ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

FRANCIS RIBAUD

A remark on the uniqueness problem for the weak solutions of Navier-Stokes equations

Annales de la faculté des sciences de Toulouse 6e série, tome 11, nº 2 (2002), p. 225-238

<http://www.numdam.org/item?id=AFST_2002_6_11_2_225_0>

Annales de la Faculté des Sciences de Toulouse Vol. XI, n° 2, 2002
pp. 225–238

A remark on the uniqueness problem for the weak solutions of Navier-Stokes equations (*)

FRANCIS RIBAUD (1)

RÉSUMÉ. — Nous donnons de nouvelles classes d'unicité pour les solutions faibles des équations de Navier-Stokes en dimension trois. Ces résultats étendent ceux de Prodi, Serrin, Sohr et von Wahl à des espaces de type $L^q([0,T],(H_p^s(\Omega))^3)$ pour des valeurs de q dans $]1,2]$.

ABSTRACT. — We give some new class of uniqueness for the weak solutions of the three dimensional Navier-Stokes equations. This extends the results of Prodi, Serrin, Sohr and von Wahl concerning this problem to some spaces of the type $L^q([0,T],(H_p^s(\Omega))^3)$ with q in $]1,2]$.

1. Introduction and main results

We consider the uniqueness problem for the weak solutions of the Navier-Stokes equations

$$\partial_t u - \Delta u + (u.\nabla)u = F + \nabla p \ ,$$
$$\nabla . u = 0 \ ,$$
$$u_{|t=0} = a \ ,$$
$$u_{|\partial\Omega} = 0 \ .$$

(*) Reçu le 23 février 2001, accepté le 4 décembre 2002

(1) Université de Marne-La-Vallée, Équipe d'Analyse et de Mathématiques Appliquées, 5 bd Descartes, Cité Descartes, Champs-sur-Marne, 77454 Marne-La-Vallée Cedex 2, France.
e-mail : ribaud@math.univ-mlv.fr

Here, a denotes a divergence free vectors field in $(L^2(\Omega))^3$ and Ω is either $\mathbb{R}^3$, $\mathbb{R}^3_+$ or an open set in $\mathbb{R}^3$ with smooth compact boundary $\partial\Omega$. For simplicity, we will assume that the external force F vanishes but our results can be extended to the case of nonzero external force F with $F \in L^2(\mathbb{R}^+, (H^{-1}(\Omega))^3)$ (see for instance [T] for this question) and where (H4) below would be replaced by

$$\text{(H4')}\ \forall t \geqslant 0,\ \|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds \leqslant \|a\|_{L^2}^2 + \int_0^t \langle u(s), F(s)\rangle\, ds\ .$$

Since the works of Prodi [P], Serrin [S], Sohr and von Wahl [SW] it is well known that there exists a tight relation between the uniqueness problem and the regularity problem for the weak solutions of (NS). Indeed, the existence of one weak solution satisfying some suitable regularity properties would imply the uniqueness of weak solutions of (NS). More precisely, if there exists one weak solution in $L^q([0,T], (L^p(\Omega))^3)$ with $2/q + 3/p = 1$, $2 \leqslant q < +\infty$, then this solution is the unique weak solution of (NS). This result was first proved in [P] and extended in [S] to the dimension $d = 4$ when $2/q + 4/p = 1$, $2 \leqslant q < +\infty$. The generalization to arbitrary dimensions was given in [SW] $(2/q + d/p = 1,\ 2 \leqslant q < +\infty)$ where the authors also considered the exceptional case of weak solutions in $C^0([0,T], (L^d(\Omega))^d)$. We refer also to the recent papers [FLR] and [LM] for different proofs of the well-posedness of (NS) in $C^0([0,T], (L^3(\mathbb{R}^3))^3)$ with initial data $a \in (L^3(\mathbb{R}^3))^3$ and to [GP] for a weak-strong uniqueness result for initial data in $(L^2(\mathbb{R}^3))^3 \cap (\dot{B}_{r,q}^{-1+3/r}(\mathbb{R}^3))^3$ with $3/r + 2/q \geqslant 1$ ($\dot{B}_{r,q}^{-1+3/r}(\mathbb{R}^3)$ being a Besov space).

In this paper our aim is to obtain similar uniqueness results for some other classes of weak solutions. We consider some classes of weak solutions which are more regular in space but less integrable in time than in [P], [S], [SW] and we prove that they are also some classes of uniqueness. In particular, we obtain that if there exists a weak solution u of (NS) such that u and ∇u belong to $L^q([0,T], (L^p(\Omega))^3)$ with $2/q + 3/p = 2$, $1 < q < \infty$ then u is the unique weak solution of (NS).

In the rest of this section we recall first some notations and definitions. Next, in Theorem 1, we recall the results of Prodi, Serrin, Sohr and von Wahl concerning the uniqueness problem and then, in Theorem 2, we give our main result.

Following some usual notation, for s integer $H^s_p(\Omega)$ will denote the usual Sobolev space and for non integer values of s, $s = \theta k + (1-\theta)(k+1)$, $H^s_p(\Omega)$ is the complex interpolation space $[H^k_p(\Omega), H^{k+1}_p(\Omega)]_\theta$ (see [A] for further details). For $p = 2$, we let $H^s(\Omega) = H^s_2(\Omega)$ and $H^1_0(\Omega)$ is the completion of

$C_c^\infty(\Omega)$ the set of smooth compactly supported functions in Ω with respect to the usual norm of $H^1(\Omega)$. Also, $E^2(\Omega)$ will denote the set of vector fields $u = (u_1, u_2, u_3)$ in $(L^2(\Omega))^3$ such that $\nabla.u = div\, u = 0$ in $\mathcal{D}'(\Omega)$. For u and v in $(L^2(\Omega))^3$, (u, v) is the usual inner product of u and v and X^* is the dual space of X.

According to Leray's definition [L], we will say that u is a weak solution of the Navier-Stokes equations if u satisfies the following properties

(H1) u is weakly continuous from $\mathbb{R}^+$ to E^2.

(H2) $u \in L^\infty(\mathbb{R}^+, E^2(\Omega)) \cap L^2(\mathbb{R}^+, (H_0^1(\Omega))^3)$.

(H3) For all φ in $(C_c^\infty(\mathbb{R}^+ \times \Omega))^3$ with $\nabla.\varphi = 0$,

$$\int_0^{+\infty} (u, \partial_t \varphi) d\tau + \int_0^{+\infty} (u, \Delta\varphi) d\tau + \int_0^{+\infty} (u, (u.\nabla)\varphi) d\tau + (a, \varphi(0)) = 0 \ ,$$

for all ψ in $(C_c^\infty(\mathbb{R}^+ \times \Omega))^3$,

$$\int_0^{+\infty} (u, \nabla.\psi) d\tau = 0 \ ,$$

and $u = 0$ on $\mathbb{R}^+ \times \partial\Omega$.

(H4) $\forall t \geqslant 0,\ \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leqslant \|a\|_{L^2}^2$.

For all a in E^2, there always exists such a weak solution, see [L] when $\Omega = \mathbb{R}^3$ and, among the huge literature on this subject, we refer to [T] and [Li] for the existence of weak solutions in more general situations and also for more references concerning (NS). The uniqueness of weak solutions remains an open problem in the three-dimensional case but, as explained previously, the existence of one weak solution with suitable regularity would imply the uniqueness :

THEOREM 1 (Prodi, Serrin, Sohr, von Walh).— *Let u and v be two weak solutions of (NS) for the same initial data $a \in E^2$. If*

$$u \in L^q([0,T], (L^p(\Omega))^3) \ \text{with}\ \frac{2}{q} + \frac{3}{p} = 1 \ ,$$
$$2 \leqslant q < +\infty \ , \ 3 < p \leqslant +\infty \ , \tag{1}$$

then $u = v$ on $[0,T]$. Furthermore, $C([0,T], (L^3(\Omega))^3)$ is also a uniqueness class.

Our aim is to prove the following extension of Theorem 1.

THEOREM 2.— *Let u and v be two weak solutions of (NS) for the same initial data $a \in E^2$. If*

$$u \in L^q([0,T],(H_p^s(\Omega))^3) \ \ with \ \frac{2}{q} - s + \frac{3}{p} = 1 \ , \tag{2}$$

$$1 < p, q < +\infty \ , \ s \geqslant 0 \ , \tag{3}$$

then $u = v$ on $[0,T]$.

Remark 1.— Let u be a weak solution of (NS) and assume that u belongs to $L^q([0,T],(H_p^s(\Omega))^3)$ with $2/q - s + 3/p = 1$, $s \geqslant 0$. When $s < 3/p$ (i.e when $q > 2$), by the Sobolev embedding theorem u fulfills condition (1) of Theorem 1 and so u is the unique weak solution of (NS). Hence, when $s < 3/p$, Theorem 2 is just a corollary of Theorem 1. However, when $3/p < s$, then u belongs to $L^q([0,T],(H_p^s(\Omega))^3)$ with $q \in (1,2)$ and our uniqueness result can not be recovered from those of [P], [S] and [SW] which all required $q \geqslant 2$. Thus, roughly speaking, Theorem 2 extends the uniqueness results of Prodi, Serrin, Sohr and von Wahl to some classes of weak solutions which are more regular in space but less integrable in time.

Remark 2.— From Theorem 1, $L^2([0,T],(L^\infty(\Omega))^3)$ is also a uniqueness class (this is one of the two borderline cases in Theorem 1). Our corresponding result is slightly different since we obtain that $L^2([0,T],(H_p^{3/p}(\Omega))^3)$ is a uniqueness class. This result can not follow from Theorem 1 because the embedding $H_p^{3/p}(\Omega) \hookrightarrow L^\infty(\Omega)$ does not hold.

2. Proof of Theorem 2

The proof of Theorem 2 follows closely those given in [S], [SW] and [T] (see also [G] and [FJR] for initial data in $(L^p(\Omega))^3$, $p \geqslant 3$). First we give a differential inequality on the L^2 norm of $w(t) = v(t) - u(t)$ (see Proposition 1). Next, from this differential inequality and from some estimates on the nonlinear term $(w, w.\nabla u)$ (see Proposition 2 and also Proposition 3), we prove that $w(t) = 0$. For the convenience of the reader, we will give the proofs of Proposition 1 and 2 in Section 3.

PROPOSITION 1.— *Let u and v be as in Theorem 2 and let $w = u - v$. Then we have*

$$\|w(t)\|_{L^2}^2 + 2\int_0^t \|\nabla w(\tau)\|_{L^2}^2 d\tau \leqslant 2|B(w,w,u)| \tag{4}$$

where

$$B(u^1,u^2,u^3) = \int_0^t b(u^1(\tau),u^2(\tau),u^3(\tau))\, d\tau \ ,$$

and

$$b(u^1, u^2, u^3) = (u^1, (u^2.\nabla)u^3) = \int_\Omega \sum_{i,k=1}^{3} u_i^1\, u_k^2\, \frac{\partial u_i^3}{\partial x_k}\, dx \ .$$

Remark 3. — Inequality (4) is proved in numerous works under the assumptions of Theorem 1 (see [P], [S] and [T] among others).

PROPOSITION 2. — *Let Ω be either $\mathbb{R}^3$, $\mathbb{R}^3_+$ or an open set in $\mathbb{R}^3$ with smooth compact boundary $\partial\Omega$. Let $s > 0$ and p such that*

$$1 < p < +\infty \ , \ \frac{3}{p} - 1 \leqslant s < \frac{3}{p} + 1 \ . \tag{5}$$

Then for

$$f \in E^2(\Omega) \cap (H_0^1(\Omega))^3 \ , \ u \in E^2(\Omega) \cap (H_0^1(\Omega))^3 \cap (H_p^s(\Omega))^3$$

we have

$$|b(f,f,u)| \leqslant C\, \|f\|_{L^2}^{1+s-3/p} \|\nabla f\|_{L^2}^{1-s+3/p} \|u\|_{H_p^s} \ . \tag{6}$$

We return now to the proof of Theorem 2. Let q, p and s be as in (2)-(3). If $s = 0$ the result follows from Theorem 1. When $s > 0$, it follows from (2) that (5) holds and from Proposition 1 and Proposition 2, for all $t \in [0,T]$,

$$\|w(t)\|_{L^2}^2 + 2\int_0^t \|\nabla w(\tau)\|_{L^2}^2 d\tau \leqslant 2\int_0^t |b(w(\tau), w(\tau), u(\tau))|\, d\tau$$

$$\leqslant C\int_0^t \|w(\tau)\|_{L^2}^{1+s-3/p} \|\nabla w(\tau)\|_{L^2}^{1-s+3/p} \|u(\tau)\|_{H_p^s} d\tau \ .$$

By the Young inequality we obtain

$$\|w(t)\|_{L^2}^2 + 2\int_0^t \|\nabla w(\tau)\|_{L^2}^2 d\tau$$

$$\leqslant \int_0^t \|\nabla w(\tau)\|_{L^2}^2 d\tau + C\int_0^t \|w(\tau)\|_{L^2}^2 \|u(\tau)\|_{H_p^s}^q d\tau$$

where q is given by

$$\frac{2}{q} = 1 + s - \frac{3}{p} \ .$$

Hence

$$\|w(t)\|_{L^2}^2 \leqslant C\int_0^t \|w(\tau)\|_{L^2}^2 \|u(\tau)\|_{H_p^s}^q d\tau$$

and from the Gronwall's lemma we deduce that $w = 0$ on $[0,T]$.

3. Proof of Propositions 1 and 2

3.1. Proof of Proposition 2

We assume first that $\Omega = \mathbb{R}^3$. In order to prove (6) we introduce now the homogeneous Sobolev space $\dot{H}_p^\gamma(\mathbb{R}^3)$ which is defined as the set of functions $f \in L^z(\mathbb{R}^3)$, $1/z = 1/p - \gamma/3$, such that $(-\Delta)^{\frac{\gamma}{2}} f \in L^p(\mathbb{R}^3)$. This space is endowed with the norm

$$\|f\|_{\dot{H}_p^\gamma} = \|(-\Delta)^{\frac{\gamma}{2}} f\|_{L^p} \ ,$$

and when $p = 2$, we just let $\dot{H}^\gamma(\mathbb{R}^3) = \dot{H}_2^\gamma(\mathbb{R}^3)$. Recall that

$$\|f\|_{\dot{H}_p^\gamma} \leqslant \|f\|_{H_p^\gamma} \ , \ \gamma \geqslant 0 \ , \ f \in H_p^\gamma(\mathbb{R}^3) \ , \tag{7}$$

and that

$$\|f\|_{\dot{H}^\gamma} \leqslant \|f\|_{L^2}^{1-\gamma} \|\nabla f\|_{L^2}^\gamma \ , \ \gamma \in [0,1] \ , \ f \in H^1(\mathbb{R}^3) \ . \tag{8}$$

Also recall the standard bilinear estimate

$$\|fg\|_{L^2} \leqslant C \|f\|_{\dot{H}^{3/p-\gamma}} \|g\|_{\dot{H}_p^\gamma} \ \ , \ 0 < \gamma < 3/p \ \ , \ 1 < p < +\infty \ , \tag{9}$$

where $f \in \dot{H}^{3/p-\gamma}(\mathbb{R}^3)$ and $g \in \dot{H}_p^\gamma(\mathbb{R}^3)$. We refer to [RS] for the proof of (9) and for the proofs of (7)-(8) to [Tr] where further details concerning the spaces $\dot{H}_p^\gamma(\mathbb{R}^3)$ can be found.

When $1 < s$ and $3/p \leqslant s$, from the Cauchy-Schwarz inequality and (9) with $\gamma = s - 1$ we obtain

$$|b(f,f,u)| \leqslant C \|f\|_{L^2} \|f\|_{\dot{H}^{1-s+3/p}} \|\nabla u\|_{\dot{H}_p^{s-1}} \ .$$

Since $0 < 1 - s + 3/p \leqslant 1$, from (8) we get

$$|b(f,f,u)| \leqslant C \|f\|_{L^2}^{1+s-3/p} \|\nabla f\|_{L^2}^{1-s+3/p} \|u\|_{H_p^s} \ . \tag{10}$$

We consider now the case $0 < s < 3/p$ and $0 < s < 1$. Since both f and u belong to $E^2(\mathbb{R}^3) \cap (H_0^1(\mathbb{R}^3))^3$, integrating by parts we see that

$$b(f,f,u) = -b(u,f,f) \ .$$

Next by the Cauchy-Schwarz inequality and (9) we have

$$|b(f,f,u)| \leqslant \|\nabla f\|_{L^2} \|f\|_{\dot{H}^{3/p-s}} \|u\|_{\dot{H}_p^s} \ .$$

Since $0 < 3/p - s \leqslant 1$, it follows from (8) that

$$|b(f,f,u)| \leqslant C\|f\|_{L^2}^{1+s-3/p}\|\nabla f\|_{L^2}^{1-s+3/p}\|u\|_{H_p^s} \quad . \tag{11}$$

In the other cases, we consider the linear form defined by $u \to b(f,f,u)$. In this case there exists s_1, s_2 and θ such that H_p^s is the complex interpolation space $[H_p^{s_1}, H_p^{s_2}]_\theta$ where

$$0<\theta<1 \ , \ 0<s_1<\frac{3}{p} \ , \ 0<s_1<1 \ , \ 1<s_2 \ , \ \frac{3}{p}\leqslant s_2 \ .$$

By interpolation, (10) and (11) it follows that

$$|b(f,f,u)| \leqslant C\|f\|_{L^2}^{1+s-3/p}\|\nabla f\|_{L^2}^{1-s+3/p}\|u\|_{H_p^s}$$

which ends the proof in the case $\Omega = \mathbb{R}^3$.

We deal now with the case where Ω is $\mathbb{R}^3_+$ or a domain with smooth compact boundary. Assume first that $0 < s < 3/p$ and $0 < s < 1$. Since both f and u belong to $E^2(\Omega) \cap (H_0^1(\Omega))^3$, integrating by parts we see that

$$b(f,f,u) = -b(u,f,f) \ ,$$

and by the Cauchy-Schwarz inequality this yields

$$|b(f,f,u)| \leqslant \|\nabla f\|_{L^2(\Omega)}\|f\,u\|_{L^2(\Omega)} \ .$$

We consider now an extension operator K build by reflexions (see [A] p. 65 for instance). It is well known that K satisfies $\|K(u)\|_{L^p(\mathbb{R}^3)} \leqslant C\|u\|_{L^p(\Omega)}$ and $\|\nabla K(u)\|_{L^p(\mathbb{R}^3)} \leqslant C\|\nabla u\|_{L^p(\Omega)}$ which implies that, for $0 < s < 1$ and $f \in H_p^s(\Omega)$,

$$\|K(f)\|_{H_p^s(\mathbb{R}^3)} \leqslant C\,\|f\|_{H_p^s(\Omega)} \ .$$

By the above calculations we have

$$|b(f,f,u)| \leqslant C\,\|\nabla f\|_{L^2(\Omega)}\|K(f)\,K(u)\|_{L^2(\mathbb{R}^3)} \ .$$

$$\leqslant C\|\nabla f\|_{L^2(\Omega)}\|K(f)\|_{L^2(\mathbb{R}^3)}^{1+s-3/p}\|\nabla K(f)\|_{L^2(\mathbb{R}^3)}^{3/p-s}\|K(u)\|_{\dot{H}_p^s(\mathbb{R}^3)}$$

$$\leqslant C\|\nabla f\|_{L^2(\Omega)}^{1+3/p-s}\|f\|_{L^2(\Omega)}^{1+s-3/p}\|K(u)\|_{\dot{H}_p^s(\mathbb{R}^3)}$$

$$\leqslant C\|\nabla f\|_{L^2(\Omega)}^{1+3/p-s}\|f\|_{L^2(\Omega)}^{1+s-3/p}\|K(u)\|_{H_p^s(\mathbb{R}^3)}$$

$$\leqslant C\|\nabla f\|_{L^2(\Omega)}^{1+3/p-s}\|f\|_{L^2(\Omega)}^{1+s-3/p}\|u\|_{H_p^s(\Omega)} \ .$$

Note that we have used $div(u) = div(f) = 0$ only at the very beginning of the proof and before introducing K. This is why there is no problems to work with $K(f)$ and $K(u)$ instead of u and v even if $div(K(u)) \neq 0$ and $div(K(f)) \neq 0$.

We consider now the case $1 < s$ and $3/p < s$. Then by Cauchy-Schwarz inequality,

$$\begin{aligned}
|b(f,f,u)| &\leqslant \|f\|_{L^2(\Omega)} \|f \nabla u\|_{L^2(\Omega)} \\
&= \|f\|_{L^2(\Omega)} \|K(f) K(\nabla u)\|_{L^2(\Omega)} \\
&\leqslant \|f\|_{L^2(\Omega)} \|K(f) K(\nabla u)\|_{L^2(\mathbb{R}^3)} \\
&\leqslant C\, \|f\|_{L^2(\Omega)} \|K(f)\|_{\dot{H}^{1-s+3/p}(\mathbb{R}^3)} \|K(\nabla u)\|_{\dot{H}_p^{s-1}(\mathbb{R}^3)} \\
&\leqslant C\, \|f\|_{L^2(\Omega)} \|K(f)\|_{L^2(\mathbb{R}^3)}^{s-3/p} \|\nabla K(f)\|_{L^2(\mathbb{R}^3)}^{1-s+3/p} \|K(\nabla u)\|_{\dot{H}_p^{s-1}(\mathbb{R}^3)} \\
&\leqslant C\, \|f\|_{L^2(\Omega)}^{1+s-3/p} \|\nabla f\|_{L^2(\Omega)}^{1-s+3/p} \|K(\nabla u)\|_{H_p^{s-1}(\mathbb{R}^3)} \\
&\leqslant C\, \|f\|_{L^2(\Omega)}^{1+s-3/p} \|\nabla f\|_{L^2(\Omega)}^{1-s+3/p} \|\nabla u\|_{H_p^{s-1}(\Omega)} \\
&\leqslant C\, \|f\|_{L^2(\Omega)}^{1+s-3/p} \|\nabla f\|_{L^2(\Omega)}^{1-s+3/p} \|u\|_{H_p^{s}(\Omega)} \,.
\end{aligned}$$

In the other cases the proof follows by interpolation.

3.2. Proof of Proposition 1

The main step in the proof of Proposition 1 is to obtain the identity

$$(u(t), v(t)) - \|a\|_{L^2}^2 = -2 \int_0^t (\nabla u(\tau), \nabla v(\tau))\, d\tau + B(w, w, u) \ . \tag{12}$$

Once that (12) is proved, using the energy inequality expressed through (H4) there is no difficulty to obtain Proposition 1 : this follows from a straightforward computation of $(w(t), w(t)) = (v(t) - u(t), v(t) - u(t))$ and this is left to the reader. To prove (12) under the regularity assumptions of Theorem 2 it will be convenient to use the following notations :

$$X = L^2((H_0^1(\Omega))^3) \cap L^\infty(E^2(\Omega)) \ ,$$

$$Y = L^2((H_0^1(\Omega))^3) \cap L^\infty(E^2(\Omega)) \cap L^q([0,T], (H_p^s(\Omega))^3) \ ,$$

and for f in Y we define $\|f\|_Y$ by

$$\|f\|_Y = \|f\|_{L^\infty(L^2)} + \|f\|_{L^2(H^1)} + \|f\|_{L^q(H_p^s)} \ ,$$

where s, p and q fulfill (2) and (3).

Next we will need the two following results.

PROPOSITION 3. — *For all $f \in X$ and $u \in Y$ we have*

$$B(f,f,u) = -B(u,f,f) \ , \tag{13}$$

$$|B(f,f,u)| \leqslant C\|f\|_{L^\infty(L^2)}^{1-s+3/p}\|f\|_{L^2(H^1)}^{1+s-3/p}\|u\|_{L^q(H_p^s)} \ , \tag{14}$$

and

$$|B(f,u,u)| \leqslant C\|f\|_{L^\infty(L^2)}^{1-\beta}\|f\|_{L^2(H^1)}^{\beta}\|u\|_Y^2 \tag{15}$$

where $\beta = \beta(s,p)$ belongs to $]0,1]$.

Proof of Proposition 3. — Let us consider $f \in X$, $g \in X$ and $h \in Y$. When $0 < s < 3/p$ and $0 < s < 1$, we notice first that $B(f,g,h) = -B(h,g,f)$. Then from the Cauchy-Schwarz inequality and (9) with $\gamma = 3/p - s$ we have

$$|B(f,g,h)| \leqslant \int_0^t \|\nabla f(\tau)\|_{L^2}\|g(\tau)\|_{\dot H^{3/p-s}}\|h(\tau)\|_{H_p^s}\,d\tau \ .$$

Since $0 < 3/p - s \leqslant 1$, using (8) we obtain

$$|B(f,g,h)| \leqslant C\|g\|_{L^\infty(L^2)}^{1+s-3/p}\int_0^t \|\nabla f(\tau)\|_{L^2}\|g(\tau)\|_{H^1}^{3/p-s}\|h(\tau)\|_{H_p^s}\,d\tau \ ,$$

and by the Cauchy Schwarz inequality,

$$|B(f,g,h)| \leqslant C\|g\|_{L^\infty(L^2)}^{1+s-3/p}\|f\|_{L^2(H^1)}\left(\int_0^t \|g(\tau)\|_{H^1}^{2(3/p-s)}\|h(\tau)\|_{H_p^s}^2\,d\tau\right)^{\frac12} \ .$$

Now we apply the Hölder inequality which leads to

$$|B(f,g,h))| \leqslant C\|f\|_{L^2(H^1)}\|g\|_{L^\infty(L^2)}^{1+s-3/p}\|g\|_{L^2(H^1)}^{3/p-s}\|h\|_{L^q(H_p^s)} \ . \tag{16}$$

When $1 < s$ and $3/p \leqslant s$, by the Cauchy-Schwarz inequality and (9) with $\gamma = 3/p + 1 - s$ we get

$$|B(f,g,h)| \leqslant C\int_0^t \|f(\tau)\|_{\dot H^{1-s+3/p}}\|g(\tau)\|_{L^2}\|\nabla h(\tau)\|_{H_p^{s-1}}\,d\tau \ .$$

Since $0 < 1 - s + 3/p \leqslant 1$, using (8) we obtain

$$|B(f,g,h)| \leqslant C\|g\|_{L^\infty(L^2)}\|f\|_{L^\infty(L^2)}^{s-3/p}\int_0^t \|f(\tau)\|_{H^1}^{1-s+3/p}\|h(\tau)\|_{H_p^s}\,d\tau \ .$$

Hence, by Hölder inequality

$$|B(f,g,h)| \leqslant C\|f\|_{L^\infty(L^2)}^{s-3/p}\|f\|_{L^2(H^1)}^{1-s+3/p}\|g\|_{L^\infty(L^2)}\|h\|_{L^q(H_p^s)} \ . \tag{17}$$

In the other cases we consider now the linear form $h \to B(f,g,h)$. In this situation, there exists q_1, q_2, s_1, s_2 and θ such that $L^q(H^s_p)$ is the complex interpolation space $[L^{q_1}(H^{s_1}_p)\ ;\ L^{q_2}(H^{s_2}_p)]_\theta$ with

$$0<\theta<1\ ,\ 0<s_1<\frac{3}{p}\ 0<s_1<1\ ,\ 1<s_2\ ,\ \frac{3}{p}\leqslant s_2\ ,$$

and

$$\frac{2}{q_1}=1+s_1-\frac{3}{p}\ ,\ \frac{2}{q_2}=1+s_2-\frac{3}{p}\ .$$

Then by interpolation, (16) and (17) we obtain that there exists $\theta\in[0,1]$ such that

$$|B(f,g,h)|$$
$$\leqslant C\|f\|_{L^\infty(L^2)}^{(1-\theta)(s-\frac{3}{p})}\|f\|_{L^2(H^1)}^{\theta+(1-\theta)(1-s+\frac{3}{p})}\|g\|_{L^\infty(L^2)}^{1-\theta(\frac{3}{p}-s)}\|g\|_{L^2(H^1)}^{\theta(\frac{3}{p}-s)}\|h\|_{L^q(H^s_p)}\ .$$

Thus, for $g=f$ and $h=u$, we obtain (14) and for $g=h=u$ we obtain

$$|B(f,u,u)|\leqslant C\|f\|_{L^\infty(L^2)}^{1-\beta}\|f\|_{L^2(H^1)}^{\beta}\|u\|_Y^2\ ,$$

where $\beta=\theta+(1-\theta)(1-s+3/p)>0$ since $s<3/p+1$ and $0\leqslant\theta\leqslant1$ which ends the proof when $\Omega=\mathbb{R}^3$.

Again, if Ω is $\mathbb{R}^3_+$ or a domain with smooth compact boundary, we just have to introduce an extension operator K to obtain the proof.

LEMMA 1.— *Let p' and q' defined by $1/p+1/p'=1$ and $1/q+1/q'=1$.*

1) If $v\in X$ is a weak solution of (NS) then

$$\partial_t v\in Y^*=L^2(\mathbb{R}^+,(H^{-1}(\Omega)))+\left[L^\infty(\mathbb{R}^+,E^2)\right]^*+L^{q'}(\mathbb{R}^+,(H^{-s}_{p'}(\Omega))^3)\ .$$

2) If $u\in Y$ is a weak solution of (NS) then

$$\partial_t u\in X^*=L^2(\mathbb{R}^+,(H^{-1}(\Omega))^3)+\left[L^\infty(\mathbb{R}^+,E^2)\right]^*\ .$$

Proof of Lemma 1.— We prove only the second part, the proof of the first one is similar. From (H2) we see that $\partial_t u$ is well defined in the distributional sense. Furthermore, from (H3), for all smooth function φ with compact support on $[0,+\infty[\times\Omega$ such that $\nabla.\varphi=0$, we have

$$<\partial_t u,\varphi>=\int_0^{+\infty}(\partial_t u(\tau),\varphi(\tau))d\tau=-\int_0^{+\infty}(\nabla u(\tau),\nabla\varphi(\tau))d\tau+$$
$$\int_0^{+\infty}(u(\tau),(u(\tau).\nabla)\varphi(\tau))d\tau+(a,\varphi(0))\ .$$

Now, from Proposition 3 we get

$$\int_0^{+\infty} (\partial_t u(\tau), \varphi(\tau))d\tau = -\int_0^{+\infty} (\nabla u(\tau), \nabla\varphi(\tau))d\tau -$$

$$\int_0^{+\infty} (\varphi(\tau), (u(\tau).\nabla)u(\tau))d\tau + (a, \varphi(0))$$

and since $u \in Y$, from (15) of Proposition 3 we obtain

$$| < \partial_t u, \varphi > | \leqslant \|u\|_{L^2(H^1)} \|\varphi\|_{L^2(H^1)}$$
$$+ \|u\|_Y^2 \|\varphi\|_{L^\infty(L^2)}^{1-\beta} \|\varphi\|_{L^2(H^1)}^{\beta} + \|a\|_{L^2} \|\varphi\|_{L^\infty(L^2)}$$

which ends the proof.

Now, in view of Lemma 1, for all $v \in X$ a weak solution of (NS) there exists a subsequence of smooth functions v_ε with compact support in $[0, +\infty[\times\Omega$ such that

(P1)

1) $\nabla.v_\varepsilon = 0$

2) $\lim\limits_{\varepsilon\to 0} v_\varepsilon = v$ in $L^2((H_0^1(\Omega))^3)$.

3) For all $\varepsilon > 0$, $\|v_\varepsilon\|_{L^\infty(L^2)} \leqslant \|v\|_{L^\infty(L^2)}$.

4) For all φ in Y, $\lim\limits_{\varepsilon\to 0} < \partial_t v_\varepsilon, \varphi > = < \partial_t v, \varphi >$.

In the same way, for all $u \in Y$ a weak solution of (NS), there exists a subsequence of smooth functions u_ε with compact support in $[0, +\infty[\times\Omega$ such that

(P2)

1) $\nabla.u_\varepsilon = 0$

2) $\lim\limits_{\varepsilon\to 0} u_\varepsilon = u$ in $L^2((H_0^1(\Omega))^3) \cap L^q((H_p^s(\Omega))^3)$.

3) For all $\varepsilon > 0$, $\|u_\varepsilon\|_{L^\infty(L^2)} \leqslant \|u\|_{L^\infty(L^2)}$.

4) For all φ in X, $\lim\limits_{\varepsilon\to 0} < \partial_t u_\varepsilon, \varphi > = < \partial_t u, \varphi >$.

Consider v_ε as in (P1) and u_ε as in (P2). Then, from (H3) and from Theorem 4 p. 79 in [S] we have

$$(a)\quad \int_0^t (u, \partial_t v_\varepsilon)d\tau - \int_0^t (\nabla u, \nabla v_\varepsilon)d\tau + \int_0^t (u, (u.\nabla)v_\varepsilon)d\tau =$$

$$(u(t), v_\varepsilon(t)) - (a, v_\varepsilon(0)) \ ,$$

and

$$(b)\quad \int_0^t (v, \partial_t u_\varepsilon)d\tau - \int_0^t (\nabla v, \nabla u_\varepsilon)d\tau + \int_0^t (v, (v.\nabla)u_\varepsilon)d\tau =$$

$$(v(t), u_\varepsilon(t)) - (a, u_\varepsilon(0)) \ .$$

Now we want to add (a) and (b) and pass to the limit when $\varepsilon \to 0$. Clearly, we have

$$\int_0^t (\nabla u(\tau), \nabla v_\varepsilon(\tau))d\tau \to \int_0^t (\nabla u(\tau), \nabla v(\tau))d\tau \ ,$$

$$\int_0^t (\nabla v(\tau), \nabla u_\varepsilon(\tau))d\tau \to \int_0^t (\nabla u(\tau), \nabla v(\tau))d\tau \ ,$$

$$(a, v_\varepsilon(0)) + (a, u_\varepsilon(0)) \to 2\|a\|_{L^2}^2 \ ,$$

$$(u(t), v_\varepsilon(t)) + (v(t), u_\varepsilon(t)) \to 2(u(t), v(t)) \ .$$

This follows directly from (H1), (H2) together with (P1-2), (P1-3), (P2-2) and (P2-3). Also, using (P1-4), (P2-4) and since $v \in X$ and $u \in Y$, we see that

$$\int_0^t (u(\tau), \partial_t v_\varepsilon(\tau)) + (v(\tau), \partial_t u_\varepsilon(\tau))d\tau \to \int_0^t (u(\tau), \partial_t v(\tau)) + (u(\tau), \partial_t v(\tau))d\tau$$

$$= \int_0^t \partial_t (u(\tau), v(\tau))d\tau \ = \ (u(t), v(t)) - \|a\|_{L^2}^2 \ .$$

Next, from (P1-2), (P1-3) and (15) of Proposition 3, we have

$$\int_0^t (u(\tau), (u(\tau).\nabla)v_\varepsilon(\tau))d\tau = B(u, u, v_\varepsilon) = -B(v_\varepsilon, u, u) \to -B(v, u, u) \ ,$$

and from (P2-2) and (14) of Proposition 3,

$$\int_0^{+\infty} (v(\tau), (v(\tau).\nabla)u_\varepsilon(\tau))d\tau = B(v, v, u_\varepsilon) \to B(v, v, u) \ .$$

Now we add (a) and (b) and for $\varepsilon \to 0$ we see that

$$(u(t), v(t)) - \|a\|_{L^2}^2 - 2\int_0^t (\nabla u(\tau), \nabla v(\tau))d\tau + B(v,v,u) - B(v,u,u) =$$

$$2(u(t), v(t)) - 2\|a\|_{L^2}^2 \ .$$

Now, again from Proposition 3, $B(u,v,u)$ is well defined and vanishes which proves that

$$B(v,v,u) - B(v,u,u) = B(w,w,u).$$

Hence we obtain

$$(u(t), v(t)) - \|a\|_{L^2}^2 = -2\int_0^t (\nabla u(\tau), \nabla v(\tau))d\tau + B(w,w,u) \ .$$

Bibliography

[A] ADAMS (R.). — *Sobolev Spaces*, Pure and applied math. series, V. 65, Academic Press, 1978.

[FJR] FABES (E.B.), JONES (B.F.) and RIVIERE (N.). — *The initial boundary value problem for the Navier-Stokes equation with initial data in L^p*, Arch. Rat. Mech. Anal., V. 45 (1972), p. 222-240.

[FLR] FURIOLI (G.), LEMARIÉ-RIEUSSET (P.-G.) and TERRANEO (E.). — *Unicité dans $L^3(\mathbb{R}^3)$ et d'autres espaces fonctionnels limites pour Navier-Stokes*, Rev. Mat. Iberoamericana, V. 16(3) (2000), p. 605-667.

[G] GIGA (Y.). — *Solutions for Semilinear Equations in L^p and Regularity of Weak Solutions of the Navier-stokes System*, J. Diff. Equa., V. 62 (1986), p. 186-212.

[GP] GALLAGHER (I.) and PLANCHON (F.). — *On infinite energy solutions to the Navier-Stokes equations: global 2D existence and 3D weak-strong uniqueness*, to appear in Arch. Rational. Mech. Anal.

[L] LERAY (J.). — *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta. Math., V. 63 (1934), p. 193-248.

[Li] LIONS (J.L.). — *Quelques méthodes de résolutions des problèmes aux limites non-linéaires*, Dunod, Paris, 1969.

[LM] LIONS (P.-L.), MASMOUDI (N.). — *Uniqueness of mild solutions of the Navier-Stokes system in L^N*, Comm. Partial Differential Equations, 26(11-12) (2001), p. 2211-2226.

[P] PRODI (G.). — *Un teorama di unicita per le equazioni di Navier-Stokes*, Annali di Mat., V. 48 (1959), p. 173-182.

[RS] RUNST (T.) and SICKEL (W.). — *Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations*, de Gruyter, Berlin, 1996.

[S] SERRIN (J.). — *The initial value problem for the Navier-Stokes equations,* Nonlinear Problems (R. Langer ed.), p. 69-98, Madison : The University of Wisconsin press, 1963.

[SW] SOHR (H.) and VON WAHL (W.). — *On the singular set and the uniqueness of weak solutions of the Navier-Stokes equations,* man. math., V. 49 (1984), p. 27-59.

[T] TEMAM (R.). — *Navier-stokes Equations,*, North-Holland, Amsterdam, 1984.

[Tr] TRIEBEL (H.). — *Theory of function spaces II,* Birkhauser, 1992.