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The canonical solution operator to $\overline{\partial}$ restricted to spaces of entire functions (*)

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RÉSUMÉ. — Dans cet article, nous étudions l'opérateur canonique de solution du $\overline{\partial}$ dans les espaces $L^2(\mathbb{C}^n, e^{-p})$ a poids et discutons de ses propriétés de compacité et d'être de Hilbert-Schmidt. Dans le cas d'une seule variable complexe, nous montrons que cet opérateur solution n'est pas compact dans $L^2(\mathbb{C}, e^{-|z|^2})$ même si on se restreint au sous-espace correspondant de fonctions entières. L'opérateur solution est compact quand on le restreint au sous-espace des fonctions entières pour les poids $e^{-|z|^m}, m > 2$, mais n'est pas Hilbert-Schmidt. Dans la seconde partie, nous montrons que, dans un contexte légerement different, nous obtenons la propriété d'être de type Hilbert-Schmidt pour une classe très large d'espaces a poids de fonctions entières de plusieurs variables complexes.

ABSTRACT. — In this paper we discuss compactness and the Hilbert-Schmidt property of the canonical solution operator to $\overline{\partial}$ in weighted $L^2(\mathbb{C}^n, e^{-p})$ spaces. In the case of one complex variable we show that the solution operator is not compact on $L^2(\mathbb{C}, e^{-|z|^2})$ even when restricted to the corresponding subspace of entire functions; the solution operator is compact when restricted to the subspace of entire functions for the weights $e^{-|z|^m}$, $m > 2$, but fails to be Hilbert-Schmidt. In the second part we show that we get the Hilbert-Schmidt property in a slightly different setting for a large class of weighted spaces of entire functions in several variables.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. In [7] it is shown that the canonical solution operator S to $\overline{\partial}$ restricted to $(0,1)$-forms with holomorphic coefficients can be expressed by an integral operator using the Bergman kernel:

$$S(g)(z) = \int_\Omega K(z,w)\,\langle g(w), z-w\rangle\, d\lambda(w),$$

where $g = \sum_{j=1}^n g_j\, d\overline{z}_j \in A^2_{(0,1)}(\Omega)$ is a $(0,1)$-form with holomorphic coefficients, $\langle g(w), z-w\rangle = \sum_{j=1}^n g_j(w)(\overline{z}_j - \overline{w}_j)$ and $K(z,w)$ is the Bergman kernel of Ω. The canonical solution operator to $\overline{\partial}$ has the properties : $\overline{\partial}S(g) = g$ and $S(g) \perp A^2(\Omega)$.

The canonical solution operator to $\overline{\partial}$ restricted to $(0,1)$-forms with holomorphic coefficients can also be interpreted as the Hankel operator

$$H_{\overline{z}}(g) = (I-P)(\overline{z}g),$$

where $P : L^2(\Omega) \longrightarrow A^2(\Omega)$ denotes the Bergman projection. See [1], [2], [9], [14] and [16] for details.

For the unit disk $\mathbb{D}$ in $\mathbb{C}$ the canonical solution operator restricted to $A^2(\mathbb{D})$ is a Hilbert-Schmidt operator, whereas for the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, $n \geq 2$ the canonical solution operator fails to be Hilbert-Schmidt (see [7]).

In many cases non-compactness of the canonical solution operator already happens when the solution operator is restricted to the corresponding subspace of holomorphic functions (or $(0,1)$-forms with holomorphic coefficients, in the case of several variables.)(see [4] , [13] , [11]). In this paper we will show that this phenomenon also occurs in the Fock space in one variable.

It is pointed out in [4] that in the proof that compactness of the solution operator for $\overline{\partial}$ on $(0,1)$-forms implies that the boundary of Ω does not contain any analytic variety of dimension greater than or equal to 1, it is only used that there is a compact solution operator to $\overline{\partial}$ on the $(0,1)$-forms with holomorphic coefficients. In this case compactness of the solution operator restricted to $(0,1)$-forms with holomorphic coefficients implies already compactness of the solution operator on general $(0,1)$-forms.

A similar situation appears in [13] where the Toeplitz C^* -algebra $\mathcal{T}(\Omega)$ is considered and the relation between the structure of $\mathcal{T}(\Omega)$ and the $\overline{\partial}$-Neumann problem is discussed (see [13] , Corollary 4.6). The question of compactness of the $\overline{\partial}$-Neumann operator is of interest for various reasons (see the survey article [5]).

In this paper we show that the canonical solution operator for $\overline{\partial}$ as operator from $L^2(\mathbb{C}, e^{-|z|^2})$ into itself is not compact. This follows from the result that the canonical solution operator for $\overline{\partial}$ restricted to the weighted space of entire functions $A^2(\mathbb{C}, e^{-|z|^2})$ (Fock space) into $L^2(\mathbb{C}, e^{-|z|^2})$ already fails to be compact. Further it is shown that the restriction to $A^2(\mathbb{C}, e^{-|z|^m})$, $m > 2$, is compact but not Hilbert-Schmidt. When using the methods of [7] in this case the main difficulty is it that there are functions $g \in A^2(\mathbb{C}, e^{-|z|^m})$ such that $\overline{z}g \notin L^2(\mathbb{C}, e^{-|z|^m})$. Hence the formula for the canonical solution operator using the Bergamn kernel can't be used directly, but it will turn out that the expression $\overline{z}g(z) - P(\overline{z}g)(z)$ makes sense in $L^2(\mathbb{C}, e^{-|z|^m})$.

In the sequel we also consider the case of several complex variables in a slightly different situation and show that the canonical solution operator to $\overline{\partial}$ is a Hilbert-Schmidt operator for a wide class of weighted spaces of entire functions using various methods from abstract functional analysis (see [12]).

2. Spaces of entire functions in one variable

We consider weighted spaces on entire functions

$$A^2(\mathbb{C}, e^{-|z|^m}) = \{f : \mathbb{C} \longrightarrow \mathbb{C} \ : \ \|f\|_m^2 := \int_{\mathbb{C}} |f(z)|^2 \, e^{-|z|^m} \, d\lambda(z) < \infty\},$$

where $m > 0$. Let

$$c_k^2 = \int_{\mathbb{C}} |z|^{2k} \, e^{-|z|^m} \, d\lambda(z).$$

Then

$$K_m(z, w) = \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{c_k^2}$$

is the reproducing kernel for $A^2(\mathbb{C}, e^{-|z|^m})$.

In the sequel the expression

$$\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2}$$

will become important. Using the integral representation of the $\Gamma-$function one easily sees that the above expression is equal to

$$\frac{\Gamma\left(\frac{2k+4}{m}\right)}{\Gamma\left(\frac{2k+2}{m}\right)} - \frac{\Gamma\left(\frac{2k+2}{m}\right)}{\Gamma\left(\frac{2k}{m}\right)}.$$

For $m = 2$ this expression equals to 1 for each $k = 1, 2, \ldots$. We will be interested in the limit behavior for $k \to \infty$. By Stirlings formula the limit behavior is equivalent to the limit behavior of the expression

$$\left(\frac{2k+2}{m}\right)^{2/m} - \left(\frac{2k}{m}\right)^{2/m},$$

as $k \to \infty$. Hence we have shown the following

LEMMA 1.— *The expression*

$$\frac{\Gamma\left(\frac{2k+4}{m}\right)}{\Gamma\left(\frac{2k+2}{m}\right)} - \frac{\Gamma\left(\frac{2k+2}{m}\right)}{\Gamma\left(\frac{2k}{m}\right)}$$

tends to ∞ *for* $0 < m < 2$, *is equal to* 1 *for* $m = 2$ *and tends to zero for* $m > 2$ *as* k *tends to* ∞.

Let $0 < \rho < 1$, define $f_\rho(z) := f(\rho z)$ and $\tilde{f}_\rho(z) = \overline{z} f_\rho(z)$, for $f \in A^2(\mathbb{C}, e^{-|z|^m})$. Then it is easily seen that $\tilde{f}_\rho \in L^2(\mathbb{C}, e^{-|z|^m})$, but there are functions $g \in A^2(\mathbb{C}, e^{-|z|^m})$ such that $\overline{z} g \notin L^2(\mathbb{C}, e^{-|z|^m})$.

Let $P_m : L^2(\mathbb{C}, e^{-|z|^m}) \longrightarrow A^2(\mathbb{C}, e^{-|z|^m})$ denote the orthogonal projection. Then P_m can be written in the form

$$P_m(f)(z) = \int_{\mathbb{C}} K_m(z, w) f(w) e^{-|w|^m} \, d\lambda(w) \ , \ f \in L^2(\mathbb{C}, e^{-|z|^m}).$$

PROPOSITION 1.— *Let* $m \geq 2$. *Then there is a constant* $C_m > 0$ *depending only on* m *such that*

$$\int_{\mathbb{C}} \left| \tilde{f}_\rho(z) - P_m(\tilde{f}_\rho)(z) \right|^2 e^{-|z|^m} \, d\lambda(z) \leq C_m \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m} \, d\lambda(z),$$

for each $0 < \rho < 1$ *and for each* $f \in A^2(\mathbb{C}, e^{-|z|^m})$.

Proof.— First we observe that for the Taylor expansion of $f(z) = \sum_{k=0}^\infty a_k z^k$ we have

$$\begin{aligned} P_m(\tilde{f}_\rho)(z) &= \int_{\mathbb{C}} \sum_{k=0}^\infty \frac{z^k \overline{w}^k}{c_k^2} \left(\overline{w} \sum_{j=0}^\infty a_j \rho^j w^j \right) e^{-|w|^m} \, d\lambda(w) \\ &= \sum_{k=1}^\infty a_k \frac{c_k^2}{c_{k-1}^2} \rho^k z^{k-1}. \end{aligned}$$

Now we obtain

$$
\begin{aligned}
&\int_{\mathbb{C}} \left| \tilde{f}_\rho(z) - P_m(\tilde{f}_\rho)(z) \right|^2 e^{-|z|^m} \, d\lambda(z) \\
= &\int_{\mathbb{C}} \left(\overline{z} \sum_{k=0}^{\infty} a_k \rho^k z^k - \sum_{k=1}^{\infty} a_k \, \frac{c_k^2}{c_{k-1}^2} \, \rho^k z^{k-1} \right) \\
\times &\left(z \sum_{k=0}^{\infty} a_k \rho^k \overline{z}^k - \sum_{k=1}^{\infty} \overline{a_k} \, \frac{c_k^2}{c_{k-1}^2} \, \rho^k \overline{z}^{k-1} \right) e^{-|z|^m} \, d\lambda(z) \\
= &\int_{\mathbb{C}} (\sum_{k=0}^{\infty} |a_k|^2 \rho^{2k} |z|^{2k+2} - 2 \sum_{k=1}^{\infty} |a_k|^2 \, \frac{c_k^2}{c_{k-1}^2} \, \rho^{2k} |z|^{2k} \\
+ &\sum_{k=1}^{\infty} |a_k|^2 \, \frac{c_k^4}{c_{k-1}^4} \, \rho^{2k} |z|^{2k-2}) \; e^{-|z|^m} \, d\lambda(z) \\
= &\; |a_0|^2 \, c_1^2 + \sum_{k=1}^{\infty} |a_k|^2 \, c_k^2 \, \rho^{2k} \left(\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2} \right).
\end{aligned}
$$

Now the result follows from the fact that

$$
\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m} \, d\lambda(z) = \sum_{k=0}^{\infty} |a_k|^2 \, c_k^2,
$$

and that the sequence $\left(\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2} \right)_k$ is bounded. □

Remark 1.— Already in the last proposition the sequence $\left(\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2} \right)_k$ *plays an important role and it will turn out that this sequence is the sequence of eigenvalues of the operator* $S_m^* S_m$ *(see below).*

PROPOSITION 2.— *Let* $m \geq 2$ *and consider an entire function* $f \in A^2(\mathbb{C}, e^{-|z|^m})$ *with Taylor series expansion* $f(z) = \sum_{k=0}^{\infty} a_k z^k$. *Let*

$$
F(z) := \overline{z} \sum_{k=0}^{\infty} a_k z^k - \sum_{k=1}^{\infty} a_k \, \frac{c_k^2}{c_{k-1}^2} \, z^{k-1}
$$

and define $S_m(f) := F$. *Then* $S_m : A^2(\mathbb{C}, e^{-|z|^m}) \longrightarrow L^2(\mathbb{C}, e^{-|z|^m})$ *is a continuous linear operator, representing the canonical solution operator to* $\overline{\partial}$ *restricted to* $A^2(\mathbb{C}, e^{-|z|^m})$, *i.e.* $\overline{\partial} S_m(f) = f$ *and* $S_m(f) \perp A^2(\mathbb{C}, e^{-|z|^m})$.

Proof. — By Fatou's theorem

$$\begin{aligned}
&\int_{\mathbb{C}} \lim_{\rho\to 1} \left|\tilde{f}_\rho(z) - P_m(\tilde{f}_\rho)(z)\right|^2 e^{-|z|^m}\, d\lambda(z) \\
&\leq \sup_{0<\rho<1} \int_{\mathbb{C}} \left|\tilde{f}_\rho(z) - P_m(\tilde{f}_\rho)(z)\right|^2 e^{-|z|^m}\, d\lambda(z) \\
&\leq C_m \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m}\, d\lambda(z)
\end{aligned}$$

and hence the function

$$F(z) := \overline{z}\sum_{k=0}^{\infty} a_k z^k - \sum_{k=1}^{\infty} a_k \frac{c_k^2}{c_{k-1}^2}\, z^{k-1}$$

belongs to $L^2(\mathbb{C}, e^{-|z|^m})$ and satisfies

$$\int_{\mathbb{C}} |F(z)|^2 e^{-|z|^m}\, d\lambda(z) \leq C_m \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m}\, d\lambda(z).$$

The above computation also shows that $\lim_{\rho\to 1} \|\tilde{f}_\rho - P_m(\tilde{f}_\rho)\|_m = \|F\|_m$ and by a standard argument for L^p-spaces (see for instance [3])

$$\lim_{\rho\to 1} \|\tilde{f}_\rho - P_m(\tilde{f}_\rho) - F\|_m = 0.$$

A similar computation as in the proof of Proposition 1 in [7] shows that the function F defined above satisfies $\overline{\partial} F = f$. Let $S_m(f) := F$. Then, by the last remarks, $S_m : A^2(\mathbb{C}, e^{-|z|^m}) \longrightarrow L^2(\mathbb{C}, e^{-|z|^m})$ is a continuous linear solution operator for $\overline{\partial}$. For arbitrary $h \in A^2(\mathbb{C}, e^{-|z|^m})$ we have

$$(h, S_m(f))_m = (h, F)_m = \lim_{\rho\to 1}(h, \tilde{f}_\rho - P_m(\tilde{f}_\rho))_m = \lim_{\rho\to 1}(h - P_m(h), \tilde{f}_\rho)_m = 0,$$

where $(.,.)_m$ denotes the inner product in $L^2(\mathbb{C}, e^{-|z|^m})$. Hence S_m is the canonical solution operator for $\overline{\partial}$ restricted to $A^2(\mathbb{C}, e^{-|z|^m})$. □

Remark 2. — *The expression for the function F in the last theorem corresponds formally to the expression $\overline{z}f - P_m(\overline{z}f)$; in general $\overline{z}f \notin L^2(\mathbb{C}, e^{-|z|^m})$, for $f \in A^2(\mathbb{C}, e^{-|z|^m})$, but $f \mapsto F$ defines a bounded linear operator from $A^2(\mathbb{C}, e^{-|z|^m})$ to $L^2(\mathbb{C}, e^{-|z|^m})$.*

THEOREM 1. — *The canonical solution operator to $\overline{\partial}$ restricted to the space $A^2(\mathbb{C}, e^{-|z|^m})$ is compact if and only if*

$$\lim_{k\to\infty} \left(\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2}\right) = 0.$$

Proof. — For a complex polynomial p the canonical solution operator S_m can be written in the form

$$S_m(p)(z) = \int_{\mathbb{C}} K_m(z,w)p(w)(\overline{z} - \overline{w})e^{-|w|^m}\, d\lambda(w),$$

therefore we can express the conjugate S_m^* in the form

$$S_m^*(q)(w) = \int_{\mathbb{C}} K_m(w,z)q(z)(z-w)e^{-|z|^m}\, d\lambda(z),$$

if q is a finite linear combination of the terms $\overline{z}^k\, z^l$. This follows by considering the inner product $(S_m(p), q)_m = (p, S_m^*(q))_m$.

Now we claim that

$$S_m^* S_m(u_n)(w) = \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}\right) u_n(w) \quad , n = 1, 2, \ldots$$

and

$$S_m^* S_m(u_0)(w) = \frac{c_1^2}{c_0^2}\, u_0(w),$$

where $\{u_n(z) = z^n/c_n, k = 0, 1, \ldots\}$ is the standard orthonormal basis of $A^2(\mathbb{C}, e^{-|z|^m})$.

From [7] we know that

$$S_m(u_n)(z) = \overline{z}u_n(z) - \frac{c_n z^{n-1}}{c_{n-1}^2}, \quad n = 1, 2, \ldots.$$

Hence

$$\begin{aligned} S_m^* S_m(u_n)(w) &= \int_{\mathbb{C}} K_m(w,z)(z-w)\left(\frac{\overline{z}z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2}\right) e^{-|z|^m}\, d\lambda(z) \\ &= \int_{\mathbb{C}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2}(z-w)\left(\frac{\overline{z}z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2}\right) e^{-|z|^m}\, d\lambda(z). \end{aligned}$$

This integral is computed in two steps: first the multiplication by z

$$\int_{\mathbb{C}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2}\left(\frac{\overline{z}z^{n+1}}{c_n} - \frac{c_n z^n}{c_{n-1}^2}\right) e^{-|z|^m}\, d\lambda(z)$$

$$= \int_{\mathbb{C}} \frac{z^{n+1}}{c_n} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^{k+1}}{c_k^2} e^{-|z|^m}\, d\lambda(z) - \frac{c_n}{c_{n-1}^2} \int_{\mathbb{C}} z^n \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} e^{-|z|^m}\, d\lambda(z)$$

$$
\begin{aligned}
&= \frac{w^n}{c_n^3}\int_{\mathbb{C}} |z|^{2n+2}e^{-|z|^m}\,d\lambda(z) - \frac{w^n}{c_{n-1}^2 c_n^2}\int_{\mathbb{C}} |z|^{2n}e^{-|z|^m}\,d\lambda(z)\\
&= \left(\frac{c_{n+1}^2}{c_n^3} - \frac{c_n}{c_{n-1}^2}\right) w^n .
\end{aligned}
$$

And now the multiplication by w

$$
w\int_{\mathbb{C}}\sum_{k=0}^{\infty}\frac{w^k\overline{z}^k}{c_k^2}\left(\frac{\overline{z}z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2}\right)e^{-|z|^m}\,d\lambda(z)
$$

$$
\begin{aligned}
&= w\int_{\mathbb{C}}\frac{z^n}{c_n}\sum_{k=0}^{\infty}\frac{w^k\overline{z}^{k+1}}{c_k^2}e^{-|z|^m}\,d\lambda(z) - w\int_{\mathbb{C}}\frac{c_n z^{n-1}}{c_{n-1}^2}\sum_{k=0}^{\infty}\frac{w^k\overline{z}^k}{c_k^2}e^{-|z|^m}\,d\lambda(z)\\
&= w\left(\frac{c_n w^{n-1}}{c_{n-1}^2} - \frac{c_n w^{n-1}}{c_{n-1}^2}\right)\\
&= 0,
\end{aligned}
$$

which implies that

$$
S_m^* S_m(u_n)(w) = \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}\right) u_n(w) \quad ,n = 1,2,\dots,
$$

the case $n = 0$ follows from an analogous computation.

The last statement says that $S_m^* S_m$ is a diagonal operator with respect to the orthonormal basis $\{u_n(z) = z^n/c_n\}$ of $A^2(\mathbb{C}, e^{-|z|^m})$. Therefore it is easily seen that $S_m^* S_m$ is compact if and only if

$$
\lim_{n\to\infty}\left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}\right) = 0.
$$

Now the conclusion follows, since $S_m^* S_m$ is compact if and only if S_m is compact (see for instance [15]). □

THEOREM 2. — *The canonical solution operator for $\overline{\partial}$ restricted to the space $A^2(\mathbb{C}, e^{-|z|^m})$ is compact, if $m > 2$. The canonical solution operator for $\overline{\partial}$ as operator from $L^2(\mathbb{C}, e^{-|z|^2})$ into itself is not compact.*

Proof. — The first statement follows immediately from Theorem 1 and Lemma 1 For the second statement we use Hörmander's L^2-estimate for the solution of the $\overline{\partial}$ equation [8] : for each $f \in L^2(\mathbb{C}, e^{-|z|^2})$ there is a function $u \in L^2(\mathbb{C}, e^{-|z|^2})$ such that $\overline{\partial} u = f$ and

$$
\int_{\mathbb{C}} |u(z)|^2\, e^{-|z|^2}\,d\lambda(z) \le 4\int_{\mathbb{C}} |f(z)|^2\, e^{-|z|^2}\,d\lambda(z).
$$

Hence the canonical solution operator for $\overline{\partial}$ as operator from $L^2(\mathbb{C}, e^{-|z|^2})$ into itself is continuous and its restriction to the closed subspace $A^2(\mathbb{C}, e^{-|z|^2})$ fails to be compact by Propositon 1 and Lemma 1. By the definition of compactness this implies that the canonical solution operator is not compact as operator from $L^2(\mathbb{C}, e^{-|z|^2})$ into itself. □

Remark 3.— In the case of the Fock space $A^2(\mathbb{C}, e^{-|z|^2})$ the composition $S_2^* S_2$ equals to the identity on $A^2(\mathbb{C}, e^{-|z|^2})$, which follows from the proof of Theorem 1.

THEOREM 3.— *Let $m \geq 2$. The canonical solution operator for $\overline{\partial}$ restricted to $A^2(\mathbb{C}, e^{-|z|^m})$ fails to be Hilbert Schmidt.*

Proof. — By Proposition 2 we know that the canonical solution operator is continuous and we can use the techniques from [7]

$$
\begin{aligned}
\|S_m(u_n)\|^2 &= \frac{1}{c_n^2}\int_{\mathbb{C}} \left| \overline{z}\, z^n - \frac{c_n^2}{c_{n-1}^2}\, z^{n-1} \right|^2 e^{-|z|^m}\, d\lambda(z) \\
&= \frac{1}{c_n^2}\int_{\mathbb{C}} |z|^{2n-2}\left(|z|^4 - \frac{2c_n^2|z|^2}{c_{n-1}^2} + \frac{c_n^4}{c_{n-1}^4} \right) e^{-|z|^m}\, d\lambda(z) \\
&= \frac{1}{c_n^2}\int_{\mathbb{C}} |z|^{2n+2}\, e^{-|z|^m}\, d\lambda(z) - \frac{2}{c_{n-1}^2}\int_{\mathbb{C}} |z|^{2n}\, e^{-|z|^m}\, d\lambda(z) \\
&+ \frac{c_n^2}{c_{n-1}^4}\int_{\mathbb{C}} |z|^{2n-2}\, e^{-|z|^m}\, d\lambda(z) \\
&= \frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}.
\end{aligned}
$$

Hence

$$\sum_{n=0}^{\infty} \|S_m(u_n)\|^2 < \infty$$

if and only if

$$\lim_{n\to\infty} \frac{c_{n+1}^2}{c_n^2} < \infty.$$

By [12] , 16.8, S_m is a Hilbert Schmidt operator if and only if

$$\sum_{n=0}^{\infty} \|S_m(u_n)\|^2 < \infty.$$

In our case we have

$$\frac{c_{n+1}^2}{c_n^2} = \Gamma\left(\frac{2n+4}{m}\right) / \Gamma\left(\frac{2n+2}{m}\right),$$

which, by Stirling's formula, implies that the corresponding canonical solution operator to $\overline{\partial}$ fails to be Hilbert Schmidt. $\square$

In the case of several variables the corresponding operator S^*S is more complicated, nevertheless we can handle a sligthly different situation with different methods from functional analysis (see next section).

3. Weighted spaces of entire functions in several variables

In this part we show that the canonical solution operator to $\overline{\partial}$ is a Hilbert-Schmidt operator for a wide class of weighted spaces of entire functions.

The weight functions we are considering are of the form $z \mapsto \tau p(z)$, where $\tau > 0$ and $p : \mathbb{C}^n \longrightarrow \mathbb{R}$. We suppose that p is a plurisubharmonic function satisfying

$$p^*(w) := \sup\{\Re < z, w > -p(z) \ : \ z \in \mathbb{C}^n\} < \infty.$$

Then $p^{**} = p$ and

$$\lim_{|z|\to\infty} \frac{p(z)}{|z|} = \infty$$

(see Lemma 1.1. in [6]). And it is easily seen that

$$\int_{\mathbb{C}^n} \exp[(\tau - \sigma)\, p(z)]\, d\lambda(z) < \infty,$$

whenever $\tau - \sigma < 0$.

We further assume that

$$\lim_{|z|\to\infty} \frac{\tilde{p}(z)}{p(z)} = 1,$$

where $\tilde{p}(z) = \sup\{p(z+\zeta) \ : \ |\zeta| \leq 1\}$.

It follows that the last property is equivalent to the following condition: for each $\sigma > 0$ and for each $\tau > 0$ with $\tau < \sigma$ there is a constant $C = C(\sigma, \tau) > 0$ such that

$$\tau\, \tilde{p}(z) - \sigma\, p(z) \leq C,$$

for each $z \in \mathbb{C}^n$.

Let $A^2(\mathbb{C}^n, \sigma p)$ denote the Hilbert space of all entire functions $h : \mathbb{C}^n \longrightarrow \mathbb{C}$ such that

$$\int_{\mathbb{C}^n} |h(z)|^2 \exp(-2\sigma p(z))\, d\lambda(z) < \infty.$$

THEOREM 4. — *Suppose that p is a weight function with the properties listed above. Then for each $\sigma > 0$ there exists a number $\tau > 0$ with $\tau < \sigma$ such that the canonical solution operator S_1 to $\overline{\partial}$ is a Hilbert-Schmidt operator as a mapping*

$$S_1 : A^2_{(0,1)}(\mathbb{C}^n, \tau p) \longrightarrow L^2(\mathbb{C}^n, \sigma p).$$

Proof. — By Lemma 28.2 from [12] we have to show that

$$\left[\int_{\mathbb{C}^n} |S_1(f)(z)|^2 \exp(-2\sigma p(z))\, d\lambda(z)\right]^{1/2} \leq \int_{\mathcal{U}} |(f,g)|\, d\mu(g),$$

where $(.,.)$ denotes the inner product of the Hilbert space $A^2_{(0,1)}(\mathbb{C}^n, \tau p)$, $\mathcal{U}$ is the unit ball of $A^2_{(0,1)}(\mathbb{C}^n, \tau p)$, μ is a Radon measure on the weakly compact set $\mathcal{U}$ and $f = \sum_{j=1}^n f_j\, d\overline{z}_j$ and $g = \sum_{j=1}^n g_j\, d\overline{z}_j$.

We first show that for $0 < \tau < \tau_1 < \tau_2 < \tau_3 < \sigma$ we have

$$\begin{aligned}
&\left[\int_{\mathbb{C}^n} |f_j(z)|^2 \exp(-2\tau_3 p(z))\, d\lambda(z)\right]^{1/2} \\
\leq\ & C_{\tau_3,\tau_2} \sup\{|f_j(z)| \exp(-\tau_2 p(z)) \ :\ z \in \mathbb{C}^n\} \\
\leq\ & C_{\tau_2,\tau_1} \int_{\mathbb{C}^n} |f_j(z)| \exp(-\tau_1 p(z))\, d\lambda(z) \\
\leq\ & C_{\tau_1,\tau} \left[\int_{\mathbb{C}^n} |f_j(z)|^2 \exp(-2\tau p(z))\, d\lambda(z)\right]^{1/2},
\end{aligned}$$

for each $f \in A^2_{(0,1)}(\mathbb{C}^n, \tau p)$.

To show this assertion we make use of the assumption that the coefficients of the $(0,1)$-form f are entire functions:

The first inequality follows from the fact that

$$\int_{\mathbb{C}^n} \exp((2\tau_2 - 2\tau_3)p(z))\, d\lambda(z) < \infty.$$

For the second inequality use Cauchy's theorem for the coefficients f_j of f to show that for $B_z = \{\zeta \in \mathbb{C}^n \ :\ |z - \zeta| \leq 1\}$ we have

$$\begin{aligned}
|f_j(z)| &\leq C \int_{B_z} |f_j(\zeta)|\, d\lambda(\zeta) \\
&= C \int_{B_z} |f_j(\zeta)| \exp(-\tau_1 p(\zeta)) \exp(\tau_1 p(\zeta))\, d\lambda(\zeta)
\end{aligned}$$

$$
\begin{aligned}
&\le C \int_{B_z} |f_j(\zeta)| \exp(-\tau_1 p(\zeta))\, d\lambda(\zeta)\ \sup\{\exp(\tau_1 p(\zeta)) : \zeta \in B_z\} \\
&\le C' \int_{\mathbb{C}^n} |f_j(\zeta)| \exp(-\tau_1 p(\zeta))\, d\lambda(\zeta)\ \exp(\tau_2 p(z)),
\end{aligned}
$$

where we used the properties of the weight function p.

The third inequality is a consequence of the Cauchy-Schwarz inequality:

$$
\begin{aligned}
&\int_{\mathbb{C}^n} |f_j(\zeta)| \exp(-\tau_1 p(\zeta))\, d\lambda(\zeta) \\
&\quad = \int_{\mathbb{C}^n} |f_j(\zeta)| \exp(-\tau p(\zeta)) \exp((\tau - \tau_1) p(\zeta))\, d\lambda(\zeta) \\
&\quad \le \left[\int_{\mathbb{C}^n} |f_j(\zeta)|^2 \exp(-2\tau p(\zeta))\, d\lambda(\zeta) \right]^{1/2} \\
&\qquad\quad \left[\int_{\mathbb{C}^n} \exp((2\tau - 2\tau_1) p(\zeta))\, d\lambda(\zeta) \right]^{1/2} .
\end{aligned}
$$

By Hörmander's L^2-estimates ([8], Theorem 4.4.2]) we have for $\tau < \tau_3 < \sigma$ and the properties of the weight function

$$
\begin{aligned}
&\int_{\mathbb{C}^n} |S_1(f)(z)|^2 \exp(-2\sigma p(z))\, d\lambda(z) \\
&\quad \le \int_{\mathbb{C}^n} |S_1(f)(z)|^2 \exp(-2\tau_3 p(z))\, (1 + |z|^2)^{-2}\, d\lambda(z) \\
&\quad \le \int_{\mathbb{C}^n} |f(z)|^2 \exp(-2\tau_3 p(z))\, d\lambda(z).
\end{aligned}
$$

Here we used the fact that the canonical solution operator S_1 can be written in the form $S_1(f) = v - P(v)$, where v is an arbitrary solution to $\overline{\partial} u = f$ belonging to the corresponding Hilbert space and that $\|S_1(f)\| = \|v - P(v)\| = \min\{\|v - h\| : h \in A^2\} \le \|v\|$.

Now choose τ_2 such that $\tau < \tau_2 < \tau_3 < \sigma$, then we obtain from the above inequalities

$$
\begin{aligned}
&\left[\int_{\mathbb{C}^n} |S_1(f)(z)|^2 \exp(-2\sigma p(z))\, d\lambda(z) \right]^{1/2} \\
&\quad \le D \int_{\mathbb{C}^n} |f(z)|_1 \exp(-\tau_2 p(z))\, d\lambda(z),
\end{aligned}
$$

where $D > 0$ is a constant and $|f(z)|_1 := |f_1(z)| + \ldots + |f_n(z)|$.

Now define for $z \in \mathbb{C}^n$ and $\tau < \tau_1 < \tau_2$

$$\delta_z^{\tau_1,\tau}(f_j) := C_{\tau_1,\tau}^{-1}\, f_j(z)\, \exp(-\tau_1 p(z)).$$

Then

$$\begin{aligned} & C_{\tau_1,\tau}^{-1}\, \sup\{|f_j(z)|\, \exp(-\tau_1 p(z)) \, : \, z \in \mathbb{C}^n\} \\ \leq \; & \left[\int_{\mathbb{C}^n} |f_j(z)|^2 \exp(-2\tau p(z))\, d\lambda(z)\right]^{1/2}, \end{aligned}$$

which, by the Riesz representation theorem for the Hilbert space $A^2_{(0,1)}$ $(\mathbb{C}^n, \tau p)$, means that each $\delta_z^{\tau_1,\tau}$ can be viewed as an element of $\mathcal{U}$.

For $\phi \in \mathcal{C}(\mathcal{U})$ the expression

$$\mu(\phi) = D \int_{\mathbb{C}^n} \phi(\delta_z^{\tau_1,\tau})\, \exp((\tau_1 - \tau_2)p(z))\, d\lambda(z)$$

defines a Radon measure on the weakly compact set $\mathcal{U}$.

This follows from the fact that

$$\mu(\phi) \leq D\, \sup\{|\phi(g)| \, : \, g \in \mathcal{U}\} \int_{\mathbb{C}^n} \exp((\tau_1 - \tau_2)p(z))\, d\lambda(z).$$

Now take for ϕ the continuous functions $\phi_j(g_j) = |(f_j, g_j)|$, where f_j is fixed. Then

$$\begin{aligned} & \int_{\mathcal{U}} |(f,g)|\, d\mu(g) \\ & \quad = \; D \int_{\mathbb{C}^n} |f(z)|_1\, \exp(-\tau_1 p(z))\, \exp((\tau_1 - \tau_2)p(z))\, d\lambda(z) \\ & \quad = \; D \int_{\mathbb{C}^n} |f(z)|_1\, \exp(-\tau_2 p(z))\, d\lambda(z) \end{aligned}$$

and hence

$$\left[\int_{\mathbb{C}^n} |S_1(f)(z)|^2 \exp(-2\sigma p(z))\, d\lambda(z)\right]^{1/2} \leq \int_{\mathcal{U}} |(f,g)|\, d\mu(g). \qquad \square$$

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