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F. D. ARARUNA
G. O. ANTUNES
L. A. MEDEIROS

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Semilinear wave equation on manifolds (*)

F. D. Araruna, G. O. Antunes and L. A. Medeiros (1)

Dedicated to M. Milla Miranda in the occasion of his 60th. anniversary.

Résumé. — Dans ce travail nous étudions un problème pour les équations des ondes non linéaire définies dans une varieté. Ce problème a été motivé par J.L.Lions [8], p. 134. Pour l'existence de solutions nous avons appliqué la méthode de Galerkin. Le comportement asymptotique des solutions a été examiné aussi.

Abstract. — In this paper, we study a type of second order evolution equation on the lateral boundary Σ of the cylinder $Q = \Omega \times]0, T[$, with Ω an open bounded set of $\mathbb{R}^n$. In this problem is fundamental that the unknown function solves an elliptic problem on Ω. This results are motivated by Lions [8], pg. 134 where he works with another type of nonlinearity.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$ $(n \geqslant 1)$ with smooth boundary Γ. Let ν be the outward normal unit vector to Γ and $T > 0$ a real number. We consider the cylinder $Q = \Omega \times]0, T[$ with lateral boundary $\Sigma = \Gamma \times]0, T[$.

We investigate existence and asymptotic behaviour of weak solution for the problem

$$\left| \begin{array}{l} w'' + \dfrac{\partial w}{\partial \nu} + F(w) + \beta(x) w' = 0 \quad \text{on} \quad \Sigma, \\ w(0) = w_0, \quad w'(0) = w_1 \quad \text{on} \quad \Gamma, \end{array} \right. \tag{1.1}$$

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Instituto de Matemática, Universidade Federal do Rio de Janeiro, Caixa Postal 68530, 21945-970, Rio de Janeiro-RJ, Brasil.
E-mail: araruna@pg.im.ufrj.br, gladson@email.com.br, lmedeiros@abc.org.br

where the prime means the derivative with respect to t, $\frac{\partial w}{\partial \nu}$ normal derivative of w and $F : \mathbb{R} \to \mathbb{R}$ is a function that satisfies

$$F \text{ continuous and } sF(s) \geqslant 0,\ \forall s \in \mathbb{R}. \tag{1.2}$$

It is important to call the attention to the reader that the idea employed in this work comes from Lions [8], pg. 134. The main point consists in adding to (1.1) an elliptic equation in Ω to reduce the problem to a canonical model of Mathematical Physics, but in this case on a manifold which is the lateral boundary Σ of the cylinder Q . A Similar type of problem, also motivated by Lions [8], can be seen in Cavalvanti and Domingos Cavalcanti [2].

The plan of this article is the following: In the section 2, we give notations, terminology and we treat the linear case associated to (1.1). In the section 3, we prove existence for weak solution when F satisfies the condition (1.2), approximating F by Lipschtz functions. In this Lipschitz case, we employ Picard's successive approximations and then we apply the Strauss' method [9]. Finally in the section 4, we obtain the asymptotic behaviour by the method of pertubation of energy as in Zuazua [10].

2. Notations, Assumptions and Results

Denote by $|\cdot|$, $(\cdot,\cdot)$ and $\|\cdot\|$, $((\cdot,\cdot))$ the inner product and norm, respectively, of $L^2(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$.

For

$$G(s) = \int_0^s F(\sigma) d\sigma$$

we will denote a primitive of F.

We consider the following assumption on β in (1.1) :

$$\beta \in L^\infty(\Gamma) \text{ such that } \beta(x) \geqslant \beta_0 > 0, \text{ a.e. on } \Gamma. \tag{2.1}$$

As was said in the introduction, for $\lambda > 0$, let us consider the problem

$$\left| \begin{array}{l} -\Delta w + \lambda w = 0 \quad \text{in} \quad Q, \\ w'' + \dfrac{\partial w}{\partial \nu} + F(w) + \beta(x) w' = 0 \quad \text{on} \quad \Sigma, \\ w(0) = w_0, \quad w'(0) = w_1 \quad \text{on} \quad \Gamma. \end{array} \right. \tag{2.2}$$

From elliptic theory, we know that for $\varphi \in H^{\frac{1}{2}}(\Gamma)$, the solution Φ of the boundary value problem

$$\left| \begin{array}{ll} -\Delta \Phi + \lambda \Phi = 0 & \text{in} \quad \Omega, \\ \Phi = \varphi & \text{on} \quad \Gamma, \end{array} \right. \tag{2.3}$$

belongs to $H^1(\Omega,\Delta)=\left\{u\in H^1(\Omega);\Delta u\in L^2(\Omega)\right\}$. By the trace theorem, it follows that $\dfrac{\partial\Phi}{\partial\nu}\in H^{-\frac{1}{2}}(\Gamma)$.

Formally, we have by (2.3) that

$$0=\int_\Omega \nabla\Phi\nabla\Psi dx+\lambda\int_\Omega \Phi\Psi dx-\int_\Gamma \frac{\partial\Phi}{\partial\nu}\Psi d\Gamma.$$

We take $\Psi\in H^1(\Omega,\Delta)$ and we define

$$a(\Phi,\Psi)=\int_\Omega \nabla\Phi\nabla\Psi dx+\lambda\int_\Omega \Phi\Psi dx \tag{2.4}$$

Thus, by (2.4)

$$a(\Phi,\Psi)=\langle\gamma_1\Phi,\gamma_0\Psi\rangle,$$

where γ_0 and γ_1 are the traces of order zero and one, respectively, and $\langle\cdot,\cdot\rangle$ represents the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$.

We consider the scheme

$$\begin{array}{ccccc} \varphi\in H^{\frac{1}{2}}(\Gamma) & & \xrightarrow{\gamma_0^{-1}} & & \Phi\in H^1(\Omega,\Delta) \\ & \searrow_A & & \swarrow_{\gamma_1} & \\ & & \dfrac{\partial\Phi}{\partial\nu}\in H^{-\frac{1}{2}}(\Gamma) & & \end{array}$$

Thus

$$A=\gamma_1\circ\gamma_0^{-1}:H^{\frac{1}{2}}(\Gamma)\to H^{-\frac{1}{2}}(\Gamma),\ A\varphi=\frac{\partial\Phi}{\partial\nu}.$$

Therefore A is self-adjoint and $A\in\mathcal{L}\left(H^{\frac{1}{2}}(\Gamma),H^{-\frac{1}{2}}(\Gamma)\right)$.

Moreover, we have

$$\langle A\varphi,\varphi\rangle=a(\Phi,\Phi) \tag{2.5}$$

and so by (2.4) we get

$$\langle A\varphi,\varphi\rangle=\int_\Omega |\nabla\Phi|^2 dx+\lambda\int_\Omega |\Phi|^2 dx\geqslant\min\{1,\lambda\}\|\Phi\|^2_{H^1(\Omega)}\geqslant$$
$$\geqslant\alpha\|\gamma_0\Phi\|^2=\alpha\|\varphi\|^2,$$

proving that A is positive.

We formulate now the problem on Σ. For this, we define

$$w(t)|_\Gamma = u(t) \quad \text{and} \quad \frac{\partial w(t)}{\partial \nu}|_\Gamma = Au(t).$$

In this way, the problem (1.2) is reduced to find a function $u : \Sigma \to \mathbb{R}$ such that

$$\left| \begin{array}{l} u'' + Au + F(u) + \beta(x) u' = 0 \quad \text{on} \quad \Sigma, \\ u(0) = u_0, \quad u'(0) = u_1 \quad \text{on} \quad \Gamma, \end{array} \right.$$

which will be investigated in the section 3.

Firstly we will state a result that guarantees the existence and uniqueness of solution for the linear problem associated the (1.1).

THEOREM 2.1.— *Given* $(u_0, u_1, f) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma) \times L^2(\Sigma)$, *there exists a unique function* $u : \Sigma \to \mathbb{R}$ *such that*

$$u \in C^0\left(0, T; H^{\frac{1}{2}}(\Gamma)\right) \cap C^1\left(0, T; L^2(\Gamma)\right), \tag{2.6}$$

$$u'' + Au + \beta u' = f \quad \text{in} \quad L^2\left(0, T; H^{-\frac{1}{2}}(\Gamma)\right), \tag{2.7}$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad \text{on} \quad \Gamma. \tag{2.8}$$

Moreover we have the energy inequality

$$\frac{1}{2}|u'(t)|^2 + \frac{\alpha}{2}\|u(t)\|^2 \leqslant \frac{1}{2}|u_1|^2 + \frac{\alpha}{2}\|u_0\|^2 + \int_0^T (f(s), u'(s))\, ds, \quad \text{a.e in} \quad [0, T]. \tag{2.9}$$

Proof. — In the proof of this linear case, we employ the Faedo-Galerkin's method. □

3. Existence of Solution

The goal of this section is to obtain existence of solutions for the problem (1.1).

THEOREM 3.1.— *Consider* F *satisfying* (1.2) *and suppose*

$$(u_0, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma) \quad \text{and} \quad G(u_0) \in L^1(\Gamma).$$

Then there exists a function $u : \Sigma \to \mathbb{R}$ *such that*

$$u \in L^\infty\left(0, T; H^{\frac{1}{2}}(\Gamma)\right), \tag{3.1}$$

$$u' \in L^\infty\left(0, T; L^2(\Gamma)\right), \tag{3.2}$$

$$u'' + Au + F(u) + \beta u' = 0 \quad in \quad L^1\left(0,T;H^{-\frac{1}{2}}(\Gamma) + L^1(\Gamma)\right), \tag{3.3}$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad on \quad \Gamma. \tag{3.4}$$

To prove the Theorem 3.1, the following Lemma will be used:

LEMMA 3.1.— *Assume that $(u_0, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma)$ and suppose that the function F satisfies*

$$F : \mathbb{R} \to \mathbb{R} \text{ be Lipschitz function such that } sF(s) \geqslant 0,\ \forall s \in \mathbb{R}. \tag{3.5}$$

Then there exists only one function $u : \Sigma \to \mathbb{R}$ satisfying the conditions

$$u \in L^\infty\left(0,T;H^{\frac{1}{2}}(\Gamma)\right), \tag{3.6}$$

$$u' \in L^\infty\left(0,T;L^2(\Gamma)\right), \tag{3.7}$$

$$u'' + Au + F(u) + \beta u' = 0 \quad in \quad L^2\left(0,T;H^{-\frac{1}{2}}(\Gamma)\right), \tag{3.8}$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad on \quad \Gamma. \tag{3.9}$$

Furthermore

$$\begin{aligned} &\frac{1}{2}\left|u'(t)\right|^2 + \frac{\alpha}{2}\left\|u(t)\right\|^2 + \int_\Gamma G(u(x,t))\,d\Gamma \leqslant \frac{1}{2}\left|u_1\right|^2 + \\ &+\frac{\alpha}{2}\left\|u_0\right\|^2 + \int_\Gamma G(u_0(x))\,d\Gamma, \ a.e \ in \ [0,T]. \end{aligned} \tag{3.10}$$

Proof of Lemma 3.1.— The proof will be done employing the Picard successive approximations method. Let us consider the sequence of successives approximations

$$u_0,\ u_1,\ u_2,\ \dots,\ u_n,\ \dots \tag{3.11}$$

defined as the solutions of the linear problems

$$\left| \begin{array}{l} u_n'' + Au_n + F(u_{n-1}) + \beta u_n' = 0 \quad \text{on} \quad \Sigma, \\ u_n(0) = u_0, \quad u_n'(0) = u_1 \quad \text{on} \quad \Gamma. \end{array} \right. \tag{3.12}$$

Using that F is Lipschitz and from Theorem 2.1, one can prove, using induction, that (3.12) has a solution for each $n \in \mathbb{N}$ with the regularity claimed in the Theorem 2.1. We will prove now that the sequence (3.11) converges to a function $u : \Sigma \to \mathbb{R}$ in the conditions of the Lemma 3.1.

For this end, we define $v_n = u_n - u_{n-1}$ which is the unique solution of the problem

$$\left| \begin{array}{l} v_n'' + Av_n + F(u_{n-1}) - F(u_{n-2}) + \beta v_n' = 0 \quad \text{on} \quad \Sigma, \\ v_n(0) = 0, \quad v_n'(0) = 0 \quad \text{on} \quad \Gamma. \end{array} \right. \tag{3.13}$$

By the energy inequality (2.9), we have

$$\frac{1}{2}|v_n'(t)|^2 + \frac{\alpha}{2}\|v_n(t)\|^2 \leqslant -\int_0^t (F(u_{n-1}) - F(u_{n-2}), v_n'(s))\,ds. \tag{3.14}$$

Set

$$e_n(t) = \underset{s\in]0,t[}{ess\sup}\left\{\frac{1}{2}|v_n'(s)|^2 + \frac{\alpha}{2}\|v_n(s)\|^2\right\}. \tag{3.15}$$

Thus, since F is Lipschitz, we have

$$-\int_0^t (F(u_{n-1}) - F(u_{n-2}), v_n'(s))\,ds \leqslant C\int_0^t |v_{n-1}(s)|^2\,ds + \frac{1}{2}e_n(t). \tag{3.16}$$

We have also

$$|v_{n-1}(s)|^2 \leqslant Ce_{n-1}(s). \tag{3.17}$$

Combining $(3.14)-(3.17)$, we get

$$e_n(t) \leqslant C\int_0^t e_{n-1}(s)\,ds,$$

and, by interation, we obtain, for $n = 1, 2, \ldots$, that

$$e_n(t) \leqslant e_0 C_T \frac{(Ct)^n}{n!},$$

hence, we conclude that the series $\sum_{n=1}^{\infty} e_n(t)$ is uniformly convergent on $]0,T[$. By the definition of $e_n(t)$, see (3.15), it follows that the series $\sum_{n=1}^{\infty}(u_n' - u_{n-1}')$ and $\sum_{n=1}^{\infty}(u_n - u_{n-1})$ are convergents in the norms of $L^\infty(0,T;L^2(\Gamma))$ and $L^\infty\left(0,T;H^{\frac{1}{2}}(\Gamma)\right)$, respectively. Therefore, there exists $u:\Sigma\to\mathbb{R}$ such that

$$u_n \to u \text{ strong in } L^\infty\left(0,T;H^{\frac{1}{2}}(\Gamma)\right), \tag{3.18}$$

$$u_n' \to u' \text{ strong in } L^\infty\left(0,T;L^2(\Gamma)\right). \tag{3.19}$$

Since F is Lipschitz, we have by (3.18) that

$$F(u_n) \to F(u) \text{ strong in } L^\infty\left(0,T;L^2(\Gamma)\right). \tag{3.20}$$

Then, by the convergences (3.18) – (3.20), we can pass to the limit in (3.12) and we obtain, by standard procedure, a unique function u satisfying (3.6) – (3.10). □

We will prove now the main result.

Proof of Theorem 3.1.— By Strauss [9], there exists a sequence of functions $(F_\nu)_{\nu\in\mathbb{N}}$, such that each $F_\nu : \mathbb{R} \to \mathbb{R}$ is Lipschitz and $(F_\nu)_{\nu\in\mathbb{N}}$ approximates F uniformly on bounded sets of $\mathbb{R}$. Since the initial data u_0 is not necessarily bounded, we have to approximate u_0 by bounded functions of $H^{\frac{1}{2}}(\Gamma)$. We consider the functions $\xi_j : \mathbb{R} \to \mathbb{R}$ defined by

$$\xi_j(s) = \left| \begin{array}{lll} -j, & \text{if} & s < -j, \\ s, & \text{if} & |s| \leqslant j, \\ j, & \text{if} & s > j. \end{array} \right.$$

Considering $\xi_j(u_0) = u_{0j}$, we have by Kinderlehrer and Stampacchia [5] that the sequence $(u_{0j})_{j\in\mathbb{N}} \subset H^{\frac{1}{2}}(\Gamma)$ is bounded a.e. in Γ and

$$u_{0j} \to u_0 \text{ strong in } H^{\frac{1}{2}}(\Gamma). \tag{3.21}$$

Thus, for $(u_{0j}, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma)$, the Lemma 3.1 says that there exists only one solution $u_{j\nu} : \Sigma \to \mathbb{R}$ satifying (3.6) – (3.9) and the energy inequality

$$\begin{aligned} &\frac{1}{2}\left|u'_{j\nu}(t)\right|^2 + \frac{\alpha}{2}\left\|u_{j\nu}(t)\right\|^2 + \int_\Gamma G_\nu\left(u_{j\nu}(x,t)\right) d\Gamma \leqslant \frac{1}{2}\left|u_1\right|^2 + \\ &+\frac{\alpha}{2}\left\|u_{0j}\right\|^2 + \int_\Gamma G_\nu\left(u_{0j}(x)\right) d\Gamma. \end{aligned} \tag{3.22}$$

We need an estimate for the term $\int_\Gamma G_\nu(u_{0j}(x))\, d\Gamma$. Since u_{0j} is bounded a.e. in Γ, $\forall j \in \mathbb{N}$, it follows that

$$F_\nu(u_{0j}) \to F(u_{0j}) \text{ uniform in } \Gamma.$$

So

$$\int_\Gamma G_\nu(u_{0j}(x))\, d\Gamma \to \int_\Gamma G(u_{0j}(x))\, d\Gamma \text{ uniform in } \mathbb{R}. \tag{3.23}$$

From (3.21), there exists a subsequence of $(u_{0j})_{j\in\mathbb{N}}$, which will still be denoted by $(u_{0j})_{j\in\mathbb{N}}$, such that

$$u_{0j} \to u_0 \text{ a.e. in } \Gamma.$$

Hence, by continuity of G, we have that $G(u_{0j}) \to G(u_0)$ a.e. in Γ. We also have that $G(u_{0j}) \leqslant G(u_0) \in L^1(\Gamma)$. Thus, by the Lebesgue's dominated convergence theorem, we get

$$G(u_{0j}) \to G(u_0) \text{ strong in } L^1(\Gamma). \tag{3.24}$$

Then, by (3.23) and (3.24), we obtain that

$$\int_\Gamma G_\nu(u_{0j}(x))\, d\Gamma \leqslant C, \tag{3.25}$$

where C is independent of j and ν. In this way, using (3.21) and (3.25), we have from (3.22) that

$$\left|u'_{j\nu}\right|^2 + \|u_{j\nu}\|^2 + \int_\Gamma G(u_{j\nu}(x,t))\, d\Gamma \leqslant C, \tag{3.26}$$

where C is independent of j, ν and t.

From (3.26), we obtain that

$$(u_{j\nu}) \text{ is bounded in } L^\infty\left(0,T;H^{\frac{1}{2}}(\Gamma)\right), \tag{3.27}$$

$$\left(u'_{j\nu}\right) \text{ is bounded in } L^\infty\left(0,T;L^2(\Gamma)\right). \tag{3.28}$$

We have that (3.27) and (3.28) are true for all pairs $(j,\nu) \in \mathbb{N}^2$, in particular, for $(i,i) \in \mathbb{N}^2$. Thus, there exists a subsequence of (u_{ii}), which we denote by (u_i), and a function $u : \Sigma \to \mathbb{R}$, such that

$$u_i \to u \text{ weak star in } L^\infty\left(0,T;H^{\frac{1}{2}}(\Gamma)\right), \tag{3.29}$$

$$u'_i \to u' \text{ weak in } L^\infty\left(0,T;L^2(\Gamma)\right). \tag{3.30}$$

We also have by (3.8) that

$$u''_i + Au_i + F_i(u_i) + \beta u'_i = 0 \quad \text{in} \quad L^2\left(0,T;H^{-\frac{1}{2}}(\Gamma)\right). \tag{3.31}$$

From (3.29), (3.30) and observing that the injection of $H^1(\Sigma)$ in $L^2(\Sigma)$ is compact, there exists a subsequence of (u_i), which we still denote by (u_i), such that

$$u_i \to u \text{ a.e. in } \Sigma.$$

Since F is continuous

$$F(u_i) \to F(u) \text{ a.e. in } \Sigma.$$

Furthermore, since $u_i(x,t)$ is bounded in $\mathbb{R}$,

$$F_i(u_i) - F(u_i) \to 0 \text{ a.e. in } \Sigma.$$

Therefore, we conclude

$$F_i(u_i) \to F(u) \text{ a.e. in } \Sigma. \tag{3.32}$$

Taking duality between (3.31) and u_i we obtain

$$\begin{aligned} \int_0^T (F_i(u_i), u_i(t))\,dt &= \int_0^T |u_i'(t)|^2\,dt - \alpha \int_0^T \|u_i(t)\|^2\,dt - \\ &- (u_i'(T), u_i(T)) + (u_1, u_{0j}) - \int_0^T (\beta u_i'(t), u_i(t))\,dt. \end{aligned} \tag{3.33}$$

Using (2.1), (3.6) and (3.7), we have by (3.33) that

$$\int_0^T (F_i(u_i), u_i(t))\,dt \leqslant C, \tag{3.34}$$

where C is independent of i.

Thus, from (3.32) and (3.34), it follows by Strauss' theorem, see Strauss [9], that

$$F_i(u_i) \to F(u) \text{ strongly in } L^1(\Sigma). \tag{3.35}$$

By (3.29), (3.30) and (3.35) it is permissible to pass to the limit in (3.31) obtaining a function $u : \Sigma \to \mathbb{R}$ satisfying $(3.1) - (3.4)$. □

4. Asymptotic Behaviour

In this section we study the exponential decay for the energy $E(t)$ associated to the weak solution u given by the Theorem 3.1. This energy is given by

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{\alpha}{2}\|u(t)\|^2 + \int_\Gamma G(u(x,t))\,d\Gamma,\ t \geqslant 0. \tag{4.1}$$

We consider the followings additional hypothesis:

$$0 \leqslant G(s) \leqslant sF(s),\ \forall s \in \mathbb{R} \tag{4.2}$$

THEOREM 4.1.— *Let F satisfying* (1.2) *and* (4.2). *Then the energy* (4.1) *satisfies*

$$E(t) \leqslant 4E(0)e^{-\frac{\epsilon}{2}t}, \tag{4.3}$$

where ϵ is a positive constant.

Proof.— For an arbitrary $\epsilon > 0$, we define the perturbed energy

$$E_{\nu\epsilon}(t) = E_\nu(t) + \epsilon\eta(t) \tag{4.4}$$

where $E_\nu(t)$ is the energy similar to (4.1) associated to the solution obtained in the Lemma 3.1 and

$$\eta(t) = (u_\nu(t), u_\nu'(t)).$$

Note that

$$|\eta(t)| \leqslant C_2 E_\nu(t),$$

where $C_2 = \max\left\{C_1, \dfrac{1}{\alpha}\right\}$, and C_1 is the immersion constant of $H^{\frac{1}{2}}(\Gamma)$ into $L^2(\Gamma)$.

Then,

$$|E_{\nu\epsilon}(t) - E_\nu(t)| \leqslant \epsilon C_2 E_\nu(t),$$

or

$$(1 - \epsilon C_2) E_\nu(t) \leqslant E_{v\epsilon}(t) \leqslant (1 + \epsilon C_2) E_v(t).$$

Taking $0 < \epsilon \leqslant \dfrac{1}{2C_2}$, we get

$$\frac{E_\nu(t)}{2} \leqslant E_{\nu\epsilon}(t) \leqslant 2E_\nu(t),\ \forall t \geqslant 0. \tag{4.5}$$

Multiplying the equation in (3.8) for u_ν', using (2.1) and the fact of A to be positive, we obtain

$$E_\nu'(t) \leqslant -\beta_0 |u_\nu'(t)|^2 \leqslant 0. \tag{4.6}$$

Differentiating the function $\eta(t)$ and using (3.8), (4.2) and the fact of A to be positive comes that

$$\eta'(t) \leqslant \left(1 + \frac{\beta_1}{2\mu}\right) \left|u_\nu'(t)\right|^2 + \left(\frac{\beta_1 \mu C_1}{2} - \alpha\right) \|u_\nu(t)\|^2 - \int_\Gamma G_\nu(u_\nu) d\Gamma, \tag{4.7}$$

where $\beta_1 = \|\beta\|_{L^\infty(\Gamma)}$ and $\mu > 0$ to be chosen.

It follows by (4.4), (4.6) and (4.7) that

$$E'_{\nu\epsilon}(t) \leqslant \left[\epsilon\left(1+\frac{\beta_1}{2\mu}\right)-\beta_0\right]\left|u'_\nu(t)\right|^2-\epsilon\left(\alpha-\frac{\beta_1\mu C_1}{2}\right)\|u_\nu(t)\|^2- \\ -\epsilon\int_\Gamma G_\nu(u_\nu)d\Gamma. \tag{4.8}$$

Taking $\mu=\dfrac{\alpha}{\beta_1 C_1}$ and $0<\epsilon\leqslant\dfrac{2\alpha\beta_0}{3\alpha+\beta_1^2C_1}$ we get

$$E'_{\nu\epsilon}(t)\leqslant -E_\nu(t). \tag{4.9}$$

Choosing $\epsilon\leqslant\min\left\{\dfrac{1}{2C_2},\dfrac{2\alpha\beta_0}{3\alpha+\beta_1^2C_1}\right\}$ then (4.5) and (4.9) occur simultaneously, therefore

$$E'_{\nu\epsilon}(t)+\frac{\epsilon}{2}E_{\nu\epsilon}(t)\leqslant 0,$$

that is,

$$E_\nu(t)\leqslant 4E_\nu(0)e^{-\frac{\epsilon}{2}t}. \tag{4.10}$$

From (3.29) , (3.30) and since G_ν is continuous, we have

$$G_\nu(u_\nu(\cdot,t))-G_\nu(u(\cdot,t))\to 0 \text{ a.e. in } \Gamma,\ \forall t\geqslant 0. \tag{4.11}$$

But we know that $F_\nu\to F$ uniformly on bounded sets of $\mathbb{R}$. Then

$$G_\nu\left(u\left(\cdot,t\right)\right)\to G\left(u\left(\cdot,t\right)\right) \text{ a.e. in } \Gamma,\ \forall t\geqslant 0. \tag{4.12}$$

Thus, by (4.11) and (4.12)

$$G_\nu\left(u_\nu\left(\cdot,t\right)\right)\to G\left(u\left(\cdot,t\right)\right) \text{ a.e. in } \Gamma,\ \forall t\geqslant 0. \tag{4.13}$$

Moreover, we have, by (4.10) , that

$$\int_\Gamma G_\nu\left(u_\nu\left(x,t\right)\right)d\Gamma\leqslant 4E_\nu\left(0\right).$$

Therefore, using (3.21) , (3.23) and (3.24) , we get

$$\liminf_{\nu\to\infty}\int_\Gamma G_\nu\left(u_\nu\left(x,t\right)\right)d\Gamma\leqslant 4E\left(0\right). \tag{4.14}$$

By (4.13), (4.14) and Fatou's lemma, we have

$$\int_{\Gamma} G(u(x,t))\,d\Gamma \leqslant \liminf_{\nu\to\infty} \int_{\Omega} G_{\nu}(u_{\nu}(x,t))\,d\Gamma.$$

Hence, passing $\liminf_{\nu\to\infty}$ in (4.10), we get (4.3). □

Remark. — In the existence we can take $\lambda = 0$. For this end, we define in $H^1(\Omega)$ the norm

$$[v]^2 = \int_{\Omega} |\nabla v|^2\,dx + \int_{\Gamma} |\gamma_0 v|^2\,d\Gamma, \quad \forall v \in H^1(\Omega),$$

obtaining now the positivity of operator $A + \zeta I$, for $\zeta > 0$ arbitrary, like in Lions [8]. For the asymptotic behaviour, we need the additional hypothesis $\beta_0 > \zeta$.

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