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Dispersive limits in the homogenization of the wave equation (*)

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ABSTRACT. — We address the homogenization of a scalar wave equation with a large potential in a periodic medium (sometime called the Klein-Gordon equation). Denoting by ϵ the period, the potential is scaled as ϵ^{-2}. The homogenized limit depends on the sign of the first cell eigenvalue λ_1. If $\lambda_1 = 0$, then the homogenized problem is a standard wave equation. If $\lambda_1 \neq 0$, then, upon changing the time scale to focus on large times of order ϵ^{-1}, we obtain dispersive homogenized problems, i.e. equations which are not of the second order in time. If $\lambda_1 < 0$, the homogenized equation is parabolic, while for $\lambda_1 > 0$, the homogenized equation is of Schrödinger type.

RÉSUMÉ. — Nous étudions l'homogénéisation d'une équation scalaire des ondes avec un fort potentiel dans un milieu périodique (ou équation de Klein-Gordon). Si l'on désigne par ϵ la période, le potentiel est de l'ordre de ϵ^{-2}. Le comportement homogénéisé limite dépend du signe de la première valeur propre λ_1 du problème de cellule. Si $\lambda_1 = 0$, alors le problème homogénéisé est une équation des ondes usuelle. Si $\lambda_1 \neq 0$, alors, sous réserve de changer l'échelle de temps afin d'observer les grands temps d'ordre ϵ^{-1}, on obtient des limites dispersives, c'est-à-dire des équations homogénéisées qui ne sont pas du deuxième ordre en temps. Si $\lambda_1 < 0$, l'équation homogénéisée est parabolique, tandis que si $\lambda_1 > 0$, l'équation homogénéisée est du type de Schrödinger.

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1. Introduction

We study the homogenization of a scalar wave equation with a large potential (the so-called Klein-Gordon equation) and periodically oscillating coefficients

$$\begin{cases} \dfrac{\partial^2 u_\epsilon}{\partial t^2} - \operatorname{div}\left(A\left(\dfrac{x}{\epsilon}\right)\nabla u_\epsilon\right) + \left(\epsilon^{-2}c\left(\dfrac{x}{\epsilon}\right) + d\left(x,\dfrac{x}{\epsilon}\right)\right)u_\epsilon & \\ \qquad = 0 & \text{in } \Omega\times(0,T), \\ u_\epsilon = 0 & \text{on } \partial\Omega\times(0,T), \\ u_\epsilon(t=0,x) = u_\epsilon^0(x) & \text{in } \Omega, \\ \dfrac{\partial u_\epsilon}{\partial t}(t=0,x) = u_\epsilon^1(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open set and $T > 0$ a final time. The potential term in (1.1), i.e. the zero-order term, is used to model some repelling or attracting effects like springs attaching an elastic membrane to a fixed support. The coefficients $A(y)$, $c(y)$ and $d(x,y)$ are real and bounded functions defined for $x \in \Omega$ and $y \in \mathbb{T}^N$ (the unit torus). More precisely, the entries of $A(y)$ and $c(y)$ belongs to $L^\infty(\mathbb{T}^N)$, while $d(x,y)$ is a Carathéodory function in $L^\infty(\Omega; C(\mathbb{T}^N))$. Furthermore, the matrix $A(y)$ is symmetric, uniformly positive definite, $d(x,y) \geqslant 0$ is non-negative while $c(y)$ does not satisfy any positivity assumption. Throughout this paper we assume that the initial data are $u_\epsilon^0 \in H_0^1(\Omega)$ and $u_\epsilon^1 \in L^2(\Omega)$, so there exists a unique solution $u_\epsilon \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$.

Of course, if there is no large potential term, namely if $c \equiv 0$, the homogenization of (1.1) is classical (see e.g. [5], [6], [7], [14]). When $c \neq 0$ it is a more difficult problem of homogenization mixed with singular perturbations. Nevertheless, the parabolic or elliptic version of (1.1), as well as the corresponding eigenvalue problem, are well understood (see e.g. [2], [3], [4], [10], [16]). However, the methods of these articles do not apply to the wave equation (1.1). In order to show the differences, we first describe the main results and ideas of these previous works in the following parabolic case

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} - \operatorname{div}\left(A\left(\dfrac{x}{\epsilon}\right)\nabla u_\epsilon\right) + \left(\epsilon^{-2}c\left(\dfrac{x}{\epsilon}\right) + d\left(x,\dfrac{x}{\epsilon}\right)\right)u_\epsilon & \\ \qquad = 0 & \text{in } \Omega\times(0,T), \\ u_\epsilon = 0 & \text{on } \partial\Omega\times(0,T), \\ u_\epsilon(t=0,x) = u_\epsilon^0(x) & \text{in } \Omega. \end{cases} \tag{1.2}$$

Introduce the first eigencouple of the spectral cell problem

$$-\operatorname{div}_y\left(A(y)\nabla_y\psi_1\right) + c(y)\psi_1 = \lambda_1\psi_1 \quad \text{in } \mathbb{T}^N, \tag{1.3}$$

which, by the Krein-Rutman theorem, is simple and satisfies $\psi_1(y) > 0$ in $\mathbb{T}^N$ (recall it is a scalar problem). By standard regularity results, the coefficients of (1.3) belonging to $L^\infty(\mathbb{T}^N)$, the eigenfunction ψ_1 is at least continuous, so it is uniformly bounded away from 0 on the compact set $\mathbb{T}^N$. As usual we normalize the eigenfunction by assuming $\int_{\mathbb{T}^N} \psi_1(y)^2 dy = 1$. The first eigenvalue λ_1 is interpreted as a measure of the balance between diffusion and reaction caused by the potential term. Then, one can change the unknown by writing a so-called factorization principle

$$v_\epsilon(t,x) = e^{\frac{\lambda_1 t}{\epsilon^2}} \frac{u_\epsilon(t,x)}{\psi_1\left(\frac{x}{\epsilon}\right)}, \tag{1.4}$$

and check easily after some algebra (see the proof of Lemma 2.1) that the new unknown v_ϵ is a solution of a simpler equation

$$\left\{\begin{array}{lr} |\psi_1|^2\left(\frac{x}{\epsilon}\right)\frac{\partial v_\epsilon}{\partial t} - \operatorname{div}\left((|\psi_1|^2 A)\left(\frac{x}{\epsilon}\right)\nabla v_\epsilon\right) & \\ \qquad\qquad + (|\psi_1|^2 d)\left(x,\frac{x}{\epsilon}\right) v_\epsilon = 0 & \text{in } \Omega\times(0,T), \\ v_\epsilon = 0 & \text{on } \partial\Omega\times(0,T), \\ v_\epsilon(t=0,x) = \dfrac{u_\epsilon^0(x)}{\psi_1\left(\frac{x}{\epsilon}\right)} & \text{in } \Omega. \end{array}\right. \tag{1.5}$$

The new parabolic equation (1.5) is simple to homogenize since it does not contain any singularly perturbed term, and we thus obtain the following result.

THEOREM 1.1. — *Consider the scalar parabolic problem (1.2). The new unknown v_ϵ, defined by (1.4), converges weakly in $L^2\left((0,T);H_0^1(\Omega)\right)$ to the solution v of the following homogenized problem*

$$\left\{\begin{array}{ll} \dfrac{\partial v}{\partial t} - \operatorname{div}(A^*\nabla v) + d^*(x)\,v = 0 & \text{in } \Omega\times(0,T), \\ v = 0 & \text{on } \partial\Omega\times(0,T), \\ v(t=0,x) = v^0(x) & \text{in } \Omega, \end{array}\right. \tag{1.6}$$

where $d^*(x) = \int_{\mathbb{T}^N} d(x,y)|\psi_1|^2(y)\,dy$, A^* is the classical homogenized matrix of $(|\psi_1|^2 A)$ (see formula (4.13)), and v^0 is the weak limit in $L^2(\Omega)$ of $u_\epsilon^0(x)\psi_1\left(\frac{x}{\epsilon}\right)$.

It is clear from the above brief summary of the parabolic case that the main idea, namely the factorization principle (1.4), is not going to work in the hyperbolic case without some improvement. Let us try a naive adaptation of this idea to convince the reader. Since (1.1) is of second order in time, the analogous time renormalization of the unknown is

$$v_\epsilon(t,x) = e^{-i\frac{\sqrt{\lambda_1}t}{\epsilon}} \frac{u_\epsilon(t,x)}{\psi_1\left(\frac{x}{\epsilon}\right)}, \tag{1.7}$$

where i is the square root of -1 and $\sqrt{\lambda_1}$ is possibly imaginary if $\lambda_1 < 0$. After some algebra we obtain that v_ϵ is a solution of

$$\begin{cases} |\psi_1|^2\left(\frac{x}{\epsilon}\right)\left(\frac{\partial^2 v_\epsilon}{\partial t^2} + 2i\frac{\sqrt{\lambda_1}}{\epsilon}\frac{\partial v_\epsilon}{\partial t}\right) & \\ \quad - \operatorname{div}\left((|\psi_1|^2 A)\left(\frac{x}{\epsilon}\right)\nabla v_\epsilon\right) + (|\psi_1|^2 d)\left(x,\frac{x}{\epsilon}\right) v_\epsilon = 0 & \text{in } \Omega\times(0,T), \\ v_\epsilon = 0 & \text{on } \partial\Omega\times(0,T), \\ v_\epsilon(t=0,x) = u_\epsilon^0(x)/\psi_1\left(\frac{x}{\epsilon}\right) & \text{in } \Omega, \\ \frac{\partial v_\epsilon}{\partial t}(t=0,x) = \left(u_\epsilon^1(x) - i\frac{\sqrt{\lambda_1}}{\epsilon}u_\epsilon^0(x)\right)/\psi_1\left(\frac{x}{\epsilon}\right) & \text{in } \Omega. \end{cases} \tag{1.8}$$

There is an additional difficulty in (1.8), compared to (1.5), which is the very large first-order time derivative. Actually it is not possible to pass to the limit in (1.8) (or to obtain uniform a priori estimates) because of this term which scales as ϵ^{-1}, except if $\lambda_1 = 0$, of course.

Therefore, the only obvious case in the homogenization of the wave equation (1.1) occurs when $\lambda_1 = 0$ (it is treated in section 2). The main new idea to treat the remaining cases $\lambda_1 \neq 0$ is to scale the time variable, i.e. to look at large times of order ϵ^{-1}. In other words, we replace the original wave equation (1.1) by the following rescaled version

$$\begin{cases} \epsilon^2\frac{\partial^2 u_\epsilon}{\partial t^2} - \operatorname{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) & \\ \qquad + \left(\epsilon^{-2}c\left(\frac{x}{\epsilon}\right) + d\left(x,\frac{x}{\epsilon}\right)\right)u_\epsilon = 0 & \text{in } \Omega\times(0,T), \\ u_\epsilon = 0 & \text{on } \partial\Omega\times(0,T), \\ u_\epsilon(t=0,x) = u_\epsilon^0(x) & \text{in } \Omega, \\ \frac{\partial u_\epsilon}{\partial t}(t=0,x) = u_\epsilon^1(x) & \text{in } \Omega. \end{cases} \tag{1.9}$$

The homogenization of (1.9) when $\lambda_1 < 0$ yields a parabolic limit equation (the imaginary root i cancels out in the factorization principle (1.7)): this case is analyzed in section 3. On the other hand, if $\lambda_1 > 0$, the new unknown v_ϵ, defined by (1.7), is complex-valued and the homogenized limit of (1.9) is a Schrödinger equation. These two limit regimes are called dispersive since they differ from the usual wave equation. Let us mention that a parabolic limit was already obtained in the homogenization of a different damped wave equation [13].

Coming back to the scaling of the original wave equation (1.1) our results can be summarized as follows. The asymptotic behavior of the solution $u_\epsilon(t,x)$ of (1.1) is:

1. if $\lambda_1 = 0$, $u_\epsilon(t,x) \approx \psi_1\left(\frac{x}{\epsilon}\right) v(t,x)$, where v is the solution of an homogenized wave equation,
2. if $\lambda_1 < 0$, $u_\epsilon(t,x) \approx e^{\frac{\sqrt{-\lambda_1}t}{\epsilon}} \psi_1\left(\frac{x}{\epsilon}\right) v(\epsilon t,x)$, where v is the solution of an homogenized parabolic equation,
3. if $\lambda_1 > 0$, $u_\epsilon(t,x) \approx e^{i\frac{\sqrt{\lambda_1}t}{\epsilon}} \psi_1\left(\frac{x}{\epsilon}\right) v(\epsilon t,x)$, where v is the solution of an homogenized Schrödinger equation.

Of course, the two last asymptotic behaviors make sense for large times, i.e. when t is of order ϵ^{-1}. Finally, we conclude this introduction by emphasizing that our results apply only for purely periodic coefficients $A(y)$ and $c(y)$. If they also depends on the slow variable x, concentration and localization effects are expected as already obtained for the parabolic problem in [4].

Notation. — For any function $\phi(x,y)$ defined on $\mathbb{R}^N \times \mathbb{T}^N$, we denote by ϕ^ϵ the function $\phi(x, \frac{x}{\epsilon})$.

2. Hyperbolic homogenized limit

We first consider the case when the first cell eigenvalue, defined in (1.3), is $\lambda_1 = 0$. In such a case we homogenize the original wave equation (1.1). Since the homogenization process is classical, we merely sketch the main arguments. When $\lambda_1 = 0$, there is no time renormalization and the factorization principle (1.7) reduces to

$$v_\epsilon(t,x) = \frac{u_\epsilon(t,x)}{\psi_1\left(\frac{x}{\epsilon}\right)}. \tag{2.1}$$

LEMMA 2.1. — *Assume $\lambda_1 = 0$. If u_ϵ is a solution of (1.1), then v_ϵ, defined by (2.1), is a solution of*

$$\begin{cases} |\psi_1^\epsilon|^2 \dfrac{\partial^2 v_\epsilon}{\partial t^2} - \operatorname{div}\left(|\psi_1^\epsilon|^2 A^\epsilon \nabla v_\epsilon\right) + |\psi_1^\epsilon|^2 d^\epsilon v_\epsilon = 0 & \text{in } \Omega\times(0,T), \\ v_\epsilon = 0 & \text{on } \partial\Omega\times(0,T), \\ v_\epsilon(t=0,x) = u_\epsilon^0(x)/\psi_1\left(\dfrac{x}{\epsilon}\right) & \text{in } \Omega, \\ \dfrac{\partial v_\epsilon}{\partial t}(t=0,x) = u_\epsilon^1(x)/\psi_1\left(\dfrac{x}{\epsilon}\right) & \text{in } \Omega. \end{cases} \tag{2.2}$$

Proof. — We briefly sketch the proof since it is by now classical [2], [3], [4], [16]. To pass from (1.1) to (2.2) it is sufficient to replace $u_\epsilon(t,x)$ by $v_\epsilon(t,x)\psi_1(x/\epsilon)$ and to multiply (1.1) by $\psi_1(x/\epsilon)$. Then, using equation (1.3), defining ψ_1, and the fact that

$$\psi_1^\epsilon \operatorname{div}\left(A^\epsilon \nabla(\psi_1^\epsilon v_\epsilon)\right) = \psi_1^\epsilon v_\epsilon \operatorname{div}\left(A^\epsilon \nabla \psi_1^\epsilon\right) + \operatorname{div}\left(|\psi_1^\epsilon|^2 A^\epsilon \nabla v_\epsilon\right)$$

yields the equivalence between the two equations. Note also that the same computation shows that the application $u(x) \to u(x)/\psi_1(x/\epsilon)$ is linear continuous in $H_0^1(\Omega)$. □

THEOREM 2.2. — *Assume $\lambda_1 = 0$ and that the initial data satisfy*

$$\begin{aligned} v_\epsilon(0,x) &\rightharpoonup v^0(x) \quad \textit{weakly in } H_0^1(\Omega), \\ |\psi_1|^2\left(\frac{x}{\epsilon}\right)\frac{\partial v_\epsilon}{\partial t}(0,x) &\rightharpoonup v^1(x) \quad \textit{weakly in } L^2(\Omega). \end{aligned} \tag{2.3}$$

Then, v_ϵ, solution of (2.2), converges weakly in $L^2\left((0,T);H_0^1(\Omega)\right)$ to the solution v of the following homogenized problem

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - \operatorname{div}\left(A^*\nabla v\right) + d^*(x)v = 0 & \textit{in } \Omega \times (0,T), \\ v = 0 & \textit{on } \partial\Omega \times (0,T), \\ v(t=0,x) = v^0(x) & \textit{in } \Omega, \\ \dfrac{\partial v}{\partial t}(t=0,x) = v^1(x) & \textit{in } \Omega, \end{cases} \tag{2.4}$$

with $d^*(x) = \int_{\mathbb{T}^N} d(x,y)|\psi_1|^2(y)\,dy$ and A^* is the classical homogenized matrix of $(|\psi_1|^2 A)$ (see formula (4.13)).

The proof of Theorem 2.2 is classical [5], [6], so we omit it. Remark that one can improve the convergence in Theorem 2.2 by introducing so-called corrector results if the initial data are well-prepared. However, in the general case, convergence of the energy density can be obtained only by means of geometric optics, WKB asymptotic expansions or H-measures [5], [8], [9], [15]. We shall not discuss these issues here.

Remark 2.3. — Theorem 2.2 still holds true if we add to equation (1.1) a source term $f(t,x) \in L^2\left((0,T)\times\Omega\right)$. It yields a source term $\left(\int_{\mathbb{T}^N}\psi_1(y)\,dy\right) f(t,x)$ in the homogenized equation (2.4).

3. Parabolic homogenized limit

We now consider the case when the first cell eigenvalue, defined in (1.3), is negative $\lambda_1 < 0$. In this case we homogenize the rescaled wave equation

(1.9). To do so, we perform a time renormalization of the unknown analogous to (1.7), namely we define

$$v_\epsilon(t,x) = e^{-\frac{\sqrt{-\lambda_1}t}{\epsilon^2}} \frac{u_\epsilon(t,x)}{\psi_1\left(\frac{x}{\epsilon}\right)}, \tag{3.1}$$

which is still a real-valued function since $\lambda_1 < 0$. The next lemma gives the equation satisfied by the new unknown v_ϵ.

LEMMA 3.1. — *Assume $\lambda_1 < 0$. If u_ϵ is a solution of (1.9), then v_ϵ, defined by (3.1), is a solution of*

$$\begin{cases} \epsilon^2|\psi_1^\epsilon|^2\dfrac{\partial^2 v_\epsilon}{\partial t^2} + 2\sqrt{-\lambda_1}|\psi_1^\epsilon|^2\dfrac{\partial v_\epsilon}{\partial t} & \\ \qquad\qquad - \operatorname{div}\left(|\psi_1^\epsilon|^2 A^\epsilon \nabla v_\epsilon\right) + |\psi_1^\epsilon|^2 d^\epsilon v_\epsilon = 0 & \text{in } \Omega\times(0,T), \\ v_\epsilon = 0 & \text{on } \partial\Omega\times(0,T), \\ v_\epsilon(t=0,x) = u_\epsilon^0(x)/\psi_1\left(\dfrac{x}{\epsilon}\right) & \text{in } \Omega, \\ \dfrac{\partial v_\epsilon}{\partial t}(t=0,x) = \left(u_\epsilon^1(x) - \dfrac{\sqrt{-\lambda_1}}{\epsilon^2}u_\epsilon^0(x)\right)/\psi_1\left(\dfrac{x}{\epsilon}\right) & \text{in } \Omega. \end{cases} \tag{3.2}$$

The proof of Lemma 3.1 is just a simple computation similar to that in Lemma 2.1, so we safely leave it to the reader. Remark that the time scaling of (1.9) is precisely chosen such that the first-order time derivative in (3.2) is of order 1 with respect to ϵ (compare with equation (1.8) in the introduction). The main advantage of the new problem (3.2) is that its solution satisfies uniform a priori estimates.

LEMMA 3.2. — *Assume that the initial data satisfy the following uniform bounds*

$$\|u_\epsilon^0\|_{L^2(\Omega)} + \epsilon\|\nabla u_\epsilon^0\|_{L^2(\Omega)^N} + \epsilon^2\|u_\epsilon^1\|_{L^2(\Omega)} \leqslant C. \tag{3.3}$$

Then the solution of (3.2) satisfies

$$\|v_\epsilon\|_{L^\infty((0,T);L^2(\Omega))} + \|\nabla v_\epsilon\|_{L^2((0,T)\times\Omega)^N} + \epsilon\|\frac{\partial v_\epsilon}{\partial t}\|_{L^2((0,T)\times\Omega)} \leqslant C \tag{3.4}$$

where $C > 0$ is a constant that does not depend on ϵ.

THEOREM 3.3. — *Assume that $\lambda_1 < 0$ and that the initial data satisfy the a priori estimates (3.3) as well as*

$$\psi_1\left(\frac{x}{\epsilon}\right)\left(\epsilon^2 u_\epsilon^1(x) + \sqrt{-\lambda_1}u_\epsilon^0(x)\right) \rightharpoonup \sqrt{-\lambda_1}v^0(x) \text{ weakly in } L^2(\Omega). \tag{3.5}$$

Then v_ϵ, solution of (3.2), converges weakly in $L^2\left((0,T);H_0^1(\Omega)\right)$ to the solution v of the following parabolic equation

$$\begin{cases} 2\sqrt{-\lambda_1}\dfrac{\partial v}{\partial t} - \operatorname{div}(A^*\nabla v) + d^*(x)v = 0 & \text{in } \Omega\times(0,T), \\ v = 0 & \text{on } \partial\Omega\times(0,T), \\ v(t=0,x) = \dfrac{1}{2}v^0(x) & \text{in } \Omega, \end{cases} \tag{3.6}$$

with $d^*(x) = \int_{\mathbb{T}^N} d(x,y)|\psi_1|^2(y)\,dy$ and A^* is the classical homogenized matrix of $(|\psi_1|^2 A)$.

Remark 3.4. — A special case of initial data satisfying (3.3) and (3.5) is that of well-prepared initial data, i.e. $u_\epsilon^0(x) = \psi_1\left(\frac{x}{\epsilon}\right)v^0(x)$ and $u_\epsilon^1(x) = \psi_1\left(\frac{x}{\epsilon}\right)v^1(x)$. In such a case, the initial velocity v^1 disappear at the limit and the factor $1/2$ in front of the initial condition for the homogenized problem is quite surprising. One possible explanation is the existence of an initial layer in time corresponding to a very fast decay of half of the initial data. This initial layer would correspond to the alternative time renormalization

$$w_\epsilon(t,x) = e^{\frac{\sqrt{-\lambda_1}t}{\epsilon^2}}\frac{u_\epsilon(t,x)}{\psi_1\left(\frac{x}{\epsilon}\right)}, \tag{3.7}$$

which has the opposite sign in the exponential compared to (3.1). Formally, w_ϵ would admit as an homogenized limit a backward heat equation which is ill-posed (so all this reasoning is purely formal).

Remark also that assumption (3.5) is much weaker than the usual assumption (2.3) for the homogenization of the wave equation without large potential. In particular, it allows for very large initial velocity $u_\epsilon^1(x)$, of the order of ϵ^{-2}.

Remark 3.5. — Theorem 3.3 still holds true if we add to equation (1.9) a source term of the type

$$f_\epsilon(t,x) = e^{\frac{\sqrt{-\lambda_1}t}{\epsilon^2}} f\left(t,x,\frac{x}{\epsilon}\right).$$

It yields a source term $f^*(t,x) = \int_{\mathbb{T}^N} f(t,x,y)\psi_1(y)\,dy$ in the homogenized equation (3.6).

Proof of Lemma 3.2. — We multiply equation (3.2) by $\frac{\partial v_\epsilon}{\partial t}$ and we integrate by parts to obtain the usual energy estimate

$$\frac{dE_\epsilon}{dt} + 2\sqrt{-\lambda_1}\int_\Omega |\psi_1^\epsilon|^2 \left|\frac{\partial v_\epsilon}{\partial t}\right|^2 dx = 0, \tag{3.8}$$

with

$$E_\epsilon(t) = \frac{1}{2}\int_\Omega |\psi_1^\epsilon|^2 \left(\epsilon^2\left|\frac{\partial v_\epsilon}{\partial t}\right|^2 + A^\epsilon\nabla v_\epsilon\cdot\nabla v_\epsilon + d^\epsilon|v_\epsilon|^2\right) dx.$$

By assumption we have $E_\epsilon(0) \leqslant C\epsilon^{-2}$ so that, upon integrating in time, we deduce from (3.8) and the fact that $d^\epsilon \geqslant 0$

$$\epsilon\|\nabla v_\epsilon\|_{L^\infty((0,T);L^2(\Omega))} + \epsilon^2\|\frac{\partial v_\epsilon}{\partial t}\|_{L^\infty((0,T);L^2(\Omega))} \leqslant C,$$

and

$$\epsilon\|\frac{\partial v_\epsilon}{\partial t}\|_{L^2((0,T)\times\Omega)} \leqslant C.$$

Now, we multiply equation (3.2) by v_ϵ to get

$$\begin{aligned}
&\sqrt{-\lambda_1}\int_\Omega |\psi_1^\epsilon|^2\,|v_\epsilon(T)|^2\,dx + \int_0^T\int_\Omega |\psi_1^\epsilon|^2\left(A^\epsilon\nabla v_\epsilon\cdot\nabla v_\epsilon + d^\epsilon|v_\epsilon|^2\right)dx\,dt = \\
&\sqrt{-\lambda_1}\int_\Omega |\psi_1^\epsilon|^2\,|v_\epsilon(0)|^2\,dx + \epsilon^2\int_0^T\int_\Omega |\psi_1^\epsilon|^2\left|\frac{\partial v_\epsilon}{\partial t}\right|^2 dx\,dt \\
&+\epsilon^2\int_\Omega |\psi_1^\epsilon|^2\,v_\epsilon(0)\frac{\partial v_\epsilon}{\partial t}(0)\,dx - \epsilon^2\int_\Omega |\psi_1^\epsilon|^2\,v_\epsilon(T)\frac{\partial v_\epsilon}{\partial t}(T)\,dx.
\end{aligned}$$

From the previous estimates, the assumption on the initial data and the non-negativeness of d^ϵ we deduce

$$\begin{aligned}
\sqrt{-\lambda_1}\int_\Omega |\psi_1^\epsilon|^2\,|v_\epsilon(T)|^2\,dx + \int_0^T\int_\Omega |\psi_1^\epsilon|^2\,A^\epsilon\nabla v_\epsilon\cdot\nabla v_\epsilon\,dx\,dt \\
\leqslant C\left(1 + \|v_\epsilon(T)\|_{L^2(\Omega)}\right),
\end{aligned}$$

which gives the desired result since by the Krein-Rutman theorem and standard regularity there exists two positive constants m, M such that $0 < m \leqslant \psi_1(y) \leqslant M$ in $\mathbb{T}^N$. □

Proof of Theorem 3.3. — In view of the a priori estimates of Lemma 3.2 there exist a subsequence and limits $v(t,x)$ and $v_1(t,x,y)$ such that v_ϵ

converges weakly to v in $L^2\left((0,T);H_0^1(\Omega)\right)$ and ∇v_ϵ two-scale converges to $\nabla_x v(t,x)+\nabla_y v_1(t,x,y)$ with $v_1(t,x,y)\in L^2\left((0,T)\times\Omega;H^1(\mathbb{T}^N)\right)$ (see [1], [12] for the notion of two-scale convergence). We define an oscillating test function

$$\phi_\epsilon(t,x)=\phi(t,x)+\epsilon\phi_1\left(t,x,\frac{x}{\epsilon}\right),$$

where ϕ and ϕ_1 are smooth test functions defined on $[0,T]\times\Omega\times\mathbb{T}^N$ with compact support in $[0,T[\times\Omega$. We multiply (3.2) by ϕ_ϵ and we integrate by parts to obtain

$$\begin{aligned}
&-\epsilon^2\int_0^T\int_\Omega|\psi_1^\epsilon|^2\frac{\partial v_\epsilon}{\partial t}\frac{\partial\phi_\epsilon}{\partial t}\,dx\,dt-2\sqrt{-\lambda_1}\int_0^T\int_\Omega|\psi_1^\epsilon|^2\,v_\epsilon\frac{\partial\phi_\epsilon}{\partial t}\,dx\,dt\\
&+\int_0^T\int_\Omega|\psi_1^\epsilon|^2\left(A^\epsilon\nabla v_\epsilon\cdot\nabla\phi_\epsilon+d^\epsilon v_\epsilon\phi_\epsilon\right)dx\,dt\\
&-\epsilon^2\int_\Omega|\psi_1^\epsilon|^2\,\phi_\epsilon(0)\frac{\partial v_\epsilon}{\partial t}(0)\,dx-2\sqrt{-\lambda_1}\int_\Omega|\psi_1^\epsilon|^2\,\phi_\epsilon(0)v_\epsilon(0)\,dx=0.
\end{aligned}\tag{3.9}$$

The two last terms in (3.9), corresponding to the initial conditions, can be rewritten in terms of the initial data for (1.9) as

$$-\int_\Omega\phi_\epsilon(0)\psi_1^\epsilon\left(\epsilon^2u_\epsilon^1+\sqrt{-\lambda_1}u_\epsilon^0\right)dx.$$

We pass to the limit in (3.9): the first term goes to zero because of estimate (3.4), we use two-scale convergence for the third one, and usual weak convergence for the other ones. Recalling that $\int_{\mathbb{T}^N}|\psi_1|^2\,dy=1$ we obtain

$$\begin{aligned}
&-2\sqrt{-\lambda_1}\int_0^T\int_\Omega v\frac{\partial\phi}{\partial t}\,dx\,dt+\int_0^T\int_\Omega d^*v\phi\,dx\,dt\\
&+\int_0^T\int_\Omega\int_{\mathbb{T}^N}|\psi_1|^2\,A(\nabla v+\nabla_y v_1)\cdot(\nabla\phi+\nabla_y\phi_1)\,dx\,dy\,dt\\
&-\sqrt{-\lambda_1}\int_\Omega\phi(0)v^0\,dx=0.
\end{aligned}\tag{3.10}$$

Eliminating v_1 and ϕ_1 in (3.10) gives the usual formula for A^* as the homogenized matrix of $(|\psi_1|^2A)$ (see e.g. [1]) and delivers a variational formulation for the homogenized problem (3.6). By uniqueness of the solution of the homogenized problem (3.6), we deduce that the entire sequence v_ϵ converges to v. □

4. Schrödinger homogenized limit

We finally consider the case when the first cell eigenvalue, defined in (1.3), is positive $\lambda_1 > 0$. In this case we homogenize the rescaled wave equation (1.9). Once again we perform a time renormalization of the unknown, namely we define

$$v_\epsilon(t,x) = e^{-i\frac{\sqrt{\lambda_1}t}{\epsilon^2}} \frac{u_\epsilon(t,x)}{\psi_1\left(\frac{x}{\epsilon}\right)}, \tag{4.1}$$

which is now a complex-valued function. The next lemma gives the equation satisfied by the new unknown v_ϵ.

LEMMA 4.1. — *Assume $\lambda_1 > 0$. If u_ϵ is a solution of (1.9), then v_ϵ, defined by (4.1), is a solution of*

$$\left\{\begin{array}{ll} \epsilon^2|\psi_1^\epsilon|^2\dfrac{\partial^2 v_\epsilon}{\partial t^2} + 2i\sqrt{\lambda_1}|\psi_1^\epsilon|^2\dfrac{\partial v_\epsilon}{\partial t} & \\ \qquad\qquad - \operatorname{div}\left(|\psi_1^\epsilon|^2 A^\epsilon \nabla v_\epsilon\right) + |\psi_1^\epsilon|^2 d^\epsilon v_\epsilon = 0 & \textit{in } \Omega\times(0,T), \\ v_\epsilon = 0 & \textit{on } \partial\Omega\times(0,T), \\ v_\epsilon(t=0,x) = u_\epsilon^0(x)/\psi_1\left(\dfrac{x}{\epsilon}\right) & \textit{in } \Omega, \\ \dfrac{\partial v_\epsilon}{\partial t}(t=0,x) = \left(u_\epsilon^1(x) - i\dfrac{\sqrt{\lambda_1}}{\epsilon^2}u_\epsilon^0(x)\right)/\psi_1\left(\dfrac{x}{\epsilon}\right) & \textit{in } \Omega. \end{array}\right. \tag{4.2}$$

The proof of Lemma 4.1 is again a simple computation similar to those in Lemmas 2.1 and 3.1, so we omit it. Remark that the time scaling of (1.9) is precisely chosen such that the first-order time derivative in (4.2) is of order 1 with respect to ϵ. Notice also that changing the sign of the exponential in the time renormalization (4.1) simply amounts to take the complex conjugate of the new unknown v_ϵ.

The main advantage of the new problem (4.2) is that its solution satisfies uniform a priori estimates. Note however that they are weaker than the ones obtained in the previous parabolic case ($\lambda_1 < 0$, see Lemma 3.2).

LEMMA 4.2. — *Assume that the initial data satisfy the following uniform bounds*

$$\|u_\epsilon^0\|_{L^2(\Omega)} + \epsilon\|\nabla u_\epsilon^0\|_{L^2(\Omega)^N} + \epsilon^2\|u_\epsilon^1\|_{L^2(\Omega)} \leqslant C. \tag{4.3}$$

Then the solution of (4.2) satisfies

$$\|v_\epsilon\|_{L^\infty((0,T);L^2(\Omega))} + \epsilon\|\nabla v_\epsilon\|_{L^\infty((0,T);L^2(\Omega)^N)} + \epsilon^2\|\frac{\partial v_\epsilon}{\partial t}\|_{L^\infty((0,T);L^2(\Omega))} \leqslant C \tag{4.4}$$

where $C > 0$ is a constant that does not depend on ϵ.

THEOREM 4.3. — *Assume that $\lambda_1 > 0$ and that the initial data satisfy the a priori estimates (4.3) as well as*

$$\psi_1\left(\frac{x}{\epsilon}\right)\left(\epsilon^2 u_\epsilon^1(x) + i\sqrt{\lambda_1}u_\epsilon^0(x)\right) \rightharpoonup i\sqrt{\lambda_1}v^0(x) \text{ weakly in } L^2(\Omega). \quad (4.5)$$

Then, v_ϵ, solution of (4.2), converges weakly in $L^2((0,T)\times\Omega)$ to the solution v of the following Schrödinger equation

$$\begin{cases} 2i\sqrt{\lambda_1}\dfrac{\partial v}{\partial t} - \operatorname{div}(A^*\nabla v) + d^*(x)v = 0 & \text{in } \Omega\times(0,T), \\ v = 0 & \text{on } \partial\Omega\times(0,T), \\ v(t=0,x) = \dfrac{1}{2}v^0(x) & \text{in } \Omega, \end{cases} \quad (4.6)$$

with $d^*(x) = \int_{\mathbb{T}^N} d(x,y)|\psi_1|^2(y)\,dy$ and A^* is the classical homogenized matrix of $(|\psi_1|^2 A)$.

Remark 4.4. — The convergence of v_ϵ in Theorem 4.3 is weaker than in the previous sections. In particular, it does not allow us to recover the Dirichlet boundary condition in the homogenized Schrödinger equation. We shall need another argument (based on the existence theory for the wave equation with non-smooth initial data) to obtain the boundary condition.

Remark 4.5. — Theorem 4.3 still holds true if we add to equation (1.9) a source term of the type

$$f_\epsilon(t,x) = e^{i\frac{\sqrt{\lambda_1}t}{\epsilon^2}} f\left(t,x,\frac{x}{\epsilon}\right).$$

It yields a source term $f^*(t,x) = \int_{\mathbb{T}^N} f(t,x,y)\psi_1(y)\,dy$ in the homogenized equation (4.6).

Proof of Lemma 4.2. — We multiply equation (4.2) by $\dfrac{\partial\overline{v}_\epsilon}{\partial t}$, we integrate by parts and take the real part to obtain the usual energy estimate

$$\frac{dE_\epsilon}{dt} = 0, \quad (4.7)$$

with

$$E_\epsilon(t) = \frac{1}{2}\int_\Omega |\psi_1^\epsilon|^2\left(\epsilon^2\left|\frac{\partial v_\epsilon}{\partial t}\right|^2 + A^\epsilon\nabla v_\epsilon\cdot\nabla\overline{v}_\epsilon + d^\epsilon|v_\epsilon|^2\right)dx.$$

By assumption we have $E_\epsilon(0) \leqslant C\epsilon^{-2}$ so that we deduce from (4.7) and the non-negative character of d^ϵ

$$\epsilon\|\nabla v_\epsilon\|_{L^\infty((0,T);L^2(\Omega))} + \epsilon^2\|\frac{\partial v_\epsilon}{\partial t}\|_{L^\infty((0,T);L^2(\Omega))} \leqslant C.$$

Now, we multiply equation (4.2) by $\overline{v}_\epsilon$ to get

$$\int_0^T \int_\Omega |\psi_1^\epsilon|^2 \left(-\epsilon^2 \left|\frac{\partial v_\epsilon}{\partial t}\right|^2 + A^\epsilon \nabla v_\epsilon \cdot \nabla \overline{v}_\epsilon + d^\epsilon |v_\epsilon|^2\right) dx\, dt$$
$$+\epsilon^2 \int_\Omega |\psi_1^\epsilon|^2\, \overline{v}_\epsilon(T) \frac{\partial v_\epsilon}{\partial t}(T)\, dx - \epsilon^2 \int_\Omega |\psi_1^\epsilon|^2\, \overline{v}_\epsilon(0) \frac{\partial v_\epsilon}{\partial t}(0)\, dx$$
$$+2i\sqrt{\lambda_1} \int_0^T \int_\Omega |\psi_1^\epsilon|^2 \frac{\partial v_\epsilon}{\partial t} \overline{v}_\epsilon\, dx\, dt = 0.$$

Taking the imaginary part yields

$$\sqrt{\lambda_1} \int_\Omega |\psi_1^\epsilon|^2 |v_\epsilon(T)|^2\, dx - \sqrt{\lambda_1} \int_\Omega |\psi_1^\epsilon|^2 |v_\epsilon(0)|^2\, dx =$$
$$-\epsilon^2 \mathcal{I} \left(\int_\Omega |\psi_1^\epsilon|^2\, \overline{v}_\epsilon(T) \frac{\partial v_\epsilon}{\partial t}(T)\, dx - \int_\Omega |\psi_1^\epsilon|^2\, \overline{v}_\epsilon(0) \frac{\partial v_\epsilon}{\partial t}(0)\, dx\right).$$

From the previous estimates and the assumption on the initial data we deduce

$$\sqrt{\lambda_1} \int_\Omega |\psi_1^\epsilon|^2 |v_\epsilon(T)|^2\, dx \leqslant C\left(1 + \|v_\epsilon(T)\|_{L^2(\Omega)}\right),$$

which gives the desired result. □

Proof of Theorem 4.3. — From the a priori estimates of Lemma 4.2, there exist a subsequence and a limit $v(t,x,y) \in L^2\left((0,T)\times\Omega; H^1(\mathbb{T}^N)\right)$ such that v_ϵ two-scale converges to $v(t,x,y)$ and $\epsilon\nabla v_\epsilon$ two-scale converges to $\nabla_y v(t,x,y)$ [1], [12]. Since the convergence of v_ϵ is weaker than in the previous parabolic case (see section 3), the proof is different from that of Theorem 3.3.

In a first step, we multiply (4.2) by a test function $\epsilon^2\phi^\epsilon \equiv \epsilon^2\phi\left(t,x,\frac{x}{\epsilon}\right)$ where $\phi(t,x,y)$ is a smooth function defined on $[0,T]\times\Omega\times\mathbb{T}^N$ with compact support in $]0,T[\times\Omega$. After integrating by parts and because of Lemma 4.2 we obtain

$$\int_0^T \int_\Omega |\psi_1^\epsilon|^2 A^\epsilon \epsilon\nabla v_\epsilon \cdot \epsilon\nabla\overline{\phi}^\epsilon\, dx\, dt = \mathcal{O}\left(\epsilon^2\right). \tag{4.8}$$

Passing to the two-scale limit in (4.8) yields

$$\int_0^T \int_\Omega \int_{\mathbb{T}^N} |\psi_1|^2 A\nabla_y v \cdot \nabla_y \overline{\phi}\, dx\, dy\, dt = 0,$$

which, for a.e. $(t,x) \in (0,T) \times \Omega$, is the variational formulation for

$$-\mathrm{div}_y \left(|\psi_1|^2 A \nabla_y v \right) = 0 \text{ in } \mathbb{T}^N.$$

By uniqueness of the solution in $H^1(\mathbb{T}^N)/\mathbb{R}$, we deduce that

$$v(t,x,y) \equiv v(t,x). \tag{4.9}$$

In a second step, we multiply (4.2) by another test function

$$\phi_\epsilon(t,x) = \phi(t,x) + \epsilon \sum_{j=1}^{N} \frac{\partial \phi}{\partial x_j}(t,x) \chi_j\left(\frac{x}{\epsilon}\right),$$

where ϕ is a smooth test function with compact support in $[0,T[\times\Omega$, and, denoting by $(e_j)_{1\leqslant j \leqslant N}$ the canonical basis of $\mathbb{R}^N$, $\chi_j(y)$ is the unique solution in $H^1(\mathbb{T}^N)/\mathbb{R}$ of the cell problem

$$-\mathrm{div}_y \left(|\psi_1|^2 A \left(e_j + \nabla_y \chi_j\right) \right) = 0 \text{ in } \mathbb{T}^N. \tag{4.10}$$

After integrating by parts (twice in space), we obtain

$$\begin{aligned}
&-\epsilon^2 \int_0^T \int_\Omega |\psi_1^\epsilon|^2 \frac{\partial v_\epsilon}{\partial t} \frac{\partial \overline{\phi}_\epsilon}{\partial t}\, dx\, dt - \int_0^T \int_\Omega v_\epsilon \mathrm{div} \left(|\psi_1^\epsilon|^2 A^\epsilon \nabla \overline{\phi}_\epsilon \right) dx\, dt \\
&+ \int_0^T \int_\Omega |\psi_1^\epsilon|^2 d^\epsilon v_\epsilon \overline{\phi}_\epsilon\, dx\, dt - 2i\sqrt{\lambda_1} \int_0^T \int_\Omega |\psi_1^\epsilon|^2 v_\epsilon \frac{\partial \overline{\phi}_\epsilon}{\partial t}\, dx\, dt \\
&- \int_\Omega |\psi_1^\epsilon|^2 \overline{\phi}_\epsilon(0) \left(\epsilon^2 \frac{\partial v_\epsilon}{\partial t}(0) + 2i\sqrt{\lambda_1} v_\epsilon(0) \right) dx = 0.
\end{aligned} \tag{4.11}$$

Let us explain how to pass to the limit in (4.11). For the first term, we notice that $\epsilon^2 |\psi_1^\epsilon|^2 \frac{\partial v_\epsilon}{\partial t}$, being bounded in $L^\infty\left((0,T); L^2(\Omega)\right)$, it converges weakly in this space to a limit which is necessarily 0 since $|\psi_1^\epsilon|^2 v_\epsilon$ is bounded in the same space. On the other hand $\frac{\partial \overline{\phi}_\epsilon}{\partial t}$ converges strongly to $\frac{\partial \overline{\phi}}{\partial t}$ in $L^2\left((0,T) \times \Omega\right)$, so the first term of (4.11) goes to zero. We can use two-scale convergence in the second term of (4.11) since, by using equation (4.10), we have

$$\begin{aligned}
\mathrm{div} \left(|\psi_1^\epsilon|^2 A^\epsilon \nabla \overline{\phi}_\epsilon \right) &= \mathrm{div}_x \left(|\psi_1|^2 A \left(\nabla_x \overline{\phi} + \sum_{j=1}^{N} \nabla_y \chi_j \frac{\partial \overline{\phi}}{\partial x_j} \right) \right)^\epsilon \\
&\quad + \mathrm{div}_y \left(|\psi_1|^2 A \sum_{j=1}^{N} \chi_j \nabla_x \frac{\partial \overline{\phi}}{\partial x_j} \right)^\epsilon + \mathcal{O}(\epsilon).
\end{aligned}$$

We also use two-scale convergence to pass to the limit in all other terms and the last one converges thanks to (4.5). Recalling that $\int_{\mathbb{T}^N} |\psi_1|^2\, dy = 1$ we obtain

$$-\int_0^T \int_\Omega v \mathrm{div}_x \left(\int_{\mathbb{T}^N} |\psi_1|^2 A \left(\nabla_x \overline{\phi} + \sum_{j=1}^N \nabla_y \chi_j \frac{\partial \overline{\phi}}{\partial x_j}) \right) dy \right) dx\, dt$$
$$+ \int_0^T \int_\Omega d^* v \overline{\phi}\, dx\, dt - 2i\sqrt{\lambda_1} \int_0^T \int_\Omega v \frac{\partial \overline{\phi}}{\partial t}\, dx\, dt - i\sqrt{\lambda_1} \int_\Omega \phi(0) v^0\, dx = 0. \tag{4.12}$$

We recognize in the first term of (4.12) the homogenized matrix A^* defined by

$$A^* e_j = \int_{\mathbb{T}^N} |\psi_1|^2 A\, (e_j + \nabla_y \chi_j)\, dy. \tag{4.13}$$

Thus, (4.12) is nothing but an ultra-weak variational formulation of the homogenized problem (4.6) which does not allow to recover variationally the Dirichlet boundary condition for v.

To obtain the boundary condition we introduce a regularized version of (4.2) (following a classical trick that can be found, e.g., in [11] and that I learned from E. Zuazua). Define

$$w_\epsilon(t, x) = \int_0^t v_\epsilon(s, x)\, dx + \chi_\epsilon(x), \tag{4.14}$$

where χ_ϵ is the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\mathrm{div}\left(|\psi_1^\epsilon|^2 A^\epsilon \nabla \chi_\epsilon\right) + |\psi_1^\epsilon|^2 d^\epsilon \chi_\epsilon = i\sqrt{\lambda_1}\psi_1^\epsilon u_\epsilon^0 + \epsilon^2 \psi_1^\epsilon u_\epsilon^1 & \text{in } \Omega, \\ \chi_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.15}$$

Then, upon time integration (4.2) is formally equivalent to

$$\begin{cases} \epsilon^2 |\psi_1^\epsilon|^2 \dfrac{\partial^2 w_\epsilon}{\partial t^2} + 2i\sqrt{\lambda_1} |\psi_1^\epsilon|^2 \dfrac{\partial w_\epsilon}{\partial t} & \\ \qquad - \mathrm{div}\left(|\psi_1^\epsilon|^2 A^\epsilon \nabla w_\epsilon\right) + |\psi_1^\epsilon|^2 d^\epsilon w_\epsilon = 0 & \text{in } \Omega \times (0, T), \\ w_\epsilon = 0 & \text{on } \partial\Omega \times (0, T), \\ w_\epsilon(t = 0, x) = \chi_\epsilon(x) & \text{in } \Omega, \\ \dfrac{\partial w_\epsilon}{\partial t}(t = 0, x) = u_\epsilon^0(x) / \psi_1\left(\dfrac{x}{\epsilon}\right) & \text{in } \Omega. \end{cases} \tag{4.16}$$

The initial data of (4.16) are less oscillating than those of (4.2). Indeed, it is easily seen that χ_ϵ is uniformly bounded in $H_0^1(\Omega)$ and that, up to a subsequence, it converges weakly in $H_0^1(\Omega)$ to χ^* which is a solution of

$$\begin{cases} -\mathrm{div}\,(A^* \nabla \chi^*) + d^*(x) \chi^* = i\sqrt{\lambda_1} v^0 & \text{in } \Omega, \\ \chi^* = 0 & \text{on } \partial\Omega. \end{cases}$$

With such smoother initial data, the energy estimates of Lemma 4.2 imply that w_ϵ is uniformly bounded in $L^2\left((0,T);H_0^1(\Omega)\right)$ and in $L^\infty\left((0,T);L^2(\Omega)\right)$. Thus it is classical to show that, up to a subsequence, w_ϵ converges weakly in these spaces to w which is the unique solution of

$$\begin{cases} 2i\sqrt{\lambda_1}\dfrac{\partial w}{\partial t} - \operatorname{div}\left(A^*\nabla w\right) + d^*(x)w = 0 & \text{in } \Omega \times (0,T), \\ w = 0 & \text{on } \partial\Omega \times (0,T), \\ w(t=0,x) = \chi^*(x) & \text{in } \Omega. \end{cases} \tag{4.17}$$

By standard regularity results the solution w is smooth enough to differentiate (4.17) with respect to the time variable t and, using the equation satisfied by χ^* (which belongs to the domain $H_0^1(\Omega) \cap H^2(\Omega)$ of the generator of the semi-group associated to (4.17)), we deduce that $\dfrac{\partial w}{\partial t}$ is the solution of the homogenized system (4.6). Since $\dfrac{\partial w_\epsilon}{\partial t} = v_\epsilon$, passing to the limit we obtain that $\dfrac{\partial w}{\partial t} = v$ in the sense of distributions. In particular, this implies that v satisfies the Dirichlet boundary condition. By uniqueness of the solution of (4.17) we also deduce that there was no need to extract subsequences and that all previous sequences were converging to the same limits. $\square$

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Bibliography

[1] ALLAIRE (G.), Homogenization and two-scale convergence, *SIAM J. Math. Anal.* **23(6)**, p. 1482-1518 (1992).

[2] ALLAIRE (G.), CAPDEBOSCQ (Y.), Homogenization of a spectral problem in neutronic multigroup diffusion, *Comput. Methods Appl. Mech. Engrg.* **187**, p. 91-117 (2000).

[3] ALLAIRE (G.), MALIGE (F.), Analyse asymptotique spectrale d'un problème de diffusion neutronique, *C. R. Acad. Sci. Paris Série I* **324**, p. 939-944 (1997).

[4] ALLAIRE (G.), PIATNITSKI (A.), Uniform Spectral Asymptotics for Singularly Perturbed Locally Periodic Operators, *Com. in PDE* **27**, p. 705-725 (2002).

[5] BENSOUSSAN (A.), LIONS (J.-L.), PAPANICOLAOU (G.), *Asymptotic analysis for periodic structures,* North-Holland, Amsterdam, 1978.

[6] BRAHIM-OTSMANE (S.), FRANCFORT (G.), MURAT (F.), Correctors for the homogenization of the wave and heat equations, *J. Math. Pures Appl.* (9) **71**, p. 197-231 (1992).

[7] CASTRO (C.), ZUAZUA (E.), Low frequency asymptotic analysis of a string with rapidly oscillating density, *SIAM J. Appl. Math.* **60(4)**, p. 1205-1233 (2000).

[8] FRANCFORT (G.), MURAT (F.), Oscillations and energy densities in the wave equation, *Comm. Partial Differential Equations* **17**, p. 1785-1865 (1992).

[9] GÉRARD (P.), Microlocal defect measures, *Comm. Partial Diff. Equations* **16**, p. 1761-1794 (1991).

[10] KOZLOV (S.), Reducibility of quasiperiodic differential operators and averaging, *Transc. Moscow Math. Soc.*, **2**, p. 101-126 (1984).

[11] LIONS J.-L., *Contrôlabilité exacte, perturbations et stabilisation des systèmes distribués,* Masson, Paris (1988).

[12] NGUETSENG (G.), A general convergence result for a functional related to the theory of homogenization, *SIAM J. Math. Anal.* **20(3)**, p. 608-623 (1989).

[13] ORIVE (R.), ZUAZUA (E.), PAZOTO (A.), Asymptotic expansion for damped wave equations with periodic coefficients, *Math. Mod. Meth. Appl. Sci.* **11**, p. 1285-1310 (2001).

[14] SANCHEZ-PALENCIA (E.), *Non homogeneous media and vibration theory,* Lecture notes in physics **127**, Springer Verlag (1980).

[15] TARTAR (L.), H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations, *Proc. Roy. Soc. Edinburgh* **115 A**, p. 193-230 (1990).

[16] VANNINATHAN (M.), Homogenization of eigenvalue problems in perforated domains, *Proc. Indian Acad. Sci. Math. Sci.* **90**, p. 239-271 (1981).