :

$$v^{-p}P_{2,D_+}(v)_p - v^pP_{2,D_-}(v)_p \equiv G(v)+H(v).$$

By the fact that components D_2,D_3,...,D_{p+1} are identical and cyclically permuted by the action of $\mathbb{Z}/p\mathbb{Z}$ we can write:

$$G(v) = v^{2\lambda}(v^{-1}-v)^{p-1}(\sum_{j=2}^{p+1}v^{-2\lambda_{1,j}}P_{1,D_{1,j}}(v)\prod_{k=2,k\neq j}^{p+1}P_{0,D_k}(v)$$
$$+\sum_{1<i<j}(v^{-2\lambda_{i,j}}P_{1,D_{i,j}}(v)\prod_{k\neq i,j}P_{0,D_k}(v)).$$

Using the fact that $\lambda_{1,2}=\lambda_{1,j}$ and that $D_{1,2}=D_{1,j}$ for all $2\leqslant j\leqslant p+1$, we get:

$$\sum_{j=2}^{p+1}v^{-2\lambda_{1,j}}P_{1,D_{1,j}}(v)\prod_{k=2,k\neq j}^{p+1}P_{0,D_k}(v) = pv^{-2\lambda_{1,2}}P_{1,D_{1,2}}(v)(P_{0,D_2}(v))^{p-1}.$$

On the other hand :

$$\begin{aligned}\sum_{1<i<j}(v^{-2\lambda_{i,j}}P_{1,D_{i,j}}(v)\prod_{k\neq i,j}P_{0,D_k}(v))&\\ &=\sum_{1<i<j}(v^{-2\lambda_{i,j}}P_{1,D_{i,j}}(v)P_{0,D_1}(v)(P_{0,D_2}(v))^{p-2}\\ &=P_{0,D_1}(v)(P_{0,D_2}(v))^{p-2}\sum_{1<i<j}v^{-2\lambda_{i,j}}P_{1,D_{i,j}}(v).\end{aligned}$$

One may check easily that $\sum_{1<i<j}v^{-2\lambda_{i,j}}P_{1,D_{i,j}}(v)\equiv 0$ modulo p. Thus G(v) is zero modulo p. A similar computation shows that:

$$\begin{aligned}\sum_{i=1}^{p+1}(P_{2,D_i}(v)\prod_{j\neq i}P_{0,D_j}(v))&\\ &=P_{2,D_1}(v)(P_{0,D_2}(v))^p+\sum_{i=2}^{p+1}P_{2,D_2}(v)P_{0,D_1}(v)(P_{0,D_2}(v))^{p-2}\\ &\equiv P_{2,D_1}(v)(P_{0,D_2}(v))^p \mod p.\end{aligned}$$

Therefore: $H(v) \equiv v^{2\lambda}(v^{-1} - v)^p P_{2,D_1}(v)(P_{0,D_2}(v))^p$ modulo p. This ends the proof of Proposition 5.2.

LEMMA 5.3. — *$P_{2,D_+} \in \Gamma'_{p,q}$ if and only if $P_{2,D_-} \in \Gamma'_{p,q}$.*

Proof. — It is easy to see that the result is true in the case D_0 has two components. If D_0 has $p+1$ components D_1, D_2, ..., D_{p+1}, then D_1 is a (p,q)-lens diagram with less crossings than D_0. Thus $P_{2,D_1} \in \Gamma'_{p,q}$ by the induction assumption. Moreover, an easy computation shows that p divides the total linking number λ. According to Proposition 5.2 we have:

$$v^{-p}P_{2,D_+}(v)_p - v^p P_{2,D_-}(v)_p = v^{2\lambda}(v^{-1} - v)^p P_{2,D_1}(v)(P_{0,D_2}(v))^p.$$

Obviously, $v^{2\lambda}(v^{-1} - v)^p (P_{0,D_2}(v))^p$ belongs to $\Gamma'_{p,q}$. Therefore, the second term in the previous identity belongs to the module $\Gamma'_{p,q}$. Consequently, $P_{2,D_+} \in \Gamma'_{p,q}$ if and only if $P_{2,D_-} \in \Gamma'_{p,q}$.

Notation. — Throughout the rest of this paper, we denote by $D_+ \leftrightarrow D_-$ the operation that consists of modifying p crossings to transform the diagram D_+ into the diagram D_- or vice-versa.

The following two lemmas, explain how to use the operation $D_+ \leftrightarrow D_-$ to transform a lens diagram D into a torus knot. Details about these techniques can be found in [4].

LEMMA 5.4. — *Every (p,q)-lens diagram may be transformed into a (p,q)-lens closed braid by a series of operations $D_+ \leftrightarrow D_-$ without increasing the number of crossings.*

LEMMA 5.5. — *Let B be an n-braid. The $(p,q)-$lens braid $B^p\Omega_n^q$ may be transformed into the torus knot $T(n, nq+p)$ by a series of operations $D_+ \leftrightarrow D_-$.*

It remains now to explain how to extract a finite set of generators for the module $\Gamma'_{p,q}$. Our approach here will be based on some combinatorial elementary properties of torus knots. Namely, we shall adapt the $D_+ \leftrightarrow D_-$ operation to diagrams of torus knots. Therefore, an easy induction will end the proof of Theorem 2.1. We refer the reader to [4] for more details.

6. More applications

This section is devoted to some applications of Theorem 2.1. As explained earlier in this paper, we can use the criterion Theorem 2.1 provides to decide if a knot K is not a (p,q)-lens knot. Let us first recall that in [4], we introduced a criterion for free periodicity using the first coefficient of the HOMFLY polynomial $P_{0,K}$ (this criterion will be called the P_0-criterion). In the case $p = 5$, this criterion writes as follows:

The P_0-criterion

If K is a (5,1)-lens knot, then $P_K = \sum a_{2i}v^{2i}$ *with* $a_{10k+4} = 2a_{10k+2}$ *and* $a_{10k+6} = 2a_{10k+8}$ *for all* $k \in \mathbb{Z}$.

Our aim here is to understand how powerful is the criterion introduced in section 2 in detecting free periodicity. Namely, we shall compare the condition obtained in the present paper to the P_0-criterion. To do, let us apply both of them to the 84 knots with less than 9 crossings. This is explained in the following table where:

The first column gives the prime knot according to the notations used in [12].

The second column (resp. the third) provides informations about the P_0-criterion (resp. the P_2-criterion introduced by Corollary 2.2) as follows:

D means that the criterion decides that the knot is not a (5,1)-lens knot.

ND means that the criterion does not decide that the knot is not a (5,1)-lens knot.

Knot	P_0 − criterion	P_2 − criterion	*Knot*	P_0 − criterion	P_2 − criterion
3_1	*ND*	*D*	9_8	*D*	*D*
4_1	*D*	*ND*	9_9	*ND*	*D*
5_1	*D*	*D*	9_{10}	*ND*	*D*
5_2	*D*	*D*	9_{11}	*D*	*D*
6_1	*D*	*D*	9_{12}	*D*	*D*
6_2	*ND*	*D*	9_{13}	*ND*	*D*
6_3	*D*	*D*	9_{14}	*D*	*D*
7_1	*ND*	*ND*	9_{15}	*D*	*D*
7_2	*D*	*D*	9_{16}	*ND*	*D*
7_3	*D*	*D*	9_{17}	*D*	*D*
7_4	*D*	*D*	9_{18}	*D*	*D*
7_5	*D*	*D*	9_{19}	*D*	*D*
7_6	*D*	*D*	9_{20}	*D*	*D*
7_7	*ND*	*D*	9_{21}	*D*	*D*
8_1	*D*	*D*	9_{22}	*D*	*D*
8_2	*D*	*D*	9_{23}	*D*	*D*
8_3	*D*	*D*	9_{24}	*D*	*D*
8_4	*D*	*D*	9_{25}	*ND*	*D*
8_5	*D*	*D*	9_{26}	*D*	*D*
8_6	*D*	*D*	9_{27}	*D*	*D*

Knot	P_0 − criterion	P_2 − criterion	*Knot*	P_0 − criterion	P_2 − criterion
8_7	D	D	9_{28}	D	D
8_8	D	D	9_{29}	D	D
8_9	D	D	9_{30}	D	D
8_{10}	D	D	9_{31}	ND	D
8_{11}	D	D	9_{32}	D	D
8_{12}	D	D	9_{33}	ND	D
8_{13}	ND	D	9_{34}	D	D
8_{14}	ND	D	9_{35}	ND	D
8_{15}	D	D	9_{36}	D	D
8_{16}	ND	D	9_{37}	D	D
8_{17}	D	D	9_{38}	ND	D
8_{18}	D	D	9_{39}	D	D
8_{19}	ND	ND	9_{40}	ND	D
8_{20}	ND	D	9_{41}	D	D
8_{21}	D	D	9_{42}	D	D
9_1	ND	ND	9_{43}	ND	D
9_2	D	D	9_{44}	D	D
9_3	ND	D	9_{45}	D	D
9_4	D	D	9_{46}	D	D
9_5	D	D	9_{47}	D	D
9_6	ND	D	9_{48}	D	D
9_7	D	D	9_{49}	ND	D

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